Adaptive Regulation with Flow and Stock Externalities

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Abstract

In confronting a consumer good whose production process is associated with both flow and stock externalities, a corrective tax is introduced to restore efficiency. The objective is to maximize social welfare over time when the stock pollutant obeys an arbitrary dynamic process. The model makes it possible to derive the optimal corrective tax as a closed form feedback control law. This feedback rule can be applied for qualitative purposes such as parameter analysis or studying the time path of the corrective tax. It can also be used for quantitative purposes, for example evaluating an actual policy or assessment of the optimal tax for a certain case. It is here used to study how the optimal corrective tax, both as a function of time and as a function of the pollution level, depends upon the decay function. It is shown that, depending upon the initial conditions and the structure of the economy and the decay function, most outcomes are possible.

Keywords: Pigovian taxation, flow and stock externalities.

1. Introduction[1]

Global warming through emission of CO_2 in the atmosphere, the thinning of the ozone layer, toxic waste, etc., are all examples of externalities in the economy. There exists already a vast literature on externalities that at least dates back to Pigou who was one of the first to suggest a corrective tax which could restore efficiency. Most of the literature on pigovian taxation, however, is about flow externalities whose harmful effects dissipate more or less immediately and not about stock externalities whose harmful effects remain for a long period. CO_2, thinning of the ozone layer due to chlorofluorocarbons, toxic waste and pollution in general are all examples of stock externalities. In fact, it is hard to think of examples of pure flow externalities, except noise and strong light, as most physical emissions tend to accumulate to some extent.

Stock externalities are obviously important and problems of stock externalities were addressed already in the early 1970s (Keeler, Spence and Zeckhauser, 1971, d'Arge and Kogiku, 1973, and Forster, 1975). With increasing concern

[1]Submitted to *Natural Resource Modeling*.

about global warming due to accumulation of CO_2 in the atmosphere there has emerged a literature on the economics of accumulation of so-called greenhouse gases which is a typical example of a stock externality. Nordhaus (1982) was one of the first to address this area in the economic literature by asking how fast the global economy should allow a buildup of atmospheric CO_2. Brito and Intriligator (1987) addressed the problem of steady state analysis in the presence of stock externalities and the use of a constant pigovian tax. Ko, Lapan and Sandler (1992) examine the use of an inflexible pigovian tax compared to a first-best path of a pigovian tax that varies continuously. Nordhaus (1991a) has formulated an economic model that links emissions of greenhouse gases and climate changes and analyzes the trade-off between the cost of reducing emissions and the damage from global warming. Wirl (1994a) uses a Nordhaus model and discusses the time path for taxation of energy in the presence of flow and stock externalities as part of his study of the dynamic and strategic interactions between producers and consumers of fossil energy. Peck and Teisberg (1992) determine optimal time paths of emissions control and corresponding carbon taxes under alternative assumptions about global warming costs. Their main result is that an optimal carbon tax will tend to increase monotonically over time. Sinclair (1992 and 1994) and Ulph and

Ulph (1994) consider the optimal time path of a carbon tax when this is linked to the extraction of fossil fuels as nonrenewable resources and discusses whether this tax should rise or decrease over time. They find that in most cases the rule is to let the tax decrease over time in order to postpone extraction.

In most of the literature on pollution a monotonically increasing decay function is applied. Among the exceptions are Forster (1975) who uses a nonconstant exponential decay function and Tahvonen and Salo (1996) who use a concave-convex decay function. These are, however, special cases of the decay function applied here which is completely general. Therefore, most of the results referred to above can be found as special cases of the model applied here.

The aim of the present paper is to provide an analytical expression for the optimal path of the corrective tax in the presence of a general decay function. The optimal path for the pigovian tax in the presence of both flow and stock externalities is derived as a feedback control. Feedback controls have several advantages. They are relatively easy to implement as the optimal tax is a function of the current level of pollution only, and sporadic shocks, e.g. sudden uncontrolled emissions, are automatically taken care of. Analytical expressions are emphasized in order to allow for parameter analysis and avoid the drawbacks of numerical analysis such as aggregation of numerical errors. The model can be used quantitatively both to assess the optimal corrective tax and to evaluate policies that are already in effect. Examples of assessment of the tax for particular cases are shown. In this paper the model is also applied qualitatively to see how the optimal tax, as a function both of time and of the pollution level, depends upon the decay function. It is therefore important for us to be able to use general decay functions.

2. The model

In the standard textbook model for determining corrective taxes there is an inverse demand function for the product x, $P = P(x)$, a private marginal cost function representing supply, $MC^P = MC^P(x)$, and a social marginal cost function, $MC^S = MC^S(x)$. Market equilibrium is given by $P(x) = MC^P(x)$. The social welfare optimum is given by $P(x^*) = MC^S(x^*)$. This is the production level that maximizes the sum of consumers' and producers' surplus corrected for the difference between private and social costs (externality). The optimal corrective tax, τ, is given once and for all by $\tau^* = P(x^*) - MC^P(x^*)$. This is illustrated in Figure 1. If, however, each unit of x is associated with a certain amount of pollution, a, and the pollutant tends to accumulate, we are faced with a dynamic optimization problem and static analysis is no longer sufficient.

Consider a sector that produces the output x to which there is a fixed quantity of pollution associated with each unit of output given by δx. The aggregate level of pollution is denoted a, and the time change in a is given by $\dot{a} = \delta x - f(a)$ where f is a general, but known, decay function.[2]

Assume that the inverse compensated demand function for x can be approximated by

$$P(a,x) = p_0(a) - p_1(a)x, \;\; p_i \geq 0, \; i \in \{0,1\}\,. \tag{2.1}$$

In this paper P will always represent the consumer price. Henceforth, the term demand function and demand curve will refer to the function in Eq.(2.1).

Further, assume that the private and social marginal costs of production can be approximated by

$$\begin{aligned} MC^S(a,x) &= c_{S0}(a) + c_{S1}(a)x, \;\; c_{\mathrm{Si}} \geq 0, \\ MC^P(a,x) &= c_{P0}(a) + c_{P1}(a)x, \;\; c_{Pi} \geq 0, \end{aligned} \tag{2.2}$$

and MC^P will always represent the producer price. The parameters c_{Si}, c_{Pi} and p_i

[2]Dots are used to denote time derivatives.

can be completely general functions in a. The social marginal costs of producing x are assumed higher than the corresponding private costs, i.e., $c_{s0} > c_{p0}$ and $c_{s1} > c_{p1}$. In other words, we are only dealing with negative externalities of production in this paper. Positive externalities can be analyzed accordingly.

There are many reasons why the parameters in the demand function or in the private or social cost functions may depend upon the level of stock pollution. The demand for a product depends on many factors, also pressure exerted by environmentalist groups. This pressure will probably - if anything - lead to a downward shift in demand for products associated with emission of pollution. It is also reasonable to think that the concern about the environment is greater the higher the level of the stock pollutant, and thus the downward shift in demand will be greater the more pollution there is already. Therefore the constant term in the demand function, p_0 in Eq.(2.1), can be modelled as a decreasing function in a.

Private costs of production may also depend upon the level of pollution. This relationship can be highlighted through an example: If clean water is needed in the production process, then the cost of purification may be higher the higher the stock of pollution. This may affect both the slope and position of the marginal cost curve. If anything, these terms, i.e. the parameters c_{p0} and c_{p1} in Eq.(2.2), will probably be increasing functions in a. Private costs will probably be decreasing as a function of emissions of pollution. Higher emission levels mean lower costs as some costs associated with purification are avoided. It is important to note, however, that this regards costs as a function of $\dot{a}$ and not as a function of a.

The social cost function is the cost to society of producing the product when external effects are taken into account. The difference between the private and the social cost functions can therefore be interpreted as a measure of the external effects. The private costs are partly included in the social costs, and these may depend positively upon the stock of pollution. It is reasonable to believe that the external effects of pollution will be higher the higher the level of pollution. Therefore it is assumed here that the parameters in the social cost function, c_{s0} and c_{s1} in Eq.(2.2), will be increasing functions in a. The assumptions above are made only for the economic interpretation and are not necessary for mathematical purposes. The feedback rule derived in this paper can be implemented no matter how the functions involved depend upon a.

Flow externality is defined in monetary terms as the difference between private and social costs at any point in time, that is for a fixed level of a and x. It is the externality associated with δx, and it exists also when a is fixed at a certain level and treated as a parameter. For any given level of a and x the magnitude of the flow externality is $MC^s - MC^p$.

Stock externality is defined as the externality in the dynamic setting that comes in addition to the flow externality. In this paper $D(a) \geq 0$ is used to denote the stock externality, that is, the disutility or damage associated with the stock pollutant. It is clear from these definitions that flow externality is a static concept whereas stock externality is a dynamic concept. If there are no dynamics, there is no stock externality and only flow externality exists.

Both flow and stock externalities can be internalized in the dynamic setting by implementing a corrective tax, τ, representing the difference between the consumer price and the producer price:

$$\tau(a, x) = P(a, x) - MC^P(a, x). \tag{2.3}$$

This corrective tax ensures market equilibrium for different combinations of a and x. The question, to which we will return in the Section 3, is how to determine the *optimal* corrective tax.

The omniscient, benevolent government's objective function is to maximize the social welfare function given by the sum of the consumers' surplus, CS, the producers' surplus, PS, and the government's surplus, GS. The consumers' surplus is defined as the difference between what the consumers actually pay (taxes included) and what they would be willing to pay under the demand curve. The producers' surplus is defined as total revenue for the producers less total private cost. The government's surplus take into account tax revenues, the difference between social and private costs and any disutility, D. These surpluses are formally defined as:

$$\begin{aligned} CS &= \int_0^x P(a,z)dz - (MC^P(a,x)+\tau)x \equiv \hat{P}(a,x) - (MC^P(a,x)+\tau)x, \\ PS &= MC^P(a,x)x - \int_0^x MC^P(a,z)dz \equiv MC^P(a,x)x - M\hat{C}^P(a,x), \\ GS &= -D(a) + \tau x - \int_0^x [MC^S(a,z) - MC^P(a,z)]dz \\ &\equiv -D(a) + \tau x - \left[M\hat{C}^S(a,x) - M\hat{C}^P(a,x)\right] \end{aligned} \tag{2.4}$$

where z is an integration variable and hats are used to denote integrals as indicated. The benevolent government wishes to maximize the sum of all these surpluses. In so doing many of the terms cancel out and what remains as the social welfare function is

$$W(a,\tau) = -D(a) + \hat{P}(a,x) - M\hat{C}^S(a,x). \tag{2.5}$$

When it comes to practical use of the model, it is important to specify the appropriate a-dependence correctly and avoid double counting. For example, inclusion of a in MC^s and MC^p only shows how the flow externality may be influenced by a and must not be confused with the stock externality, $D(a)$.

It is evident from Eqs. (2.1), (2.2) and (2.3) that x can be written as a linear function in τ :

$$x(a,\tau) = x_0(a) - x_1(a)\tau \tag{2.6}$$

where

$$x_0(a) = \frac{p_0 - c_{P0}}{p_1 + c_{P1}} > 0, \quad x_1(a) = \frac{1}{p_1 + c_{P1}} > 0.$$

Eq. (2.6) defines x as a function of τ given that $\tau = P - MC^P$, that is τ values that result in market equilibrium, see Figure 1. The term x_0 can be interpreted as the market equilibrium without any policy measures. Major definitions in this article, such as the definitions of x_0 and x_1 above, are summarized in Appendix 1. We are only interested in non-negative levels of production, meaning that τ has an upper bound given by $\tau \leq x_0/x_1$.

The dynamic optimization problem can be formulated as

$$\max_{\tau} \int_0^{\infty} e^{-rt} W(a, \tau) dt \tag{2.7}$$

subject to the constraint

$$\dot{a} = \delta x - f(a) \tag{2.8}$$

and $a \geq 0$, $x \geq 0$. The discount rate, r, is the social discount rate. The case of a zero discount rate is of particular interest and will be given special attention.

In order to simplify the following calculations and interpretations it will be preferable to change the scale along which a is measured from physical units to monetary units. This is done as follows: a is measured by $A(a)$ where A is a known, monotone but not necessarily linear rescaling function. By definition $\dot{A} = A'(a)\dot{a}$ and we choose A to have the property $A' = 1/\delta x_1 > 0$. These two measures of pollution will be used interchangeably in the rest of the paper, so that W can be a function of a or A. The variable A is interpreted as pollution in monetary terms whereas a is the pollution measured in physical units. This means that the dynamic problem can be rewritten

$$\max_{\tau} \int_0^{\infty} e^{-rt} W(A, \tau) dt \tag{2.9}$$

subject to

$$\dot{A} = F(A) - \tau \tag{2.10}$$

where $F = x_0/x_1 - f/\delta x_1$. Eq. (2.10) represents a translation of the dynamic constraint into monetary terms such that F and τ are comparable. The function F is the actual increase in pollution, A, without any policy measures, and this can be either positive or negative. The policy measure, τ, is negatively related to the change in pollution, and $F - \tau$ therefore represents the net change in pollution when policy measures are applied. This can also be seen by writing

$$F(A) = \frac{\delta x_0(a) - f(a)}{\delta x_1(a)} \tag{2.11}$$

remembering that a is a function $a(A)$. By comparing the numerator, $\delta x_0 - f$, with (2.8) and recalling the definition of x_0, it is seen that the numerator is the actual change in pollution in physical terms without policy measures. Dividing by δx_1 we get the actual change in pollution in monetary terms. Note that F

and f may have opposite signs, and usually have. The intuition behind this is that f measures the actual decay of pollution. If this decay is large, it calls for a relatively small corrective tax because only a minor reduction in output is needed and vice versa.

From Eqs. (2.4), (2.5) and (2.6) it is evident that after the rescaling of a into A, W can be written as a function in A and τ as follows:

$$W(A,\tau) = -\alpha(A) + \beta(A)\tau - \gamma(A)\tau^2 \tag{2.12}$$

where

$$\begin{aligned}
\alpha &= D + \frac{1}{2}x_0(x_0p_1 + x_0c_{s1} - 2p_0 + 2c_{s0}), \\
\beta &= \frac{(c_{s0} - c_{p0})(p_1 + c_{p1}) + (p_0 - c_{p0})(c_{s1} - c_{p1})}{(p_1 + c_{p1})^2} > 0, \\
\gamma &= \frac{(p_1 + c_{s1})}{2(p_1 + c_{p1})^2} > 0.
\end{aligned} \tag{2.13}$$

This will be used in the following to derive the optimal corrective tax.

3. The feedback control law

The objective of this section is to derive the optimal corrective tax, τ, as a feedback control law; that is, as an explicit function of the pollution level, A. The easiest way to do this is to go by dynamic programming, see e.g. Kamien and Schwartz (1991). With a constant social discount rate, the problem of maximizing (2.9) subject to (2.10) yields the value function

$$V(t,A) = \max \int_t^T e^{-rs} W(A,\tau) ds \tag{3.1}$$

and the associated Jacobi-Bellman equation[3]

$$V_t + \max_\tau \left\{ \left[-\alpha(A) + \beta(A)\tau - \gamma(A)\tau^2 \right] e^{-rt} + V_A \left[F(A) - \tau \right] \right\} = 0, \tag{3.2}$$

Let us try a solution of the form

$$V(t,A) = -K + \left[\frac{W_0}{r} + \psi(A) \right] e^{-rt}, \quad \psi(A(T)) + \frac{W_0}{r} = Ke^{rT}, \quad \tau(A) = \frac{\beta(A) - \psi'(A)}{2\gamma(A)}$$

[3] $V_j \equiv \frac{\partial V}{\partial j}$.

where K and W_0 are arbitrary constants. Further, use the following relationships and definitions:

$$\begin{aligned} V_A = \psi'(A) &\equiv 2\gamma(A)\left[\tau^*(A) - \tau(A)\right], \\ W_0 - S(A) &= \gamma(A)\left[F(A) - \tau_0(A)\right]^2, \\ V_A - \mu(A) &= 2\gamma(A)\left[F(A) - \tau(A)\right] \end{aligned} \tag{3.3}$$

where $\tau^* = \frac{\beta}{2\gamma}$ is the optimal tax in the static equivalent of the model derived from maximizing W with respect to τ treating A as a constant. Let S be defined as the level of welfare associated with sustaining any level of pollution, that is,

$$S(A) \equiv W(A, F(A)) = -\alpha(A) + \beta(A)F(A) - \gamma(A)F(A)^2 \tag{3.4}$$

and μ be defined as

$$\mu(A) \equiv \left.\frac{\partial W(A,\tau)}{\partial \tau}\right|_{\tau=F(A)} = \beta(A) - 2\gamma(A)F(A).$$

Economically μ can be interpreted as the marginal increase in welfare due to a marginal change in the tax given that the present level of pollution is preserved. The term τ_0 is defined as the optimal corrective tax when the discount rate is zero, and W_0 represents a certain target value equivalent to the welfare in optimal steady state to be derived soon. Inserted into the Jacobi-Bellman equation (3.2) this yields

$$r\psi(A) = \gamma(A)\left[\tau(A) - \tau_0(A)\right]\left[\tau(A) + \tau_0(A) - 2F(A)\right]. \tag{3.5}$$

In the special case when either $r = 0$ or $\psi = 0$,

$$\tau(A) = \begin{cases} \tau_0(A) \\ 2F(A) - \tau_0(A) \end{cases}, \quad \tau_0(A) = F(A) + \sqrt{\frac{W_0 - S(A)}{\gamma(A)}}. \tag{3.6}$$

In the general case it is necessary to apply perturbation methods in order to derive analytical expressions for the feedback control law, see Appendix 2. The feedback rule derived in Appendix 2 for the case of a positive discount rate is the following

$$\begin{aligned} \tau(A) &= \tau_0(A) + r\,\tau_1(A) + O(r^2), \\ \tau_1(A) &= \frac{\psi_0(A)}{2\gamma(A)\left[\tau_0(A) - F(A)\right]}, \; \psi_0'(A) = -2\gamma(A)\left[\tau_0(A) - \tau^*(A)\right] \end{aligned} \tag{3.7}$$

solved to second order. The term τ_0 from (3.6) is the optimal tax with zero discounting which will be given special attention in the next section. The correction

$r\tau_1$ will usually be small compared to τ_0

One question remains, namely what is the optimal steady state (or target) level of the state variable. With an infinite time horizon this is given by solving

$$S'(A) = r\mu(A) \tag{3.8}$$

with respect to A. The validity of (3.8) can be seen from (3.3) and (3.5) by noting that in steady state $F = \tau$ such that $V_A = \psi' = \mu$. Eq. (3.5) is an integrated version of (3.8). The interpretation of (3.8) is that the marginal benefit of a unit of A should in optimum be the same whether it is left alone (left hand side) or it is taxed away (right hand side).

3.1. The case of zero discounting

The case of a zero social discount rate is of particular interest in this context because then the social welfare of future generations will not be less important than the welfare of the present generation. Furthermore, all the important features of this approach are clearly seen in the case of zero discounting. Sensitivity analysis also shows that the optimal time path for τ is not very sensitive to changes in the discount rate whereas it is quite sensitive to changes in the degree of nonlinearity through changes in γ.

An analytical expression for the optimal corrective tax at any point in time is given by

$$\tau(A) = F(A) \pm \sqrt{\frac{W_0 - S(A)}{\gamma(A)}} \tag{3.9}$$

from (3.6) where W_0 is a constant. With an infinite time horizon, the appropriate choice of W_0 is $\max S(A)$. The constant W_0 is the target level of welfare derived from (3.8) when $r = 0$. In other words, the optimal steady state with zero discounting is determined as the global maximum of S, assuming that an interior solution exists. By substitution, it can easily be confirmed that the rule applied here conforms to the Pontryagin maximum principle. In Eq. (3.9), the solution with the plus sign, making $\dot{A} = F - \tau < 0$, is chosen in (3.9) when A is higher than the optimal steady state level, and the solution with the minus sign is chosen when A is below the optimal steady state.

As Eq.(3.9) represents a feedback control, it is an easy task to set the optimal corrective tax at any point in time as a function of the current level of pollution when the compensated demand function as well as the private and social cost functions and the decay function of pollution are known. In this model τ is a continuous function. How often τ should be updated in practical policy depends on the conventions of the fiscal institutions and on how often new estimates for the pollution level are available. Along an optimal path the time change in A corresponding to (3.9) is given by

$$\dot{A} = \mp\sqrt{\frac{W_0 - S(A)}{\gamma(A)}} \tag{3.10}$$

Equations (3.9) and (3.10) describe the phase plane for any value of A no matter how far it is from the optimal steady state.

Eq.(3.9) shows that the question whether a carbon tax should increase or decrease over time, which has been discussed by Peck and Teisberg (1992), Sinclair (1992 and 1994) and Ulph and Ulph (1994), depends totally on the characteristics of the decay function in combination with the economic parameters in the model. We will return to this question in more detail in the next section.

There are still some interesting characteristics to note about the structure of the model. Note, for example, that for any curve, $S(A)$, a local maximum corresponds to a saddle point steady state, a local minimum corresponds to a center and an inflection point corresponds to a cusp in the phase-space. Thus the complete topology of the phase-space is given by the S-curve. If W_0 is given by the global maximum of the S-curve, as it is with zero discounting, then the steady state corresponding to this is a saddle point and therefore by nature unstable. This makes the feedback rule even more useful. Given that the pollution level corresponding to W_0 is known and that the current level of pollution is known by monitoring, then it is an easy task to set the corrective tax at the optimal level that eventually leads to the welfare optimum.

Also sporadic distortions and shocks become less important with a feedback control. Exogenous distortions will automatically be taken care of no matter whether these distortions take place far from the optimal steady state or locally around it. The model is just as applicable far away from steady state as it is locally.

Although it is obvious that the optimal policy derived from static analysis is not globally optimal when the true model is dynamic, it is often believed that a static policy is approximately correct very close to the optimal steady state. Unfortunately, this is not the case as both the optimal path and the optimal steady state are different in the two approaches. A quasi-dynamic policy based on $\tau(A) = \frac{\beta(A)}{2\gamma(A)}$, where A is treated as a parameter that is continuously updated, will never lead to the steady state found by dynamic optimization.

4. Qualitative analysis: The time path of the corrective tax

The purpose of the present section is to make a complete analysis of the time path of the optimal corrective tax, that is; what factors does it depend on and how. In particular the following question is asked: How does the time path of the optimal tax look, and how does it depend upon the initial situation and the decay function. This question has also been raised by Peck and Teisberg (1992), Sinclair (1992 and 1994) and Ulph and Ulph (1994) using somewhat similar models.

As the optimal corrective tax can be written as a function of the pollution level, $\tau(A)$, the time derivative is

$$\dot{\tau} = \tau'(A)\dot{A}.$$

As $\dot{A} > 0$ when we are below the optimal steady state and vice versa, it is important to determine the sign of τ' on each side of the optimal steady state. In the following we look at the case with zero discounting, and where γ in Eq. (2.12) does not depend upon the pollution level, A. This corresponds to the case where A-dependence can only occur through the disutility term, D, or through the intercepts of the demand and marginal cost curves (2.1) and (2.2), not through the slope of these curves. The derivative can then be written (recall (3.10))

$$\tau'(A) = F'(A) + \frac{S'(A)}{2\gamma \dot{A}}. \tag{4.1}$$

In the case of zero discounting optimal steady state is determined by $S' = 0$, and $W_0 = S(A^*)$ where A^* is the optimal steady state level of pollution. The time derivative of τ is given by

$$\dot{\tau} = F'(A)\dot{A} + \frac{S'(A)}{2\gamma}.$$

The sign of $\dot{\tau}$ depends upon the sign of the three terms F', S' and $\dot{A}$. Whether $\dot{A}$ is positive or negative depends on whether we are above or below the optimal steady state level. Thus we have eight possibilities to investigate. These are listed in the following schedule.

1) $\dot{A} > 0$

$$a)\ S' > 0 \ \&\ F' > 0 \Rightarrow \dot{\tau} > 0$$

$$b)\ S' > 0 \ \&\ F' < 0 \ \&\ W_0 - S \gtrless \frac{1}{\gamma}\left(\frac{S'}{2F'}\right)^2 \Rightarrow \dot{\tau} \lessgtr 0$$

$$c)\ S' < 0 \ \&\ F' > 0 \ \&\ W_0 - S \gtrless \frac{1}{\gamma}\left(\frac{S'}{2F'}\right)^2 \Rightarrow \dot{\tau} \gtrless 0$$

$$d)\ S' < 0 \ \&\ F' < 0 \Rightarrow \dot{\tau} < 0$$

2) $\dot{A} < 0$

$$a)\ S' > 0 \ \&\ F' > 0 \ \&\ W_0 - S \gtrless \frac{1}{\gamma}\left(\frac{S'}{2F'}\right)^2 \Rightarrow \dot{\tau} \lessgtr 0$$

$$b)\ S' > 0 \ \&\ F' < 0 \Rightarrow \dot{\tau} > 0$$

$$c)\ S' < 0 \ \&\ F' > 0 \Rightarrow \dot{\tau} < 0$$

$$d)\ S' < 0 \ \&\ F' < 0 \ \&\ W_0 - S \gtrless \frac{1}{\gamma}\left(\frac{S'}{2F'}\right)^2 \Rightarrow \dot{\tau} \gtrless 0$$

From the schedule above it is seen that there are three factors affecting the time path of the corrective tax: the sign of S', the sign of F' and whether the difference between W_0 and S is greater or smaller than some value given by $\frac{1}{\gamma}\left(\frac{S'}{2F'}\right)^2$. How are these factors interpreted? It is useful to start by recalling the interpretations of S and F. The expression for S is given by (3.4), and it is interpreted as the sustainable level of social welfare. The maximum sustainable welfare (MSW) is W_0. The sustainable welfare can be both increasing and decreasing in A on either side of the optimal steady state due to the nonlinearity.

The expression for F is given by (2.11). From this it is seen that whether F' is positive or negative depends upon whether $\delta x_0' - f'$ is positive or negative. Recall that $\delta x_0 - f$ is the actual increase in pollution without policy measures as x_0 is the production level without policy measures (under market equilibrium). Therefore δx_0 is the increase in pollution determined from the market equilibrium production level whereas f is the natural decay. In the typical case that private costs are shifted upward and demand is shifted downward with the pollution level, $x_0' < 0$. In this case there are three forces working towards reduced emissions of pollution when the level of pollution increases, the market, nature itself and the policy measure. If $f' > 0$ then $F' < 0$. The requirement for $F' > 0$ is that

$f' < 0$ and $\delta x_0' > f'$. As x_0' typically is negative, $F' > 0$ requires that the natural decay decreases with the level of the stock pollutant and that the market is less sensitive to a change in the pollution level than the environment. An example of pollution with a decreasing decay functions is acid rain where the natural purification process becomes less efficient the higher the pollution level.

This enables us to interpret the different cases in the schedule above. Case 1a) is interpreted as follows. If the initial level of pollution is below the optimal steady state, sustainable welfare is increasing in the pollution level and the free market emission level is less sensitive to changes in pollution than the environment, the corrective tax should definitely be increasing over time. Case 1b) tells us that if the initial level of pollution is below the optimal steady state, sustainable welfare is increasing in the pollution level and the free market emission level is more sensitive to changes in pollution than the environment, the corrective tax should only be increasing over time if the difference between MSW and the present sustainable welfare level, $W_0 - S$, is less than a certain value. This value can be calculated as $\frac{1}{\gamma}(\frac{S'}{2F'})^2$. The distance $W_0 - S$ can be interpreted as a measure of how far we are from the optimum, or rather how much one can gain from approaching optimum. The remaining six cases can be interpreted accordingly.

How does the interpretation of these cases confirm with intuition? Note that when S' goes from negative to positive, everything else equal, this calls for an increase in $\dot{\tau}$. In other words, if the sustainable welfare is increasing in the pollution level, the optimal tax should be more increasing over time than in the opposite case. This result is rather counterintuitive. Next we note that when F' goes from negative to positive, everything else equal, this calls for an increase in $\dot{\tau}$ when the pollution level is below the steady state level and a decrease in $\dot{\tau}$ when the pollution level is above steady state. In other words, when the decay decreases with the level of pollution, and is more sensitive to the pollution level than the market is, the optimal corrective tax should increase over time for small pollution levels and decrease over time for high pollution levels. This too seems somewhat counterintuitive.

Some of the cases in the schedule are more relevant than other. For example, if S is concave, as it certainly is in the neighbourhood of steady state, only the cases 1a) and b) and 2c) and d) are relevant. Assuming that S is concave, $\dot{\tau}$ is typically positive below optimal steady state and negative above optimal steady state. The only cases in which the opposite may happen is in 1b) and 2d) when $W_0 - S > \frac{1}{\gamma}\left(\frac{S'}{2F'}\right)^2$, that is when F' is negative and large in magnitude. However, to say that F' is negative and large in magnitude is to say that either x_0' is large negative or f' is large positive. If x_0' is large negative it means that the private market equilibrium without policy measures is quite sensitive to the level of pollution; that is, the equilibrium level of production decreases with the level of pollution due to decreased demand (environmental awareness) or higher production costs. If f' is large positive it means that decay process is efficient. Both are examples of a case in which policy measures are not very essential because the net increase in emissions is reduced due to market and environmental effects. In other words, in some neighbourhood of the optimal steady state, the only case in which the tax will be increasing over time for pollution levels higher than the optimal one and decreasing over time for pollution levels lower than the optimal one, is when the tax plays a minor role.

A small positive discount rate will not change the qualitative conclusions in this section. With discounting $\dot{\tau} = \dot{\tau}_0 + r\dot{\tau}_1$ where $r\dot{\tau}_1$ represents a small perturbation. Only if $\dot{\tau}_0$ is close to zero, will the last term possibly affect the sign of $\dot{\tau}$, but if so, the corrective tax will be more or less constant anyway. If, however, γ is made A-dependent, the time path of the optimal tax becomes even more complex, and it becomes more difficult to draw general conclusions than with a constant γ.

5. Quantitative analysis: Some numerical examples

In this section some examples based on a stylized numerical model is given for illustrative purposes. In particular we look at how the optimal tax changes as a function of the pollution level, A, with different decay functions. It is assumed that 0.1 units of pollution are associated with each unit of production, that is $\delta = 0.1$. People are more concerned about the environment the higher the aggregate level of pollution such that demand for this product is shifted down as the aggregated level of pollution rises. Private marginal costs are shifted up as the level of pollution rises. Social marginal costs are both shifted up and become steeper as the level of pollution rises. The numerical specification is given by:

$$
\begin{aligned}
p_0 &= 3 - a, \quad p_1 = 1, \\
c_{p0} &= \frac{1}{2} + a, \quad c_{p1} = \frac{3}{2}, \\
c_{S0} &= 1 + a, \quad c_{S1} = \frac{3 + a}{2}.
\end{aligned}
$$

With these assumptions, private equilibrium is given by $x = 1$ and social optimum by $x = 0.8$ when $a = 0$. When $a = 0.5$ these figures are $x = 0.6$ and $x = 0.36$ respectively, and when $a = 1$ they are $x = 0.2$ and $x = 0$. The transformation of

a into A is given by $a = A/25$ such that $0 \leq A \leq 25$ corresponds to $0 \leq a \leq 1$.

Three different possibilities for the function $f(a)$ are examined and the results are illustrated in Figures 2 - 4. In Figure 2 f is given by an equation similar to the normal distribution, that is

$$f(a) = \frac{1}{10\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{a-\mu}{\sigma})^2}$$

where $\mu = 0.5$ and $\sigma = 0.25$. The optimal corrective tax given by (3.9) and the corresponding production level, x, as functions of A are illustrated. The optimal a-level is 0.13 and $\max S = S(0.13) = 0.61$. Note the u-shaped form of the optimal tax, and that there is no production when a is greater than 0.72 ($A > 18$). As negative production is impossible, $x = 0$ for $a > 0.72$. Any prohibitive tax will do when $a > 0.72$.

Figure 3 illustrates the optimal tax and corresponding production when $f(a)$ is linearly increasing, $f = 0.16a$. For the purpose of comparability all f-curves examined here have a maximum of 0.16 in the range $0 \leq a \leq 1$. With this choice of f, the optimal a-level is 0.24 and $S(0.24) = 0.39$. Note that in this case the optimal corrective tax is monotonically decreasing in the range $0 \leq a \leq 1$ and production is choked when $a > 0.84$ ($A > 21$).

The third case examined here is the case of a logistic f-function given by $f = 0.64a(1-a)$. This case is illustrated in Figure 4. The optimal a-level is 0.1 and $S(0.1) = 0.61$. In this case the optimal corrective tax is monotonically increasing and a prohibitive tax is needed when $a > 0.75$.

Depending upon the shape of the f-curve, the optimal corrective tax may be u-shaped, decreasing or increasing in a. However, in all cases examined here, the corresponding optimal production level, x, is a decreasing function in a. All illustrations here are based upon zero discounting. The optimal tax is not very sensitive to changes in the discount rate, and the positions of the curves are virtually unchanged for reasonable values of the social discount rate, r.

6. Conclusions

The model described in this article is designed to analyze the dynamic problem of maximizing social welfare from the production and consumption of a consumer good whose production process is associated with an externality, in this case emission of pollution. Social welfare is defined as the sum of the consumer and producer surplus from the production, corrected for the externality associated with the emission of pollution, the flow externality, and corrected for any further disutility associated with the stock pollutant, the stock externality. The stock of pollution obeys a dynamic process by which the stock increases as a result of production of the good and decreases due to assimilation in the environment represented by a general, but known, decay function. A corrective tax is introduced to internalize the externalities and restore efficiency. It is possible to derive the optimal corrective tax as an explicit function (feedback control law) of the current stock of pollution and the discount rate.

In particular, it is found that the properties of the optimal corrective tax, both as a function of time and as a function of the pollution level, depend critically upon the shape of the decay function. To study whether there is decay or not is not sufficient, it is the shape of the decay as a function of the pollution that counts. Typically the corrective tax will be an increasing function of time for pollution levels less than the optimal steady state and a decreasing function of time for pollution levels greater than the optimal steady state. The opposite will usually only be the case when policy measures are not highly required because net accumulation of pollution to a large extent is reduced by increasing natural decay and by the response of the market.

7. Acknowledgments

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8. Appendix 1. List of major symbols and definitions.

This appendix gives brief definitions of some of the major symbols. For more thorough definitions, see the text.

x : Some product causing emissions

a : The stock pollutant measured in physical units

τ : A corrective (pigovian) tax

$D(a)$: Disutility caused by the presence of the stock pollutant ($D > 0$)

$f(a)$: The decay function for pollution

A : The stock pollutant measured in monetary units (cost of the stock pollutant)

$F(A)$: Actual increase ($F > 0$) or decrease ($F < 0$) in A in monetary terms

$P(a, x) = p_0(a) - p_1 x$: Inverse demand function for x

$MC^P(a, x) = c_{p0}(a) + c_{p1}(a)x$: Private cost of producing x

$MC^S(x) = c_{s0}(a) + c_{s1}(a)x$: Social cost of producing x

$x(a,\tau) = x_0(a) - x_1(a)\tau$: The market equilibrium with respect to x for a given corrective tax τ

$x_0 = \frac{p_0 - c_{P0}}{p_1 + c_{P1}} > 0$

$x_1 = \frac{1}{p_1 + c_{P1}} > 0$

$W(a,\tau) = -\alpha(a) + \beta(a)\tau - \gamma(a)\tau^2$: Social welfare from the production and consumption of x

$S(A) = -\alpha(A) + \beta(A)F(A) - \gamma(A)F(A)^2$: Sustainable welfare

$\mu = \frac{\partial W}{\partial \tau}\Big|_{\tau = F(A)} = \beta(A) - 2\gamma(A)F(A)$

$\alpha = D + \frac{1}{2}x_0(x_0 p_1 + x_0 c_{s1} - 2p_0 + 2c_{s0})$

$\beta = \frac{(c_{s0} - c_{p0})(p_1 + c_{p1}) + (p_0 - c_{p0})(c_{s1} - c_{p1})}{(p_1 + c_{p1})^2} > 0$

$\gamma = \frac{(p_1 + c_{s1})}{2(p_1 + c_{p1})^2} > 0$

9. Appendix 2.

In order to derive the expression for the optimal feedback control law in (3.7) it is necessary to introduce perturbation theory, see, e.g., Nayfeh (1973) for a more thorough introduction to this theory. This enables us to compare the implications of a constant discount rate with zero discounting for the nonlinear model and still get explicit analytical solutions. Thus it is not necessary to resort to numerical methods. In the following the same symbols denote both current values and present values. Straightforward perturbation yields the following scheme:

$$
\begin{aligned}
\psi = \psi_0 + r\psi_1 + r^2\psi_2 + \cdots,\ \ \tau = \tau_0 + r\tau_1 + r^2\tau_2 + \cdots, \\
\tfrac{d\psi_0}{dA} - \mu = \pm 2\sqrt{\gamma(W_0 - S)},\ \ \tau_0 = F \mp \sqrt{\tfrac{W_0 - S}{\gamma}}, \\
2\left[\tfrac{d\psi_0}{dA} - \mu\right]\tfrac{d\psi_1}{dA} = 4\gamma\psi_0,\ \ 2\gamma\tau_1 = -\tfrac{d\psi_1}{dA}, \\
2\left[\tfrac{d\psi_0}{dA} - \mu\right]\tfrac{d\psi_2}{dA} = 4\gamma\psi_1 - \left[\tfrac{d\psi_1}{dA}\right]^2,\ \ 2\gamma\tau_2 = -\tfrac{d\psi_2}{dA}, \\
2\left[\tfrac{d\psi_0}{dA} - \mu\right]\tfrac{d\psi_3}{dA} = 4\gamma\psi_2 - 2\tfrac{d\psi_1}{dA}\tfrac{d\psi_2}{dA},\ \ 2\gamma\tau_3 = -\tfrac{d\psi_3}{dA}.
\end{aligned}
\tag{9.1}
$$

Note that for any order the necessary correction is assessed using *at most* one direct integration of the solution to the previous order. Note also that the value function developed in the discount rate yields

$$
\begin{aligned}
V(t, A) = -K + \left[\tfrac{W_0}{r} + \psi(A)\right]e^{-rt} = -K + \left[\tfrac{W_0}{r} + \psi(A)\right][1 - rt + \cdot\cdot] \\
= -W_0 t + \hat{\psi}(A) + O(r).
\end{aligned}
$$

The constant term is included in $\hat{\psi}(A)$. To the lowest order this is identical to the value function for the case with zero discounting yielding the rule in Eq.(3.9).

The general solution to order three is given by

$$\begin{aligned}\tau &= \tau_0 + \frac{1}{2\gamma(\tau_0 - F)}\left[r\psi_0 + r^2\psi_1 - \gamma r^2 \tau_1^2\right] + O(r^3), \\ \tau_1 &= \frac{\psi_0}{2\gamma(\tau_0 - F)},\ \psi_0' = -2\gamma(\tau_0 - \tau^*),\ \psi_1' = -\frac{\psi_0}{\tau_0 - F}.\end{aligned} \tag{9.2}$$

Usually, however, the desired order is low, i.e. less than three, and therefore the solution to second order is applied in the text body.

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10. Figure legends

Figure 1. Static analysis: The optimal corrective tax, τ^*, the corresponding production level, x^*, and the market equilibrium, x_0, for a given pollution level, a, when P is inverse demand, MC^P is the private marginal cost function and MC^S is the social marginal cost function.

Figure 2. The optimal corrective tax and the corresponding production level as functions of A when f is similar to a normal distribution with $\mu = 0.5$ and $\sigma = 0.25$ divided by ten. $A = 25a$.

Figure 3. The optimal corrective tax and the corresponding production level as functions of A when f is linearly increasing, $f = 0.16a$. $A = 25a$.

Figure 4. The optimal corrective tax and the corresponding production level as functions of A when f is given by the logistic function $f = 0.64a(1-a)$. $A = 25a$.