Closed form spread option valuation

Petter Bjerksund and Gunnar Stensland

Department of Finance, NHH
Helleveien 30, N-5045 Bergen, Norway

e-mail: petter.bjerksund@nhh.no
Phone: +47 55 959548, Fax: +47 55 959650

This version: October 31, 2006

Abstract

This paper considers the valuation of a spread call when asset prices are lognormal. The implicit strategy of the Kirk formula is to exercise if the price of the long asset exceeds a given power function of the price of the short asset. We derive a formula for the spread call value, conditional on following this feasible but non-optimal exercise strategy. Numerical investigations indicate that the lower bound produced by our formula is extremely accurate. The precision is much higher than the Kirk formula. Moreover, optimizing with respect to the strategy parameters (which corresponds to the Carmona-Durrleman procedure) yields only a marginal improvement of accuracy (if any).

Keywords: Spread option, closed form, valuation formula, lognormal asset prices.

JEL code: G12, G13, D81, C63.

Closed form spread option valuation

Abstract

This paper considers the valuation of a spread when asset prices are lognormal. The implicit strategy of the Kirk formula is to exercise if the price of the long asset exceeds a given power function of the price of the short asset. We derive a formula for the spread call value, conditional on following this feasible but non-optimal exercise strategy. Numerical investigations indicate that the lower bound produced by our formula is extremely accurate. The precision is much higher than the Kirk formula. Moreover, optimizing with respect to the strategy parameters (which corresponds to the Carmona-Durrleman procedure) yields only a marginal improvement of accuracy (if any).

Keywords: Spread option, closed form, valuation formula, lognormal asset prices.

JEL code: G12, G13, D81, C63.

1   Introduction

This paper considers the valuation of a spread call when the asset prices are lognormal. The starting point is the observation that the implicit strategy of the Kirk formula is to exercise if the price of the long asset exceeds a given power function of the price of the short asset.

We derive a formula that evaluates the spread call, conditional on following this exercise strategy. The formula consists of three terms, one for each of the two assets, and one for the strike. A standard normal cumulative probability enter into each term, and each argument is a function of the forward prices, time to exercise, volatilities, and correlation. The formula fits well in to the tradition of Black-Scholes, Black76, and Margrabe.

Numerical investigations indicate that our formula is extremely accurate. The precision is much higher than the Kirk formula. Furthermore, the accuracy of our formula is comparable with the precision of the lower bound procedure suggested by Carmona and Durrleman, which requires a two-dimensional optimization scheme.

2   Assumptions

Consider a frictionless market with no arbitrage opportunities and with a constant riskless interest rate r. Assume two assets where the prices at the future date T are

$$S_1(T) = F_1 \exp\left\{-\tfrac{1}{2}\sigma_1^2 T + \sigma_1 \sqrt{T}\varepsilon_1\right\} \tag{1}$$

$$S_2(T) = F_2 \exp\left\{-\tfrac{1}{2}\sigma_2^2 T + \sigma_2 \sqrt{T}\varepsilon_2\right\} \tag{2}$$

with respect to the Equivalent Martingale Measure (EMM), where F_1 and F_2 are the current forward prices for delivery at the future date T, σ_1 and σ_2 are volatilities, and ε_1 and ε_2 are standard normal random variables with correlation ρ. It follows from above that the two asset prices are lognormal, and that the expected future price for each asset (wrt. the EMM) coincides with the current forward price.

3 The spread call

Consider a European call option on the price spread $S_1(T) - S_2(T)$ with strike $K \geq 0$ and time to exercise T. The call option pay-off at time T is

$$C(T) = (S_1(T) - S_2(T) - K)^+ \tag{3}$$

where $()^+$ denotes the positive part. The call value at time 0 can be represented by[1]

$$C = e^{-rT} E_0\left[(S_1(T) - S_2(T) - K)^+\right] \tag{4}$$

where the expectation is taken with respect to the EMM, and r is the riskless interest rate. It follows from the put-call parity that the value of a European put option on the price spread $S_1(T) - S_2(T)$ with strike $K \geq 0$ and time to exercise T is given by $P = C - e^{-rT}(F_1 - F_2 - K)$.

With both $S_1(T)$ and $S_2(T)$ being lognormal, there is no known general formula for the spread call value. However, closed form solutions are available for the following limiting cases: Firstly, if $F_2 = 0$, the call spread collapses into a standard call on $S_1(T)$, and the value is given by the Black76 formula (c.f. Black (1976)). And secondly, if $K = 0$, the call spread collapses into an option to exchange one asset for another. The option value in this case is given by the Margrabe formula (see Margrabe (1978)).

4 The Kirk formula

In the general case, however, we have to rely on either approximation formulas or extensive numerical methods. Approximation formulas allow quick calculations and facilitate analytical tractability, whereas numerical methods typically

---

[1] See, e.g., Cox and Ross (1976), Harrison and Kreps (1979), and Harrison and Pliska (1981).

produces more accurate results. Practitioners are very focused on simple calculations and real time solutions, hence a closed form approximation formula is typically the preferred alternative.

Kirk (1995) suggests the following approximation to the spread call[2]

$$c_K = e^{-rT} \left\{ F_1 N\left(d_{K,1}\right) - (F_2 + K) N\left(d_{K,2}\right) \right\} \tag{5}$$

where $N\left(\right)$ denotes the standard normal cumulative probability function, and $d_{K,1}$ and $d_{K,2}$ are given by

$$d_{K,1} = \frac{\ln\left(F_1 / \left(F_2 + K\right)\right) + \frac{1}{2}\sigma_K^2 T}{\sigma_K \sqrt{T}} \tag{6}$$

$$d_{K,2} = d_1 - \sigma_K \sqrt{T} \tag{7}$$

$$\sigma_K = \sqrt{\sigma_1^2 - 2\frac{F_2}{F_2 + K}\rho\sigma_1\sigma_2 + \left(\frac{F_2}{F_2 + K}\right)^2 \sigma_2^2} \tag{8}$$

5   The Carmona-Durrleman procedure

Carmona and Durrleman (2003a, 2003b) represent the future spot prices by two independent state variables and model the correlation by using trigonometric functions. In particular, Eqs.(1) and (2) above translate into

$$S_1\left(T\right) = F_1 \exp\left\{-\tfrac{1}{2}\sigma_1^2 T + \left(z_1 \sin\phi + z_2 \cos\phi\right)\sigma_1\sqrt{T}\right\} \tag{9}$$

$$S_2\left(T\right) = F_2 \exp\left\{-\tfrac{1}{2}\sigma_2^2 T + \sigma_2\sqrt{T}z_2\right\} \tag{10}$$

where z_1 and z_2 are standard normal and independent random variables, and $\cos\phi = \rho$ where $\phi \in [0, \pi]$. The authors consider the value from exercising the spread option according to a feasible, but non-optimal strategy conditional on the two state variables. In particular, the strategy is to exercise when

$$Y_{\theta^*} \equiv z_1 \sin\theta^* - z_2 \cos\theta^* \leq d^* \tag{11}$$

where $\theta^* \in [\pi, 2\pi]$ and d^* are found numerically by maximising the option value.[3]

---

[2] By the put-call parity, the Kirk approximation of a put on the price spread $S_1\left(T\right) - S_2\left(T\right)$ with strike $K \geq 0$ and time to exercise T is $p_K = c_K - e^{-rT}\left(F_1 - F_2 - K\right)$.

[3] $\phi \in [0, \pi]$ and $\theta^* \in [\pi, 2\pi]$ translate into $\sin\phi \geq 0$ and $\cos\theta^* \leq 0$. To motivate this, observe from Eqs. (9) and (11) that an increase in z_1 will increase the pay-off from asset 1, and push the call more in-the-money (less out-of-the-money).

The value from following this strategy, which represents a lower bound to the true spread option value, is[4]

$$
\begin{aligned}
c_{CD} &= e^{-rT} E_0 \left[(S_1(T) - S_2(T) - K) I (z_1 \sin\theta^* - z_2 \cos\theta^* \leq d^*) \right] \\
&= e^{-rT} \left\{ F_1 N \left(d^* + \sigma_1 \sqrt{T} \cos(\theta^* + \phi) \right) \right. \\
&\quad \left. - F_2 N \left(d^* + \sigma_2 \sqrt{T} \cos\theta^* \right) - K N (d^*) \right\}
\end{aligned} \tag{12}
$$

see Appendix B.[5] Numerical investigations in Carmona and Durrleman (2003a) indicate that their lower bound optimization procedure produces very accurate estimates to the true option value.

6 A closed form spread option formula

It can be verified (see Appendix C) that the Kirk formula follows from the expectation

$$
c_K = e^{-rT} E_0 \left[\left(S_1(T) - \frac{a \cdot (S_2(T))^b}{E \left[(S_2(T))^b \right]} \right)^+ \right] \tag{13}
$$

where $a = F_2 + K$, $b = F_2/(F2 + K)$, and

$$
E_0 \left[(S_2(T))^b \right] = \exp \left\{ \tfrac{1}{2} b(b-1) \sigma_2^2 T \right\} F_2^b
$$

Note that $0 \leq b < 1$ when $K \geq 0$. Observe from above that the implicit strategy is to exercise if and only if $S_1(T)$ exceeds a scaled power function of $S_2(T)$.

We want to use the insight above to obtain an alternative spread option approximation formula. Now, consider the future spread call pay-off conditional on exercising if and only if $S_1(T)$ exceeds a power function of $S_2(T)$, with exponent b and scalar $a/E_0 \left[(S_2(T))^b \right]$. We can express the future pay-off from following this strategy as

$$
c(T) = (S_1(T) - S_2(T) - K) \cdot I \left(S_1(T) \geq \frac{a \cdot (S_2(T))^b}{E \left[(S_2(T))^b \right]} \right) \tag{14}
$$

where $I()$ represents the indicator function, assuming unity whever the argument is true and zero otherwise. Compare Eq.(14) with Eq.(3), and observe

---

[4] There is a typo in Eq.(20) of Carmona and Durleman (2003a) as well as in Eq.(6.3) of Carmona and Durrleman (2003b). The trigonometric function entering the second term should read cos, and not sin.

[5] By the put-call parity, the Carmona-Durrleman approximation of a put on the price spread $S_1(T) - S_2(T)$ with strike $K \geq 0$ and time to exercise T is $p_{CD} = c_{CD} - e^{-rT}(F_1 - F_2 - K)$.

that the specified strategy is feasible but not optimal. Consequently, the spread option value from following such a strategy represents a lower bound to the true spread option value.

Proposition: Approximate the spread call value by the following formula

$$
\begin{aligned}
c\left(a,b\right) &= e^{-rT}E_0\left[\left(S_1\left(T\right)-S_2\left(T\right)-K\right)\cdot I\left(S_1\left(T\right)\geq\frac{a\cdot\left(S_2\left(T\right)\right)^b}{E\left[\left(S_2\left(T\right)\right)^b\right]}\right)\right] \\
&= e^{-rT}\left\{F_1N\left(d_1\right)-F_2N\left(d_2\right)-KN\left(d_3\right)\right\} \quad (15)
\end{aligned}
$$

where d_1, d_2, and d_3 are defined by

$$
d_1 = \frac{\ln\left(F_1/a\right)+\left(\frac{1}{2}\sigma_1^2-b\rho\sigma_1\sigma_2+\frac{1}{2}b^2\sigma_2^2\right)T}{\sigma\sqrt{T}} \quad (16)
$$

$$
d_2 = \frac{\ln\left(F_1/a\right)+\left(-\frac{1}{2}\sigma_1^2+\rho\sigma_1\sigma_2+\frac{1}{2}b^2\sigma_2^2-b\sigma_2^2\right)T}{\sigma\sqrt{T}} \quad (17)
$$

$$
d_3 = \frac{\ln\left(F_1/a\right)+\left(-\frac{1}{2}\sigma_1^2+\frac{1}{2}b^2\sigma_2^2\right)T}{\sigma\sqrt{T}} \quad (18)
$$

$$
\sigma = \sqrt{\sigma_1^2-2b\rho\sigma_1\sigma_2+b^2\sigma_2^2} \quad (19)
$$

and where the constants a and b are

$$
a = F_2+K \quad (20)
$$

$$
b = \frac{F_2}{F_2+K} \quad (21)
$$

By the put-call parity, the put option on the price spread $S_1(T)-S_2(T)$ with strike K and time to exercise T is approximated by $p = c - e^{-rT}\left(F_1-F_2-K\right)$.

Proof: See Appendix D.

The Black-Scholes, the Black76, and the Margrabe formulas consist of one term for each component that enters into the future option pay-off. Eqs.(15)-(19) conforms with this tradition. This form is similar to Eq.(6.3) in Carmona and Durrleman (2003b). However, by comparing with our Eqs.(15)-(19), there should be no doubt that our representation of the arguments d_1, d_2, d_3 is more along the lines of the Black-Scholes, Black76, and Margrabe than the corresponding arguments found in Carmona and Durrleman (2003b).

In order to obtain a stricter lower bound, one could optimise the spread call value $c\left(a,b\right)$ above with respect to a and b. Optimal parameters a^* and b^* can be obtained by expanding first order conditions and applying the Newton-Raphson iterative procedure, using Eqs.(20) and (21) as the initial guess. The first order conditions, as well as the second order partials needed for Newton-Raphson, are provided in Appendix E.

Optimizing our formula with respect to a and b is in fact equivalent to the Carmona and Durrleman procedure (see Appendix F). Extensive numerical

investigations, however, indicate that with our initial choice a and b, there is very little to gain from implementing such an optimization procedure. Put differently, the formula stated in Eqs.(15)-(21) above represents a very tight lower bound to the true spread option value.

7   Numerical results

In the following, we compare the accuracy of the Kirk approximation in Eqs.(5)-(8), our formula in Eqs.(15)-(21) above, and the optimal lower bound following from maximizing the formula with respect to the parameters a and b (which is similar to the Carmona-Durrleman optimization procedure). To approximate the true spread option value, we apply a Monte Carlo simulation procedure using the first 100,000 pair of numbers from a two-dimensional Halton sequence. In order to reduce the simulation error, we use our representation of the spread call as control variate.[6]

We adopt the numerical example in Carmona and Durrleman (2003a), where the annual riskless interest rate is $r = 0.05$ and the time horizon is $T = 1$ year. Their numerical case translate into forward prices $F_1 = e^{(0.05-0.03)\cdot 1}110 \approx 112.22$ and $F_2 = e^{(0.05-0.02)\cdot 1}100 \approx 103.05$. The annualized volatilities are $\sigma_1 = 0.10$ and $\sigma_2 = 0.15$.

We consider different combinations of strike K and correlation ρ. In case of $K = 0$, the spread option reduces to the Margrabe exchange option (c.f. Margrabe (op.cit.)). With $K > 0$, the option corresponds to a call on the price spread $S_1(T) - S_2(T)$. In case of $K < 0$, the option represents a put on the opposite price spread, i.e. $S_2(T) - S_1(T)$. The put values are obtained by the put-call parity.

---

[6] Rewrite Eq.(4) as

$$E\left[e^{-rT}C(T)\right] = c + E\left[e^{-rT}(C(T) - c(T))\right]$$

where the pay-offs $C(T)$ and $c(T)$ are defined in Eqs.(3) and (14), and c is our spread option formula in Eqs.(15)-(21). Clearly, the two pay-offs $C(T)$ and $c(T)$ are highly correlated. Consequently, the simulation error from evaluating the expectation on the RHS is much lower than the simulation error from evaluating the expectation on the LHS.

ρ K	-1	-0.5	0	0.3	0.8	1
Table 1. Spread option value approximation						
-20	29.6752	29.0056	28.3848	28.0709	27.7704	27.7538
	29.6561	*28.9948*	*28.3811*	*28.0701*	*27.7701*	*27.7538*
	29.6561	28.9948	28.3811	28.0701	27.7701	27.7538
	29.6561	28.9948	28.3811	28.0701	27.7701	27.7538
-10	21.8787	20.9114	19.8917	19.2710	18.3816	18.2444
	21.8686	*20.9050*	*19.8889*	*19.2701*	*18.3811*	*18.2439*
	21.8686	20.9050	19.8889	19.2701	18.3811	18.2439
	21.8686	20.9049	19.8888	19.2701	18.3811	18.2438
0	15.1332	13.9180	12.5237	11.5618	9.6325	8.8212
	15.1332	*13.9180*	*12.5237*	*11.5618*	*9.6325*	*8.8212*
	15.1332	13.9180	12.5237	11.5618	9.6325	8.8212
	15.1332	13.9180	12.5237	11.5618	9.6325	8.8212
5	12.2425	10.9543	9.4431	8.3649	5.9628	4.4420
	12.2441	*10.9562*	*9.4453*	*8.3674*	*5.9670*	*4.4542*
	12.2441	10.9562	9.4453	8.3674	5.9670	4.4542
	12.2441	10.9562	9.4453	8.3674	5.9670	4.4542
15	7.5376	6.2559	4.7562	3.6907	1.3545	0.0724
	7.5218	*6.2422*	*4.7445*	*3.6798*	*1.3425*	*0.0488*
	7.5218	6.2422	4.7444	3.6797	1.3422	0.0488
	7.5217	6.2421	4.7443	3.6796	1.3421	0.0479
25	4.2475	3.1686	1.9923	1.2441	0.1124	0.0000
	4.2014	*3.1300*	*1.9621*	*1.2200*	*0.1041*	*0.0000*
	4.2014	3.1300	1.9620	1.2198	0.1039	0.0000
	4.2013	3.1298	1.9617	1.2194	0.1032	0.0000
Number on top of each box: Kirk's formula. Second number (in italics) from top of each box: Simulation result (100,000 trials). Third number from the top of each box: Optimizing our formula wrt. a and b. Number on the bottom of each box: Our formula (a and b fixed)).						

Table 1 shows the results of the spread option value approximations. The number on top in each box represents the result from the Kirk formula. The second number (in italics) from top in each box is the spread option value obtained by Monte Carlo computation with 100,000 trials. We use the simulation results as the benchmark for the true spread option value. The third number from top of each box is the result from optimizing our formula with respect to a and b, which is similar to the Carmona-Durrleman optimization procedure. The number at the bottom of each box represents the result from our formula.

It is interesting to observe from Table 1 that the Kirk formula violates the lower bound (provided by our formula) for all correlations when the strike is $K = 5$.

Table 2. Pricing error						
ρ K	-1	-0.5	0	0.3	0.8	1
-20	0.0191 0.0000	0.0108 0.0000	0.0037 -0.0001	0.0008 0.0000	0.0003 0.0000	0.0001 0.0000
-10	0.0101 0.0000	0.0065 0.0000	0.0029 0.0000	0.0010 0.0000	0.0005 0.0000	0.0006 0.0000
0	0.0000 0.0000	0.0000 0.0000	0.0000 0.0000	0.0000 0.0000	0.0000 0.0000	0.0000 0.0000
5	- 0.0016 0.0000	-0.0019 0.0000	- 0.0023 0.0000	- 0.0025 0.0000	- 0.0042 0.0000	- 0.0122 0.0000
15	0.0158 -0.0001	0.0137 - 0.0001	0.0118 -0.0001	0.0109 - 0.0002	0.0120 - 0.0004	0.0236 - 0.0009
25	0.0461 - 0.0001	0.0386 - 0.0002	0.0301 -0.0004	0.0241 - 0.0006	0.0083 - 0.0009	0.0000 0.0000
Number on top of each box: Approximation error following from the Kirk formula. Number on the bottom of each box: Approximation error following from our formula.						

Table 2 shows the approximation error associated with the Kirk formula (number on top of each box) and our formula (number on bottom of each box), as compared to the benchmark. Note that the Kirk formula seems to underprice the spread option when the strike is closer to zero, and to overprice the spread option when the strike is further away from zero. Our formula represents a lower bound, hence the approximation error (if any) is negative. Observe that for all relevant cases in Table 2,[7] our formula performs much better than the Kirk formula. In our view, practitioners looking for a pricing formula are better off using our formula than the Kirk formula when evaluating spread options.

Table 3. Improved accuracy obtained by optimization						
ρ K	-1	-0.5	0	0.3	0.8	1
-20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15	0.0001	0.0001	0.0001	0.0001	0.0002	0.0009
25	0.0001	0.0001	0.0003	0.0004	0.0007	0.0000
Number in each box: Improved accuracy obtained by optimization.						

Table 3 shows the improved accuracy by optimizing our formula with respect to a and b, as compared to using our formula with parameter values for a and b in Eqs.(20) and (21). Recall that optimizing our formula corresponds to the Carmona-Durrleman optimization procedure. Hence, we may interpret the

---

[7] When $K = 0$, both formulas degenerate to the Margrabe exhange option formula, which represents the true value in this case. Hence, the pricing errors are zero in these cases.

results in Table 3 as the gain form using their numerical optimization procedure as compared to our formula which is closed form. The results in the table indicate that the improved accuracy from implementing numerical optimization is either marginal or zero. For practical purposes, the benefits of a closed form solution are obvious. In our view, the numerical results indicate that the accuracy of our formula is comparable with the accuracy of using an optimization procedure. Consequently, practioners should settle for our formula rather than the Carmona-Durrleman procedure when evaluating spread options.

8   Conclusions

This paper considers the valuation of a European spread option when the asset prices are lognormal. We derive a spread option formula that consists of three terms, one for each of the two assets and one for the strike. A standard normal cumulative probability enters into each term, and each argument is a function of the forward prices, time to exercise, volatilities, and correlation. The formula fits well in to the tradition of Black-Scholes, Black76, and Margrabe.

Numerical investigations indicate that our formula is extremely accurate. The precision is much better than the Kirk formula, which is the current market standard in practice. Moreover, the precision of our formula is comparable the lower bound procedure of Carmona and Durrleman, which requires a two-dimensional optimization scheme.

Black, F. (1976): "The pricing of commodity contracts," *Journal of Financial Economics* 3 (1976), pp. 167-179.

Carmona, R. and V. Durrleman (2003a): "Pricing and Hedging spread options in a log-normal model," Tech. report, Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ, 2003.

Carmona, R. and V. Durrleman (2003b): "Pricing and hedging spread options," *SIAM Review* Vol. 45, No. 4, pp. 627-685.

Cox, J. and S. Ross (1976): "The valuation of options for alternative stochastic processes," *Journal of Financial Economics*, 3 (January-March), pp. 145-166.

Harrison, J.M. and D. Kreps (1979): "Martingales and arbitrage in multiperiod security markets," *Journal of Economic Theory* 20 (July). pp. 381-408.

Harrison, J.M. and S. Pliska (1981): "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and Their Applications*, 11, pp. 313-316.

Kirk, E. (1995): "Correlation in the energy markets," In *Managing Energy Price Risk* (First Edition). London: Risk Publications and Enron, pp. 71-78.

Margrabe, W. (1978): "The value of an option to exchange one asset for another," *Journal of Finance*, 33 (1978), pp. 177-186.

A Bivariate normal variables - a useful result

The standard bivariate normal density function is defined by

$$m(x, y; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\}$$

where ρ is correlation. The density function satisfies the identity

$$\exp\left\{(ax+by) - \tfrac{1}{2}\left(a^2 + 2\rho ab + b^2\right)\right\} m(x, y; \rho) = m(x - (a + \rho b), y - (\rho a + b); \rho)$$

where $E\left[\exp\left\{(ax+by) - \tfrac{1}{2}\left(a^2 + 2\rho ab + b^2\right)\right\}\right] = 1$.
   Consequently,

$$
\begin{aligned}
&E\left[\exp\left\{(ax+by) - \tfrac{1}{2}\left(a^2 + 2\rho ab + b^2\right)\right\} h(x, y)\right] \\
=\ & \int\int \exp\left\{(ax+by) - \tfrac{1}{2}\left(a^2 + 2\rho ab + b^2\right)\right\} h(x, y) m(x, y; \rho) dy dx \\
=\ & \int\int h(x, y) m(x - (a + \rho b), y - (\rho a + b); \rho)\, dy dx \\
=\ & \int\int h(x + (a + \rho b), y + (\rho a + b)) m(x, y; \rho)\, dy dx \\
=\ & E\left[h(x + (a + \rho b), y + (\rho a + b))\right]
\end{aligned}
$$

B The Carmona-Durrleman result

For notational convenience, write

$$
\begin{aligned}
X_1 &= F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1(z_1 \sin\phi + z_2 \cos\phi)\right\} \\
X_2 &= F_2 \exp\left\{-\tfrac{1}{2}v_2^2 + v_2 z_2\right\}
\end{aligned}
$$

where z_1 and z_2 are independent and standard normal. Consider the expectation

$$
\begin{aligned}
c_{CD} &= E\left[(X_1 - X_2 - K) I\left(z_1 \sin\theta^* - z_2 \cos\theta^* \leq d^*\right)\right] \\
&= E\left[X_1 I\left(z_1 \sin\theta^* - z_2 \cos\theta^* \leq d^*\right)\right] \\
&\quad - E\left[X_2 I\left(z_1 \sin\theta^* - z_2 \cos\theta^* \leq d^*\right)\right] \\
&\quad - E\left[K I\left(z_1 \sin\theta^* - z_2 \cos\theta^* \leq d^*\right)\right]
\end{aligned}
$$

Observe that due to the identity

12

$$(\sin \phi)^2 + (\cos \phi)^2 = 1$$

both $z \equiv (z_1 \sin \phi + z_2 \cos \phi)$ and $Y_{\theta^*} \equiv (z_1 \sin \theta^* - z_2 \cos \theta^*)$ are standard normal. Evaluate the last term

$$
\begin{aligned}
& E\left[KI\left(z_1 \sin \theta^* - z_2 \cos \theta^* \le d^*\right)\right] \\
= \quad & KE\left[I\left(z \le d^*\right)\right] \\
= \quad & KN\left(d^*\right)
\end{aligned}
$$

Evaluate the second term

$$
\begin{aligned}
& E\left[X_2 I\left(z_1 \sin \theta^* - z_2 \cos \theta^* \le d^*\right)\right] \\
= \quad & E\left[F_2 \exp\left\{-\tfrac{1}{2}v_2^2 + v_2 z_2\right\} I\left(z_1 \sin \theta^* - z_2 \cos \theta^* \le d^*\right)\right] \\
= \quad & F_2 E\left[I\left(z_1 \sin \theta^* - (z_2 + v_2)\cos \theta^* \le d^*\right)\right] \\
= \quad & F_2 E\left[I\left(z_1 \sin \theta^* - z_2 \cos \theta^* \le d^* + v_2 \cos \theta^*\right)\right] \\
= \quad & F_2 E\left[I\left(z \le d^* + v_2 \cos \theta^*\right)\right] \\
= \quad & F_2 N\left(d^* + v_2 \cos \theta^*\right)
\end{aligned}
$$

using the result in Appendix A. And finally, evaluate the first term

$$
\begin{aligned}
& E\left[X_1 I\left(z_1 \sin \theta^* - z_2 \cos \theta^* \le d^*\right)\right] \\
= \quad & E\left[F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1 (z_1 \sin \phi + z_2 \cos \phi)\right\} I\left(z_1 \sin \theta^* - z_2 \cos \theta^* \le d^*\right)\right] \\
= \quad & F_1 E\left[I\left(\sin \theta^* (z_1 + v_1 \sin \phi) - \cos \theta^* (z_2 + v_1 \cos \phi) \le d^*\right)\right] \\
= \quad & F_1 E\left[I\left(z_1 \sin \theta^* - z_2 \cos \theta^* \le d^* + v_1 (\cos \theta^* \cos \phi - \sin \theta^* \sin \phi)\right)\right] \\
= \quad & F_1 E\left[I\left(z \le d^* + v_1 \cos (\theta^* + \phi)\right)\right] \\
= \quad & F_1 N\left(d^* + v_1 \cos (\theta^* + \phi)\right)
\end{aligned}
$$

where we use the result in Appendix A and the identity

$$\cos \theta^* \cos \phi - \sin \theta^* \sin \phi = \cos (\theta^* + \phi)$$

Now, collect the results, apply riskless discounting, and translate $v_1 = \sigma_1 \sqrt{T}$ and $v_2 = \sigma_2 \sqrt{T}$, to obtain the result in Eq.(12).

C   The implicit Kirk strategy

For notational convenience, write

$$X_1 = F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1\varepsilon_1\right\}$$
$$X_2 = F_2 \exp\left\{-\tfrac{1}{2}v_2^2 + v_2\varepsilon_2\right\}$$
$$\frac{aX_2^b}{E\left[X_2^b\right]} = a\exp\left\{-\tfrac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}$$

and consider the expectation

$$E\left[\left(X_1 - \frac{aX_2^b}{E\left[X_2^b\right]}\right)^+\right] = E\left[X_1 I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right] - E\left[aX_2^b I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right]$$

The two terms are evaluated as follows:

$$
\begin{aligned}
&E\left[X_1 I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right] \\
=\ & E\left[F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1\varepsilon_1\right\} I\left(F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\tfrac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}\right)\right] \\
=\ & F_1 E\left[I\left(F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1(\varepsilon_1 + v_1)\right\} \geq a\exp\left\{-\tfrac{1}{2}b^2v_2^2 + bv_2(\varepsilon_2 + \rho v_1)\right\}\right)\right] \\
=\ & F_1 E\left[I\left(v_1\varepsilon_1 - bv_2\varepsilon_2 \geq -\ln(F_1/a) - \tfrac{1}{2}v_1^2 + b\rho v_1 v_2 - \tfrac{1}{2}b^2v_2^2\right)\right] \\
=\ & F_1 E\left[I\left(\varepsilon \geq -\frac{\ln(F_1/a) + \tfrac{1}{2}v_1^2 - b\rho v_1 v_2 + \tfrac{1}{2}b^2v_2^2}{\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2}}\right)\right] \\
=\ & F_1 N\left(\frac{\ln(F_1/a) + \tfrac{1}{2}v_1^2 - b\rho v_1 v_2 + \tfrac{1}{2}b^2v_2^2}{\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[\frac{aX_2^b}{E\left[X_2^b\right]} I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right] \\
=\ & E\left[a\exp\left\{-\tfrac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\} I\left(F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\tfrac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}\right)\right] \\
=\ & aE\left[\exp\left\{-\tfrac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\} I\left(F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\tfrac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}\right)\right] \\
=\ & aE\left[I\left(F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1(\varepsilon_1 + \rho bv_2)\right\} \geq a\exp\left\{-\tfrac{1}{2}b^2v_2^2 + bv_2(\varepsilon_2 + bv_2)\right\}\right)\right] \\
=\ & aE\left[I\left(v_1\varepsilon_1 - bv_2\varepsilon_2 \geq -\ln(F_1/a) + \tfrac{1}{2}v_1^2 - b\rho v_1 v_2 + \tfrac{1}{2}b^2v_2^2\right)\right] \\
=\ & aE\left[I\left(\varepsilon \geq -\frac{\ln(F_1/a) - \tfrac{1}{2}v_1^2 + b\rho v_1 v_2 - \tfrac{1}{2}b^2v_2^2}{\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2}}\right)\right] \\
=\ & aN\left(\frac{\ln(F_1/a) - \tfrac{1}{2}v_1^2 + b\rho v_1 v_2 - \tfrac{1}{2}b^2v_2^2}{\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2}}\right)
\end{aligned}
$$

Now, collect the results, apply riskless discounting, choose the constant a such that

$$a = F_2 + K$$

and translate $v_1 = \sigma_1\sqrt{T}$; $v_2 = \sigma_2\sqrt{T}$; $\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2} = \sigma\sqrt{T}$, to obtain the Kirk formula stated in Eqs.(5) - (8) above.

D   Derivation of the spread option formula

Consider the expectation

$$
E\left[(X_1 - X_2 - K)\,I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right]
$$
$$
= E\left[X_1 I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right] - E\left[X_2 I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right] - E\left[I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right]
$$

The first term is evaluated in Appendix C. The two remaining terms are evaluated as follows

$$
E\left[X_2 I\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right]
$$
$$
= E\left[F_2 \exp\left\{-\tfrac{1}{2}v_2^2 + v_2\varepsilon_2\right\} I\left(F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\tfrac{1}{2}b^2 v_2^2 + bv_2\varepsilon_2\right\}\right)\right]
$$
$$
= F_2 E\left[I\left(F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1(\varepsilon_1 + \rho v_2)\right\} \geq a\exp\left\{-\tfrac{1}{2}b^2 v_2^2 + bv_2(\varepsilon_2 + v_2)\right\}\right)\right]
$$
$$
= F_2 E\left[I\left(v_1\varepsilon_1 - bv_2\varepsilon_2 \geq -\ln(F_1/a) + \tfrac{1}{2}v_1^2 - \rho v_1 v_2 - \tfrac{1}{2}b^2 v_2^2 + bv_2^2\right)\right]
$$
$$
= F_2 E\left[I\left(\varepsilon \geq -\frac{\ln(F_1/a) + \tfrac{1}{2}v_1^2 - \rho v_1 v_2 - \tfrac{1}{2}b^2 v_2^2 + bv_2^2}{\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2}}\right)\right]
$$
$$
= F_2 N\left(\frac{\ln(F_1/a) - \tfrac{1}{2}v_1^2 + \rho v_1 v_2 + \tfrac{1}{2}b^2 v_2^2 - bv_2^2}{\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2}}\right)
$$

$$
E\left[KI\left(X_1 \geq \frac{aX_2^b}{E\left[X_2^b\right]}\right)\right]
$$
$$
= KE\left[I\left(F_1 \exp\left\{-\tfrac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\tfrac{1}{2}b^2 v_2^2 + bv_2\varepsilon_2\right\}\right)\right]
$$
$$
= KE\left[v_1\varepsilon_1 - bv_2\varepsilon_2 \geq -\ln(F_1/a) + \tfrac{1}{2}v_1^2 - \tfrac{1}{2}b^2 v_2^2\right]
$$
$$
= KE\left[\varepsilon \geq -\frac{\ln(F_1/a) - \tfrac{1}{2}v_1^2 + \tfrac{1}{2}b^2 v_2^2}{\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2}}\right]
$$
$$
= KN\left(\frac{\ln(F_1/a) - \tfrac{1}{2}v_1^2 + \tfrac{1}{2}b^2 v_2^2}{\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2}}\right)
$$

Collect the results, and translate $v_1 = \sigma_1 \sqrt{T}$; $v_2 = \sigma_2 \sqrt{T}$; $\sqrt{v_1^2 - 2b\rho v_1 v_2 + b^2 v_2^2} = \sqrt{(\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2)\,T} = \sigma\sqrt{T}$, to obtain the result stated as Eqs.(15)-(19) 1 above.

E   Optimizing wrt. the exercise strategy

Define

$$
\begin{aligned}
H(a,b) &= e^{-rT}\left\{ F_1 N(d_1) - F_2 N(d_2) - K N(d_3) \right\} \\
d_1 &= \frac{\ln(F_1/a) + \left(\frac{1}{2}\sigma_1^2 - b\rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2\right) T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln(F_1/a) + \left(-\frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2 - b\sigma_2^2\right) T}{\sigma\sqrt{T}} \\
d_3 &= \frac{\ln(F_1/a) + \left(-\frac{1}{2}\sigma_1^2 + \frac{1}{2}b^2\sigma_2^2\right) T}{\sigma\sqrt{T}} \\
\sigma &= \sqrt{\sigma_1^2 - 2b\sigma_1\sigma_2 + b^2\sigma_2^2}
\end{aligned}
$$

First, establish

$$
\begin{aligned}
\frac{\partial d_1}{\partial a} &= \frac{\partial d_2}{\partial a} = \frac{\partial d_3}{\partial a} = \frac{-1}{a\sigma\sqrt{T}} \\
\frac{\partial \sigma}{\partial b} &= \widehat{\rho}\sigma \\
\frac{\partial d_1}{\partial b} &= C_1 - d_1\widehat{\rho} \\
\frac{\partial d_2}{\partial b} &= C_2 - d_2\widehat{\rho} \\
\frac{\partial d_3}{\partial b} &= C_3 - d_3\widehat{\rho}
\end{aligned}
$$

where we for notational convenience define

$$
\begin{aligned}
\widehat{\rho} &= \frac{b\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma^2} \\
C_1 &= \frac{-\rho\sigma_1\sigma_2 T + b\sigma_2^2 T}{\sigma\sqrt{T}} \\
C_2 &= \frac{b\sigma_2^2 T - \sigma_2^2 T}{\sigma\sqrt{T}} \\
C_3 &= \frac{b\sigma_2^2 T}{\sigma\sqrt{T}}
\end{aligned}
$$

16

Consequently, the first order partials of H are

$$
\begin{aligned}
H_a &= -e^{-rT}\left\{F_1 n\left(d_1\right) - F_2 n\left(d_2\right) - K n\left(d_3\right)\right\}\frac{1}{a\sigma\sqrt{T}}\\
H_b &= -e^{-rT}\left\{F_1 n\left(d_1\right)d_1 - F_2 n\left(d_2\right)d_2 - K n\left(d_3\right)d_3\right\}\widehat{\rho}\\
&\quad +e^{-rT}\left\{F_1 n\left(d_1\right)C_1 - F_2 n\left(d_2\right)C_2 - K n\left(d_3\right)C_3\right\}
\end{aligned}
$$

Next, obtain the second order partials

$$
\begin{aligned}
\frac{\partial^2 d_1}{\partial a^2} &= \frac{\partial^2 d_2}{\partial a^2} = \frac{\partial^2 d_3}{\partial a^2} = \frac{1}{a^2\sigma\sqrt{T}}\\
\frac{\partial^2 d_1}{\partial a\partial b} &= \frac{\partial^2 d_2}{\partial a\partial b} = \frac{\partial^2 d_3}{\partial a\partial b} = \frac{\widehat{\rho}}{a\sigma\sqrt{T}}\\
\frac{\partial^2 d_1}{\partial b^2} &= \frac{\sigma_2^2}{\sigma} - 2C_1\widehat{\rho} + \left(3\widehat{\rho}^2 - \frac{\sigma_2^2}{\sigma^2}\right)d_1\\
\frac{\partial^2 d_2}{\partial b^2} &= \frac{\sigma_2^2}{\sigma} - 2C_2\widehat{\rho} + \left(3\widehat{\rho}^2 - \frac{\sigma_2^2}{\sigma^2}\right)d_2\\
\frac{\partial^2 d_2}{\partial b^2} &= \frac{\sigma_2^2}{\sigma} - 2C_3\widehat{\rho} + \left(3\widehat{\rho}^2 - \frac{\sigma_2^2}{\sigma^2}\right)d_3
\end{aligned}
$$

and the second order partials of H are

$$
\begin{aligned}
H_{aa} &= -e^{-rT}\left\{F_1 n\left(d_1\right)d_1 - F_2 n\left(d_2\right)d_2 - K n\left(d_3\right)d_3\right\}\frac{1}{a^2\sigma^2 T}\\
&\quad +e^{-rT}\left\{F_1 n\left(d_1\right) - F_2 n\left(d_2\right) - K n\left(d_3\right)\right\}\frac{1}{a^2\sigma\sqrt{T}}\\
H_{ab} &= H_{ba} = -e^{-rT}\left\{F_1 n\left(d_1\right)d_1^2 - F_2 n\left(d_2\right)d_2^2 - K n\left(d_3\right)d_3^2\right\}\frac{\widehat{\rho}}{a\sigma\sqrt{T}}\\
&\quad +e^{-rT}\left\{F_1 n\left(d_1\right)C_1 d_1 - F_2 n\left(d_2\right)C_2 d_2 - K n\left(d_3\right)C_3 d_3\right\}\frac{1}{a\sigma\sqrt{T}}\\
&\quad +e^{-rT}\left\{F_1 n\left(d_1\right) - F_2 n\left(d_2\right) - K n\left(d_3\right)\right\}\frac{\widehat{\rho}}{a\sigma\sqrt{T}}\\
H_{bb} &= -e^{-rT}\left\{F_1 n\left(d_1\right)d_1^3 - F_2 n\left(d_2\right)d_2^3 - K n\left(d_3\right)d_3^3\right\}\widehat{\rho}^2\\
&\quad +e^{-rT}\left\{F_1 n\left(d_1\right)C_1 d_1^2 - F_2 n\left(d_2\right)C_2 d_2^2 - K n\left(d_3\right)C_3 d_3^2\right\}2\widehat{\rho}\\
&\quad -e^{-rT}\left\{F_1 n\left(d_1\right)C_1^2 d_1 - F_2 n\left(d_2\right)C_2^2 d_2 - K n\left(d_3\right)C_3^2 d_3\right\}\\
&\quad +e^{-rT}\left\{F_1 n\left(d_1\right)d_1 - F_2 n\left(d_2\right)d_2 - K n\left(d_3\right)d_3\right\}\left(3\widehat{\rho}^2 - \frac{\sigma_2^2}{\sigma^2}\right)\\
&\quad -e^{-rT}\left\{F_1 n\left(d_1\right)C_1 - F_2 n\left(d_2\right)C_2 - K n\left(d_3\right)C_3\right\}2\widehat{\rho}\\
&\quad +e^{-rT}\left\{F_1 n\left(d_1\right) - F_2 n\left(d_2\right) - K n\left(d_3\right)\right\}\frac{\sigma_2^2}{\sigma}
\end{aligned}
$$

We now have the necessary results to implement the Newton-Raphson iterative procedure, using $a = F_2 + K$ and $b = F_2/(F_2 + K)$ as our initial guess.

F   Optimizing our formula and the Carmona-Durrleman procedure

Compare Eq.(12) with Eqs.(15)-(19), assuming that a and b are optimal. Note that it is sufficient to show that the arguments of $N()$ equal for each of the three terms. Firstly, let $d^* = d_3$. Secondly, let $\sigma_2\sqrt{T}\cos\theta^* = d_2 - d_3$ which leads to $\cos\theta^* = (\rho\sigma_1 - b\sigma_2)/\sigma$.

Hence, we need to show that $\sigma_1\sqrt{T}\cos(\theta^* + \phi) = d_1 - d_3$. Recall that $\cos\phi = \rho$, and that $\phi \in [0, \pi]$ and $\theta^* \in [\pi, 2\pi]$ (see footnote 3). Obtain the result as follows:

$$
\begin{aligned}
\cos(\theta^* + \phi) &= \cos(\theta^*)\cos(\phi) - \sin(\theta^*)\sin(\phi) \\
&= \frac{\rho\sigma_1 - b\sigma_2}{\sigma}\rho - (-1)\sqrt{1 - \left(\frac{\rho\sigma_1 - b\sigma_2}{\sigma}\right)^2}\sqrt{1 - \rho^2} \\
&= \frac{\rho^2\sigma_1 - b\rho\sigma_2}{\sigma} + \sqrt{\frac{\sigma_1^2 - \rho^2\sigma_1^2}{\sigma^2}}\sqrt{1 - \rho^2} \\
&= \frac{\rho^2\sigma_1 - b\rho\sigma_2}{\sigma} + \frac{\sigma_1\left(1 - \rho^2\right)}{\sigma} \\
&= \frac{\sigma_1 - b\rho\sigma_2}{\sigma} \\
&= \frac{1}{\sigma_1\sqrt{T}}\frac{\left(\sigma_1^2 - b\rho\sigma_1\sigma_2\right)T}{\sigma\sqrt{T}} \\
&= \frac{d_1 - d_3}{\sigma_1\sqrt{T}}
\end{aligned}
$$

Consequently, optimizing our formula with respect to a and b is equivalent to the Carmona-Durrleman procedure.