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# The Nash Bargaining Solution vs. Equilibrium in a Reinsurance Syndicate

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## Abstract

We compare the Nash bargaining solution in a reinsurance syndicate to the competitive equilibrium allocation, focusing on uncertainty and risk aversion. Restricting attention to proportional reinsurance treaties, we find that, although these solution concepts are very different, one may just appear as a first order Taylor series approximation of the other, in certain cases. This may be good news for the Nash solution, or for the equilibrium allocation, all depending upon one's point of view.

Our model also allows us to readily identify some properties of the equilibrium allocation not shared by the bargaining solution, and vice versa, related to both risk aversions and correlations.

*KEYWORDS: Nash's Bargaining Solution, Equilibrium, Pareto Optimal Risk Exchange, Reinsurance Treaties, Uncertainty, Risk Aversion, Correlations, Multinormal Universe*

## I Introduction

We consider the situation of two or more parties who negotiate with the view of concluding a reciprocal reinsurance treaty. We assume that the agents are

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under no compulsion to reach an agreement. This means that if the parties conclude a treaty, it must be such that all parties consider themselves better off than without any treaty. We further assume that no outside party can break into the negotiations. This means that the agents either have to come to terms, or be without any reinsurance.

Cooperative game theory come into play in situations like these, where a set of Pareto optimal allocations are characterized, termed the core, but this theory alone is generally unable to provide a unique solution to a bargaining situation. An equilibrium can be considered as such a solution, although an equilibrium need not always be unique. Another principle we shall discuss in this paper is the Nash bargaining solution, which leads to a unique solution of a bargaining problem.

The core of the standard reinsurance syndicate was analyzed by Baton and Lemaire (1981a), where it was characterized in the case of negative exponential utility, and shown to be nonempty. Different aspects of the core were further developed in Aase (2002). The bargaining set, first introduced by Aumann and Maschler (1964), was analyzed in Baton and Lemaire (1981b) for a reinsurance syndicate. The bargaining set contains the core, and is accordingly not related to Nash's bargaining solution.

Nash (1950) proposed a set of axioms which leads to a determinate solution in the general case. In addition to the axioms that guarantee an expected utility representation, Nash requires (weak) Pareto optimality, an assumption about "symmetry" and a third assumption we may call "independence of unchosen alternatives", all of which seem plausible in most situations.

In Nash's terminology, consider two player bargaining games defined by a pair  $(S, d)$  where  $d$  is a point in the plane, and  $S$  is a compact convex subset of the plane containing  $d$  and at least one point  $x$  such that  $x > d$ . The interpretation is that  $S$  is the set of feasible expected utility payoffs to the players, any one of which will result if agreed to by both players. If no agreement is reached, the disagreement point  $d$  results.

Nash proposed that bargaining between rational players be modeled by a function called a *solution*, which selects a feasible outcome for every bargaining game. If  $B$  denotes the class of all two-player bargaining games, a solution is a function  $c : B \rightarrow R^2$  such that  $c(S, d)$  is in  $S$ . His last two assumptions are then: (i) If  $(S, d)$  is a symmetric game (i.e., if  $(x_1, x_2) \in S$  implies  $(x_2, x_1) \in S$  and if  $d_1 = d_2$ ), then  $c_1(S, d) = c_2(S, d)$ . (ii) If  $T$  contains a set  $S$  and  $c(T, d)$  is in  $S$ , then  $c(T, d) = c(S, d)$ .

The assumption of "symmetry" was asserted by Nash (1950) to "express equality of bargaining skills" but later (1953) he disavows this interpretation. It says that the labels of the agents do not matter: if switching the labels of the members leaves the bargaining problem unchanged, then it should

leave the solution unchanged. The "independence of unchosen alternatives" assumption was initially termed "independence of irrelevant alternatives". Nash (1950) then shows that these axioms imply that the solution is the one that maximizes the product of the utility gains.

The Nash solution of the bargaining problem is essentially an elegant mathematical derivation of the solution from a few, simple and apparently acceptable axioms. Harsanyi (1965) has pointed out that Nash's solution is identical with a solution offered by Zeuthen (1930) more than 20 years earlier. Zeuthen reaches his result after an analysis of how the actual bargaining takes place.

Nash showed that this procedure and no others satisfies the assumption of "independence of unchosen alternatives"<sup>1</sup>, meaning that the point chosen remains the same if unchosen points are removed from the feasible set.

In the context of fair division it has been pointed out (see Pratt (2007)) that unchosen points may not be irrelevant, because they represent what agents have given up. This criticism is relevant if the objective is to infer preferences from observations of the choices that people make. In the present model revealed preferences is not the issue, since the utility functions are exogenous, so we find this assumption acceptable in our setting. The Nash-framework is, however, used in several other situations in economics, where this assumption is not considered innocuous. These assumptions have been discussed amply elsewhere (cf. Nash (1950), Luce and Raiffa (1957), Harsanyi (1977) and Roth (1977-79)).

Just to illustrate some of the arguments from the lively debate that has been going on around the axioms of Nash, let us cite a passage from Luce and Raiffa (1957) regarding the "independence of unchosen alternatives" axiom. The discussion is centered around an example where the solution is the midpoint of the two players feasibility sets, and then one of the players gets his set halved. Is it now reasonable that the same point is still the solution? Or, starting with this solution point, and then one of the players gets his feasibility set doubled: Is it reasonable for this player to argue that he now deserves more? If so, this axiom is violated. Luce and Raiffa write (p 133)

*"We feel at this time - the implications being that we have changed our minds in the past - that this argument against assuming independence of irrelevant alternatives loses its appeal when applied to bargaining problems; the reason is that the naturally distinguished trade, the status quo, serves to point*

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<sup>1</sup>This axiom should not be confused with one of the key assumptions behind the expected utility representation of von Neumann and Morgenstern, also part of Nash's assumptions, called the substitution or the independence axiom.

*out that certain aspirations are merely empty dreams.”*

They also point out about this axiom that although it may itself be reasonable, there are numerous related assumptions which appear to be equally plausible at first glance but which are not. The typical problem with these alternatives is that they are internally inconsistent with the axiom of Pareto optimality.

Our description of the bargaining problem so far follows the usual custom in describing bargaining games solely in terms of the feasible utility payoffs available to the members (players), without specifying the particular bargains which yield those utilities. To consider the effects of uncertainty and e.g., risk aversion, we need to consider alternatives over which bargaining is conducted.

In the model that we present below, the status quo is represented by a random vector  $X = (X_1, X_2, \dots, X_I)$ , and the space of feasible outcomes are realizations of random vectors  $Y = (Y_1, Y_2, \dots, Y_I)$ . The significance of these variables will be explained in the next section. Each random variable  $Y_i$  is some function of  $X$ , i.e.,  $Y_i = f_i(X_1, X_2, \dots, X_I)$ , where  $f_i$  are Borel-measurable functions. The random variables  $Y_i$  are members of the infinite-dimensional vector space  $L^2(\Omega, \mathcal{F}, P)$ . Here  $\Omega$  signifies the set of states and  $\mathcal{F}$  the set of events, where  $\mathcal{F} = \sigma(X_1, X_2, \dots, X_I)$ , i.e.,  $\mathcal{F}$  is generated by the "status quo" random vector  $X$ . Finally the probability measure  $P$  is intimately connected to the probability distribution function  $F$  of  $X$ :

Formally this connection is the following: If  $F(x_1, x_2, \dots, x_I)$  is a cumulative probability distribution function, it gives rise to the probability space  $(R^I, \mathcal{B}, \mu_F)$  on Euclidian  $R^I$ -space, where  $\mathcal{B}$  is the Borel sets in  $R^I$ , and  $\mu_F$  is the associated probability measure on  $\mathcal{B}$  generated by  $F$ , i.e., the one in which  $\mu_F((-\infty, x_1] \times \dots \times (-\infty, x_I]) = F(x_1, \dots, x_I)$  for all  $x \in R^I$ . Then we know that there exists a random vector  $X$  and a probability space  $(\Omega, \mathcal{F}, P)$  such that  $X : (\Omega, \mathcal{F}) \rightarrow (R^I, \mathcal{B})$  and  $P(A) := PX^{-1}(B) = \mu_F(B)$  for any  $A \in \mathcal{F}$  and  $B \in \mathcal{B}$ . We then consider  $L^2(\Omega, \mathcal{F}, P)$  as the bargaining set in this paper.

In the model that we analyze,  $F$  is the joint cumulative normal distribution function, in which case situations of the kind discussed in Luce and Raiffa (1957) simply can not arise.

The paper is organized as follows; Section 2 characterizes Nash's bargaining solution in an exchange economy, interpreted as a reinsurance syndicate, and focuses on proportional reinsurance treaties. Section 3 develops new results for the competitive equilibrium under joint normality of the initial portfolios, providing the computational basis for the rest of the paper. Section 4 extends the results obtained for the equilibrium model to the Nash bargaining model, and focus in particular on the effects of risk aversion, and

correlations. In this section we compare the two types of solutions to the bargaining problem. Section 5 concludes.

The conditions of the Nash solution follow next.

## II The Bargaining Solution

We consider a one-period model of a syndicated market with two time points, zero and one. The initial portfolio allocation of the members is denoted by  $X = (X_1, X_2, \dots, X_I)$ , i.e., the one which realizations would result at time one if no reinsurance exchanges took place. At time zero  $X$  is a random vector with a probability distribution  $F(x) = P[X_1 \leq x_1, \dots, X_I \leq x_i]$ . As indicated above, the random vector  $X$  corresponds to the disagreement point  $d$  in Nash's terminology, or the point of status quo ("the naturally distinguished trade"). After reinsurance at time zero the random vector  $Y = (Y_1, Y_2, \dots, Y_I)$  results, the final portfolio. The bargaining solution is the allocation  $Y$  which solves the problem

$$\max_{Z_1, \dots, Z_I} \prod_{i=1}^I (E(u_i(Z_i)) - E(u_i(X_i))) \quad (1)$$

and satisfies

$$\lambda_1 u'_1(Y_1) = \lambda_2 u'_1(Y_2) = \dots = \lambda_I u'_1(Y_I) \quad (\text{Pareto optimality}) \quad (2)$$

and

$$\sum_{i=1}^I Y_i = \sum_{i=1}^I X_i := X_M \quad (\text{market clearing}). \quad (3)$$

Here  $u_i(\cdot)$ ,  $i = 1, 2, \dots, I$  are the utility functions of the agents, or the members of the syndicate. We assume that these are all strictly increasing and concave.

The criterion (1) represents the maximization of the utility gains, which becomes the objective in the Nash bargaining solution. In order to explain the constraint (2), let us first define what is meant by a Pareto optimum. The concept of *Pareto optimality* offers a minimal and uncontroversial test that any social optimal economic outcome should pass. In words, an economic outcome is Pareto optimal if it is impossible to make some individuals better off without making some other individuals worse off.

Let us call a treaty  $Y$  *feasible* if it satisfies  $\sum_{i=1}^I Y_i \leq \sum_{i=1}^I X_i := X_M$ , where by  $X_M$  we mean the "market portfolio", which is just the aggregate of the initial portfolios of the members. Formally our definition of (strong) Pareto optimality is the following

**Definition 1** A feasible allocation  $Y = (Y_1, Y_2, \dots, Y_I)$  is called Pareto optimal if there is no feasible allocation  $Z = (Z_1, Z_2, \dots, Z_I)$  with  $Eu_i(Z_i) \geq Eu_i(Y_i)$  for all  $i$  and with  $Eu_j(Z_j) > Eu_j(Y_j)$  for some  $j$ .

We can then give the following characterization of Pareto optimal allocations (see e.g., Aase (2002-04)):

**Proposition 1** Suppose  $u_i$  are concave and increasing for all  $i$ . Then  $Y$  is a Pareto optimal allocation if and only if there exists a nonzero vector of agent weights  $\lambda \in R_+^I$  such that  $Y = (Y_1, Y_2, \dots, Y_I)$  solves the problem

$$\sup_{(Z_1, \dots, Z_I)} \sum_{i=1}^I \lambda_i Eu_i(Z_i) \quad \text{subject to} \quad \sum_{i=1}^I Z_i \leq X_M. \quad (4)$$

Pareto optimal allocations can now be further characterized under the above conditions. The next result is known as Borch's Theorem (see e.g., Borch (1960-62)):

**Proposition 2** A Pareto optimum  $Y$  is characterized by the existence of non-negative agent weights  $\lambda_1, \lambda_2, \dots, \lambda_I$  and a real function  $\lambda(\cdot) : R \rightarrow R$ , such that

$$\lambda_1 u'_1(Y_1) = \lambda_2 u'_2(Y_2) = \dots = \lambda_I u'_I(Y_I) := \lambda(X_M) \quad a.s. \quad (5)$$

Proposition (2) can be proven from Proposition (1) by the Kuhn-Tucker theorem and a variational argument (see e.g., Aase (2002)). We notice that the constraint (2) is the same as (5).

Karl Borch's characterization of a Pareto optimum  $Y = (Y_1, Y_2, \dots, Y_I)$  simply says that there exist positive "agent" weights  $\lambda_i$  such that the marginal utilities at  $Y$  of all the agents are equal modulo these constants.

The constraint (3) that we have called "market clearing" is part of the definition of Pareto optimal allocations which are assumed to be feasible as we have just seen. The equality sign in (3) is a consequence of strict monotonicity of the utility functions, since there are no "satiation points" under this condition. Finally, it is a result of Ruohonen (1979) that the concepts of weak Pareto optimality and strong Pareto optimality are the same under strict monotonicity and strict concavity. Thus Definition 1 is consistent with Nash's condition of (weak) Pareto optimality under the present conditions.

## II-A The Affine Model

In this section we characterize Nash's bargaining solution for proportional reinsurance treaties.

Borch (1960a-b) discuss the Nash solution in the standard reinsurance model with two parties only, and illustrate for various utility functions, in particular for the quadratic case.

We choose a situation not discussed by Borch. It is the setting where we have a full characterization of the Pareto optimal solutions for the case of affine contracts with zero sum side-payments. These contracts are sufficiently general to be of interest in the present setting, and are also in use in reinsurance markets, often referred to as proportional reinsurance treaties. The contracts originate when the agents have CARA-utility functions with constant risk tolerances, or negative exponential utility functions, in which case the Pareto optimal exchanges satisfying (2) are affine and given by

$$Y_i = \frac{a_i}{A} X_M + b_i, \quad \text{for all } i, \quad (6)$$

where  $a_i$  is agent  $i$ 's risk tolerance,  $A = \sum_j a_j$  and  $b_i$  are the zero-sum side payments, here represented by constants  $b_i$  satisfying  $\sum_j b_j = 0$ . The problem is then to find these constants  $b_i$  consistent with a solution to (1) and (3).

If the ranges of values, or the supports of the random variables  $Y_i$  and  $X_i$  are the same for all  $i$ , the "levels of aspiration" of the members are the same before and after reinsurance, indicating that the axiom "independence of unchosen alternatives" will not be restrictive. This will hold in the present context, since we later make an assumption about joint normality of  $X$ . Under this assumption also  $Y_i$  is normal, since it is a linear function of normal variables.

Since the logarithm is a monotonically, strictly increasing function, the problem can be reformulated as follows: Solve

$$\max_{b_1, b_2, \dots, b_I} \mathcal{L}(b_1, b_2, \dots, b_I; \lambda), \quad (7)$$

where

$$\mathcal{L}(b_1, b_2, \dots, b_I; \lambda) = \sum_{i=1}^I \ln (E(u_i(Y_i)) - E(u_i(X_i))) - \lambda \left( \sum_{i=1}^I b_i \right), \quad (8)$$

and

$$E(u_i(Y_i)) - E(u_i(X_i)) = a_i (E(e^{-X_i/a_i}) - E(e^{-X_M/A})e^{-b_i/a_i}), \quad (9)$$

where the marginal utilities of the members are given by  $u'_i(x) = e^{-x/a_i}$  for  $i = 1, 2, \dots, I$ , satisfying strict monotonicity and strict concavity.

The first order conditions of an optimum are given by

$$\frac{\partial \mathcal{L}(b_1, b_2, \dots, b_I; \lambda)}{\partial b_i} = \frac{E(e^{-X_M/A})e^{-b_i/a_i}}{a_i(E(e^{-X_i/a_i}) - E(e^{-X_M/A})e^{-b_i/a_i})} = \lambda,$$

for  $i = 1, 2, \dots, I$ .

In terms of the Lagrange multiplier  $\lambda$ , the side payments are given by

$$b_i = a_i \left( \ln(E(e^{-X_M/A})(1 + a_i\lambda)) - \ln(a_i\lambda E(e^{-X_i/a_i})) \right) \quad (10)$$

for all  $i$ , and from the constraint that the side payments sum to zero we get that

$$\prod_{i=1}^I \left( \frac{1}{\lambda} + a_i \right)^{a_i} = \frac{\prod_{i=1}^I (a_i E(e^{-X_i/a_i}))^{a_i}}{(E(e^{-X_M/A}))^A}. \quad (11)$$

We have shown the following:

**Theorem 1** *The bargaining solution for proportional reinsurance, or affine reinsurance contracts of the form (6), is the solution to the problem (7)-(9), and is given by the equations (10) and (11).*

In order to gain some insights of the bargaining solution, consider the special case of  $I = 2$  in the above. If the attitudes towards risk are the same for both parties,  $a_1 = a_2 := a$  and  $A = 2a$ , the side payments are

$$b_1 = a \ln \left( \sqrt{\frac{E(e^{-X_2/a_2})}{E(e^{-X_1/a_1})}} \right) \quad \text{and} \quad b_2 = a \ln \left( \sqrt{\frac{E(e^{-X_1/a_1})}{E(e^{-X_2/a_2})}} \right).$$

Suppose  $X_1 \geq X_2$  a.s., i.e., agent 1 is at least as wealthy as agent 2 in (almost) all contingencies of the world, then it is easy to see from the above two expressions that  $b_1 \geq b_2$ , which implies that the final exchange satisfies  $Y_1 \geq Y_2$  a.s., which seems reasonable. If the probability distributions of  $X_1$  and  $X_2$  are the same (but they need not be independent), then  $b_1 = b_2 = 0$ , in which case both parties end up with  $Y_1 = \bar{X}^{(2)} = \frac{1}{2}(X_1 + X_2)$ , i.e., with the mean of the two initial portfolios. This mean is, of course, more stable, or less dispersed than the individual risks, which seems like a reasonable solution under risk aversion and full symmetry between the two parties.

### III The Competitive Equilibrium Solution

In this section we develop new results for the competitive solution under joint normality.

Classical economics sought to explain the way markets coordinate the activities of many distinct individuals each acting in their own self-interest. An elegant synthesis of two hundred years of classical thought was achieved by the general equilibrium theory. The essential message of this theory is that when there are markets and associated prices for all goods and services in the economy, no externalities or public goods and no informational asymmetries or market power, then competitive markets allocate resources efficiently.

Let us briefly describe what we mean by a competitive equilibrium in the reinsurance syndicate: First, the problem each member is supposed to solve is the following:

$$\sup_{Z_i \in L^2} Eu_i(Z_i) \quad \text{subject to} \quad \pi(Z_i) \leq \pi(X_i). \quad (12)$$

The formal definition is:

**Definition 2** *A competitive equilibrium is a collection  $(\pi; Y_1, Y_2, \dots, Y_I)$  consisting of a price functional  $\pi$  and a feasible allocation  $Y = (Y_1, Y_2, \dots, Y_I)$  such that for each  $i$ ,  $Y_i$  solves the problem (12) and markets clear;  $\sum_{i=1}^I Y_i = \sum_{i=1}^I X_i$ .*

Since we do not have any restrictions on contract formation in this model, it can be shown that (e.g., Aase (2002)) *the pricing functional  $\pi$  must be linear and strictly positive if and only if there does not exist any arbitrage*. Since any positive, linear functional on  $L^2 := L^2(\Omega, \mathcal{F}, P)$  is also continuous, by the Riesz Representation Theorem there exists a unique random variable  $\xi \in L^2_+$ , the positive cone of  $L^2(\Omega, \mathcal{F}, P)$ , such that

$$\pi(Z) = E(Z\xi) \quad \text{for all} \quad Z \in L^2.$$

Notice that the system is closed by assuming *rational expectations*. This means that the market clearing price  $\pi$  implied by agent behavior is assumed to be the same as the price functional  $\pi$  on which agent decisions are based. The main analytic issue is then the determination of equilibrium price behavior.

Assume that  $\pi(X_i) > 0$  for each  $i$ . It seems reasonable that each member of the syndicate is required to bring to the market an initial portfolio of positive value. In this case we have the following:

**Theorem 2** *Suppose the preferences of the agents are strictly monotonic and convex, i.e.,  $u'_i > 0$  and  $u''_i \leq 0$  for all  $i \in \mathcal{I}$ , and assume that a competitive equilibrium exists, where  $\pi(X_i) > 0$  for each  $i$ . The equilibrium is then*

characterized by the existence of positive constants  $\alpha_i$ ,  $i \in \mathcal{I}$ , such that for the equilibrium allocation  $(Y_1, Y_2, \dots, Y_I)$

$$u'_i(Y_i) = \alpha_i \xi, \quad a.s. \quad \text{for all} \quad i \in \mathcal{I}, \quad (13)$$

where  $\xi$  is the Riesz representation of the pricing functional  $\pi$ .

Proof: The proof can be found in Aase (2002).  $\square$

In order to explain the variational argument used in this theorem, for problems like (12) we can use the *Kuhn-Tucker Theorem* which says that, granted a suitable constraint qualification, any optimal solution  $Y_i$  will be supported by a Lagrange multiplier  $\alpha_i$ : That is, there exists  $\alpha_i \geq 0$  such that the Lagrangian

$$\mathcal{L}_i(Z_i; \alpha_i) = Eu_i(Z_i) - \alpha_i h(Z_i)$$

is maximal in  $Z_i$  at  $Z_i = Y_i$ . Moreover, complementary slackness holds:  $\alpha_i h(Y_i) = 0$ , where  $h(Z_i) := \pi(Z_i) - \pi(X_i)$ .

In order to find the first order condition, we compute the directional derivative of  $\mathcal{L}_i$  at  $Y_i$  in the "direction"  $Z$ , here denoted  $(\nabla \mathcal{L}_i(Y_i))(Z)$ . It can here be shown to be given by

$$(\nabla \mathcal{L}_i(Y_i))(Z) = E\{(u'_i(Y_i) - \alpha_i \xi)Z\}. \quad (14)$$

A necessary condition for a maximum of  $\mathcal{L}_i$  at  $Y_i$  is that the linear functional in equation (14) is zero in all directions  $Z$ , which leads directly to the condition (13). This condition is also sufficient for an optimum due to the concavity of  $u_i$ .

As for the existence issue of a CE, it is usually a delicate mathematical matter, which we need not consider here (see e.g., Bühlmann H. (1984), Aase (1993) among others).

For the affine model it is known that the equilibrium solution is also of the form (6), i.e., the optimal portfolios  $Y_i$  are given by the expressions

$$Y_i = \frac{a_i}{A} X_M + b_i^c, \quad i = 1, 2, \dots, I,$$

where the side-payments, here denoted by  $b_i^c$ , are given by (see e.g., Aase (1993) and (2002))

$$b_i^c = \frac{E(X_i e^{-X_M/A}) - \frac{a_i}{A} E(X_M e^{-X_M/A})}{E(e^{-X_M/A})}, \quad i = 1, 2, \dots, I. \quad (15)$$

These we found by applying the budget constraints  $\pi(Y_i) = \pi(X_i)$  of each of the syndicate members  $i$ , given in (12), where equality follows from strict monotonicity of preferences.

An equilibrium exists in this situation provided the expectations in these expressions are well-defined. When  $X$  is jointly normal as we shall assume below, the relevant expectations will be shown to exist, so an equilibrium both exists and is unique in this case.

Since the market premium in this syndicate is given by

$$\pi(X_i) = \frac{E(X_i e^{-X_M/A})}{E(e^{-X_M/A})},$$

we notice that the side-payment of agent  $i$ 's reinsured portfolio is this portfolio's value in excess of the agent's "proportion" of the value of the market portfolio, where the "proportion" is the ratio of agent  $i$ 's risk tolerance to the risk tolerance of the market. Although there is no price mechanism in the bargaining solution, the fact that there exist side-payments  $b_i$  makes a comparison between these two solutions both possible and meaningful, since these side-payments can be interpreted as some sort of risk premiums.

In the paper on "fair (and not so fair) division", Pratt (2007) introduces precisely a "geometric-mean" pricing mechanism derived from the participants utilities, and the resulting solution is compared to, among others, the Nash solution. Uncertainty did not play any major role in this paper, and the problems discussed are fair divisions of

... *"everything from a box of bonbons to all the fish in the sea."*

A central theme is that for each item  $i$ , the agents have different valuations for this item, which calls for the setting of some kind of an auction. In our framework, "item"  $i$  is simply the risk  $X_i$  which the agents may have different valuations (utilities) of because of different attitudes towards risk. Their initial allocations may be far from optimal according to the preferences of the agents, which is what gives rise to seeking out a reinsurance arrangement. We are able to use the market prices in a comparison between the two different solution concepts, where uncertainty plays an essential role. The issue of a "fair division" is thus relative to the agents' initial, random endowments  $(X_1, \dots, X_I)$ , and we assume the agents are content in comparing two different Pareto optimal sharing rules.

Returning to the Nash bargaining solution of the last section for the symmetric case when  $I = 2$ , we notice from the above formula (15) that the Nash solution coincides with the competitive equilibrium. From the difference between the two sets of formulas, one may conjecture that this is about the only case where these allocations coincide exactly. However, below we shall see that this is true only to a certain extent.

First notice how different these two principles appear to deal with stochastic dependence, at least in the case where  $I = 2$ . Suppose the probability

distributions of  $X_1$  and  $X_2$  are the same, but that they are negatively correlated. Normally risk averse agents would be further motivated for a reinsurance exchange due to a decreasing correlation from a diversification point of view, whereas an increasing correlation would similarly lead to less demand for reinsurance. We shall return to this conjecture below as well.

Only the fact that the marginal distributions are different will have an effect on the Nash solution, but the side-payments does not seem to depend on the degree of stochastic dependence, or at least this is how it appears from these formulas. This may be different for the competitive solution, where the side payments apparently depend on the correlations as well. In order to analyze these loose observations more closely, we need to specify the probability distribution  $F(x)$  for the random vector  $X$ . Here we choose the multivariate normal universe for an investigation.<sup>2</sup>

### III-A Binormal Initial Portfolio

To start with, we assume that  $I = 2$  and that  $(X_1, X_2)$  is jointly normally distributed with mean vector  $(\mu_1, \mu_2)$ , variances  $\sigma_1^2$  and  $\sigma_2^2$  and  $\text{cov}(X_1, X_2) = \rho\sigma_1\sigma_2$ . Here  $X_M = X_1 + X_2$ ,  $\mu_M := E(X_M) = \mu_1 + \mu_2$ ,  $\sigma_M^2 := \text{var}(X_M) = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$ , and the covariance between  $X_M$  and  $X_1$  is given by  $\text{cov}(X_M, X_1) = \sigma_1^2 + \rho\sigma_1\sigma_2$ . The correlation coefficient  $\rho_{1,M}$  between  $X_1$  and  $X_M$  is

$$\rho_{1,M} = \frac{\sigma_1^2 + \rho\sigma_1\sigma_2}{\sigma_1\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}},$$

and similarly for the correlation coefficient between  $X_2$  and  $X_M$ ,  $\rho_{2,M}$ . Let us denote the joint probability density of  $X_1$  and  $X_2$  by  $f_{1,2}(x_1, x_2)$ , the joint density of  $X_i$  and  $X_M$  by  $f_{i,M}(x_i, y)$  and the conditional density of  $X_M$  given  $X_i = x_i$  by  $f_{M|i}(y|x_i)$ . While the latter density is univariate normal with mean  $\mu_M + \rho_{i,M}\sigma_M(x_i - \mu_i)/\sigma_i$  and variance  $\sigma_M^2(1 - \rho_{i,M}^2)$ ,  $i = 1, 2$ , the three former ones are all bivariate normal. We then have the following result:

**Proposition 3** *Under the above distributional assumption, we have that*

$$E\left(e^{-\frac{X_M}{A}}\right) = e^{-\frac{\mu_M}{A} + \frac{1}{2}\frac{\sigma_M^2}{A^2}}, \quad (16)$$

$$E\left(X_M e^{-\frac{X_M}{A}}\right) = e^{-\frac{\mu_M}{A} + \frac{1}{2}\frac{\sigma_M^2}{A^2}} \left(\mu_M - \frac{\sigma_M^2}{A}\right), \quad (17)$$

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<sup>2</sup>This is a relevant choice for the study of the theoretical properties of these solutions, in particular since correlations are easy to interpret. We are, however, not claiming anything about the realism of this joint distribution in a reinsurance market.

and

$$E\left(X_i e^{-\frac{x_M}{A}}\right) = e^{-\frac{\mu_M}{A} + \frac{1}{2} \frac{\sigma_M^2}{A^2}} \left(\mu_i - \frac{1}{A}(\sigma_i^2 + \rho\sigma_1\sigma_2)\right), \quad (18)$$

for  $i = 1, 2$ . Furthermore, the side-payments are given by the expressions

$$b_i^c = \mu_i - \frac{1}{A}(\sigma_i^2 + \rho\sigma_1\sigma_2) - \frac{a_i}{A}(\mu_M - \frac{\sigma_M^2}{A}), \quad i = 1, 2. \quad (19)$$

Proof: The expression (16) follows from the known formula for the moment generating function of the normal distribution, namely  $E(e^{\theta X}) = \exp(\mu\theta + \sigma^2\theta^2/2)$  when  $X$  is normal with mean  $\mu$  and variance  $\sigma^2$ , and where  $\theta$  is a real parameter; here use  $\theta = -\frac{1}{A}$ .

Consider the expression in (18). We focus on  $i = 1$  and have to compute

$$\begin{aligned} E\left(X_1 e^{-\frac{x_M}{A}}\right) &= \int_{-\infty}^{\infty} x \left( \int_{-\infty}^{\infty} e^{-\frac{y}{A}} f_{1,M}(x, y) dy \right) dx \\ &= \int_{-\infty}^{\infty} x \left( \int_{-\infty}^{\infty} e^{-\frac{y}{A}} f_{M|1}(y|x) dy \right) f_1(x) dx \end{aligned} \quad (20)$$

where  $f_1(x)$  is the marginal probability density of  $X_1$ . Using the moment generating function of the univariate normal distribution, the expression in parentheses in (20) is

$$E\left(e^{-\frac{x_M}{A}} | X_1 = x\right) = \exp\left(-\frac{1}{A}(\mu_M + \rho_{1,M}\sigma_M(x - \mu_1)/\sigma_1) + \frac{1}{2A^2}\sigma_M^2(1 - \rho_{1,M}^2)\right).$$

It follows that

$$E\left(X_1 e^{-\frac{x_M}{A}}\right) = \int_{-\infty}^{\infty} E\left(e^{-\frac{x_M}{A}} | X_1 = x\right) \frac{x}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\left(\frac{x - \mu_1}{\sigma_1}\right)^2} dx.$$

The key to the proof is now to form a full square in the exponent of this integral. Doing this, we obtain

$$E\left(X_1 e^{-\frac{x_M}{A}}\right) = e^{-\frac{1}{A}\mu_M + \frac{1}{2A^2}\sigma_M^2(1 - \rho_{1,M}^2)} \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\left(\frac{x - \kappa}{\sigma_1}\right)^2} e^{\sigma_M^2\rho_{1,M}^2/2A^2} dx,$$

where  $\kappa = \mu_1 - \frac{1}{A}(\sigma_1^2 + \rho\sigma_1\sigma_2)$ . From our last expression we get

$$E\left(X_1 e^{-\frac{x_M}{A}}\right) = \kappa \cdot e^{-\frac{1}{A}\mu_M + \frac{1}{2A^2}\sigma_M^2}$$

which is the conclusion in (18) for  $i = 1$ , the case  $i = 2$  following similarly.

Using the same technique as above, we can also compute the following:

$$E\left(X_M e^{-\frac{X_M}{A}}\right) = \int_{-\infty}^{\infty} y e^{-y/A} \frac{1}{\sqrt{2\pi\sigma_M^2}} e^{-\frac{1}{2}\left(\frac{y-\mu_M}{\sigma_M}\right)^2} dy =$$

$$e^{-\frac{1}{A}\mu_M + \frac{1}{2A^2}\sigma_M^2} \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi\sigma_M^2}} e^{-\frac{1}{2}\left(\frac{y-b}{\sigma_M}\right)^2} dy,$$

where  $b = \mu_M - \frac{\sigma_M^2}{A}$ . This gives (17). Using the expression (15) for the side-payments, and the three formulas just derived, the conclusion in (19) follows.  $\square$

Looking into one of the conjectures presented above, we can now state the following:

**Theorem 3** *Consider the situation of a bivariate normal distribution where  $\mu_1 = \mu_2$  and  $\sigma_1 = \sigma_2$ . For the most risk tolerant agent, the corresponding side-payment decreases when the correlation coefficient  $\rho$  decreases. For the most risk averse agent, the corresponding side-payment increases when  $\rho$  decreases.*

Proof: This can best be seen by rewriting the side-payments in (19) as follows:

$$b_i^c = \mu_i + \frac{\rho\sigma_1\sigma_2}{A} \left(\frac{2a_i}{A} - 1\right) - \frac{1}{A}\sigma_i^2 - \frac{a_i}{A} \left(\mu_M - \frac{1}{A}(\sigma_1^2 + \sigma_2^2)\right).$$

Suppose  $a_1 > a_2$ . Then  $2a_1/A > 1$  since  $A = a_1 + a_2$ , and the conclusion for the most risk tolerant agent follows from the second term above. If  $a_1 < a_2$  then  $2a_1/A < 1$ , so now the second term is negative for the most risk averse agent, so the side-payment increases when  $\rho$  decreases.  $\square$

The theorem says that the most risk averse agent benefits after reinsurance from e.g., a negative correlation between  $X_1$  and  $X_2$ . This conclusion agrees with the above conjecture.

We also have the following corollary:

**Corollary 1** *When  $(\mu_M > \frac{\sigma_M^2}{A})$ , then the most risk averse agent obtains the largest side-payment under the conditions of Theorem 3. This conclusion is reversed when  $(\mu_M < \frac{\sigma_M^2}{A})$ ; then the most risk tolerant participant gets the highest side-payment.*

Proof: This follows from (19) of Proposition 3 and Theorem 3.  $\square$

One way to interpret this is to consider the inequality  $(\mu_M > \frac{\sigma_M^2}{A})$  for given  $\mu_M$  and  $\sigma_M$ . This inequality holds for large enough  $A$ , so the most risk averse agent has an advantage after pooling in a group that is typically risk tolerant. The inequality  $(\mu_M < \frac{\sigma_M^2}{A})$  holds when  $A$  is small enough, which says that in a group which is typically risk averse, the most risk tolerant has an advantage after reinsurance.

### III-B Multinormal Initial portfolio

The above proposition can be generalized to  $I > 2$ . Suppose the number of agents  $I$  is arbitrary, and that  $(X_1, X_2, \dots, X_I)$  is multivariate normally distributed. Then  $(X_i, X_M)$  is bivariate normally distributed for any  $i$ , where  $X_M = \sum_{i=1}^I X_i$ . Here

$$\sigma_M^2 = \sum_{i=1}^I \sigma_i^2 + 2 \sum_{i>j} \rho_{i,j} \sigma_i \sigma_j$$

where  $\rho_{i,j}$  is the correlation coefficient between  $X_i$  and  $X_j$ , and  $\mu_M = \sum_{i=1}^I \mu_i$ . Furthermore

$$\text{cov}(X_i, X_M) = \sum_{j=1}^I \sigma_{i,j} = \sigma_i^2 + \sum_{j \neq i} \rho_{i,j} \sigma_i \sigma_j.$$

We have the following corollary to Proposition 3:

**Corollary 2** *Under the above distributional assumption, we have that*

$$E\left(e^{-\frac{X_M}{A}}\right) = e^{-\frac{\mu_M}{A} + \frac{1}{2} \frac{\sigma_M^2}{A^2}}, \quad (21)$$

$$E\left(X_M e^{-\frac{X_M}{A}}\right) = e^{-\frac{\mu_M}{A} + \frac{1}{2} \frac{\sigma_M^2}{A^2}} \left(\mu_M - \frac{\sigma_M^2}{A}\right), \quad (22)$$

and

$$E\left(X_i e^{-\frac{X_M}{A}}\right) = e^{-\frac{\mu_M}{A} + \frac{1}{2} \frac{\sigma_M^2}{A^2}} \left(\mu_i - \frac{1}{A} (\sigma_i^2 + \sum_{j \neq i} \rho_{i,j} \sigma_i \sigma_j)\right), \quad (23)$$

for  $i = 1, 2, \dots, I$ . Furthermore, the side-payments are given by

$$b_i^c = \mu_i - \frac{1}{A} (\sigma_i^2 + \sum_{j \neq i} \rho_{i,j} \sigma_i \sigma_j) - \frac{a_i}{A} \left(\mu_M - \frac{\sigma_M^2}{A}\right), \quad i = 1, 2, \dots, I. \quad (24)$$

Proof: By going through the same steps as in the proof of Proposition 3, only minor changes need to be carried out.  $\square$

Consider any two agents  $i$  and  $j$ , with the same attitudes towards risk, so that  $a_i = a_j$ , where the initial portfolios  $X_i$  and  $X_j$  have a correlation structure where one is positively correlated, the other is negatively correlated with the market portfolio. In this case, ceteris paribus, the agent with the negative correlations has the highest side-payment after reinsurance. More generally we have:

**Theorem 4** Consider two agents  $i$  and  $j$  in the multi agent model where  $a_i = a_j$ , and the initial portfolios satisfy  $\mu_i = \mu_j$  and  $\sigma_i = \sigma_j$ . If  $\rho_{i,k} \leq \rho_{j,k}$  for  $k \neq i, j$ , then  $b_i^c \geq b_j^c$ .

Proof: This follows from a closer inspection of the expression for the side-payments in (24) for agents  $i$  and  $j$ , which can be rewritten as:

$$b_i^c = \mu_i + \frac{\sum_{k \neq i} \rho_{i,k} \sigma_i \sigma_k}{A} \left( \frac{a_i}{A} - 1 \right) - \frac{1}{A} \sigma_i^2 - \frac{a_i}{A} \left( \mu_M - \frac{1}{A} \left( \sum_{k=1}^I \sigma_k^2 \right) \right),$$

and

$$b_j^c = \mu_j + \frac{\sum_{k \neq j} \rho_{j,k} \sigma_j \sigma_k}{A} \left( \frac{a_j}{A} - 1 \right) - \frac{1}{A} \sigma_j^2 - \frac{a_j}{A} \left( \mu_M - \frac{1}{A} \left( \sum_{k=1}^I \sigma_k^2 \right) \right).$$

The second terms on the right hand side have the same negative factors  $\left( \frac{a_i}{A} - 1 \right) < 0$  since  $a_i = a_j$  and  $A = \sum_k a_k$ . Since all other terms are equal in these two expressions by assumption, the highest side-payment is the one with the lowest correlations.  $\square$

This theorem partly transforms the property of a market value of the initial portfolios to statistical properties of the underlying risks. Notice that if the market values satisfy  $\pi(X_i) > \pi(X_j)$ , then  $\pi(Y_i) > \pi(Y_j)$ , which holds in equilibrium because of the budget constraints. To check this, we see that

$$\pi(X_i) = \frac{E(X_i e^{-X_M/A})}{E(e^{-X_M/A})} = \mu_i - \frac{1}{A} (\sigma_i^2 + \sum_{j \neq i} \rho_{i,j} \sigma_i \sigma_j)$$

by the result of Corollary 2, and

$$\pi(Y_i) = \frac{E(Y_i e^{-X_M/A})}{E(e^{-X_M/A})} = \frac{E\left(\left(\frac{a_i}{A} X_M + b_i^c\right) e^{-X_M/A}\right)}{E(e^{-X_M/A})} = \frac{a_i}{A} \left( \mu_M - \frac{\sigma_M^2}{A} \right) + b_i^c,$$

which by equation (24) is seen to imply that  $\pi(X_i) = \pi(Y_i)$  as the case should be.

The result of the theorem points out how valuable a security (portfolio) is compared to the rest of the securities (portfolios) provided it has, for example, a negative correlation with the rest of the market. Recall that for most securities in any market it is usually the case that covariances are non-negative with the rest of the market, or that for any  $j$ ,  $\rho_{j,k} \geq 0$  for any  $k$ .

Next we explore the effects of risk aversion. We consider two otherwise identical individuals  $i$  and  $j$ , except from having different risk aversions.

Assume in particular that that member  $i$  is more risk averse than member  $j$ , i.e., that  $a_i < a_j$ . In this situation  $i$ 's share of the total market portfolio  $X_M$  is smaller than  $j$ 's share, since  $Y_i = \frac{a_i}{A} X_M + b_i^c$ . Thus it is sufficient to investigate the effects on the side-payments under the above assumptions:

**Theorem 5** (a) Suppose  $\mu_M > \frac{\sigma_M^2}{A}$ ; then the corresponding side-payments satisfy  $b_i^c > b_j^c$ .

(b) Suppose  $\mu_M < \frac{\sigma_M^2}{A}$ ; then the corresponding side-payments satisfy  $b_i^c < b_j^c$ .

(c) Suppose  $\mu_M = \frac{\sigma_M^2}{A}$ ; then the corresponding side-payments satisfy  $b_i^c = b_j^c$ .

Proof: This follows from an inspection of the expressions for the side-payments  $b_i^c$  in (24).  $\square$

The intuition for the case (a) of the theorem is that if the syndicate performs well, since the more risk averse syndicate member holds a relatively smaller share of the market portfolio, the better the terms of the treaty which he must be offered in order to induce him to reach an agreement.

On the other hand, if the syndicate does not perform all that well as in case (b), again since the more risk averse agent still holds a relatively smaller share of the market portfolio, he needs not be offered quite that good terms of the treaty in order to induce him to participate.

In case (c) of the theorem, risk aversion is seen to have no influence on the side-payments, only on the fractions held of the market portfolio.

We notice that in a relatively risk tolerant syndicate ( $A$  is large), the risk averse agent needs to be offered an advantage in terms of the side payment to induce him to participate. In a relatively risk averse syndicate there is no need to compensate the risk averse agent in this way, meaning that the "odd" agent gains after pooling, as observed in the previous section when  $I = 2$ .<sup>3</sup>

## IV A Comparison Between Nash and the CE

The competitive solution is not always the best in predicting the outcomes of negotiations between parties. This is particularly the case when strategic considerations play a major role. However, the competitive equilibrium has the pleasant feature of providing prices, on which the merit of the model can readily be judged in many situations.

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<sup>3</sup>An analogue from the property market could be that a deaf person could benefit from buying a home close to an airport, or a blind person could get a bargain by buying a house without a view.

One would presume that there must exist a connection between the static axiomatic theory of bargaining and the sequential strategic approach to bargaining. And yes, there is such a connection. Under certain conditions it has been demonstrated that the strategic solution approaches the Nash bargaining solution, which provide a guide for the application of the bargaining solution in economic modeling (cf. Binmore, Rubinstein and Wolinsky (1986)).

On the other hand, the connection between the Nash bargaining solution and the competitive equilibrium is also a topic of independent interest. In this section we discuss this issue, relying on the results developed in the previous sections. In particular we are in a position where we may explore the effects of risk aversion.

Several investigators have considered how risk aversion influences the outcome of bargaining, as modeled by Nash's model, and related models. For example, Kannai (1977) noted that when bargaining concerns distribution of a divisible commodity between two risk averse individuals, then Nash's solution assigns a larger share of the commodity to a bargainer as his utility function displays less risk aversion. Thus, risk aversion is a disadvantage in this situation, according to Nash's model.

Roth and Rothblum (1982) consider a more general case, in which bargaining may be over risky as well as riskless outcomes. However they only analyze the case in which the "disagreement outcome"  $X$  is riskless. We, on the other hand are in a position to study the case where also  $X$  is a random vector. In fact, all the randomness, or uncertainty, in our model stems from the randomness of  $X$ . According to Roth and Rothblum (1982), in some cases, risk aversion continues to be a disadvantage in bargaining; in some cases it has no influence; and in some cases, risk aversion turns out to be an advantage. As we have seen in the last section, this description is general enough to be satisfied also by the competitive equilibrium. Our model differs so much from the one by Roth and Rothblum, that it is difficult to compare results any further.

#### IV-A The symmetric two agent case

Starting with the case of two agents, we notice that the side-payments of the competitive solution will in general depend on the correlation coefficient  $\rho$  between the initial portfolios  $X_1$  and  $X_2$ , but in the special case that the risk tolerances are equal,  $a_1 = a_2$ , the correlation coefficient  $\rho$  drops out of the expressions for the side-payments. This follows since

$$b_1^c = \frac{1}{2}(\mu_1 - \mu_2) + \frac{1}{2A}(\sigma_2^2 - \sigma_1^2), \quad (25)$$

and

$$b_2^c = \frac{1}{2}(\mu_2 - \mu_1) + \frac{1}{2A}(\sigma_1^2 - \sigma_2^2). \quad (26)$$

Notice that if  $\mu_1 > \mu_2$  and  $\sigma_2 > \sigma_1$ , then  $b_1^c > 0$  and  $b_2^c < 0$ , i.e., agent 1 acts as a "net lender" and agent 2 as a "net borrower" in the exchange. As long as  $\sigma_2 > \sigma_1$  this effect is strengthened when the risk tolerance  $A$  decreases, or put differently, when the risk aversion increases.

Curiously enough, the same conclusions follow from the Nash solution in the case when  $a_1 = a_2$ . To see this, consider the Nash side-payments in this case

$$b_1 = a \ln \left( e^{\frac{1}{2a}(\mu_1 - \mu_2)} e^{\frac{1}{4a^2}(\sigma_2^2 - \sigma_1^2)} \right), \quad (27)$$

and

$$b_2 = a \ln \left( e^{\frac{1}{2a}(\mu_2 - \mu_1)} e^{\frac{1}{4a^2}(\sigma_1^2 - \sigma_2^2)} \right). \quad (28)$$

Using a first order Taylor series approximation of  $b_1$  we obtain  $b_1^c$  of equation (25) and similarly for  $b_2$ . We formulate this as

**Theorem 6** *For proportional reinsurance treaties, the Nash solution and the equilibrium solution differ only by the side-payments, the leading term  $\frac{a_i}{A} X_M$  being identical for both these solutions. In the symmetric case where the risk tolerances are equal, the side-payments for the two solutions coincide to a first order Taylor series approximation.*

The symmetric case is easiest to study when the Nash solution is analyzed, due to the symmetry assumption. We may conclude that the two solutions are very similar for this model when  $I = 2$ .

## IV-B More than two agents

We now turn to a comparison in the general case where  $a_i \neq a_j$  and with more than two agents. At first glance this case seems more complicated analytically. This involves to investigate the situation where correlations matter, which here means situations when the attitudes towards risk differ among the members. The results of theorems 3 - 5 are typical equilibrium properties, which we can not expect carry over to the bargaining solution with no caveats added.

First notice that from the expression (10) for the Nash side-payments and the results of Corollary 2 that

$$b_i = \left( \mu_i - \frac{1}{2} \frac{\sigma_i^2}{a_i} \right) - \frac{a_i}{A} \left( \mu_M - \frac{1}{2} \frac{\sigma_M^2}{A} \right) + a_i \ln \left( \frac{1 + a_i \lambda}{a_i \lambda} \right), \quad (29)$$

where the Lagrange multiplier  $\lambda$  is given by equation (11). We observe that the first and third terms on the right-hand side of this expression are increasing functions of  $a_i$  for a given value of  $\lambda$ , while for the second term this property depends on the sign of  $(\mu_M - \frac{1}{2} \frac{\sigma_M^2}{A})$ .

Let us define the quantity  $B_{i,j}$  by

$$B_{i,j} = \frac{A}{\mu_M - \frac{1}{2} \frac{\sigma_M^2}{A}} \left( \frac{1}{2} \sigma_i^2 \left( \frac{1}{a_i} - \frac{1}{a_j} \right) + a_j \ln \left( \frac{1 + a_j \lambda}{a_j \lambda} \right) - a_i \ln \left( \frac{1 + a_i \lambda}{a_i \lambda} \right) \right),$$

and consider two otherwise identical individuals  $i$  and  $j$  except that  $i$  is more risk averse than  $j$ , i.e.,  $a_j > a_i$ . In particular this means that in (29)  $\mu_i = \mu_j$  and  $\sigma_i = \sigma_j$ , and  $\lambda$ , which depends on all the parameters of the problem in a symmetrical manner, is a given constant, the same for side-payment  $b_i$  as for  $b_j$  (since  $\lambda$  does not depend on the index  $i$ ). It is easy to see in this situation that  $B_{i,j} > 0$  provided  $\mu_M > \frac{1}{2} \frac{\sigma_M^2}{A}$ . Also in this situation  $i$ 's share of the total market portfolio  $X_M$  is smaller than  $j$ 's share for the same reason as in the competitive case.

We are now in position to explore the effects on the side-payments, corresponding to Theorem 5 for the competitive solution:

**Theorem 7** (a) *Suppose  $\mu_M > \frac{1}{2} \frac{\sigma_M^2}{A}$ . If member  $i$  is more risk averse than member  $j$ , and in addition  $a_j > a_i + B_{i,j}$ , the corresponding Nash side-payments satisfy  $b_i > b_j$ .*

*If member  $i$  is more risk averse than member  $j$ , but  $a_i < a_j \leq a_i + B_{i,j}$ , the corresponding Nash side-payments satisfy  $b_i \leq b_j$ .*

(b) *Suppose  $\mu_M \leq \frac{1}{2} \frac{\sigma_M^2}{A}$ . If member  $i$  is more risk averse than member  $j$ , the corresponding Nash side-payments satisfy  $b_i < b_j$ .*

Proof: Consider first (b). From the expression for the side-payments  $b_i$  in (29) we notice that all three terms on the right-hand side are increasing functions of the risk tolerance parameter, given  $\lambda$ , and this shows (b).

In the case (a) only the second term on the right-hand side of (29) is a decreasing function of the risk tolerance parameter  $a$ , the two other terms are still increasing. Starting with  $b_i > b_j$  in this situation, we see that this is equivalent to  $a_j - a_i > B_{i,j} > 0$ , which proves (a).  $\square$

Comparing this result to the corresponding result of Theorem 5 (b) for the competitive solution, we notice that the conclusions are the same when the syndicate is not doing so well, except that "not doing so well" in the Nash solution involves a more restricted region than for the competitive solution. Also notice that when  $\mu_M = \frac{1}{2} \frac{\sigma_M^2}{A}$ , then  $b_i < b_j$  in the Nash solution, and similarly is  $b_i^c < b_j^c$  in the competitive solution since this case satisfies requirement (b) of Theorem 5.

In a relatively risk averse syndicate (small  $A$ ), there is accordingly less need to compensate the risk averse agent than the more risk tolerant one. In this case risk aversion continues to be a disadvantage in the Nash solution also under uncertainty and with an uncertain "disagreement point".

Turning to (a) where "the syndicate is doing well", we first notice that "doing well" here is not so demanding as "doing well" in the competitive solution. However in order for the Nash side-payment of the most risk averse member to be the largest in this situation, the difference between the risk tolerances must in addition be large enough, i.e.,  $a_j > a_i + B_{i,j}$ . Merely the statement that member  $i$  is more risk averse than member  $j$  is not enough to secure this result, contrary to the case for the competitive equilibrium (Theorem 5 (a)). In fact, when  $a_i < a_j \leq a_i + B_{i,j}$  in this situation, we obtain the same conclusion as in (b).

We notice from the expression for  $B_{i,j}$  that the better the syndicate is doing and the less risky the initial portfolio of agent  $i$  is, the larger is the parameter region in which the most risk averse agent benefits from increased risk aversion in terms of the Nash side-payments. From the last two terms in  $B_{i,j}$  we observe that this conclusion is further strengthened both for large as well as small values of the Lagrange multiplier  $\lambda$ .

Finally, consider two agents  $i$  and  $j$  in the multi agent model where  $a_i = a_j$ , and the initial portfolios satisfy  $\mu_i = \mu_j$  and  $\sigma_i = \sigma_j$ . Contrary to the case of the competitive solution (see Theorem 4), we get that  $b_i = b_j$  in this situation. For the competitive solution we have that if  $\rho_{i,k} \leq \rho_{j,k}$  for  $k \neq i, j$ , then  $b_i^c \geq b_j^c$ . For the Nash solution correlations do not play the same role as in equilibrium.

## V Conclusions

In his discussion of the problem of reinsurance in the case of quadratic utility functions, Borch (1960a) comments:

*"That the aim of reinsurance arrangements is to reduce the variance, and at the same time retain as much as possible of the net premium is assumed, more or less explicitly, by many writers."*

From our analyses we notice that the competitive solution does more than this, and that the side-payments compensate the agent with the "best" portfolio, taking into consideration both the means, the variances, the covariance between the different portfolios and also the risk aversions of the agents. All these elements are reflected in the market value  $\pi(X_i)$  of agent  $i$ 's portfolio, and we notice from the general expression for the side-payments that the agent with the highest market value will obtain the highest side-payment as

well, *ceteris paribus*, and act as a net lender in the reinsurance exchange. Thus the aim of "retaining as much as possible of the net premium" seems more than fulfilled by the competitive solution.

In the Nash bargaining solution there are no market premiums, so a direct comparison is not possible regarding prices. However, in both solutions to the bargaining problem there exist side-payments having about the same interpretations, and the risky parts of the sharing rules are identical. This facilitates a comparison, and makes it both possible and meaningful.

Furthermore, we have demonstrated that for contracts of the affine type, the Nash solution has many of the same qualitative properties as the competitive equilibrium, in particular in the two member case with equal risk aversions.

In the multi member situation, risk aversions play a similar qualitative role for the Nash solution as for the equilibrium allocation, except for the situation of equal risk aversions, in which case correlations influence the two solutions differently.

Both models of this article are axiomatic ones. The advantage with this is that when observations of individuals behavior is contradictory with the predictions of any of these models, we can go back and check the axioms. This makes the models more readily refutable. Since no model is entirely "correct" (models are only more or less fruitful), we need models that are rejectable in order to learn something new.

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