

American Derivatives - a review

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Abstract

The paper gives an overview over the theory of pricing and hedging financial derivatives that can be exercised at any time during a fixed time interval $[0, T]$. The analysis makes use of the theory of optimal stopping, and as such it constitutes an interesting application of probability theory to the theory of financial economics.

In this paper we concentrate on the main principles involved only, which means, for example, that we abstract from derivatives where the underlying asset pays out dividends.

KEYWORDS: Optimal exercise policy, American put option, perpetual option, optimal stopping, superhedging

1 Introduction

An American derivative has the distinguishing feature that it can be exercised at any time before its expiry date. Compared to a European type instrument, which can only be exercised at expiration, this added flexibility ought to have some value in itself, which should then be reflected in the market price of the derivative.

For fairly obvious reasons, valuing American options has been given much attention in the economic and financial literature. Firstly, there has been a remarkable development of financial markets where American options are traded. And, secondly, many investment situations where both random prices and decision flexibility are present, have been phrased in terms of American options.

In this paper we start out explaining why it is never optimal to exercise early an American call option. The arguments we use do not presuppose any specific

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pricing model. Then we turn to the American put option as an example of a derivative where early exercise may be rational.

The stylized model by Black and Scholes is discussed briefly, where from the comparative statics we are also led to the conjecture that early exercise may sometimes be advantageous for the holder of an American put option.

The arguments are presented why such an instrument has a market price found as the supremum over a set of stopping times of values computed the usual way, that is, as conditional expected, discounted values under an equivalent martingale measure. A no-free-lunch-type argument is used, involving a super-replicating strategy.

We proceed with the mathematical theory of exercise and continuation regions for American instruments, ending in a set of variational inequalities. This is where the theory of optimal stopping really is used. The solution in the case of the American put option is still unknown, but some of its qualitative features, which are known, are briefly discussed.

The paper ends with the perpetual case, in which the time to expiration goes to infinity. Here we have the advantage of studying an explicit solution to the pricing problem for the American put option, which is presented for the Black and Scholes model.

References to the literature are given as we proceed, but the list of references is, however, not intended to be exhaustive.

2 The Economic Model

Given is a filtered probability space (Ω, \mathcal{F}, P) , satisfying the usual conditions, where $\{\mathcal{F}_t\}$ is the information filtration generated by a d -dimensional Brownian motion process $B(t)$, where $0 \leq t \leq T$. This latter process generates the asset processes $X(t) = (X_0(t), X_1(t), \dots, X_N(t))$ of Itô-type, where $\mathcal{F} = \mathcal{F}_T$, T being the time horizon.

Each $\omega \in \Omega$ denotes a complete description of the exogenous uncertain environment from time 0 to time T , \mathcal{F} is the sigma-field of distinguishable events at time T , and P is the common probability belief held by the agents in the economy. Process $X_0(t)$ denotes the price of a risk free security at time t , whereas the processes $(X_1, X_2 \dots, X_N)$ are the corresponding price processes of N risky securities. We assume that $X(t)$ is a $(N+1)$ -dimensional Itô-process admitting no arbitrage, and that the market is complete. Consider the following example:

Example 1. The Black and Scholes model.

Here "the bank account" is represented by $\beta(t) := X_0(t) = \beta_0 e^{rt}$, where β_0 and r are positive constants, r being the continuously compounded interest rate, and the risky asset has price process $S(t) := X_1(t) = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}$ where B_t is a Brownian motion process, and where μ and σ are two constants, $\mu > r$. Thus S_t is log normally distributed for each t , and $d = 1$. This model allows no arbitrage, and is furthermore complete. Here S can be found as the

solution to the following stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \quad (1)$$

with the initial condition $S(0) = S_0$. One could also view the price of the risk-free asset as a solution to the (ordinary) differential equation $d\beta(t) = r\beta(t)dt$ subject to the initial condition $\beta(0) = \beta_0$. \square

2.1 No Arbitrage and Complete Markets

For a risk-free asset of the form $d\beta(t) = r(t, \omega)\beta(t)dt$, $\beta(0) = \beta_0$, let us continue to denote the price process of the risky assets by S , i.e., $X(t) := (\beta(t), S(t))$ where S has N components, or $S_i(t) := X_i(t)$, $i = 1, 2, \dots, N$. The Itô security price process S in R^N satisfies the following stochastic differential equation

$$dS(t) = \mu(t, \omega)dt + \sigma(t, \omega)dB(t). \quad (2)$$

Assuming that the short-rate process r is bounded, we remind the reader that for the model to admit “no free lunch”, is sufficient that there exists an equivalent martingale measure Q such that the discounted price process $\hat{S}(t) = \frac{S(t)}{\beta(t)}$ is a martingale under Q , where the variance of the corresponding Radon-Nikodym derivative is assumed finite. A different condition is the following: Suppose there exists a process $u(t, \omega)$ of dimension d , where $\int u^2(t, \omega) dt < \infty$, $P - a.s.$, such that

$$\sigma(t, \omega)u(t, \omega) = \mu(t, \omega) - r(t, \omega)S(t) \quad (3)$$

and such that

$$E[\exp(\frac{1}{2} \int_0^T u^2(t, \omega) dt)] < \infty, \quad (4)$$

then the pricing process X has no arbitrage. Conversely, if the process X has no arbitrage, then there exists a (t, ω) -measurable process $u(t, \omega)$ such that (3) holds for a.a. (t, ω) , (but not necessarily (4)), (see Karatzas (1996) Th. 0.2.4). In terms of the process u , the equivalent martingale measure Q referred to above can be expressed as follows:

$$dQ(\omega) = \exp(-\int_0^T u(t, \omega) dB(t) - \frac{1}{2} \int_0^T u^2(t, \omega) dt)dP(\omega). \quad (5)$$

Assuming that equations (3) and (4) hold true, then $Q \sim P$ and by the Girsanov theorem the process

$$\tilde{B}(t) := \int_0^t u(s, \omega) ds + B(t) \quad (6)$$

is a Q -Brownian motion, and in terms of $\tilde{B}(t)$ we get by Itô's lemma

$$d\hat{S}_t = (-r_t\hat{S}_t + \frac{\mu_t}{\beta_t})dt + \frac{\sigma_t}{\beta_t}dB_t = \frac{\sigma_t}{\beta_t}d\tilde{B}_t; \quad (7)$$

Hence \hat{S} is a Q -martingale, so there exists an equivalent martingale measure, and by the above cited result, there can be no arbitrage.

We also remind the reader about the result that the market model is *complete* if and only if $\sigma(t, \omega)$ has a left inverse for a.a. (t, ω) , which is equivalent to the property that

$$\text{rank}(\sigma(t, \omega)) = d \quad \text{for a.a. } (t, \omega). \quad (8)$$

Let us just check these criteria for the model of Example 1. The “price of risk” $u(t, \omega)$ is clearly given by

$$u(t, \omega) = \frac{\mu S(t) - rS(t)}{\sigma S(t)} = \frac{\mu - r}{\sigma},$$

a constant, since S is strictly positive, and obviously the Novikov condition (4) holds true. Also from (5) it is seen that the variance of $\frac{dQ}{dP}$ is finite, since the log-normal distribution has a finite variance. Since $d = \text{rank}(\sigma) = 1$, the model is also complete.

The assumption of no arbitrage seems reasonable for a consistent theory, and it says essentially that it should not be possible to obtain strictly positive financial gains from investing nothing, i.e., from not taking any financial risk.

The assumption of complete markets is, on the other hand, far from obvious, at least in practice. The above technical characterizations of these properties seem rather similar, but the concepts are very different indeed. If the market is complete, it should be possible to perfectly duplicate the pay-out of any finite variance, financial instrument by forming (linear) portfolios of the primitive securities. No stock markets are so “rich” that anything like it could be possible, and if it were, derivatives would simply have no economic meaning. It also means that there is only one state price process ζ that will work, admittedly a very convenient property. Much more could be said about this topic, especially if investors and consumption are brought into the model, but we shall simply use this assumption as a practical, theoretical vehicle, without further discussion.

2.2 The Valuation Rule

Consider now a risky security with price process S , and suppose that it is a claim to a cumulative dividend process D . Then the following connection is essential:

$$S_t = E_t^Q[\exp(-\int_t^T r_s ds)S_T + \int_t^T \exp(-\int_t^s r_u du)dD_s], \quad (9)$$

where the expectation is conditional on the information at time t , under the measure Q .

The fact that we require the variance of $\xi := \frac{dQ}{dP}$ to be finite, leads to a continuous pricing functional. In order to see this, let us define $\xi_t := E_t(\xi)$. Clearly ξ_t is martingale under the given measure P . Letting Y be the pay-off of a security at time T , represented as a finite variance random variable, its price at any time $t \in [0, T]$ is given by

$$V_t(Y) = \frac{E_t[\zeta_T Y]}{\zeta_t} = E_t^Q[\exp(-\int_t^T r_s ds)Y], \quad (10)$$

where $\zeta_t := \exp(-\int_0^t r_s ds)\xi_t$ is called the state price deflator. In particular the market value of Y at time zero can be written as $V_0(Y) = E(\zeta_T Y)$, so if $\text{var}(\zeta_T) < \infty$, the linear functional V is bounded, and hence continuous. Alternatively, since $\zeta > 0$, P -a.s., we know that a positive linear functional on $L^2(\Omega, \mathcal{F}, P)$ is continuous (in this case V is strictly positive). Continuity of the pricing functional V is an important economic property; changing the pay-off structure a little does not change the market price in any dramatic manner, i.e., if $\|Y - Z\|_2$ is small, then $|V(Y) - V(Z)|$ is also small, where $\|\cdot\|_2$ signifies the L^2 -norm. This is a kind of stability, or smoothness property of the pricing functional that seems reasonable in a theory of valuation.

As an illustration of the valuation rule (10), consider an European call option in the model of Example 1. This is a contract that pays out $C(S_T, T) = (S_T - K)^+$ at the expiration date T only, where the exercise price K is a given constant. Thus $Y = (S_T - K)^+$, and the market price $V_t(Y) := C(S_t, t)$ of this call option at time $t \leq T$, when the price of the underlying asset is S_t , is given as follows:

$$C(S_t, t) = S_t \Phi(a) - e^{-r(T-t)} K \Phi(a - \sigma \sqrt{(T-t)}) \quad (11)$$

where

$$a = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{(T-t)}} \quad (12)$$

and where Φ is the cumulative standard normal distribution function. This computation in the setting of Example 1 is straight-forward, as it only amounts to taking the expectation of a simple function of a log-normally distributed random variable.

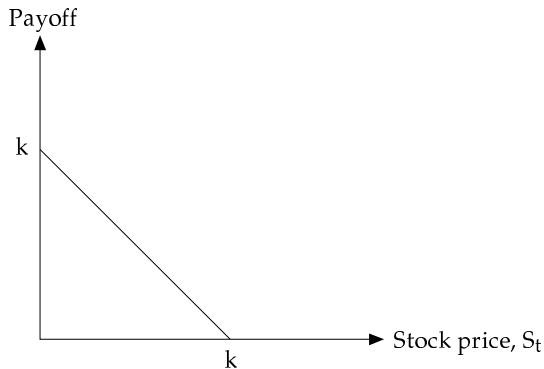
3 American Derivatives

We now turn to American derivative securities in the above setting. The distinguishing feature of such a security is that it can be exercised at any time before its expiration. We denote the payoff by $U(t)$ for any $t \leq T$. The pay-out will

take place at a stopping time $\tau \leq T$, where the event $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$. We may think of a derivative security as one which has no value in itself, but derives its value from another underlying security. This has nothing to do with our discussion of complete markets at the end of the last section, since derivatives may very well have risk allocational value in, say, incomplete markets. We are now only trying to explain what a derivative is regardless of the market structure.

As a typical example of a derivative security we pick an American put option, in which case $U(t) = (K - S_t)^+$. Here K is a fixed constant, the exercise price, upon which the parties agree in advance, and $S(t)$ is the market price at time t of a underlying stock, which may be thought of as one of the primitive securities. Thus the put only gets a positive value once the stock price drops below the fixed exercise price K . In this regard the put option can be thought of as an insurance contract. See figure (1)

Figure 1: Pay-off



Pay-off at expiration of an American put option.

3.1 The Optimal Stopping Rule

Suppose the holder of such an instrument adopts a fixed strategy τ . According to the theory of the last section, in particular equation (10), the market price of the instrument exercised according to this strategy must be

$$V_t(U(\tau)) = E_t^Q[\exp(-\int_t^\tau r_s ds)U(\tau)].$$

Denote by $\Gamma_{[t,s]} = \{\text{Stopping times in } [t, s]\}$, and consider the problem

$$V_0^* = \sup_{\tau \in \Gamma_{[0,T]}} V_0(U(\tau)). \quad (13)$$

If there exists a stopping time τ^* solving problem (13), then V_0^* is the market price of this instrument: Suppose not, denote the price by V_0 and consider two cases:

(a) $V_0 < V_0^*$. In this case an investor could use the following strategy: Buy the instrument and pay V_0 . Sell short a self-financing portfolio replicating the payoff $-U(\tau^*)$. This latter construction is possible since the market is assumed complete, and this portfolio has a market value $V_0^* = E_0^Q[\exp(-\int_0^{\tau^*} r_s ds)U(\tau^*)]$.

The cash flow at time zero is $V_0^* - V_0 > 0$, and at time τ^* the option is exercised giving $U(\tau^*)$, which can be then used to cancel the short position $-U(\tau^*)$. This strategy is thus risk-free, but gives a positive cash flow at time zero, implying a free lunch. Since we have assumed there is no arbitrage, we have reached a contradiction.

(b) $V_0 > V_0^*$. Again we recommend buying the cheapest instrument, and selling the most expensive one. This means selling the option, but here the seller has the problem that he does not know the buyer's exercise strategy τ . This is where we may use the concept of a *super-replicating* strategy θ , defined as a self-financing strategy having the property that $\theta_t X_t \geq U(t), \forall t \in [0, T]$. Suppose for the moment that such a strategy exists with initial value $\theta_0 X_0 = V_0^*$. Then do the following: Sell the American instrument and "buy" the super-replicating strategy θ . The cash flow from this is at time zero $V_0 - V_0^* > 0$. At the point in time τ , chosen by the holder of the American instrument, $\theta_\tau X_\tau \geq U(\tau)$, so this strategy will never give a loss after time zero, i.e., again a free lunch.

The conclusion must then be that the market price $V_0 = V_0^*$. It remains to verify that a super-replicating strategy exists, and for this we need some regularity conditions.

Consider a non-negative process U , let $U^* = \sup_{t \in [0, T]} U_t$, and assume that $E(U^*)^p < \infty$ for some $p > 2$. We then have the following results:

Proposition 1 *Under the above stipulated conditions*

(a) *There exists a rational exercise strategy τ^* such that*

$$V_0^* = \sup_{\tau \in \Gamma_{[0, T]}} V_0(U(\tau)) = E_0^Q[\exp(-\int_0^{\tau^*} r_s ds)U(\tau^*)]$$

Implicit in this is that the last expectation exists.

(b) *There exists a super-replicating strategy θ and some constant k such that $\theta_t X_t \geq k, \forall t$ where $\theta_0 X_0 = V_0^*$.*

Proof. For the proof of (a) we refer to Karatzas (1988). Here we concentrate on (b): Let $\hat{U}_t = \exp(-\int_0^t r_s ds)U_t$ and let W be the Snell envelope of \hat{U} under Q , i.e.,

$$W_t = \sup_{\tau \in \Gamma_{[t, T]}} E_t^Q(\hat{U}_\tau), \quad 0 \leq t \leq T.$$

Under the given conditions we know that W is a continuous super-martingale under Q . Thus $W = Z - A$, where Z is a Q -martingale and A is an increasing process, where $A_0 = 0$. Consider the quantity $Z_T \exp(\int_0^T r_s ds)$. By the assumption of complete markets there exists a self-financing strategy with associated portfolio θ in the primitive assets, such that $\theta_T X_T = Z_T \exp(\int_0^T r_s ds)$. Here $\theta_{t,n}$ is the number of shares of asset no. n in the portfolio consisting of the given assets at time t . In this theory we know that the discounted value process is a Q -martingale, i.e.,

$$\theta_t X_t \exp(-\int_0^t r_s ds) = E_t^Q(\exp(-\int_0^T r_s ds) \theta_T X_T)$$

This is equivalent to

$$\begin{aligned} \theta_t X_t &= \exp(\int_0^t r_s ds) E_t^Q(\exp(-\int_0^T r_s ds) \theta_T X_T) \\ &= \exp(\int_0^t r_s ds) E_t^Q(Z_T) = \exp(\int_0^t r_s ds) Z_t \\ &= \exp(\int_0^t r_s ds) (W_t + A_t) \geq \exp(\int_0^t r_s ds) W_t \geq U_t. \end{aligned}$$

The second equality follows from the definition of θ , the third follows since Z is a martingale, the first inequality is true since $A_t \geq 0$ and last inequality follows from the definition of W , \hat{U} and U , i.e., from the fact that

$$W_t \geq \hat{U}_t = \exp(-\int_0^t r_s ds) U_t.$$

By the definition of W_0 and V_0^* it follows that $\theta_0 X_0 = W_0 = V_0^*$, since $A_0 = 0$. Since $U_t \geq 0$, $\theta_t X_t \geq 0$ for all t , so θ is a super-replicating strategy for (U, T) . \square

For later reference we notice the following: Define

$$\tau^0 = \inf\{t : W_t = \hat{U}_t\}. \quad (14)$$

Then τ^0 is a rational exercise strategy, i.e., it solves the problem (13).

3.2 When is Early Exercise Optimal?

Consider a call option written on an underlying security having price process S , and paying no dividends. For notational simplicity, assume that the short rate

process $r(t, \omega) = r(\omega)$ does not depend upon time. Suppose the call is exercised at a possibly random stopping time $\tau \leq T$. Then it follows, by a slight extension of Jensen's inequality, that its price C_0^τ must satisfy

$$\begin{aligned} C_0^\tau &= E^Q[(S_\tau e^{-r\tau} - K e^{-r\tau})^+] \\ &\leq E^Q[(S_T e^{-rT} - K e^{-r\tau})^+] \\ &\leq E^Q[(S_T e^{-rT} - K e^{-rT})^+] = C_0^T \end{aligned}$$

where the last inequality is trivially true if $r \geq 0$ a.s. Notice that the above holds regardless of the dynamics of the pricing process of the risky asset, and also regardless of the interest rate process, as long as $r \geq 0$ for a.a. t a.s. Thus, under these circumstances it is not advantageous to exercise a call option early.

In the case where the underlying asset pays out dividends, it is known that there exist situations in which early exercise may be optimal. Consider e.g., a situation where the stock has known ex-dividend dates and the dividend amounts per share are random variables which can exceed the exercise price K , then there is a positive probability of early exercise just prior to the ex-dividend dates. Moreover early exercise is only optimal (if ever) at the instant before the stock goes ex-dividend (see e.g., Jarrow and Rudd (1983)). In order to keep the presentation from becoming too technical, we shall concentrate on situations where the underlying asset pays no dividends in the following, unless explicitly stating otherwise.

Let us try to go through the above reasoning for a put option. Its price P_0^τ must satisfy

$$\begin{aligned} P_0^\tau &= E^Q[(K e^{-r\tau} - S_\tau e^{-r\tau})^+] \\ &\leq E^Q[(K e^{-r\tau} - S_T e^{-rT})^+]. \end{aligned}$$

But

$$\begin{aligned} E^Q[(K e^{-r\tau} - S_T e^{-rT})^+] &\geq \\ E^Q[(K e^{-rT} - S_T e^{-rT})^+] &= P_0^T, \end{aligned}$$

and one can guess that early exercise of American put options is sometimes optimal, still assuming $r \geq 0$ for a.a. t a.s. Since the short interest rate is in nominal terms in this analysis, it must be positive.

Consider now the Black and Scholes model of Example 1. Here we want to see how of the price of a *European* put option varies as the parameters vary. We take as the starting point the put-call parity, which states that the price P_0 of a put option can be found in terms of the price C_0 of a call option as follows:

$$P_0 = C_0 - S_0 + K e^{-rT},$$

a relation which can easily be verified from equality of the respective payoffs at expiration. Using the formula (11) for the price of a European call option in

this model, we deduce the following:

$$\frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1 = \Phi(a) - 1 < 0 \quad (15)$$

$$\frac{\partial P}{\partial K} = \frac{\partial C}{\partial K} + e^{-rT} = e^{-rT} (1 - \phi(a - \sigma\sqrt{T})) > 0 \quad (16)$$

$$\frac{\partial P}{\partial T} = \frac{\partial C}{\partial T} - rKe^{-rT} = \frac{S\sigma}{2\sqrt{T}}\varphi(a) + Kre^{-rT} (\Phi(a - \sigma\sqrt{T}) - 1) \quad (17)$$

$$\frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} = S\sqrt{T}\varphi(a) > 0 \quad (18)$$

$$\frac{\partial P}{\partial r} = \frac{\partial C}{\partial r} - KTe^{-rT} = TKe^{-rT} (\Phi(a - \sigma\sqrt{T}) - 1) < 0, \quad (19)$$

where φ is the standard normal density function. Of particular interest in the present case is relation (17). Here the sign is ambiguous, since the first term on the right hand side is positive, but the second is negative. However, $\partial C/\partial T > 0$, so the European call option increases in value by increasing the time to expiration, consistent with our more general observation above.

Several other insights can be derived from the above comparative statics: As the stock price increases, the put value declines, which is fairly intuitive, since then the “insurance case” becomes less likely. As the exercise price increases, the “insured sum” increases, which naturally increases the put value, i.e., the “insurance premium”. Increasing the “volatility” σ of the stock, both the call and the put values increase. It may be somewhat surprising that increased uncertainty is valuable, but for the put case this simply means that “insurance” becomes more worth. For the call option, the holder of such an option has no downside risk (i.e., no matter how far the call is out-of-the-money at expiration, its value is still zero), and increasing σ increases the probability that the option will end up in-the-money at maturity.¹ Finally, as the interest rate increases, the value of the put decreases. One may explain this from looking at the present value of the exercise payment in the event of exercise declines as r increases, making the present value of the “insurance sum” smaller, and hence the insurance premium also declines.

4 Exercise and Continuation Regions

Consider an American instrument having payoff $U_t = g(Y_t, t)$, where $g : R^k \rightarrow R$ is a continuous function. Here Y is a Markov diffusion process in R^k having dynamics given by

$$dY_t = a(Y_t)dt + b(Y_t)d\hat{B}_t \quad \text{under } Q. \quad (20)$$

We maintain our previous assumptions, but let the interest rate $r = 0$ for the moment. If $Y_t = y$, the market value at time t of the instrument is

$$h(y, t) = \sup_{\tau \in \Gamma_{[t, T]}} E_t^Q [g(Y_\tau, \tau)]. \quad (21)$$

¹The issuer of the options can control the risk by hedging, since the market is complete.

By picking $\tau = t$, we see that $h(y, t) \geq g(y, t)$. From a previous remark we also know that

$$\tau^* = \inf_t \{t \in [0, T] : h(Y_t, t) = g(Y_t, t)\}. \quad (22)$$

Thus $h(Y_t, t) > g(Y_t, t)$ for $t < \tau^*$, and

$$\mathcal{E} = \{(y, t) \in R^k \times [0, T] : h(y, t) = g(y, t)\} \quad (23)$$

is called the *exercise region*, while

$$\mathcal{C} = \{(y, t) \in R^k \times [0, T] : h(y, t) > g(y, t)\} \quad (24)$$

is termed the *continuation region*. The optimal strategy is to wait in \mathcal{C} and exercise in \mathcal{E} , i.e., to exercise the first time the process Y hits \mathcal{E} .

The associated *variational inequality* of this problem is

$$h \geq g, \quad \mathcal{D}h \leq 0, \quad (h - g)(\mathcal{D}h) = 0, \quad (25)$$

having the boundary condition $h(y, T) = g(y, T)$, where

$$\mathcal{D}h(y, t) = h_t(y, t) + h_y(y, t)a(y) + \frac{1}{2}tr [h_{yy}(y, t)b(y)b^T(y)] \quad (26)$$

is the relevant differential operator. Here $tr[\cdot]$ is the trace of the matrix indicated, and $b^T(y)$ is the transpose of $b(y)$.

The problem can readily be adapted to the situation where $r \neq 0$, e.g., where $r_t = \alpha(Y_t)$ for some function $\alpha : R^k \rightarrow R$. In this case the differential operator $\mathcal{D}^r h(y, t)$ changes to $\mathcal{D}h(y, t) - \alpha(y)h(y, t)$. Also, if $a = a(y, t)$, $b = b(y, t)$ depend upon time, use the new diffusion process $\hat{Y}(t) = (t, Y(t))$ (see e.g., Øksendal (1995)). Jaillet, Lamberton, and Lapeyre (1988, 1990) review the treatment of the optimal stopping problem valuation problem as a variational inequality. Notice that we require the Markov property in the above treatment.

There exists no closed form solution for the American put option in the Black and Scholes model. However, it is known that the continuation region is a subset of $R_+ \times [0, T]$: There exists an increasing, continuously differentiable function $f(t) : [0, T] \rightarrow R$ such that

$$\mathcal{C} = \{(x, t) : x > f(t)\}, \quad (27)$$

where $f(T) = K$, and $f(t)$ behaves roughly as the square root function close to the terminal time T , having an infinite derivative in the point (T, K) (see Barles et al. (1995)). The optimal rational exercise strategy is to exercise as soon as the stock price S hits f , i.e.,

$$\tau^* = \inf\{t : S(t) = f(t)\}. \quad (28)$$

In Barles et al. (1991) it is in particular shown that

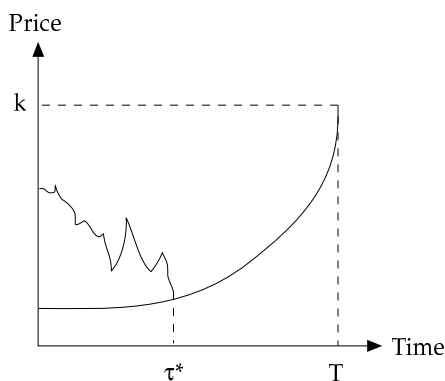
$$f(t) - K \sim -\sigma K \sqrt{(T-t) |\ln(T-t)|} \quad \text{as } t \rightarrow T^-,$$

in the sense that

$$\frac{f(t) - K}{-\sigma K \sqrt{(T-t)} |\ln(T-t)|} \rightarrow 1 \quad \text{as} \quad t \rightarrow T^-.$$

This indicates that the continuation region has the shape shown in figure (2), but the exact form is still unknown. The behavior of the optimal exercise boundary near expiration is in addition treated in Lamberton (1993), and Carretour et al. (1992). Van Moerbeke (1976) was the first to demonstrate that the critical boundary $f(t)$ is continuously differentiable.

Figure 2: Continuation Region



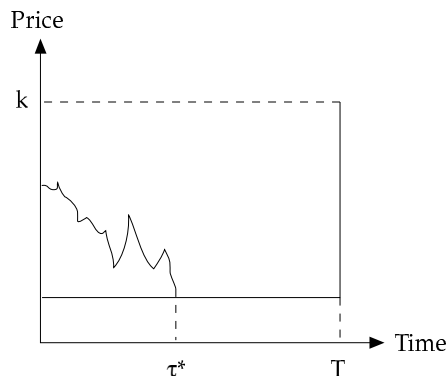
Continuation Region of an American put option.

There are several numerical techniques for approximation the value function h and exercise boundary f , one of which is based on a direct finite-difference numerical solution of the variational inequality (25) adapted to the Black and Scholes model. Another is described in Bjerksund and Stensland (1993) and consists in the following: They derive a lower bound to the option value by restricting the set of feasible exercise strategies. This they accomplish by restricting the functions f to be linear of the type $f(t) \equiv c$ for all $t \in [0, T]$ for c some constant, and this constant is then given two recipes to determine. At the point T the boundary has a vertical line from the point (T, c) to the point (T, K) , see figure (3).

Their method is compared to other numerical techniques, like the finite-difference and the binomial-tree approximations, and it seems to do reasonably well.

In Aase (1986) a similar approximation technique was attempted in a situation where the underlying asset was modeled by a jump-diffusion process, and then based on Monte Carlo simulations. Barone-Adesi and Whaley (1987) proposed a quadratic approximation, which they found accurate for short-term options. Geske and Johnson (1984) developed a compound-option approximation model. Approximate solutions to the American option price are also treated

Figure 3: Approximative Region



Approximative continuation region for the American put option.

by Allegretto, Barone-Adesi and Elliott (1993), Broadie and Detemple (1993), Carr (1994), Geske and Johnsen (1984), Gandhi, Kooros, and Salkin (1993), and Baron-Adesi and Elliott (1991) among others. Many of these these treat options written on a dividend-paying asset, often limited to a dividend rate.

Broadie and Detemple (1996) develop lower and upper bounds on the prices of American call and put options, which they compare to the binomial tree approximation.

A decomposition of the American option in terms of an early exercise premium was proposed by Jamshidian (1989), Jacka (1991), Kim (1990), and Carr, Jarrow, and Myneni (1992).

For American-style Asian options there is recent work by Hansen and Jørgensen (1997), who find analytical pricing formulas for such instruments.

Bensoussan (1984) and Harrison and Kreps (1979), among others, did important early work on American option pricing.

In the above cited literature one can find references to a large body of research that deals with American options, see in particular Duffie (1996).

5 The Perpetual Case

The formula for pricing the infinite-lived, or perpetual American call option when the underlying asset provides a continuous proportional pay-out was derived by Samulson (1965), and generalized by McDonald and Siegel (1986). A similar simple formula for the finite-lived American option is not yet found. However, some important existence and uniqueness results are given in Kim (1990) and Jacka (1991). We know from their work that there exists a unique parabolic boundary characterizing the optimal exercise strategy.

In this section we shall look at the perpetual American put option when, again, the underlying asset pays no dividends. This leads to the same mathe-

matical problem as the above mentioned call option when the underlying asset provides a continuous proportional pay-out. Firstly we remark that the market value of a *European* perpetual ($T \rightarrow \infty$) put option is zero:

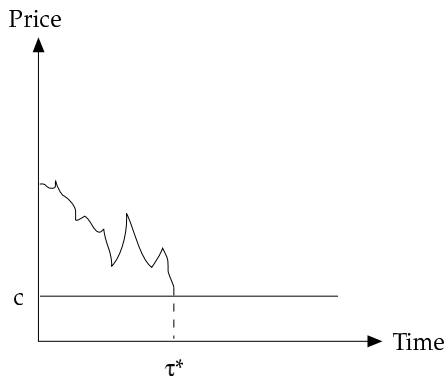
$$P_0^T = C_0^T - S_0 + Ke^{-rT} \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty,$$

since $C_0^T \rightarrow S_0$ as $T \rightarrow \infty$ follows directly from the expression in (11). However, the corresponding value of the American perpetual put option converges to a strictly positive value, demonstration at least one situation where there is a difference between these two products, in the situation with no dividends from the underlying asset. The perpetual case was first treated in Merton (1973), and has later been extended by Karatzas (1988). The value of the American perpetual put option can actually be found explicitly in the Black and Scholes model, which we now demonstrate:

For a sufficiently low stock price, it may be advantageous to exercise the put, and define the trigger price c as the largest value of the stock such that the put holder is better off exercising the put than continuing to hold it. Here the linear approximation made by Bjerksund and Stensland (1993) happens to hold exactly: Because at each time point $t \geq 0$ in the life of the put option, bought at time zero, the remaining time to maturity is the same, time can not enter into the function f of the last section, so $f(t) = c$ for all t , where c is a positive constant that must be determined as part of the solution. In other words, the continuation region is given as follows, see figure (4):

$$\mathcal{C} = \{(x, t) : x > c\}.$$

Figure 4: Continuation Region



Continuation region of the perpetual American put option.

In the continuation region \mathcal{C} the partial differential equation in one of the relations of (25) reduces to an ordinary differential equation:

$$\frac{1}{2}\sigma^2 x^2 h_{xx}(x) + rxh_x(x) - rh(x) = 0, \quad (29)$$

valid for $c \leq x < \infty$. Given the trigger price c , let us denote the market value $h(x) := h(x; c)$. The relevant boundary conditions are then

$$h(\infty; c) = 0 \quad \forall c \quad (30)$$

$$h(c; c) = K - c \quad (\text{exercise}). \quad (31)$$

The trigger price c must then be determined in accordance with our optimal stopping rule in equation (13), so that it maximizes the value of the option.

The solution can now be derived as follows: From the theory of ordinary differential equations we know that

$$h(x; c) = a_1 x + a_2 x^{-\gamma},$$

where $\gamma = \frac{2r}{\sigma^2} > 0$. The boundary condition (30) implies that $a_1 = 0$, and the boundary condition (31) implies that $a_2 = (K - c)c^\gamma$. Thus

$$h(x; c) = \begin{cases} (K - c) \left(\frac{x}{c}\right)^{-\gamma}, & \text{if } x \geq c; \\ (K - x), & \text{if } x < c. \end{cases} \quad (32)$$

In order to determine the optimal value of c , we maximize $h(x; c)$ with respect to c , leading to the unique value of the trigger price

$$c = \frac{\gamma K}{1 + \gamma}, \quad \text{where} \quad \gamma = \frac{2r}{\sigma^2} \quad (33)$$

and K is the exercise price. Inserting this expression in (32) finally leads to the closed form solution for the market value of the American perpetual put option:

$$h(x) = \begin{cases} \frac{K}{1 + \gamma} \left(\frac{(1 + \gamma)x}{\gamma K}\right)^{-\gamma}, & \text{if } x \geq c; \\ (K - x), & \text{if } x < c, \end{cases} \quad (34)$$

where c is given above in equation (33). See figure (5).

The ‘‘high-contact’’ boundary condition of Samuelson and McKean (1965)

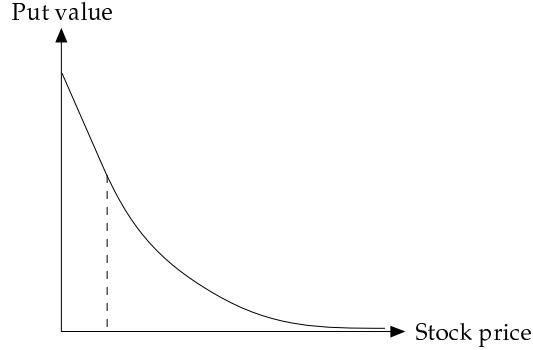
$$h_x(c; c) = -1 \quad (35)$$

yields the same value of the trigger price c as we have found above from the first order condition

$$h_c(c; c) = 0 \quad (36)$$

It rests on an observation that the function h must be a C^1 -function in the ‘‘pasting’’. Of course, in general h must be a C^2 -function in order to satisfy the second order differential equation, but in a ‘‘thin’’ enough area it is actually enough that it is C^1 (see e.g., Øksendal (1995)). In the present case the Lebesgue measure of the set of time points where the function is C^1 , is zero, which is thin enough. The high-contact condition ensures that h is smooth enough in the

Figure 5: Perpetual American put option value



Market value of a perpetual American put option as a function of stock price.

pasting point $x = c$: Here $h(x; c) |_{x=c} = (K - c)$, and $-1 = h_x(x; c) |_{x=c} = \frac{\partial}{\partial x}(K - x) |_{x=c} = -1$.

Comparative statics can be derived from the expression for the market value in (34). The results are directly comparable to the results in section 3.2 for the finite-lived European put option: The put price h increases with K , ceteris paribus, and the put price decreases as the stock price x increases, which can be seen directly from figure (5). Changes in the volatility parameter have the following effects: Let $v = \sigma^2$, then

$$\frac{\partial h}{\partial v} = \begin{cases} \frac{c}{v} \left(\frac{c}{x}\right)^\gamma \ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c. \end{cases} \quad (37)$$

Clearly this partial derivative is positive as we would expect.

Similarly, but with opposite sign, for the interest rate r :

$$\frac{\partial h}{\partial r} = \begin{cases} -\frac{2c}{\gamma v} \left(\frac{c}{x}\right)^\gamma \ln\left(\frac{x}{c}\right), & \text{if } x \geq c; \\ 0, & \text{if } x < c. \end{cases} \quad (38)$$

The effect of the interest rate on the perpetual put is the one we would expect, i.e., a marginal increase in the interest rate has, ceteris paribus, a negative effect on the perpetual put value.

We end by a few philosophical remarks about the perpetual model. It turns out to be a border line model, which only makes sense when interpreted properly. One may wonder if a call option can have a positive market value when the expiration time never materializes. We also notice that the Novikov condition does not hold in the limit, and there is no well defined equivalent martingale measure on all of R_+ , since e.g., $\hat{B}(t) = \frac{\mu-r}{\sigma}t + B(t)$ can not be a Brownian motion under Q as $t \rightarrow \infty$.

The problem can equivalently be phrased for the risky asset: The discounted price process \hat{S} is a martingale under the measure Q , but this martingale is not

uniformly integrable, so in particular $\lim_{t \rightarrow \infty} \hat{S}_t$ does not exist in the sense of L_1 , and there is no real random variable \hat{S}_∞ which is integrable such that $\hat{S}_t = E^Q(\hat{S}_\infty | \mathcal{F}_t)$ for all t . It is easy to show that under Q , $\lim_{t \rightarrow \infty} \hat{S}_t = 0$ a.s.² Now we have an asset which does not pay dividends, and never realizes any positive “scrap value” $\hat{S}_\infty > 0$; how can it possibly have a positive value S_0 at time zero, say?

The answer is that since the process $\{\hat{S}_t, 0 \leq t \leq T\}$ is well defined as a Q -martingale for all T , it follows that for some $S_0 > 0$, $S_0 = E^Q(\hat{S}_T)$ for all $T < \infty$, so it must be the case that $\lim_{T \rightarrow \infty} E^Q(\hat{S}_T) = S_0 > 0$.³ It is in this meaning the perpetual model makes sense, as a limit of market values as $T \rightarrow \infty$, and it is this interpretation we used above, when we considered the variational inequalities in the limit. Notice that

$$S_0 = \lim_{T \rightarrow \infty} E^Q(\hat{S}_T) \neq E^Q(\lim_{T \rightarrow \infty} \hat{S}_T) = 0$$

(by the lack of uniform integrability). In other words, as noticed above, it is not the case that the market value S_0 at time zero can be recovered as $E^Q(\hat{S}_\infty)$ for any real random variable \hat{S}_∞ closing the martingale \hat{S} . We may conclude that the perpetual case is well defined as a model in the limit, but not well defined as a limiting model.⁴

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²For the geometric Brownian motion it is known that $S_T \rightarrow 0$ a.s. when $T \rightarrow \infty$ if $\mu < \frac{1}{2}\sigma^2$, $S_T \rightarrow \infty$ a.s. when $T \rightarrow \infty$ if $\mu > \frac{1}{2}\sigma^2$. Under the measure Q the process \hat{S} is a geometric Brownian motion having $\mu = 0$.

³This can be shown directly, since $E(S_t) = S_0 e^{\mu t}$ for the geometric Brownian motion. Under discounting we get that $E\hat{S}_t = S_0 e^{(\mu-r)t}$, and under Q the term $\mu = r$, so $E^Q(\hat{S}_t) = S_0$ for all $t < \infty$.

⁴Some people may perhaps even prefer this latter limiting interpretation, where e.g., the market value S_0 of the risky asset is zero.

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