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Recursive utility and the equity premium puzzle: A discrete-time approach

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Recursive utility and the equity premium puzzle: A discrete-time approach

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Abstract

We study the Epstein-Zin model with recursive utility. Recognizing that recursive preferences implies that the underlying model is not Markovian, we use methods not depending upon the Markov property to solve the model. We work with the returns directly, which we approximate by Taylor series expansions, in log terms. Leaving out moments of order three and higher, we calibrate the resulting model to the data of Mehra and Prescott (1985) under various assumptions about the wealth portfolio. The results are very reasonable for the US-data. We also calibrate to a newer Norwegian data set, where we also have the relevant estimates for the national wealth portfolio. Again, the results are consistent with plausible values for the preference parameters.

KEYWORDS: Recursive utility, the Epstein-Zin model, utility gradients, calibrations, the Markov property
JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

Rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time. Questions like the following are sometimes asked: Are asset prices too volatile relative to the information arriving in the market? Is the mean risk premium on equities over the riskless

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rate too large? Is the real interest rate too low? Is the market's risk aversion too high?

Mehra and Prescott (1985) raised some of these questions in their well-known paper, using a variation of Lucas's (1978) pure exchange economy with a Kydland and Prescott (1982) "calibration" exercise. They chose the parameters of the endowment process to match the sample mean, variance and the annual growth rate of per capita consumption in the years 1889 - 1978. The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

The puzzle has been verified by many others, e.g., Hansen and Singleton (1983), Ferson (1983), Grossman, Melino, and Shiller (1987). Many theories have been suggested during the years to explain the puzzle, but to date there does not seem to be any consensus that the puzzles have been fully resolved by any single of the proposed explanations [1].

In the present paper we address these empirical deficiencies of the conventional asset pricing model in financial and macroeconomics using recursive utility. We analyzed the Epstein-Zin model with recursive utility, and recognize that the recursive agent takes into account more than just the present when evaluating utility. As a consequence, when evaluating conditional probabilities of the future state prices in the economy, the recursive agent must take into account not only the present, but also past values of the basic variables, i.e., the Markov property does not hold.

An illustration of this is the paper by Weil (1989). In this the author analyzes the Epstein-Zin (1989) discrete-time model, and employs the exact same two-state Markov process as fitted by Mehra and Prescott for the period 1989-1979 using the standard additive and separable expected utility model. Using numerical methods he calculated the equity premium and the interest rate using this two-state Markov process. While leaving the equity premium unchanged, it appeared that the ability of the model to replicate the level of the risk-free rate was deteriorated, compared to the conventional additive and separable expected utility model (in short: the Eu-model). While it is true

[1] Constantinides (1990) introduced habit persistence in the preferences of the agents. Also Campbell and Cochrane (1999) used habit formation. Rietz (1988) introduced financial catastrophes, Barro (2005) developed this further, Weil (1992) introduced non-diversifiable background risk, and Heaton and Lucas (1996) introduce transaction costs. There is a rather long list of other approaches aimed to solve the puzzles, among them are borrowing constraints (Constantinides et al. (2001)), taxes (Mc Grattan and Prescott (2003)), loss aversion (Benartzi and Thaler (1995)), survivorship bias (Brown, Goetzmann and Ross (1995)), and heavy tails and parameter uncertainty (Weitzmann (2007)).

that one can sometimes use Markov methods in non-Markovian situations, this requires a transformation of the state space, not carried out in his paper.

In the present paper we avoid methods that require the Markov property. The model is solved using directional derivatives, and the equity premium as well as the short-term interest rate are calculated using Taylor series approximations. With six term for the short rate, and two terms for risk premiums, all in log terms, we calibrate the model to the same US-data as used by Mehra and Prescott (1985). This results in rather reasonable values of the preference parameters, under various assumptions about the wealth portfolio.

When the market portfolio is a proxy of the wealth portfolio, this corresponds to reasonable values for the risk aversion and the time-preference parameters, with an the impatience rate around 4 to 5 per cent.

We then explore the situation where the market portfolio is not assumed a satisfactory proxy for the wealth portfolio. By assuming that we can view income streams as dividends of some shadow asset, the model is valid if the market portfolio is expanded to include the new assets. Since the latter are not traded, the return to the wealth portfolio is not readily observable or estimable from the available US-data. Still we can get a reasonable impression of how the model fits data, under various assumptions, and the results are really promising. We also calibrate to a newer Norwegian data set, where we do have the relevant estimates for the national wealth portfolio. Again, the results are consistent with plausible values for the preference parameters.

In a companion paper (Aase 2015a) we have analyzed the continuous-time model established by Duffie and Epstein (1992a,b) using the stochastic maximum principle, and also calibrated this model to the same data as in this paper. The results are consistent with the ones in this paper.

Besides giving new insights about these interconnected puzzles, the recursive model is likely to lead to many other results that are difficult, or impossible, to obtain using, for example, the conventional time additive Eu-model. One example included in the paper is related to empirical regularities for Government bills[2].

The paper is organized as follows: In Section 2 the basic model is formulated and the state price deflator and the stochastic discount factor are established. Section 2.4 explains the financial market. Section 2.5 applies

[2]There is by now a long standing literature that has been utilizing recursive preferences. We mention Avramov and Hore (2007), Avramov et al. (2010), Eraker and Shaliastovich (2009), Hansen, Heaton, Lee, Roussanov (2007), Hansen and Scheinkman (2009), Wacther (2012), Bansal and Yaron (2004), Campbell (1996), Bansal and Yaron (2004), Kocherlakota (1990 b), and Ai (2012) to name some important contributions. Related work is also in Browning et. al. (1999), and on consumption see Attanasio (1999). Bansal and Yaron (2004) study a richer economic environment than we employ.

the general theory to the CES specialized preference ordering of Epstein and Zin. In Section 2.6 the risk premiums and the equilibrium interest rate are derived using Taylor series expansions. Some initial calibrations are carried out in Section 2.7, and in Section 2.8 we look at Government bills. In Section 3 we carry out calibrations when the market portfolio is not a proxy for the wealth portfolio. Section 4 analyzes the Norwegian data, and Section 5 concludes. The paper contains an Appendix for some of the more technical material.

2 The discrete time development

2.1 Introduction

The conventional asset pricing model in financial economics, the consumption-based capital asset pricing model (CCAPM) of Lucas (1978) and Breeden (1979), assumes a representative agent with a utility function of consumption that is the expectation of a sum of future discounted utility functions. The model has been criticized for several reasons. First, it does not perform well empirically. Second, the conventional specification of utility can not separate the risk aversion from the elasticity of intertemporal substitution, while it would clearly be advantageous to disentangle these two conceptually different aspects of preference. Third, while this representation seems to function well in deterministic settings, and for *timeless* situations, it is not well founded for *temporal* problems (e.g., derived preferences does not satisfy the substitution axiom, see e.g., Mossin (1969)).

Recursive utility was first formulated in the setting of discrete time. The basic notions of separating time and risk preferences are roughly summarized as follows: First consider a risk-less economy, where preferences over consumption sequences $(c_0, c_1, \cdots, c_T)$ are characterized with Koopmans' (1960) time aggregator $f(\cdot)$, where

$$U_t(c_t, c_{t+1}, \cdots, c_T) = f(u(c_t), U_{t+1}(c_{t+1}, c_{t+2} \cdots, c_T)).$$

This framework is then generalized to evaluate uncertain consumption sequences essentially by replacing the second argument in $f(\cdot)$ by the period t certainty equivalent of the probability distribution over all possible consumption continuations. The resultant class of recursive preferences may be characterized as

$$U_t(c_t, c_{t+1}, \cdots, c_T) = f(u(c_t), m_{t+1}(U_{t+1}(c_{t+1}, c_{t+2} \cdots, c_T))),$$

where $m_{t+1}(\cdot)$ describes the certainty equivalent function based on the conditional probability distribution over consumption sequences beginning in period $t+1$. Such preferences are dynamically consistent (Johnsen and Donaldson (1985)). We use the notation $V_t = U_t(c_t, c_{t+1}, \cdots, c_T)$ from now on.

In addition to assumption A1: dynamic consistency, there are also A2: irrelevance of past consumption, and A3: state independence of time preference, for the standard version of recursive utility (see Skiadas (2009a)).

Recursive preferences have an axiomatic underpinning, in which the main axiom, "Recursivity" is essentially identical to the notion of dynamic consistency, see Chew and Epstein (1991).

2.2 The first order conditions

The agent is characterized by a utility function U and an endowment process $e \in L$. The agent's problem is

$$\sup_{c \in L} U(c) \text{ subject to } E\big(\sum_{s=0}^{T} p_s c_s\big) \leq E\big(\sum_{s=0}^{T} p_s e_s\big)$$

where L is the space of adapted consumption processes, and p is the state price deflator (the Arrow-Debreu state price in units of probability).

The Lagrangian of the problem is

$$\mathcal{L}(c, \lambda) = U(c) - \lambda\, E\big(\sum_{s=0}^{T} p_s (c_s - e_s)\big)$$

where $\lambda > 0$ is the Lagrangian multiplier. Assuming U to be continuously differentiable, the gradient of U at c in the direction x is denoted by $\bigtriangledown U(c; x)$. This directional derivative is a linear functional, and by the Riesz Representation Theorem and e.g., dominated convergence, it is given by

$$\bigtriangledown U(c; x) = E\big(\sum_{s=0}^{T} \pi_s x_s\big).$$

Here π is the Riesz representation of $\bigtriangledown U(c; \cdot)$. The first order condition is

$$\bigtriangledown \mathcal{L}(c, \lambda; x) = 0 \text{ for all } x \in L.$$

This is equivalent to

$$E\Big\{\sum_{s=0}^{t}(\pi_s - \lambda p_s)x_s\Big\} = 0 \text{ for all } x \in L.$$

This implies that $\pi_t = \lambda p_t$ for all $t \leq T$.

Our next task is to characterize the Riesz representation π of U. When this is done, we have the state price in the economy modulo a constant.

2.3 The state prices in the economy

In order to characterize the state price in this economy, we need to find the Riesz representation π of the utility function U as explained in the last section.

Consider the following specification of recursive utility when time is indexed by $t = 0, 1, \cdots, T$. The utility $U(c) = V_0$, $V_t = f(u(c_t), m_{t+1})$, and $V_T = f(u(c_T), 0)$. Here f is the aggregator function mentioned above, life time utility is given by $U(c) = V_0$, and m_{t+1} is the certainty equivalent, given the information $\mathcal{F}_t$ available to the agent in the planning period. We assume that $m_{t+1} = h^{-1}(E_t(h(V_{t+1}))$ for some strictly increasing and concave function h, so that the certainty equivalent is obtained via expected utility, in which case the preferences fall in the Kreps and Porteus (1978) family. The utility function u is increasing, concave and differentiable and its derivative is denoted by u'. We denote the derivative of h by h', and f_u and f_m are the partial derivatives of f with respect to the first and the second argument respectively. Present utility V_t can depend on past consumption history if V_{t+1} does.

For this class of utility functions we want to characterize $\bigtriangledown U(c;x)$, the gradient of U at c in the direction x, i.e., to find $\{\pi_t\}$ adapted to $\mathcal{F}_t$ such that

$$\bigtriangledown U(c;x) = \lim_{\alpha \downarrow 0} \frac{U(c+\alpha x) - U(c)}{\alpha} = E\Big\{\sum_{t=0}^{T} x_t \pi_t\Big\}. \tag{1}$$

The utility function U is continuously differentiable in which case $\bigtriangledown U(c;x)$ is a linear and continuous functional from the set of consumption processes L into R. By the Riesz Representation Theorem and by dominated or monotone convergence the last equality in (1) follows.

When c is an equilibrium allocation, or the aggregate endowment in a representative agent economy, π_t has the interpretation of being the state price deflator at time t (see e.g., Duffie (2001), Ch. 2).

In Appendix 1 we show that the utility gradient is given by

$$\bigtriangledown U(c;x) = \bigtriangledown V_0(c;x) =$$
$$E\Big\{ \sum_{t=0}^{T} x_t \; f_u(u(c_t), m_{t+1})\, u'(c_t) \prod_{s=0}^{t-1} \frac{f_m(u(c_s), m_{s+1})}{h'(m_{s+1})} h'(V_{s+1}) \Big\}, \quad (2)$$

from which it follows that the state price deflator is given by

$$\pi_t = f_u(u(c_t), m_{t+1})\, u'(c_t) \prod_{s=0}^{t-1} \frac{f_m(u(c_s), m_{s+1})}{h'(m_{s+1})} h'(V_{s+1}) \quad (3)$$

for $t = 0, 1, \cdots, T$. In (3) c is assumed optimal from now on.

As can be seen, π_t depends on past consumption and utility from time zero to the present time t. This implies that the economy is not Markovian. State prices are determined by consumption process c and the certainty equivalent process m. Let g be a Borel function from $\mathbb{R}^2$ to $\mathbb{R}$ and B a Borel set in $\mathbb{R}$, and consider for any $\tau \geq t > 0$ the following conditional probability:

$$(i) \qquad P\{g(c_{t+1}, m_{t+2}) \in B | c_s, m_{s+1}, s \leq t\}.$$

In order to determine from (c, m) the time t-conditional probability distribution of future state prices, this can not be achieved by restricting attention to a conditional probability of the type

$$(ii) \qquad P\{g(c_{t+1}, m_{t+2}) \in B | c_t, m_{t+1}\},$$

for some appropriate g, since π_{t+1} depends on the entire past of the variables (c, m). Thus, with recursive utility one should avoid methods requiring the Markov property[3].

The stochastic discount factor/marginal rate of substitution is $\mathcal{M}_{t+1} = \pi_{t+1}/\pi_t$, and given by

$$\mathcal{M}_{t+1} = \frac{f_u(u(c_{t+1}), m_{t+2}) u'(c_{t+1})}{f_u(u(c_t), m_{t+1}) u'(c_t)} \, f_m(u(c_t), m_{t+1}) \, \frac{h'(V_{t+1})}{h'(m_{t+1})}. \quad (4)$$

Comparing to Epstein and Zin (1991), eq. (23), Donaldson and Mehra (2008), eqn. (8), Cochrane (2008), eqn. (14), or Skiadas (2009), the latter based on directional derivatives, our expression in (4) is in agreement with this literature.

[3]This feature is the same as in the continuous-time model.

2.4 The financial market

Having established the general, homogeneous recursive utility of interest, in this section we turn our attention to pricing restrictions relative to the given optimal consumption plan. Since we consider a pure exchange economy, the aggregate consumption process in society is optimally consumed by the single agent at each time t, i.e., $c_t = e_t$, or the optimal consumption c_t equals this agent's endowment process e_t for all $t = 0, 1, \cdots, T$. There is a risk-less asset with price process β_t

$$\beta_t = \beta_0 R_1^f R_2^f \cdots R_t^f,$$

where $R_{t+1}^f := \frac{\beta_{t+1}}{\beta_t}$ is the gross rate of return on this asset over the period $(t, t+1)$ and $R_{t+1}^f := 1 + r_{t+1}^f$, with r_{t+1}^f the corresponding arithmetic return $(\beta_{t+1} - \beta_t)/\beta_t$ on the risk-less asset. This means that at any time $t < T$, one can invest one unit of account in order to receive $1 + r_{t+1}$ units of account at time $t+1$. The return rate r_{t+1}^f over the period $(t, t+1)$ can be locked in at time t by trading government bonds.

The risk-less asset is characterized by the property that its return has zero conditional covariance with the state price deflator, i.e., $\text{cov}_t(\pi_{t+1}, R_{t+1}^f) = 0$ for all t.

Suppose S_t is the (cum dividend) price process of any risky asset in this economy, with corresponding gross return $R_{t+1}^R := \frac{S_{t+1}}{S_t}$. Since we have a state price deflator π, there is no arbitrage in this economy if and only if $S_t \pi_t$ is a martingale. The martingale property implies the following pricing relation

$$S_t = \frac{1}{\pi_t} E_t\{\pi_{t+1} S_{t+1}\}$$

for any $t \in [0, T-1]$. This implies the pricing restriction

$$E_t\{\mathcal{M}_{t+1} R_{t+1}^R\} = 1. \tag{5}$$

From this it follows by the defining property of covariance that

$$-\frac{\text{cov}_t(\mathcal{M}_{t+1}, R_{t+1}^R)}{E_t(\mathcal{M}_{t+1})} = E_t(R_{t+1}^R) - R_{t+1}^f, \tag{6}$$

provided that we interpret the reciprocal of $E_t(\mathcal{M}_{t+1})$ as the gross rate of return on the riskless asset over the period $(t, t+1)$, i.e.,

$$R_{t+1}^f := \frac{1}{E_t(\mathcal{M}_{t+1})}. \tag{7}$$

This is seen from (6) by replacing R^R_{t+1} by R^f_{t+1}, in which case

$$\mathrm{cov}_t(\mathcal{M}_{t+1}, R^f_{t+1}) = 0,$$

the defining property of the risk-less asset. The right-hand side of (6) is of course the *risk premium* of the risky asset.

The main question of interest is then the determination of prices, including risk premiums and the interest rate that makes the agent's behavior optimal.

The model for the financial market is complete, but has otherwise few restrictions on returns and aggregate consumption, except that we need some kind of ergodicity of returns and the growth rate of aggregate consumption in order for statistical estimation to make sense.

Our definition of wealth W_t includes current consumption (dividend), so the gross real rate of return on the wealth portfolio over the period $(t, t+1)$ is

$$R^W_{t+1} := \frac{W_{t+1}}{W_t - c_t}. \tag{8}$$

We make the assumption that one can view exogenous income streams as dividends of some shadow asset. Then our model is valid if the market portfolio is expanded to include the new asset. While this is the most important addition, a few more portfolios must be included in order to be a reasonable proxy for a nation's wealth portfolio. We assume the latter marketed, in which case W_t is the time-t wealth required to finance the consumption plan c from time t on; in other words (c, W) is a traded contract[4].

Define the wealth-to-consumption ratio as

$$\alpha_: = \frac{W_t}{c_t},$$

in which case $1/\alpha_t$ is the dividend yield on the wealth portfolio. In our model

$$\alpha_t = \frac{1}{1-\beta}\Big(\frac{V_t}{c_t}\Big)^{1-\rho},$$

which is seen to be stochastic unless $\rho = 1$, in which case our model is not valid. This means that $E_t(\ln\big(\frac{\alpha_{t+1}}{\alpha_t - 1}\big)) > 0$ is a valid assumption. Moreover,

$$\mathrm{cov}_t(\ln \frac{\alpha_{t+1}}{\alpha_t - 1}, \ln R^M_{t+1}) > 0$$

[4]In reality the (c, W) is not traded, so the return to the wealth portfolio is not readily estimated from available data. However, see Section 4.

is also feasible in our model, where R^M is the gross return on the market portfolio. As a consequence the time-t gross return on the wealth portfolio can exceed the growth rate on aggregate consumption, and the time-t covariance between the wealth portfolio and the market portfolio can be larger than the corresponding time-t covariance between the the consumption growth rate and the market portfolio. These observations we will make use of later, where we indicate reasonable adjustments when the market portfolio is not a proxy for the wealth portfolio.

2.5 Application of the standard aggregator

In their most basic analysis, Epstein and Zin explore the CES-like specialized preference ordering that is detailed below:

$$V_t = f(u(c_t), m_{t+1}) = \Big((1-\beta)c_t^{1-\rho} + \beta\big(E_t(V_{t+1}^{1-\gamma})\big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{1}{1-\rho}} \tag{9}$$

with $0 < \beta < 1$, $1 \neq \gamma > 0$, $\rho > 0, \rho \neq 1$. The parameter γ is the relative risk aversion, ρ is time preference, the inverse of the EIS-parameter ψ, and β is the impatience factor, with impatience rate $\delta = -\ln(\beta)$. When the parameter β is large, the agent puts more weight on the future and less weight on the present, in accordance with the impatience interpretation of this parameter. The preferences represented by (9) have axiomatic underpinnings (e.g., Chew and Epstein (1991)).

In order to find the stochastic discount factor we must compute the quantities in (4). These are

$$\frac{\partial}{\partial u} f(u(c_t), m_{t+1})u'(c_t) = (1-\beta)V_t^{\rho}c_t^{-\rho}, \quad \frac{\partial}{\partial m} f(u(c_t), m_{t+1}) = \beta V_t^{\rho} m_{t+1}^{-\rho}$$

and

$$\frac{h'(V_{t+1})}{h'(m_{t+1})} = \frac{V_{t+1}^{-\gamma}}{m_{t+1}^{-\gamma}}.$$

where we have chosen the usual CRRA specification for the utility function h. This means that the stochastic discount factor takes the form

$$\mathcal{M}_{t+1} = \frac{\pi_{t+1}}{\pi_t} = \beta\Big(\frac{c_{t+1}}{c_t}\Big)^{-\rho}\Big(\frac{V_{t+1}}{m_{t+1}}\Big)^{\rho-\gamma}. \tag{10}$$

Also, from (9) the certainty equivalent can be written as

$$m_{t+1} = \Big(\frac{1}{\beta}\big((\pi_t W_t)^{1-\rho} - (1-\beta)c_t^{1-\rho}\big)\Big)^{\frac{1}{1-\rho}}. \tag{11}$$

Let c signify optimal consumption, and W_t is the agent's wealth at time t, given by

$$W_t = \frac{1}{\pi_t} E_t\Big(\sum_{s=t}^{T} \pi_s c_s\Big). \tag{12}$$

If U and V_t are both homogeneous of degree one, or scale invariant, it follows that

$$\triangledown U(c;c) = U(c).$$

Homogeneity is shown in the Appendix. Since the gradient is a linear functional, it follows from the Riesz Representation Theorem and the first order condition of optimality of c that

$$\triangledown U(c;c) = E(\sum_{s=0}^{T} \pi_s c_s).$$

By (12) it follows that

$$U = V_0 = W_0 \pi_0.$$

Similarly, since $\triangledown V_t$ is also a linear functional, the following holds as well

$$\triangledown V_t(c;c) = E(\sum_{s=t}^{T} \pi_s^{(t)} c_s) = V_t(c), \tag{13}$$

where $\pi_s^{(t)}$, s = t, t+1, $\cdots$ is the Riesz representation of this functional as of time t. The last equality in (13) follows since $V_t := V_t(c)$ is scale invariant (see the Appendix). With reference to the state price deflator π_t given in (2), let

$$Y_t = \prod_{s=0}^{t-1} \frac{f_m(u(c_s), m_{s+1})}{h'(m_{s+1})} h'(V_{s+1}), \quad t = 1, 2, \cdots$$

Under assumptions A1, A2 and A3 (see Skiadas (2009b)), it follows that

$$\pi_s^{(t)} = \frac{\pi_s}{Y_t}, \quad s = t, t+1, t+2, \cdots$$

In particular

$$\pi_t^{(t)} = f_u(u(c_t), m_{t+1}) u'(c_t) = V_t^{\rho}(1-\beta) c_t^{-\rho}. \tag{14}$$

From the expression for W_t we now get

$$E(\sum_{s=t}^{T}\pi_s^{(t)}c_s) = \frac{1}{Y_t}E(\sum_{s=t}^{T}\pi_s c_s) = \frac{1}{Y_t}W_t\pi_t = W_t\pi_t^{(t)}.$$

From (13) we then have

$$V_t = V_t(c) = W_t\pi_t^{(t)}, \tag{15}$$

and from (14) and (15) we finally obtain

$$V_{t+1} = \big(W_{t+1}(1-\beta)c_{t+1}^{-\rho}\big)^{\frac{1}{1-\rho}}.$$

Following Cochran (2008), from this and (11) the stochastic discount factor in (10) can be written

$$\mathcal{M}_{t+1} = \beta\Big(\frac{c_{t+1}}{c_t}\Big)^{-\rho}\left(\frac{\big(W_{t+1}(1-\beta)c_{t+1}^{-\rho}\big)^{\frac{1}{1-\rho}}}{\big(\frac{1}{\beta}\big((1-\beta)c_t^{-\rho}W_t - (1-\beta)c_t^{1-\rho}\big)\big)^{\frac{1}{1-\rho}}}\right)^{\rho-\gamma}. \tag{16}$$

By the definition of the gross real return on the wealth portfolio in (8), it now follows directly that

$$\mathcal{M}_{t+1} = \beta^{\frac{1-\gamma}{1-\rho}}\Big(\frac{c_{t+1}}{c_t}\Big)^{-\rho\frac{1-\gamma}{1-\rho}}\big(R_{t+1}^{W}\big)^{\frac{\rho-\gamma}{1-\rho}}. \tag{17}$$

This expression has been the starting point for much of the literature on recursive utility in discrete time models, see e.g., Donaldson and Mehra (2008) and Cochrane (2008), among many others. This is the stochastic discount factor f irst derived by Epstein and Zin (1989-91), in their seminal papers, based on dynamic programming techniques.

This is also our starting point for calibrations to the data of Mehra and Prescott (1985). When doing so, we will take into account that the variable W_t is the wealth portfolio of the agent, which may be different from the market portfolio.

As has been claimed, without any adjustments a mere calibration to the market portfolio may lead to negative equity premium for reasonably parameterized Epstein-Zin utility for the data of Mehra and Prescott (1985) (e.g., Azeredo (2007)), or to implausible values for the parameters γ and ρ for reasonable values of β. As Donaldson and Mehra (2008) write (p 52) about the (EZ) relationship in (17) ..."But, as we are aware, this alone does not, in general, solve the puzzle"...

About this we have much more to say in the next sections.

2.6 Risk premiums and the interest rate

First we derive approximate expressions for the risk premiums and the real interest rate. To this end we use the pricing restriction $E_t\{\mathcal{M}_{t+1}\, R^R_{t+1}) = 1$, valid for any risky security R in the market, together with the relationship $\ln R^f_{t+1} = -\ln(E_t(\mathcal{M}_{t+1}))$. In the pricing restriction we employ Taylor series expansions involving terms containing up to second order moments. Next we take the logarithm of the resulting equation, and use a Taylor series expansion up to the second order. In the expression for the log of the gross, risk-free interest rate we employ similar Taylor series expansions. Using this second relationship in the first, we obtain the following results. The risk premiums in log terms are given by

$$E_t(\ln R^R_{t+1}) - \ln R^f_{t+1} = \frac{\rho(1-\gamma)}{1-\rho}\,\mathrm{cov}_t\Big(\ln\big(\frac{c_{t+1}}{c_t}\big), \ln R^R_{t+1}\Big) + \frac{\gamma-\rho}{1-\rho}\,\mathrm{cov}_t\Big(\ln R^W_{t+1}, \ln R^R_{t+1}\Big), \quad (18)$$

for any asset R in the economy. The log-return on the risk-free asset takes the form

$$\ln R^f_{t+1} = \frac{1-\gamma}{1-\rho}\ln\big(\frac{1}{\beta}\big) + \frac{\rho(1-\gamma)}{1-\rho}\,E_t\ln\big(\frac{c_{t+1}}{c_t}\big) - \frac{1}{2}\frac{\rho^2(1-\gamma)^2}{(1-\rho)^2}\mathrm{var}_t(\ln\frac{c_{t+1}}{c_t}) + \frac{\gamma-\rho}{1-\rho}E_t\ln R^W_{t+1} - \frac{1}{2}\frac{(\rho-\gamma)^2}{(1-\rho)^2}\mathrm{var}_t(\ln R^W_{t+1}) + \rho\frac{1-\gamma}{1-\rho}\frac{\rho-\gamma}{1-\rho}\mathrm{cov}_t\big(\ln\big(\frac{c_{t+1}}{c_t}\big), \ln R^W_{t+1}\big). \quad (19)$$

For comparisons, in the conventional, expected utility model these relationships are

$$E_t(\ln R^R_{t+1}) - \ln R^f_t = \gamma\,\mathrm{cov}_t\Big(\ln\big(\frac{c_{t+1}}{c_t}\big), \ln R^R_{t+1}\Big), \quad (20)$$

and

$$\ln R^f_{t+1} = \ln\big(\frac{1}{\beta}\big) + \gamma\,E_t\ln\big(\frac{c_{t+1}}{c_t}\big) - \frac{1}{2}\gamma^2\,\mathrm{var}_t(\ln\frac{c_{t+1}}{c_t}), \quad (21)$$

which can be obtained from the recursive formulas by setting $\rho = \gamma$[5].

[5]Weil (1989) do not develop expressions like (18) and (19), but rather analyze (5) and

The latter term in the expression for the risk premiums, and the last three term on the right hand side of (19) distinguish the recursive model from the Eu-model. Superscript W signifies the wealth portfolio, M refers to the market portfolio of the risky securities, while superscript R refers to any (risky) asset in the market. In the application below $R = M$.

2.7 Calibrations

In Table 1 we provide the key summary statistics of the data in Mehra and Prescott (1985) of the real annual return data related to the S&P-500, denoted by M, as well as for the annualized consumption data, denoted c, and the Government bills, denoted b [6].

	Expectat.	Standard dev.	Covariances
Consumption growth	1.83%	3.57%	$\text{cov}(M, c) = .002226$
Return S&P-500	6.98%	16.54%	$\text{cov}(M, b) = .001401$
Government bills	0.80%	5.67%	$\text{cov}(c, b) = -.000158$
Equity premium	6.18%	16.67%	

Table 1: Key US-data for the time period 1889-1978. Discrete-time compounding.

In our calibrations it will be convenient to consider a log transformation and use log returns. The relevant summary statistics are given in Table 2. Notice that this table is not a mere transformation of Table 1, but developed from the the original data set used in the Mehra and Prescott (1985)-study, by taking logarithms of the relevant yearly quantities, and basing the statistical analysis on these transformed data points.[7]

Assuming for the moment that the market portfolio can be used as a proxy for the wealth portfolio, we then interpret the risky asset as the value weighted market portfolio M corresponding to the S&P-500 index. We then have two equation in two unknowns to provide estimates for the preference parameters by the "method of moments". The impatience rate $\delta = \ln(1/\beta)$. We denote the elasticity of intertemporal substitution in consumption by $\psi := 1/\rho$, and refer to it as the EIS-parameter. Under this assumption we

(7) directly directly using a stationary two state Markov process and numerical methods.

[6] There are of course newer data by now, but these retain the same basic features. If we can explain the data in Table 1, we can explain any of the newer sets as well.

[7] We have obtained the original data set from Professor R. Mehra. For example, a log return is not obtained simply adjusted as $\mu - (1/2)\sigma^2$ from Table 1, which would be (almost) true if returns and growth rates of consumption were normally distributed. We observe some deviations from normality in the data.

	Expectat.	Standard dev.	Covariances
Consumption growth	1.75%	3.55%	$\text{cov}(M, c) = .002268$
Return S&P-500	5.53%	15.84%	$\text{cov}(M, b) = .001477$
Government bills	0.64%	5.74%	$\text{cov}(c, b) = -.000149$
Equity premium	4.89%	15.95%	

Table 2: Key US-data for the time period 1889-1978 in terms of log returns of discrete-time compounding.

calibrate our model (18) and (19) for various values of β. The results are given in Table 3 when the market portfolio is assumed a proxy for the wealth portfolio.

Parameters	γ	ρ	EIS	δ
The expected utility model :				
$\beta = 1.08$	21.97	21.97	.046	-.077
The recursive model:				
$\beta = .945$	1.95	.000	-	.057
$\beta = .950$	1.68	.290	3.45	.051
$\beta = .955$	1.41	.590	1.70	.046
$\beta = .961$	1.05	.950	1.05	.040
$\beta = .962$	1.02	.980	1.02	.039
$\beta = .962$	1.01	.990	1.01	.039
$\beta = .963$	.990	1.01	.990	.038
$\beta = .964$	.895	1.10	.909	.036

Table 3: Various Calibrations Consistent with Table 2.

These results may be seen in view of the comments made above, where researchers express disappointing results for the EZ-model's ability to explain empirical observations. When the agent is as patient as commonly assumed in applied work, the results for γ may become unacceptable, in particular if δ is smaller than 2 per cent (β larger than .98). However, when β is between .945 and .975, the other two parameters actually take on rather reasonable values, considering our proxy for the wealth portfolio.

In this connection it may be of some interest to recall the study of Andersen et. al. (2008). They use controlled experiments with field subjects in Denmark to elicit the impatience rate and risk preference, ignoring the subject of time preferences. First, an estimate of δ around 25% is reached assuming risk neutrality, second, a new estimate of δ around 10% is obtained assuming risk aversion, with an associated estimate of γ around .74, both based on arithmetic averaging.

Compared to corresponding results for the continuous-time model of Epstein and Duffie (1992), which are calibrated by Aase (2015a), the latter model is consistent with reasonable values of γ and ρ for lower values of δ.[8]

These results are encouraging, and not surprising considering the background for the data. For large parts of the period this data covers, the participation in the stock market has been rather low, around 8-10 per cent on average according to Vissing-Jørgensen (1999). This may call for a heterogeneous model (e.g., Aase (2014b)). Thus most consumers are not participating in the stock market, in which case the issue of early resolution of uncertainty ($\gamma > \rho$) may not be all that important. If the average agent obtains as good return on the wealth portfolio as in the stock market, there seems to be no point in stressing with the early information issue. So, for example, the values for β above .963 yield $\gamma < \rho$, which may not be unreasonable with this point of view in mind. However, we find it more reasonable that the EIS parameter $\psi > 1$, which is violated in this range. Dagsvik et.al. (2006) estimate the ψ, and find it larger than 1 [9].

Weil (1989) obtained the values $\beta = .95, \gamma = 45, \psi = .10$ when the net risk premium is .0572 and the risk-free rate is .0085. According to Table 1 this risk premium is a bit smaller that estimated, and the risk-free rate is a bit larger, but this is the closest he got to these estimates. Nevertheless, using these values for the moment, our model produces two possible solutions for $\beta = .95$: ($\gamma = .86, \psi = .84$) and ($\gamma = 31.94, \psi = .070$). It is this last one that is closest to what Weil (1989) obtained, using the two-state Makrov model. When $\beta = .94$, the solution for this risk premium and interest rate is ($\gamma = 1.24, \psi = 1.51$), which is reasonable. In other words, assuming that the underlying model is Markovian, yields results rather different from the ones we obtain.

In Section 3 of the paper we take a different approach, and suppose we can view exogenous income streams as dividends of some shadow asset, and similarly with other non-marketed assets that make up the wealth portfolio. In this case our model is valid if the market portfolio is expanded to include the new assets. However, if the latter are not traded, then the return to the wealth portfolio is not readily observable or estimable from available data. Still we should be able to get a pretty good impression of how the two models compare. This we do in Section 3 below, where it will also be observed that this approach is close in spirit to considering a heterogeneous model. We also

[8]For example, when $\delta = .023$, the two other parameters are $\gamma = 1.15$ and $\rho = .90$. Also, the present model gives a better fit for values of δ between 3.5 and 5.5 percent.

[9]Weil (1989) assumed, for example, that $\psi = .1$, i.e., $\rho = 10$. Here this gives $\beta = 1.9$ and $\gamma = -87.9$.

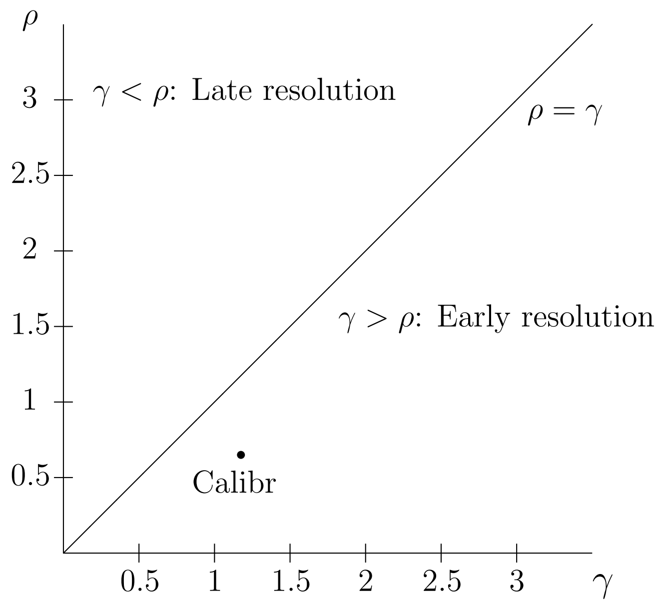

Figure 1: Calibration points in the (γ, ρ)-space

consider the example of Norway, where we have the relevant estimates and various correlations with the market portfolio and aggregate consumption.

Figure 1 illustrates the feasible region in (ρ, γ)-space. For the conventional model it is the 45°-line shown ($\rho = \gamma$). For the recursive utility it is all of the first quadrant, including the axes. The points above the 45°-line represent late resolution of uncertainty, the points below correspond to early resolution.

The larger region for the (ρ, γ)-combinations permitted by recursive utility is not a frivolous generalization of the conventional, additive model. That the richer structure of the recursive model is a modest extension is demonstrated by the interpretations and plausible results yielded in the simple expressions (18) and (19). The model is based on fundamental assumptions and axioms of rational behavior (Chew and Epstein (1991)).

2.8 Including Government bills

There is an additional problem with the conventional, additive Eu model that could be mentioned. From Table 2 we see that there is a negative correlation between Government bills and the consumption growth rate. Similarly there is a positive correlation between the return on S&P-500 and Government bills.

If we interpret Government bills as risk free in a yearly context, the former correlation should be zero for the CCAPM-model to be consistent. Since this correlation is not zero, then γ must be zero if Government bills are to be considered as risk free, in which case the conventional model breaks down. Since the Government bills used by Mehra and Prescott (1985) have duration one month, and the data are yearly, Government bills are not the same as Sovereign bonds with duration of one year. One month bills in a yearly context will contain price risk 11 months each year, and hence the real risk free rate should, perhaps, be set strictly lower that .80%.

With bills included, the conventional, additive model does not seem to have enough 'degrees of freedom' to match the data, since in this situation the model contains three basic relationships and only two 'free parameters'. Thus the conventional model can not have the correct state price deflator, or stochastic discount factor.

Exactly what risk premium bills command we can here only stipulate. With recursive utility we have a third equation, namely

$$E_t(\ln R^b_{t+1}) - \ln R^f_{t+1} = \frac{\rho(1-\gamma)}{1-\rho}\operatorname{cov}_t\Big(\ln\big(\frac{c_{t+1}}{c_t}\big), \ln R^b_{t+1}\Big) + \frac{\gamma-\rho}{1-\rho}\operatorname{cov}_t\Big(\ln R^W_{t+1}, \ln R^b_{t+1}\Big) \qquad (22)$$

that can be solved together with the equations when $R = M$, and the equation for the interest rate. With the covariance estimates provided in Table 2, and an estimate for the risk premium of Government bills of .0032 (log terms), we have three equations in three unknowns, giving the following solution

$$\beta = .955, \quad \gamma = 1.65 \quad \text{and} \quad \rho = .42$$

This is seen to be consistent with a typical value set in Table 3 in the region of β-values where the model gives an otherwise reasonable fit[10].

With reference to Section 3 below, where we do not assume that the market portfolio is a proxy for the wealth portfolio, with a growth rate of the wealth portfolio of 4.6 per cent (instead of the 5.53 per cent of the market

[10] For a risk premium larger than .00345 on Government bills, the calibration is consistent with late resolution of uncertainty (e.g., $\beta = .97, \gamma = .90, \rho = 1.08$ when the premium is .00345).

portfolio), we obtain the calibrated values

$$\beta = .970, \quad \gamma = 2.28 \quad \text{and} \quad \rho = .44,$$

for a risk premium of Government bills .0050 and a correlation coefficient between the wealth portfolio and the market portfolio of .90.

3 The market portfolio is not a proxy for the wealth portfolio

In this section we illustrate how the calibrations change when the market portfolio is not a proxy for the wealth portfolio.

Parameters	γ	ρ	EIS	δ
$E_t(\ln R^W_{t+1}) = .025$, $\sigma_W(t) = .05$; $\kappa_{W,M} = .85$				
$\beta = .994$	2.32	.85	1.18	.006
$\beta = .995$	1.46	.95	1.05	.005
$E_t(\ln R^W_{t+1}) = .025$, $\sigma_W(t) = .05$; $\kappa_{W,M} = .60$				
$\beta = .998$	2.62	.90	1.11	.002
$\beta = .999$	1.85	.95	1.05	.001
$E_t(\ln R^W_{t+1}) = .030$, $\sigma_W(t) = .08$; $\kappa_{W,M} = .80$				
$\beta = .990$	2.77	.60	1.67	.010
$\beta = .992$	1.93	.80	1.25	.008
$E_t(\ln R^W_{t+1}) = .035$, $\sigma_W(t) = .08$; $\kappa_{W,M} = .70$				
$\beta = .994$	2.64	.70	1.42	.006
$\beta = .996$	1.59	.90	1.11	.004

Table 4: Calibrations with different assumptions for the wealth portfolio.

In the conventional Eu-model with constant coefficients, the growth rate of the wealth portfolio has the same volatility as the growth rate of aggregate consumption. Here we may consider the growth rate of consumption as a lower bound for the growth rate of the wealth portfolio, and the same relationship may seem reasonable between the two volatilities. Below we assume the volatility of the wealth portfolio is .10 (in log terms). Also we set the correlation coefficient $\kappa_{W,M}$ between the return on the market portfolio and the growth rate of the wealth portfolio varies between .45 and .90. We assume a correlation coefficient $\kappa_{W,c}$ between the wealth portfolio and the growth rate of consumption to be .8, while $\kappa_{M,c} = .4$ (Table 2). Also we start in Table 4 with an annual return rate on the wealth portfolio of 2.5 per cent. The results of these calibrations are presented in tables 4 and 5.

Parameters	γ	ρ	EIS	δ
$E_t(\ln R^W_{t+1}) = .040$, $\sigma_W(t) = .10$; $\kappa_{W,M} = .80$				
$\beta = .979$	2.85	.40	2.50	.021
$\beta = .983$	1.66	.80	1.25	.017
$E_t(\ln R^W_{t+1}) = .040$, $\sigma_W(t) = .10$; $\kappa_{W,M} = .50$				
$\beta = .997$	2.94	.70	1.43	.003
$\beta = .999$	1.69	.90	1.11	.001
$E_t(\ln R^W_{t+1}) = .046$, $\sigma_W(t) = .12$; $\kappa_{W,M} = .80$				
$\beta = .976$	2.19	.50	2.00	.011
$\beta = .990$	1.48	.95	1.05	.024
$E_t(\ln R^W_{t+1}) = .046$, $\sigma_W(t) = .12$; $\kappa_{W,M} = .60$				
$\beta = .989$	2.14	.70	1.43	.011
$\beta = .990$	1.78	.80	1.25	.010

Table 5: Calibrations with different assumptions for the wealth portfolio

In Table 4 the growth rate of the wealth portfolio varies between 2.5 and 3.5 per cent, and $\kappa_{W,M}$ varies between .60 and .85. Compared to Table 3, small values of the impatience rate δ fit well, corresponding to values of β closer to one. The calibrated values for the relative risk aversion γ and the time preference ρ are plausible.

Since the majority of the population did not invest in the stock market for the period this data covers, the wealth portfolio with a growth rate of about 3.5 per cent could well reflect the average real return (log terms) that this majority received. This group also dominates in aggregate consumption because of the sheer number of people this 'agent' represents.

In Table 5 the growth rate of the wealth portfolio varies between 4.0 and 4.6 per cent, and $\kappa_{W,M}$ varies between .50 and .80. Compared to Table 3, reasonably small values of the impatience rate δ again fit well. Now values of δ around .1 to 2.1 per cent go well together with reasonable values for the other two parameters.

The illustrations in this section give an indication of how the model fits data when the market portfolio is not a proxy for the wealth portfolio. These results are consistent with similar analyses for the continuous-time model, see Aase (2014a). Many additional examples can be given, but the results presented above are fairly representative.

The main change from the standard Lucas (1978)-model is the new preference, and we have no Markov assumptions related to the dynamics of the economy.

When considering puzzles, it is desirable to alter as few features of the original model as possible. In this paper we have followed that recipe. Compared to the conventional model the difference is dramatic.

4 An empirical example

In this section we present the results of the Norwegian economy in which the central statistical agent, Statistisk sentralbyrå, has provided the data needed, also related to the wealth portfolio (from 1985 to 2013, see Hjortland (2015)). Table 6 contains the data corresponding to Table 2.

	Expectat.	Standard dev.	Covariances
Consumption growth	1.785%	1.366%	$\text{cov}(M, c) = .00068485$
Return OBX	5.588%	32.795%	$\text{cov}(M, b) = .00189848$
Government bills	2.079%	3.559%	$\text{cov}(c, b) = 8.299E\text{-}06$
Equity premium	3.508%	32.407%	

Table 6: Key Norwegian-data for the time period 1971-2014 in terms of log returns of discrete-time compounding.

The estimates provided by Statistisk sentralbyrå (2014) are restricted to include capital that is measurable in units of account: (i) human capital; (ii) real capital; (iii) financial capital (including the Sovereign Pension Fund of Norway); (iv) natural resources. For the whole period 72-75 per cent of the national wealth can be attributed to human capital.

Based on further information from the statistical agent, e.g., population growth, Hjetland (2015) has calculated the following per-capita estimates related to the wealth portfolio (log terms): $\text{var}_t(\ln(R^W_{t+1})) = .000333$, $E_t(\ln R^W_{t+1}) = .0218$, $\text{cov}_t(\ln R^W_{t+1}, \ln R^M_{t+1}) = .00142$, , $\text{cov}_t\big(\ln R^W_{t+1}, \ln\big(\frac{c_{t+1}}{c_t}\big)\big) =$ .000127. This gives the calibrated model reported in Table 7.

The values of the impatience rate vary little, so we have chosen to use the time preference parameter ρ as the variable on the left-hand side of the table. The parameter estimates are reasonable over most of the range shown.

Parameters	γ	ρ	EIS	δ
$\rho = .70$	11.73	.70	1.42	.035
$\rho = .80$	8.72	.80	1.25	.039
$\rho = .85$	7.37	.85	1.17	.042
$\rho = .90$	5.19	.90	1.11	.044
$\rho = .95$	3.18	.95	1.05	.047
$\rho = .97$	2.34	.97	1.03	.049
$\rho = .98$	1.90	.98	1.02	.049
$\rho = .99$	1.45	.99	1.01	.050

Table 7: Calibrations of the recursive model to the Norwegian economy.

When $\rho > 1$ negative values for γ appear. This is in accordance with Dagsvik et.al. (2006), who estimate EIS to be between 1 and 1.5 for the Norwegian population. When $\rho = 0.00$, $\beta = .983$ and $\gamma = 24.7$. The relative risk aversion γ varies between 2.77 and 1.90, when ρ varies between .96 and .98, with corresponding values of β between .953 and .951.

This indicates that the average Norwegian is reasonably patient with an impatience factor $\beta \in (.95, .96)$, with a time preference parameter $\rho \in (.80, .98)$, an EIS above 1 and a relative risk aversion $\gamma \in (1.9, 8.7)$, all rather plausible parameter values[11].

5 Extensions

Given the period the US-data covers, where the participation in the stock market has been low at times, a heterogeneous model seems appropriate to explain empirical regularities. Several agents have been considered by e.g., Guvenen (2009) in a discrete time model, and by Aase (2014b) in the continuous-time model analogues to the one considered in the present paper. In the latter plausible results were obtained by non-Markov methods.

The effects of recursive utility is seen clearly in applications to the life cycle model: The recursive utility maximizer is not merely myopic, but takes into account more than just the present. For the US-data, the typical consumer smoothens consumption more than the Eu-maximizer, invests more in good times, and can then consume more in bad times. This behavior goes a long way in explaining the puzzle, provided the consumer prefers early

[11]For the standard Eu-model this data set gives the solution $\beta = 1.91$ ($\delta = -.65$) and $\gamma = \rho = 51.22$. The assumption that the economy is closed is of course restrictive, since it imposes that consumption be equal to domestic output. When exports and imports balance, this could still be a reasonable assumption.

resolution of uncertainty to late.

In dealing with puzzles, it is desirable to change as few features of the original model as possible, at the time. It seems as the change from the conventional representation of Eu-preferences to recursive utility, taking into account the non-Markovian property of the economy, is just what it takes.

6 Conclusions

We have addressed the well-known empirical deficiencies of the conventional asset pricing model in financial and macroeconomics using recursive utility. We have analyzed the Epstein-Zin model, which we have calibrated to the data of Mehra and Prescott (1985) under various assumptions.

Observing that the economy is not Markovian, our formal approach is to use directional derivatives. This method can handle state dependence, which is important in dealing with recursive utility.

In this paper we first used the assumption that the market portfolio is a proxy of the wealth portfolio. This corresponded to reasonable values for the risk aversion and the time preference parameters, when the agent is moderately impatient.

We then made the assumption that the market portfolio is not a satisfactory proxy for the wealth portfolio. By assuming that we can view the various wealth components as dividends of some shadow assets, the model is valid if the market portfolio is expanded to include the new assets. Since a majority of the latter are not traded, the return to the wealth portfolio is not readily observable or estimable from available data. Still we can get a reasonable impression of how the model fits data, by calibrating under various reasonable assumptions.

In particular we assumed that the annual growth rate of the wealth portfolio is lower that then return on the market portfolio. Also the annual correlation between the wealth portfolio and the market portfolio we assume to be smaller than the variance rate of the market portfolio. The parameters indicate a less impatient and more risk averse agent compared to the situation when the market portfolio is a proxy for the wealth portfolio. In all, our analyses indicate that the recursive model fits data well.

We also pointed out that the recursive model may explain the covariances of Government bills with consumption and equity, for a moderate risk premium for the bills. Here the conventional, additive Eu-model can offer no reasonable answers.

Finally we considered newer data from Norway, where we do have the necessary summary statistics regarding the wealth portfolio. Here the recursive model fits the data surprisingly well.

7 Appendix 1

7.1 The derivation of the state price deflator

Proof of (3) in Section 2.3:

By backward induction we have that

$$\bigtriangledown V_T(c;x) = \lim_{\alpha\downarrow 0}\frac{f(u(c_T+\alpha x_T),0)-f(u(c_T),0)}{\alpha} = f_u(u(c_T),0)\,u'(c_T)\,x_T$$

and

$$\bigtriangledown V_{T-1}(c;x) = f_u(u(c_{T-1}),h^{-1}(E_{T-1}\{h(V_T)\}))\,u'(c_{T-1})\,x_{T-1}+ \frac{f_m(u(c_{T-1}),h^{-1}(E_{T-1}\{h(V_T)\}))}{h'(h^{-1}(E_{T-1}\{h(f(c_T,0))\})}E_{T-1}\{h'(V_T)\bigtriangledown V_T(c;x)\}.$$

Using induction we obtain

$$\bigtriangledown V_t(c;x) = f_u(u(c_t),m_{t+1})\,u'(c_t)\,x_t+ \frac{f_m(u(c_t),m_{t+1})}{h'(m_{t+1})}E_t\{h'(V_{t+1})\bigtriangledown V_{t+1}(c;x)\} \quad (23)$$

for $t = 0, 1, \cdots, T-1$. Using the relation (23) iteratively, this gives

$$\bigtriangledown U(c;x) = \bigtriangledown V_0(c;x) = E\big\{f_u(u(c_0),m_1)\,u'(c_0)\,x_0+$$

$$\frac{f_m(u(c_0),m_1)}{h'(m_1)}h'(V_1)\bigtriangledown V_1(c;x)\} =$$

$$
\begin{aligned}
&E\Big\{f_u(u(c_0),m_1)u'(c_0)x_0+\frac{f_m(u(c_0),m_1)}{h'(m_1)}h'(V_1)\big[f_u(u(c_1),m_2)\,u'(c_1)\,x_1\\
&\qquad+\frac{f_m(u(c_1),m_2)}{h'(m_2)}h'(V_2)\cdot\big[f_u(u(c_2),m_3)\,u'(c_2)\,x_2\\
&\qquad\quad+\frac{f_m(u(c_2),m_3)}{h'(m_3)}h'(V_3)\bigtriangledown V_3(c;x)\big]\big]\Big\}=\cdots\\
&=E\Big\{\sum_{t=0}^{T}x_t\ f_u(u(c_t),m_{t+1})\,u'(c_t)\prod_{s=0}^{t-1}\frac{f_m(u(c_s),m_{s+1})}{h'(m_{s+1})}h'(V_{s+1})\Big\}. \quad (24)
\end{aligned}
$$

From this we obtain that the state price deflator is given by (3). □.

Derivation of (13) in Section 2.5.

Towards this end, note that both utility U and future utility at time t, V_t, are homogeneous of degree one, so by the definition of directional derivatives it follows that for c optimal

$$
\begin{aligned}
\bigtriangledown U(c;c)&=\lim_{\alpha\downarrow 0}\frac{U(c+\alpha c)-U(c)}{\alpha}=\lim_{\alpha\downarrow 0}\frac{U(c(1+\alpha))-U(c)}{\alpha}\\
&=\lim_{\alpha\downarrow 0}\frac{(1+\alpha)U(c)-U(c)}{\alpha}=\lim_{\alpha\downarrow 0}\frac{\alpha U(c)}{\alpha}=U(c)
\end{aligned}
$$

where the third equality uses the homogeneity of U. It follows from the expression for the gradient of U in (1) that

$$\bigtriangledown U(c;c)=E\Big(\sum_{s=0}^{T}\pi_s c_s\Big)=W_0\pi_0=U(c),$$

where W_0 is the wealth of the representative agent at time zero, i.e., the second equality follows from the definition of W_t in (12).

Moving to time t, let $V_t(c)$ denote future utility at the optimal consumption c for our utility representation. Since V_t is continuously differentiable, by the Riesz' Representation Theorem the gradient is a linear functional, so that

$$\bigtriangledown V_t(c;c)=E_t\Big(\sum_{s=t}^{T}\pi_s^{(t)}c_s\Big),$$

where $\pi^{(t)}$ is the Riesz representation of $\bigtriangledown V_t(c)$. As explained in the text, $\pi_s^{(t)}=\pi_s/Y_t$ for all $t\le s\le T$, and by homogeneity of V_t the same, basic relationship holds for the associated directional derivatives, i.e.,

$$\triangledown V_t(c;c) = V_t(c).$$

Using (12) and $\pi_s^{(t)} = \pi_s / Y_t$, this shows (13). $\square$

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