Valuation of Asian options and commodity contingent claims

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Chapter 1 Introduction

ABSTRACT - A derivative security is a security whose value depends on the values of other more basic underlying variables. The use of derivatives have spread to non-financial markets. In this thesis we are studying issues in agricultural derivatives pricing, electricity contingent claims valuation and risk management, and average based contingent claims valuation. These areas of research may be perceived as rather unconnected at first sight. In this introductory chapter we make an effort of illuminating the points of contact between these three research areas. All are topics in the growing literature on contingent claims valuation and risk management in commodity markets. We give a short description of the papers contained in the following chapters of this thesis, and finally we suggest areas for further research.

Derivatives, or contingent claims, have been enormously successful the last 30 years, and these products have penetrated new markets. The Chicago Board Options Exchange (CBOE), founded in 1973, revolutionised options trading by organising a marketplace for options. The founding year of CBOE coincided with the publication of the seminal papers by Black and Scholes (1973) and Merton (1973) on stock option valuation. The use of derivative products quickly spread to currency and fixed income markets. Derivatives can now be bought on insurance, volatility, weather, electricity etc. The thesis deals with several issues that are important when modelling commodity markets for valuation and risk management purposes. We will investigate agricultural and electricity markets. This thesis consists of five independent essays. In this introduction we give a brief overview of the literature on the

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modelling of commodity markets and a summary of the main results. Finally we provide some suggestions for further research.

1.1 Modelling commodity markets

1.1.1 The theory of storage

In the commodity literature the theory of storage developed by Kaldor (1939), Working (1948, 1949), Telser (1958) and Brennan (1958) is the dominant model of commodity spot and futures prices. The futures and spot price differential is equal to the cost of storage (including interest) and an implicit benefit that producers and consumers receive by holding inventories of a commodity. This benefit is termed the convenience yield. Kaldor (1939) recognised the convenience yield as an explanation of the futures and spot price differential, the basis, "by enabling the producer to lay hands on them the moment they are wanted". Working (1948) and (1949) argued that the convenience yield could explain the negative basis, so-called contango, observed in agriculture markets at specific times of the year. The convenience yield is expected to depend upon the level of inventory, but in a marginal sense:

"The amount of stock which can thus be useful is, in given circumstances, strictly limited; their marginal yield falls sharply with an increase in the stock above requirements, and may rise very sharply with a reduction in stocks below requirement." Kaldor (1939) p.4.

Telser (1958) made an empirical investigation where he confirmed this negative relation when analysing the marginal convenience yield and inventories of cotton and wheat. Since holders of inventories earn the convenience yield but holders of futures contracts do not, a positive convenience yield depresses the futures price relative to the spot price.

1.1.2 Commodity contingent claims

Fischer Black introduced option pricing valuation in commodity markets. Using the dynamic hedging argument pioneered in Black and Scholes (1973)

and Merton (1973), Black (1976) derived a pricing formula for the European commodity futures option, by explicitly assuming a geometric Brownian futures price process. Brennan and Schwartz (1985) focused instead on the spot price of the commodity, and linked the dynamic hedging argument to the theory of storage. They modelled the commodity spot price as geometric Brownian motion. Assuming continuous trading opportunities in the commodity, a constant risk free rate and constant proportional convenience yield, they developed no-arbitrage valuation expressions and optimal managing policies for a real asset (copper mine). This method has been modernised, and in recent applications the stochastic dynamics of the spot price and other state variables are usually specified directly under the equivalent martingale measure. Under this probability measure the commodity earns the risk free rate of return subtracted by the net convenience yield which accrues to holder of the commodity and not to the holder of a futures contract. Gibson and Schwartz (1990) provide a generalisation in which the convenience yield is modelled as a mean reverting Ornstein-Uhlenbeck process. Schwartz (1997) adds the interest rate as a third stochastic factor.

One problem with spot-based models is that forward prices are given endogenously from the parameters governing the spot price dynamics, and so theoretical forward prices will in general not be consistent with market observed forward prices. As a response to this, a line of research has modernised the theory developed by Black (1976) which focused on a single commodity contract. A modern approach describes the dynamic evolution of the whole forward curve, taking as given the initial term structure. Examples of this research, building on the pioneering work on modelling forward interest rate by Heath et al. (1992), are Cortazar and Schwartz (1994), Miltersen and Schwartz (1998), Clewlow and Strickland (1999a), (1999b) and (2000) and Björk and Landén (2002).

Commodities constitute a large and heterogeneous group of assets. This means that one modelling approach that works in one market might need to be substantially modified in order to perform satisfactorily in another. Duffie (1989) classifies the commodity futures contracts traded on U.S. futures markets as follows; forest products (lumber), textiles (cotton), metals (gold, silver, platinum, palladium, copper, aluminium), energy (heating oil, gasoline, crude oil, propane gas), foodstuffs (cocoa, coffee, orange juice, potatoes, sugar, corn syrup), livestock (pork, beef), grains (corn, oats, rice, wheat) and oil and meal (soybean). In this thesis we conduct empirical investigations in both an agricultural market (the US wheat market) and an electricity market (the Nordic electricity market). We note that wheat is mentioned explicitly in the subgroup of agricultural products called grains. Electricity on the other hand, usually considered an energy commodity, is not included in the overview in Duffie (1989). This is due to the fact that exchange traded power contracts are a fairly new phenomenon. In the following subsections we briefly sketch the particularities of these two commodity markets, and indicate in which ways derivatives pricing models might deviate from the traditional commodity contingent claims models described above. Finally we argue that average based contingent claims are important instruments in both of these markets.

1.1.3 Agricultural derivatives

The model suggested by Black (1976) and Brennan and Schwartz (1985) implies that futures prices are lognormally distributed with variance proportional with time to maturity. Empirical investigations have indicated that this model assumption is too restrictive in the case of agricultural futures markets. Price jumps will typically occur due to abrupt changes in supply and demand conditions, and such discontinuities in the price path of futures prices will affect the prices on options written on futures contracts (see e.g. Hall et al. (1989) and Hilliard and Reis (1999)).

Other studies have investigated whether the assumption of constant volatility is valid. Samuelson (1965) claimed that the volatility of futures price returns increases as time to maturity decreases. He argued that the most important information was revealed close to the maturity of the contract. For example, the weather affecting demand or a temporary supply disruption will affect spot prices and hence short dated futures contracts. Short-term price movements are not expected to persist but rather revert back towards a normal level. This implies that long dated contracts will be less affected by spot price changes and experience lower volatility than short dated contracts. This maturity effect is sometimes referred to as the "Samuelson hypothesis". Other authors have argued that the volatility of futures prices depends on the distribution of underlying state variables. This is sometimes termed the "state variable hypothesis". For crop commodities one would typically expect the information flow to vary during the crop cycle. The most important information is revealed during growing and harvest seasons, hence seasonality in the volatility of futures prices is expected (see Anderson (1985), Milonas (1986) and Galloway and Kolb (1996)). Fackler and Tian (1999) propose

a simple one-factor spot price model with mean reversion (in the log price) and seasonal volatility. They show that futures prices consistent with this spot price model have a volatility term structure exhibiting both seasonality and maturity effects. Their empirical results indicate that both phenomena are present in the soybean futures and option markets. Both jumps and time-dependent volatility naturally affects contingent claims valuation. In chapter 2 we specify a futures price process that allows for both the possibility of jumps and time-dependent volatility. The model is estimated using eleven years of wheat futures options listed on Chicago Board of Trade.

1.1.4 Electricity derivatives

The electricity derivatives markets have grown rapidly as the restructuring of electricity supply industries is spreading around the world. Electricity differs in several respects from other commodities. Some important features are¹:

- Non-storability. There is currently no technology that can economically store electricity once generated. Therefore electricity demand and supply has to be balanced continuously in a transmission network to prevent the network from collapsing. The lack of storage technology implies that electricity cannot be considered a financial asset held purely for investment purposes. The usual cash-and-carry arbitrage relationship does not apply for electricity.
- No lower bound. Since electricity cannot be sold short there is no lower bound on electricity prices. In fact, negative prices have occurred in several electricity markets. Prices may become negative, as power plants have to get rid of excess output in periods when demand is low.
- Correlation between short- and long term pricing. Pilipović (1998) conjectures that energy prices exhibits a "split personality". This, she claims, applies especially to electricity, where short term prices are to a large extent demand driven while forward prices are determined by expectations of market productions capacity, improved technology and long run cost.

¹See Leong (1997), Clewlow and Strickland (2000) and Pilipović (1998) for a thorough discussion on several of these issues.

- Generation and transmission technology. Electricity may be generated from natural gas, coal, oil, nuclear fuel, water turbines, from alternative resources such as cogeneration and renewable sources such as wind power, solar energy and biomass. After electricity is generated, it is transmitted over high-voltage power lines before it is distributed to the end users. In periods of high demand, the electricity transmitted may come close to maximum capacity. Increased demand cannot be met by increased supply, and prices may jump to extreme levels for short periods of time. In some electricity markets "price spikes" are common (see Deng (2000) and Clewlow and Strickland (2000)).
- Seasonality. In many markets prices peak twice a year, once during winter due to demand for heating and once in summer months due to intensive use of air-condition. Electricity markets also exhibit daily and weekly price patterns.

An especially worrisome feature from the list above is the non-storability of electricity. An immediate consequence of this is that the traditional theory of storage does not apply to this commodity. Continuous dynamic hedging is impossible directly in the underlying asset. Still, spot price models have been investigated in the literature. In these models the spot price is treated as a state variable on which derivatives are written, and for valuation purposes this state variable is adjusted for risk (usually making ad hoc assumptions). Examples of spot price based electricity models are Lucia and Schwartz (2000), Knittel and Roberts (2001), Kamat and Ohren (2000), Clewlow and Strickland (1999b) and (2000), Deng (2000) and Pilipović (1998). What process then, should we adopt for the electricity price? The most common choice is the familiar geometric Brownian motion. The plain geometric version is usually modified in one way or the other to allow for the special properties of electricity. The technically most advanced of these studies is Deng (2000). He models the log of the spot price with mean reversion, regime switching, stochastic volatility and different types of jumps. Lucia and Schwartz (2000) and Knittel and Roberts (2001) consider arithmetic Brownian motion as the process driving the spot price process.² Contrary to the traditional geomet-

 $^{^{2}}$ In fact Lucia and Schwartz (2000) consider a mean reverting spot price both in the log price and in the price level. They do not make any strong judgement regarding which model is the best. Knittel and Roberts (2001) also consider arithmetic mean reverting models with GARCH-effects.

ric Brownian motion, the arithmetic spot price process allows for negative prices.

In the case of electricity, the forward curve approach seems even more profitable, since the problem of non-storability is avoided altogether. In a spot-based model of electricity we do not model the tradable assets directly. The assets introduced in such a model are all derivatives on the spot rate. In a forward curve model this is no longer an issue; the forward prices modelled are the tradable assets. In chapter 3 we conduct an empirical investigation of arithmetic and geometric multi-factor forward curve models in the Nordic electricity market. In chapter 4 we investigate the analytical tractability of the arithmetic forward curve model, with application to option pricing and risk management.

1.1.5 Asian options

When an option depends on the average price history of the underlying asset prior to maturity, it is called an *Asian option*. In the electricity market both futures contracts and spot price options are based on the arithmetic average of the underlying asset. In agricultural markets, many of the options traded over-the-counter (OTC) are of Asian style (Hilliard and Reis (1999)). When the uncertainty of the underlying asset is arithmetic Brownian motion, the arithmetic average of the underlying asset is itself normally distributed, and a closed form solution to the Asian option can be derived (see chapter 4 in this thesis). When the underlying asset is lognormally distributed, on the other hand, the arithmetic average is not itself lognormally distributed. In fact, the distribution of the arithmetic average of a lognormally distributed asset is unknown, and we must resort to different approximations and/or numerical techniques to price Asian options. Chapters 5 and 6 are devoted to valuation of Asian options when the underlying asset is lognormally distributed.

1.2 Summary of results

This section provides a brief summary of the results in the forthcoming chapters.

1.2.1 Chapter 2: Volatility and price jumps in agricultural markets - evidence from option data³

In this chapter specify a futures price process that allows for both the possibility of jumps and time-dependent volatility. The volatility captures both a seasonal and a maturity effect. A futures option pricing model is derived, and the model specification is estimated using eleven years of wheat futures options listed on Chicago Board of Trade. The market observed option prices are compared to the theoretical option prices, and the parameters of our futures price model are estimated using non-linear least squares. Several models suggested previously in the literature are nested in our model specification, and we can use standard statistical tests to determine whether jumps and time dependent volatility are present in the data. The results show that the maturity effect is especially strong in the wheat futures market. The seasonal effect is of lesser importance, but it is statistically significant. The estimated jump intensity is significantly different from zero. This result is in line with results found in the soybean futures option market reported in Hilliard and Reis (1999). When testing different models against each other, we find that simpler models are rejected in favour of our proposed model with jumps, seasonality and maturity effects. A numerical example illustrates the implications for market prices of options.

1.2.2 Chapter 3: Forward curve dynamics in the Nordic electricity market⁴

Even though the analysis in chapter 2 is concerned with the price dynamics of a wheat futures contract, the model employed is not a forward curve model. Contrary to the analysis in chapter 2, which is concerned with options written on a single futures contract, a forward curve model is concerned with the links between the stochastic processes of futures contracts with different time to maturity. In chapter 3 we adopt the forward curve approach and perform an empirical examination of the dynamics of the forward curve

³This chapter is co-authored with Gudbrand Lien. An earlier version of this paper appeared as Discussion Paper 19/2001 at Norwegian School of Economics and Business and Administration, Department of Finance and Management Science.

⁴This chapter is co-authored by Fridthjof Ollmar. An earlier version of this paper appeared as Discussion Paper 21/2001 at Norwegian School of Economics and Business and Administration, Department of Finance and Management Science.

in the Nordic electricity market using market prices on futures and forward contracts in the 1995-2001 period. We specify two different models for the evolution of the forward price of electricity in the framework of Heath et al. (1992); the geometric and the arithmetic Brownian motion. Two sets of data are constructed. For the arithmetic model forward price differences are analysed, and forward price returns are analysed in the case of the geometric model. The maturities for the contracts that constitute the data sets range from one week to two years. Following the work of Cortazar and Schwartz (1994) and Clewlow and Strickland (2000) we use principal component analysis to analyse the volatility factor structure of the forward curve. Similar to the wheat futures market investigated in chapter 2, we find a very strong maturity effect in the electricity market. In the short end of the term structure, the volatility increases sharply as time to maturity decreases. In other commodity markets one typically find that a few factors are able to explain most of the variation in the forward prices. The portion of explained variance is lower in the electricity market. We find that a two-factor model explains 75% of the price variation in our data, compared to approximately 95% in most other markets. Pilipović (1998) conjecture that electricity prices exhibit "split personalities". By this she means that the correlation between short- and long term forward prices are lower in electricity markets than in other markets. We provide some empirical support of this claim. The most important factors driving the long end of the curve have very little impact on price changes in the short end. Furthermore we find some evidence of changing volatility dynamics both seasonally and from one year to another. Finally, we are unable to decide if an arithmetic or geometric model describes the data best.

1.2.3 Chapter 4: A multi-factor forward curve model for electricity derivatives

The purpose of chapter 4 is to develop valuation formulas and hedging strategies for electricity contingent claims in a multi-factor *arithmetic* forward curve model. The proposed forward curve model is identical to the arithmetic model investigated in chapter 3. The fact that electricity cannot be stored implies that production and consumption have to balance in a power network. This property makes electricity unique compared to other commodities, and often electricity is described as a *flow commodity*. Consequently, contracts

traded in the electricity industry are typically specified with a future time period for delivery, not delivery at a future time point. The value of such a contract depends on the arithmetic average of the electricity spot price in the delivery period. In a lognormal electricity forward price model, simple closed form solutions to such derivatives do not exist, since the distribution of the sum of lognormal random variables is unknown. Hence, in a lognormal model, approximations are needed even for simple European contingent claims (see Bjerksund et al. (2000) for approximate valuation of different average based contracts in a lognormal forward curve model). Our model, being a forward price model, provides an important generalisation of the Gaussian spot price model proposed by Lucia and Schwartz (2000) and Knittel and Roberts (2001), since it is consistent with observed market prices. But the most important property of our model is that it provides simple closed form pricing formulae for arithmetic average based contingent claims. We investigate the dynamic properties of two different average based forward contracts. These contracts are of purely financial nature, hence no delivery of electricity is actually made. In the first contract we consider, the owner of the contract receives or pays, at maturity, the difference between the forward price and the average electricity price during a pre-specified delivery period. In the second contract the owner receives or pays the difference between the price of a unit of electricity and the contract price each instant during the delivery period of the contract. The contract specifications mimic the specification of the contracts trading in the Nordic electricity market. We show that both these contracts are normally distributed. Based on these results, closed form solutions to both European and Asian options and corresponding hedge ratios are calculated. We briefly discuss factor hedging in this model, and we provide some numerical examples using data from the Nordic electricity market.

1.2.4 Chapter 5: Approximate Asian option pricing in the Black '76 framework

In this chapter we derive an approximate lognormal valuation model for Asian options. The preferred model among practitioners to price arithmetic average Asian options seems to be the lognormal approximation proposed by Levy (1992). We propose a new lognormal approximation. Our model is a modification of the Black (1976) formula. Fischer Black published this modification of the original stock option model to value options on commodity forward and futures contracts. As the use of futures contracts has penetrated all major financial markets, the Black (1976) model is perhaps the most frequently used option pricing formula there is. We need two inputs for our model; a futures price and a plug-in volatility. The first step in our analysis is to calculate the "price" today of the future arithmetic average asset price. This is an easy computable conditional expectation. We then interpret this price as a financial futures contract, which delivers the value of the arithmetic average of the underlying asset price at maturity. This means that an Asian option can be reinterpreted as a European futures option. We show that this contract is actually lognormally distributed prior to the averaging period. After entering the averaging period, the arithmetic average contract is no longer lognormally distributed. We then propose a lognormal approximation of the contract inside the averaging period. Based on the analysis above, we calculate a plug-in volatility for the futures option model. In a Monte Carlo exercise, we show that our model has some advantage over the Levy (1992) model in terms of accuracy. We finally study the implicit volatility of the average rate options. We calculate "exact" market prices by Monte Carlo simulation and use the Black (1976) formula to back out implicit volatilities. An Asian call option in a standard Black-Scholes economy has an upward sloping implied volatility "smile" across maturities due to the deviation from lognormality of the arithmetic average. This smile cannot be captured by a lognormal approximation.

1.2.5 Chapter 6: Valuation of Asian options by matching moments

This chapter extends the analysis from chapter 5. The purpose of this final chapter is to develop a valuation method applied to Asian options that gives more accurate prices than a lognormal approximation. Our method utilises the information in the moments of the arithmetic average. Our analysis is linked to the work of Turnbull and Wakeman (1991). They apply the Edgeworth expansion using the lognormal density as an approximate distribution. In their paper the series expansion is truncated after the fourth term, hence information from the first four moments of the arithmetic average is utilised in their valuation approach. It is well known that this method is inaccurate when the volatility of the underlying asset is high. We take a somewhat

different approach to the method of matching moments. The analysis of the average based futures contract from chapter 5 is pushed a bit further. Having already established that the contract is lognormally distributed prior to the averaging period, we give a stochastic volatility interpretation of the futures price dynamics inside the delivery period. Unfortunately the resulting stochastic differential equation is unfamiliar and a closed form pricing formula cannot be reached. Instead we choose a lognormal futures model with stochastic variance, where the variance is modelled as a mean reverting square-root process. Heston (1993) first introduced this model, and he showed that European option pricing can be done efficiently by Fourier inversion methods. The goal is to price Asian options, but first we need to find suitable parameters in Heston's model. Valuation is done in the following steps: 1) Calculate the variance, skewness and excess kurtosis of the arithmetic average. 2) Use an optimising routine to pick parameters of the approximate model that produce variance, skewness and excess kurtosis close to (by minimising mean square error) the arithmetic average. 3) Use the Fourier inversion technique to calculate the price of a European option on the average rate contract. This procedure allows us to match the first four moments of the arithmetic average. Our method produces very accurate option prices, also when the underlying asset volatility is high. From our analyses we can conclude that the first four moments contain enough information about the density of the arithmetic average of the geometric Brownian motion to facilitate accurate option pricing. We provide some numerical examples where we compare the accuracy of our model with other methods proposed previously in the literature.

1.3 Some suggestions for future research

In this section I will outline some possible directions for future research. These suggestions are very closely related to content in each of the following chapters, and as such, may be considered extensions of these specific research areas.

In chapter 2 we model the futures price dynamics as a jump diffusion with time dependent volatility. The volatility can capture both seasonal and maturity specific effects. The moments of the returns of a contract will exhibit excess kurtosis. Some authors have documented skewness in futures price returns series, and our model does not capture this. One possible extension would be to modify our deterministic volatility specification with stochastic volatility. Bates (1996) has analysed a model with both jumps and stochastic volatility. The volatility is modelled as a mean reverting square-root process. The mean towards the volatility reverts is a constant. A possible modification of the model suggested by Bates (1996), is to let the volatility revert towards a time dependent mean. The time dependent mean can capture both maturity and seasonality effects. Such a model is richer than the one considered in chapter 2, and it can capture possible fluctuations in volatility. Especially when calibrated to prices over long periods of time, such a model might give a better fit than a deterministic volatility specification.

In chapter 3 we investigate the volatility dynamics of the forward curve using principal component analysis (PCA). The PCA analysis is able to capture the maturity effect in this market. Our analysis also indicates seasonal variation in volatility and possibly also stochastic volatility. In future research it would be desirable to be able to incorporate seasonal and/or stochastic volatility explicitly for one or each of the factors determined by the PCA analysis. One way to proceed would be to investigate a so-called orthogonal GARCH model, where seasonal variation is included in the GARCH specification.⁵ Orthogonal GARCH is essentially a two-step procedure; first run a PCA, and then fit a univariate GARCH model to each of the principal components. The univariate structure of the GARCH models is possible, since each principal component is uncorrelated.

In chapter 4 we consider a market in which there exist a continuum of forward prices. All other contracts are derived from these forward prices. In particular, the average based electricity forward contracts traded in the Nordic electricity market are derived from this continuum of basic forward prices. This modelling approach is parallel to the modelling of the forward interest rates pioneered by Heath et al. (1992). However, there are other interest rate models that may be adapted in electricity markets, even more appropriate than the Heath-Jarrow-Morton model. In the fixed income market place, the rates applicable to interest rate derivatives are typical LIBOR or swap rates. From a modelling point of view, starting with a continuum of initial forward rates, and construct a continuum of processes, typically leads to analytical intractable processes for forward LIBOR and swap rates (see Musiela and Rutkowski (1997)). New types of interest rate models, so-called market models, have appeared. Such a model concentrates on the actual

⁵See Alexander (2001) for an overview of orthogonal GARCH.

rate at hand, and models it directly, circumventing the need to model forward rates. Jamshidian (1997) develops both LIBOR and swap rate based market models, and he derives closed form solutions for different kinds of swaptions etc. in the case of lognormal forward swap or LIBOR rates.⁶ In the electricity market the same problems with forward rate based models. The forward prices of electricity are not available in the market place. They have to be estimated from average based forward contracts. The stochastic process of this average based forward is typically intractable, and no closed form solutions generally exist.⁷ These contracts may be interpreted as swap contracts. Entering into a long electricity contract means that you are swapping floating electricity prices, against the fixed price - the price of the average based forward contract. Hence, the average based contracts can be interpreted as delivering the swap rate during a specified time period the delivery period of the contract. In a swap market model, the dynamic properties of the average based contracts will stated explicitly, and model estimation and testing can be performed directly using observed market prices instead of smoothed prices.

We encounter the problem of pricing average based contingent claims in several chapters. Chapter 4 provides closed form solutions to such a claim in an arithmetic Brownian model, and chapters 5 and 6 provide approximations in a standard geometric Brownian model. The bulk of research on Asian option valuation has been conducted within this model. Chapter 2 shows that the occurrence of jumps is important in describing futures price dynamics in the wheat futures market. Hilliard and Reis (1999) investigated the price effect of jumps on Asian option prices in a Monte Carlo experiment. They showed that Asian option prices in a jump-diffusion model relative to Asian options in a lognormal model differs more than corresponding European options. The ratio of a lognormal over jump-diffusion out-of-the-money Asian put is over twice the corresponding ratio for European options. This evidence suggests that it be worth investigating Asian option pricing when the underlying asset is deviates from lognormality. The method suggested

 $^{^6 \}mathrm{See}$ Sandmann and Sondermann (1993, 1994), Goldys et al. (1994) Musiela (1994), Miltersen et al. (1995, 1997) and Brace et al. (1997) for contributions in the literature on market models.

⁷Of course, in chapter 4 we show that in an arithmetic forward based model, closed form solutions exists. In the market place, the Black-Scholes model is the preferred model, and in the case of lognormal forward rates, no closed form Black-Scholes formula exists (see Bjerksund et al. (2000)

in chapter 6 may be modified to handle underlying asset prices that exhibit independent jumps (as in chapter 2). A recursion could be set up to calculate moments for the discrete average, and those moments could be matched with say, the jump-diffusion stochastic volatility model of Bates (1996). If we rely on Monte Carlo methods, we are free to pick any underlying asset price dynamics.

I will like to stress the fact the suggestions for further research mentioned above, is by no means an exhaustive list of paths were future research may or should go. Rather they constitute neighbouring research areas for each chapter in this thesis. They may not be the most important research topics that need addressing. For example, in this thesis we concentrate on models dealing with price risk. An equally important risk to many market participants in both agricultural and electricity markets is the volumetric risk. Volumetric risk is important both when it comes to hedging decisions and to valuation of volume dependent contracts (called *time-of-use* or *swing* contracts in the electricity market). This is unquestionably a big and challenging area for future research. The use and valuation of weather derivatives is another. This shows that we are experiencing exciting times in commodity markets risk management and contingent claims valuation.

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Chapter 2

Volatility and price jumps in agricultural markets - evidence from option data

This paper is co-authored with Gudbrand Lien¹

ABSTRACT - Empirical evidence suggests that agricultural futures price movements have fat-tailed distributions and exhibit sudden and unexpected price jumps. There is also evidence that the volatility of futures prices is time dependent. It varies both as a function of calendar-time (seasonal effect) and time to maturity (maturity effect). This paper extends Bates (1991) jump-diffusion option pricing model by including both seasonal and maturity effects in the volatility specification. An in-sample fit to market option prices on wheat futures shows that the suggested model outperforms previous models considered in the literature. A numerical example indicates the economic significance of our results for option valuation.

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2.1 Introduction

Black (1976) derives a pricing model for European puts and calls on a commodity futures contract, assuming that the futures price follows a geometric Brownian motion (GBM). In the literature on agricultural futures markets, several empirical characteristics have been documented, indicating that the GBM assumption may be too simple. Research has detected leptokurtic returns in agricultural futures prices (e.g. Hudson et al. (1987) and Hall et al. (1989)), and the prices often exhibit sudden, unexpected and discontinuous changes. Price jumps will typically occur due to abrupt changes in supply and demand conditions, and such discontinuities in the price path of futures prices will affect the prices on options written on futures contracts. Hilliard and Reis (1999) used transactions data on soybean futures and futures options to test the Black (1976) diffusion model versus the jump-diffusion option pricing model of Bates (1991). Their results show that Bates' model performs considerably better than Black's model in both in-sample and out of sample tests.

A number of studies have demonstrated the presence of a volatility term structure in agricultural futures prices. Samuelson (1965) claimed that the volatility of futures price returns increases as time to maturity decreases. He argued that the most important information was revealed close to maturity of the contract. For example, the weather affecting demand or a temporary supply disruption will affect spot prices and hence short dated futures contracts. In the long term, short-term price movements are not expected to persist rather revert back towards a normal level. This implies that long dated contracts will be less affected by spot price changes and experience lower volatility than short dated contracts. This maturity effect is sometimes referred to as the "Samuelson hypothesis". Other authors have argued that the volatility of futures prices depends on the distribution of underlying state variables. This is sometimes termed the "state variable hypothesis". For crop commodities one would typically expect the information flow to vary during the crop cycle. The most important information is revealed during growing and harvests season, hence seasonality in the volatility of futures prices is expected. is that Empirical research has produced mixed evidence on the two effects. Milonas (1986) found strong support for the maturity effect after controlling for seasonality. Galloway and Kolb (1996) concluded that the maturity effect is present in markets where commodities experience seasonal demand and/or supply, but not in commodity markets where the
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cost-of-carry model works well. Anderson (1985) found support for the maturity effect, but claimed it is secondary to seasonality. Anderson (1985) also concluded that the pricing of options on futures contracts should allow for the regular pattern to the volatility of futures. Bessembinder et al. (1996) have reconciled much of the early evidence on the "Samuelson hypothesis". They have shown formally that, in markets where spot price changes include a temporary component so that investors expect some portion of a typical price change to revert in the future, the "Samuelson hypothesis" will hold. Mean reversion is more likely to occur in agricultural commodity markets than in markets for precious metals or financial assets (Bessembinder et al. (1995)), so we expect to see maturity effects in agricultural commodity markets.

Any regular pattern in the volatility is inconsistent with the underlying assumptions in Black (1976) and Bates (1991). Choi and Longstaff (1985) applied the formula of Cox and Ross (1976) for constant elasticity of variance option pricing in the presence of seasonal volatility. They found this superior to the Black (1976) model for pricing options on soybeans futures. Myers and Hanson (1993) present option-pricing models when time-varying volatility and excess kurtosis in the underlying futures price are modelled as a GARCH process. Empirical results suggest that the GARCH optionpricing model outperforms the standard Black (1976) model. Fackler and Tian (1999) proposed a simple one-factor spot price model with mean reversion (in the log price) and seasonal volatility. They show that futures prices consistent with this spot price model have a volatility term structure exhibiting both seasonality and maturity effects. Their empirical results indicate that both phenomena are present in the soybean futures and option markets.

In this paper we assume that the futures price follows a jump-diffusion process. The diffusion term includes time dependent volatility that captures (possibly) both seasonal and maturity effects. We derive a futures option pricing model given our specified futures price dynamics. The model parameters are estimated from option prices written on the futures contract. Eleven years of futures and option data is collected from Chicago Board of Trade (CBOT). The market observed option prices are compared to the theoretical option prices, and the parameters of our futures price model are estimated using non-linear least squares. Several models suggested previously in the literature are nested in our model specification, and we can use standard statistical tests to determine whether jumps and time dependent volatility are present in the data. The results show that the maturity effect is especially strong in the wheat futures market. The seasonal effect is of lesser importance, but it is statistically significant. The estimated jump intensity is significantly different from zero. This result is in line with results found in the soybean futures option market reported in Hilliard and Reis (1999). When testing different models against each other, we find that simpler models are rejected in favour of our proposed model with jumps, seasonality and maturity effects. A numerical example illustrates the economic significance of our results.

This paper is organised as follows: In the next section we present the futures price dynamics and derive a futures option pricing formula. Section 2.3 describes the data. In section 2.4 we estimate parameters in the jump-diffusion model. The economic significance of our results is illustrated in a numerical example. Section 2.5 concludes.

2.2 Model description

We assume that there exists an idealised futures market (liquid, frictionless, no taxes, limitless short selling allowed etc.) for every delivery date T^* . Denote the price of a futures contract as $F(t, T^*)$, where t is today's date and T^* is the maturity date of the contract.² The futures price is assumed to follow the following dynamics under the equivalent martingale measure (EMM):

$$\frac{dF(t,T^*)}{F(t,T^*)} = -\lambda \overline{\kappa} dt + \sigma(t,T^*) dB(t) + \kappa dq \qquad (2.1)$$

where B is standard Brownian motion under the EMM and κ is the random percentage jump conditional upon a Poisson distributed event, q, occurring and $\overline{\kappa}$ is defined as the expected value of the jump size if it in fact occurs. The counting process q is independent of κ , with $Prob(dq = 1) = \lambda dt$ and $Prob(dq = 0) = 1 - \lambda dt$. By standard no-arbitrage arguments we know that

²We present our modelling framework in a non-technical manner. Merton (1976) first introduced the jump-diffusion model of asset prices. The modern mathematical framework for modelling discontinuities in asset price is by the use of so called marked point processes, in which the Poisson distributed jump arrival process considered in this paper is one of many possible candidates. See Veredas (2000) for a nice, readable introduction on marked point processes. A very nice exposition of forward, futures and option pricing in a very general framework is given in Björk and Landén (2002). Since the focus of this paper is the empirical properties of a jump-diffusion model, we have omitted the technicalities.

since it costs nothing to enter a futures contract, the expected return on holding the contract should be zero under the EMM. We can easily check that this is the case in our model: The Brownian motion has zero expectation. The expectation of κdq during a time increment dt is $E[\kappa dq] = E[\kappa] E[dq] = \overline{\kappa} \lambda dt$, thus $E\left[\frac{dF(t,T^*)}{F(t,T^*)}\right] = 0$. We now need to specify the jump distribution and the volatility term structure. The inclusion of jumps in a model free of arbitrage raises some issues of market incompleteness. We give a brief discussion of this in the following subsection. We then describe a model for the volatility that is able to capture both seasonal and maturity effects.

2.2.1 Jumps and market incompleteness

We assume that $\ln(1+\kappa)$ is a normally distributed random variable with mean $(\gamma - \frac{1}{2}\nu^2)$ and variance ν^2 . Consequently, the expected percentage jump size is $E[\kappa] \equiv \overline{\kappa} = e^{\gamma} - 1$. These distributional assumptions are equal to those stated in Merton $(1976)^3$ and Bates (1991), but other distributions might be considered.⁴ Note that the jump parameters are constants, in particular they are independent of time to maturity. This means that if a jump occurs, a parallel shift in the term structure of futures prices will emerge. If we observe futures contracts with time to maturity spanning several years into the future, the assumption that the returns on all contracts jump with equal amounts may seem inadequate. If, for example, exceptional bad weather (such as a hurricane) partly destroys a harvest, then futures prices are likely to jump. But we would expect contracts with maturity before the next harvest to experience a greater price change than contracts with maturity preceding the next harvest, since the next harvest is likely to turn out better than the present one. This behaviour can easily be incorporated in our model by imposing time dependence on the jump amplitude. Such an extension is ignored here since the maturity of the futures contracts analysed in the empirical part of this paper never exceed one year. Hence, in our data set, imposing parallel jumps may be a satisfactory assumption.

³Merton (1976) assumed zero mean jump size, hence $\gamma = 0$.

⁴Other jump distributions are considered in the financial literature. Duffie et al. (2000) assume that abrupt changes in volatility are caused by Pareto distributed jumps, and Kou (2000) investigates option pricing in the presence of double-exponentially distributed price jumps. The literature on jumps in financial agricultural prices, as far as we know, concentrates on the lognormal jump model. Investigating other jump distribution in agricultural markets is left for further research.

Merton (1976) assumed that jumps are symmetric (zero mean) and nonsystematic. In a stock market model, this means that jumps are of no concern to an investor with a well-diversified portfolio, since jumps on average cancel out. Given such assumptions of firm specific jump risk, parameters concerning the jump part are equal under both the real world probability measure and the EMM. The assumption of non-systematic jump risk may be inappropriate many settings, and this is also the case in commodity futures markets. If, for example, bad weather results in a poor harvest, futures prices may jump. However, the occurrence of such an event is likely to move all the commodity futures prices in the same direction, and so diversifying the jump risk is impossible. In other words, jump risk is systematic. It is well known that the presence of systematic jump risk in our market makes it incomplete in the Harrison and Pliska (1981) sense. This means that it is not possible to set up a dynamic hedging strategy in the underlying asset and a risk free asset that replicates a contingent claim due to the possibility of abrupt jumps in the underlying asset price. This essentially means that under the absence of arbitrage opportunities, there are many (infinite) equivalent martingale measures. Furthermore, without explicit assumptions on preferences and technologies, each martingale measure defines an admissible price of a contingent claim (see Harrison and Kreps (1979)).

Bates (1991) derives a unique martingale measure in a jump-diffusion setting by considering a specific equilibrium model. He assumes that optimally invested wealth W follows a jump-diffusion

$$\frac{dW(t)}{W(t)} = (\mu_W - \lambda_W \overline{\kappa}_W - C/W) dt + \sigma_W(t, t) d\widetilde{B}(t) + \kappa_W dq \qquad (2.2)$$

where B is standard Brownian motion, μ_W is constant and κ_W is the random percentage jump in wealth conditional on the Poisson event q occurring. The Poisson counter has intensity λ_W . The subscript W indicates that that the model is specified under the objective (or "real world") measure. Now let $\ln(1 + \kappa_W)$ be normally distributed with mean $(\gamma_W - \frac{1}{2}\nu_W^2)$ and variance ν_W^2 , and set $E[\kappa_W] \equiv \overline{\kappa}_W = e^{\gamma_W} - 1$ and $Cov [\ln(1 + \kappa_W), \ln(1 + \kappa)] \equiv \epsilon_{FW}$, where $Cov [\bullet]$ denotes the covariance. Furthermore the representative consumer has time separable power utility U where

$$E_{\tau} \int_{\tau}^{\infty} e^{-\rho\tau} U(C_{\tau}) d\tau, \quad U(C) = \frac{C^{1-R} - 1}{1 - R}$$
(2.3)

and R is the relative risk aversion. The riskless rate is constant, and jump risk is by construction systematic, since prices and (optimally invested wealth) jump simultaneously. Bates (1991) shows that in this economy, when the representative investor optimises his utility over an infinite time horizon, there exists a unique martingale measure, and that the stochastic differential equation describing futures prices are given by (2.1). He shows the following relations between model parameters under the objective and risk neutral measure⁵

$$\sigma(t,t) = \sigma_W(t,t)$$

$$\nu = \nu_W$$

$$\lambda = \lambda_W e^{\left(-R\gamma_W + \frac{1}{2}R(1+R)\nu_W^2\right)}$$

$$E[\kappa] \equiv \overline{\kappa} = e^{\gamma_W - R\epsilon_{FW} - 1}$$
(2.4)

We see that both the diffusion term and the variance of the jumps are the same under both measures. But both the jump intensity and mean jump size is different under the two measures. Bates (1991) interprets λ as the cost per unit time of jump insurance. If the mean jump size is zero, $\overline{\kappa}_W = \gamma_W = 0$, and the representative investor is risk averse, we find that $\lambda > \lambda_W$. Mathematically this means that the probability of a jump occurring is greater under the risk neutral measure than under the objective measure. Economically it means that risk aversion increases the price of jump insurance. In the case of risk neutrality we find that $\gamma = \gamma_W$. The mean jump size will typically be downward biased under the equivalent martingale measure. The model suggested by Merton (1976) can be seen as a special case of Bates (1991) with a risk neutral agent and zero mean jump size. In this special case we have $\overline{\kappa} = \overline{\kappa}_W$ and $\gamma = \gamma_W$.

In the empirical part of this paper, we extract the jump parameters from option prices. From the discussion above it is clear that these parameters are not equal to the parameters of the actual jump process governing futures prices under the objective measure. Therefore care must be taken when interpreting parameters implicit in option prices.

⁵In his model, Bates (1991) assumes constant volatility in the diffusion term, but it is not difficult to show that the diffusion term is equal under both measures in the case of a deterministic term structure of volatility as well (see the appendix in Bates (1991)).

2.2.2 Time dependent volatility

The function $\sigma(t, T^*)$ represents the instantaneous volatility of the futures price driving the diffusion term in the futures price dynamics. We want to capture two possible effects in the specification of the volatility function; seasonality and maturity. We will concentrate on the following candidate

$$\sigma(t, T^*) = \widehat{\sigma}(t) \,\widehat{\delta}(T^* - t) \tag{2.5}$$

where seasonality is represented by $\hat{\sigma}(t)$ and the maturity effect is given by $\hat{\delta}(T^*-t)$. This multiplicative relation between the two effects nests several models suggested previously in the literature. We will specify both $\hat{\sigma}(t)$ and $\hat{\delta}(T^*-t)$ in the subsections below.

Seasonal volatility

The first term in (2.5), $\hat{\sigma}(t)$, represents the time t dependent seasonal volatility pattern. A seasonal pattern evolving gradually through a cycle leads naturally to some trigonometric representation. The trigonometric function

$$\widehat{\sigma}(t) = \sin t \tag{2.6}$$

is defined in terms of an angle, t, which is measured in radians. In a circle there are 2π radians, and therefore $\hat{\sigma}$ goes through its full complement of values as t goes from 0 to 2π . We need flexibility of the seasonal function. As a first step we can replace t with χt . The parameter, χ , is known as the (angular) frequency. The time for $\hat{\sigma}$ to go through its complete sequence of values is called the *period* of the cycle, and it is equal to $2\pi/\chi$. Multiplying the trigonometric function by c, affects the *amplitude* of the cyclical waves. Finally, in order shift the function along the time axis, we introduce the parameter, ζ , which is known as the *phase*. The expression then becomes

$$\widehat{\sigma}(t) = \overline{\sigma} + c\sin\left(\chi t - \zeta\right) \tag{2.7}$$

where $\overline{\sigma}$ represent the average volatility during a cycle. Instead of introducing the phase, the same lateral shift could be introduced by a mixture of sine and cosine functions. Furthermore the model specification in (2.7) produce a volatility function which exhibits exactly one peak and one trough during a full cycle. Summing several trigonometric functions with different phase,

Previous models	Parameter constraints
Black (1976)	$\overline{\lambda} = \delta = \alpha_i = \beta_j = 0$
Schwartz (1997)	$\lambda = \alpha_i = \beta_i = 0$
Fackler and Tian (1999)	$\lambda = 0$
Bates (1991)	$\delta = \alpha_i = \beta_j = 0$

Table 2.1: Previously suggested models Several models suggested previously in the literature are nested within our model for the futures price dynamics given in (2.1), (2.8) and (2.9). The models are given in column one, and the corresponding parameter constraints are given in column two.

amplitude and period can induce more flexibility of the seasonal pattern (for example several peak periods). This leads to

$$\widehat{\sigma}(t) = \overline{\sigma} + \sum_{j=1}^{p} \left(\alpha_j \sin 2\pi t - \beta_j \cos 2\pi t \right)$$
(2.8)

We end the discussion of seasonality here. In the empirical section we will stick to the symmetric seasonality (p = 1) in (2.8).

Maturity effect

The second term in (2.5) we model as a negative exponential

$$\widehat{\delta}\left(T^* - t\right) = e^{-\delta(T^* - t)} \tag{2.9}$$

By choosing this particular specification, we are able to nest several models suggested previously in the literature. Previous models along with the parameter constraints are listed in table 2.1. In the empirical part of this paper we will consider several of these constrained models along with our new unconstrained model.

2.2.3 Valuation of futures options

Consider a European call option, C, with maturity T and strike K written on a futures contract with maturity T^* , where $T \leq T^*$. The value is given by

$$C(F(t,T^*),t,T) = e^{-r(T-t)} \sum_{n=0}^{\infty} P(n) \left(F(t,T^*) e^{b(n)(T-t)} \Phi(d_{1n}) - K \Phi(d_{2n}) \right)$$
(2.10)

where

$$P(n) = \frac{e^{-\lambda(T-t)} (\lambda(T-t))^n}{n!}$$
$$b(n) = -\lambda\overline{\kappa} + \frac{n\gamma}{T-t}$$
$$d_{1n} = \frac{\ln\left(\frac{F(t,T^*)}{K}\right) + \frac{1}{2} (\omega^2 + n\nu^2) + b(n)(T-t)}{\sqrt{\omega^2 + n\nu^2}}$$
$$d_{2n} = d_{1n} - \sqrt{\omega^2 + n\nu^2}$$
$$\omega = \sqrt{\int_t^T \sigma (s,T^*)^2 ds}$$

and $\Phi(\bullet)$ denotes the cumulative standard normal distribution. This formula is a slight generalisation of the formula given in Bates (1991) and Merton (1976). A derivation is given in appendix A. The formula can be understood intuitively as a sum of Black-Scholes (BS) type formulas with variance $\omega^2 + n\nu^2$ and a risk free rate b(n)(T-t), with each BS formula weighted by the probability of n jumps occurring in the period [t, T]. Since there is no upper limit to the number of possible jumps occurring in this period, we are in fact summing over infinite BS formulas. In practise this is not a big problem, since, for reasonable jump parameters, very accurate prices can be obtained when truncating the infinite sum by setting n rather low.⁶ Put options can be calculated explicitly, or they can be found via the futures option put-call parity. In the empirical part of this paper, we use data on American futures options, and consequently, some modification of the above European option pricing model is required. Bates (1991) derives an approximation for an American option in the jump-diffusion framework. His approximation generalises the formula of Barone-Adesi and Whaley (1987)

⁶In our empirical investigation we set out with $n \approx \lambda T e^{\gamma}$. Then n is extended until additional terms do not increase accuracy. Following Bates (1991) we set n = 1000 at the maximum. There is a way of avoiding the truncation problem altogether. Zhu (1999) computes the characteristic function of the jump-diffusion and by inverting this using Fourier inversion technique, he propose an alternative formula without summation. This method could easily be applied in our model as well, but this is not done here.

to a jump-diffusion model of the underlying asset. We use the same approximation as described in Bates (1991), replacing the constant volatility in his setting with the time-dependent volatility given by ω above. This model is called *New* in the empirical part of the paper.

2.3 Data description

We use price quotes on wheat futures and wheat futures options collected from CBOT to estimate the parameters of the futures price dynamics. Weekly data were obtained from January 1989 until December 1999. Wheat futures contracts are available with expiration in March, May, July, September, and December. The total sample consists of fifty-five futures contracts. The futures contracts matures in March, May, July, September, and December. At each point in time, there are five contracts traded, meaning that a one-year contract, is the longest contract an investor can enter into. The options written on the contracts can be exercised prior to maturity, hence they are of American type. The last trading day for the options is the first Friday preceding the first notice day for the underlying wheat futures contract. The expiration day of a wheat futures option is on the first Saturday following the last day of trading.

We applied several exclusion filters to construct the data sample. First, our sample starts in 1989. We did not use prices prior to 1989 since market prices then were likely to be affected by government programs in the United States (price floor of market prices and government-held stocks). Second, only trades on Wednesdays were considered, yielding a panel data set with weekly frequency. Weekly sampling is simply a matter of convenience. Daily sampling would place extreme demands on computer memory and time. The reason for choosing Wednesday is that this is the day of the week least affected by holidays. Third, only settlement (closing) prices were considered. Fourth, the last six trading days of each option contract were removed to avoid the expiration related price effects (these contracts may induce liquidity related biases). Fifth, to mitigate the impact of price discreteness on option valuation, price quotes lower than 2.5 cents/bu were deleted. Sixth, assuming that there is no arbitrage in this market, option prices lower or equal to their intrinsic values were removed. Three-month Treasury bill yields were used as a proxy for the risk free discount rate. The exogenous variables for each option in our data set are strike price, K, futures spot price, F, today's date, t,

the maturity date of the option contract, T, the maturity date of the futures contract, T^* , the instantaneous risk-free interest rate, r, observed settlement option market price, C_{it} . Here i is an index over transactions (calls of assorted strike prices and maturities), and t is an index over Wednesdays in the sample.

2.4 Model estimation and performance

2.4.1 Estimation method

Besides the exogenous variables obtained from the data set, the option pricing formula requires some parameters as inputs. In the full model the season $(\overline{\sigma}, \alpha_j, \beta_j, \text{ and } \delta)$ and maturity-effect $(\gamma, \nu, \text{ and } \lambda)$ related parameters and the jump-related parameters need to be estimated. There are two main approaches to estimate these parameters; from time series analysis of the underlying asset price, or by inferring them from option prices conditional upon postulated models (Bates (1995)). There are two main drawbacks of the former approach. First, very long time-series are necessary to correctly estimate jump parameters, at least if prices jump rarely. Second, parameters obtained from this procedure correspond to the actual distribution, and hence the parameters cannot be used in an option pricing formula, since the parameters needed for option pricing are given under the EMM. The latter approach has been used in e.g. Bates (1991, 1996 and 2000), Bakshi et al. (1997) and Hilliard and Reis (1999). Implicit parameter estimation is based on the fact that options, if rationally priced, contain information of the future probability distribution under the EMM.

We infer model-specific parameters from option prices over an eleven years long time period. The model is separately estimated for March, May, July, September and December wheat futures contracts expiring in 1989 through 1999. In previous studies, implicit parameters are inferred from option prices during a very short time interval, often daily (e.g., Bates (1991, 1996) and Hilliard and Reis (1999)). However, this method can be applied to data spanning any interval that has sufficient number of trades (Hilliard and Reis (1999)). Daily re-calibrations can fail to pick up longer horizon parameter instabilities (Bates (2000)). In this study, one of the aspects we focus on is the changing volatility during the year. There are only one maturity date for options written on a specific contract. If we were to use daily data, a model with time-dependent volatility would be indistinguishable from a model with constant volatility. Information of changing volatility will be revealed as the option prices change during the course of the year. In other words, we need a long time span, in order to be able to pick up volatility term structure effects in this market.

American option prices, C_{it} , are assumed to consist of model prices plus a random additive disturbance term:

$$C_{it} = C\left(F_t, K_i, t, T, T^*, r, \overline{\kappa}, \nu, \lambda, \overline{\sigma}, \alpha_i, \beta_i, \delta\right) + \varepsilon_{it}$$
(2.11)

Equation (2.11) can be estimated using non-linear regression. The unknown implicit parameters $\overline{\kappa}$, ν , λ , $\overline{\sigma}$, α_i , β_i , δ are estimated by minimisation the sum of squared errors (SSE) for all options in the sample given by

$$SSE = \sum_{t=1}^{T} \sum_{i=1}^{N} \left[C_{it} - C(\bullet) \right]^2 = \sum_{t=1}^{T} \sum_{i=1}^{N} \left[\varepsilon_{it} \right]^2$$
(2.12)

where i is an index over option prices on a given date (calls of different strike prices), and t is a time index summing over weekly observations.

Many alternative criteria could be used to evaluate performance of option pricing models. The overall sum of squared errors (SSE) is used as a broad summary measure to determine how well each alternative option pricing model fits actual market prices. Assuming normality of the error term, nested models can be tested using F-tests. Bates (1996, 2000) points out that his option pricing models are poorly identified. By this he means that in a jump-diffusion quite different parameter values can yield virtually identical *SEE*. This applies to our model as well, hence, parameter estimates should be interpreted with care.

2.4.2 Implied parameters

The following models were estimated (abbreviations used later in the paper are in parentheses): The diffusion model of Black (1976) with constant volatility (*Black76*), the jump-diffusion model of Bates (1991) (*Bates91*), the model suggested by Fackler and Tian (1999) with a seasonal and maturity dependent diffusion term (*Fackler99*) and our unrestricted model with both time dependence and jumps (*New*). Table 2.2 or shows implicit parameter

estimates for March, May, July,⁷ September and December wheat futures call options.⁸ In all the estimations reported we have set p = 1 which implies symmetric yearly seasonality. Experimenting with higher order lags resulted in only marginally better fit, and the results are not reported here.⁹ As a result of forcing eleven years of data into one option pricing model with constant parameters, the SSE is quite large. From table 2.2 we also see indication that both volatility term structure effects and jumps are important. The unrestricted model (*New*) produces the lowest SSE for all contracts. This is not surprising, since more parameters necessarily means better fit.

Comparing SSE for each model we find that Bates91 gives a better fit than Black76 for all contracts. This is in accordance with the conclusion in Hilliard and Reis (1999). When comparing the pure volatility term structure specification in Fackler99 with the pure jump specification in Bates91 we find mixed results. Bates91 produce lower SSE than Fackler99 for December, May and March (marginally). The opposite is true for the July contract.

We have formally tested the models against each other using F-statistics. The F-statistic is computed as $F[J, n - L] = \frac{(SSE_R - SSE_U)/J}{SSE_U/(n-L)}$ where SSE_U and SSE_R are sum of squared errors for the unrestricted and restricted models respectively, J is the number of restrictions, n is number of observations in the sample, and L is number of parameters in the unrestricted model. The test statistic is asymptotically F-distributed with J and (n - L) degrees of freedom.¹⁰ We ran the following tests

H_0		H_1
Black 76		Fackler 99
Black76	vs.	Bates 91
Bates 91		New

⁷For July contracts we had problems minimising (2.12) for the New model, so the parameters for this model is estimated in two steps. In step one all parameters except α_1 and β_1 were estimated. In step two, σ and δ_1 were fixed from the first estimation, and the rest of the parameters were (re)estimated.

⁸We were unable to get sensible parameter estimates in the *Fackler99* model for the September contract.

⁹Generally speaking, using SSE as performance criteria there are only small improvements when including several trigonometric terms compared to the more restrictive order 1 seasonal effect. The results are available from authors upon request.

¹⁰See for example chapter 5 in Davidson and MacKinnon (1993) for a description of different tests available in non-linear lest squares regression. Since the test statistics is F-distributed only asymptotically, they term it a *pseudo-F* test.

2.4. MODEL ESTIMATION AND PERFORMANCE

March contracts								
Parms.	Bl	ack76	Fac	kler99	Bates91		New	
σ	0.21	(514.7)	0.85	(1072)	0.15	(132.1)	1.18	(955.0)
γ					0.04	(51.5)	0.04	(47.9)
ν					0.19	(542.8)	0.19	(215.4)
λ					0.57	(61.3)	0.59	(45.2)
δ			2.85	(247.3)			3.98	(812.6)
α_1			-0.11	(-22.6)			-0.11	(-10.2)
β_1			-0.57	(-223.4)			-1.00	(-151.8)
SSE	2	3006	2	0356	20	0166	18226	
			M	ay contra	ncts			
Parms.	Bl	ack76	Fac	kler99	Bates91		New	
σ	0.20	(1388)	0.25	(2897)	0.18	(2146)	0.23	(11.4)
γ					0.08	(6.4)	0.05	(5.9)
ν					0.26	(673.8)	0.17	(466.9)
λ					0.14	(21.4)	0.60	(8.4)
δ			0.36	(3935)			0.71	(3.3)
α_1			-0.02	(-74.0)			-0.03	(-1.9)
β_1			-0.02	(-121.3)			-0.05	(-7.0)
SSE	1	5140	1	.4582 139		3991	12990	
			Ju	ly contra	icts			
Parms.	Bl	ack76	Fac	kler99	Bates91		New	
σ	0.21	(1102)	0.22	(889.7)	0.13	(598.0)	0.39	(183.2)
γ					0.04	(89.4)	0.02	(71.5)
ν					0.05	(206.5)	0.15	(225.2)
λ					6.49	(578.8)	1.52	(93.8)
δ			0.01	(0.9)			4.49	(177.0)
α_1			-0.03	-(26.0)			-0.15	-(5.8)
β_1			-0.08	-(76.7)			-0.10	-(6.1)
SSE	47931		38481		46099		38409	

Table 2.2: Implicit parameter estimates for various models. The table shows parameter estimates from non-linear least squares regressions on wheat futures call option prices. Estimations are made separately on March [4264], May [3859], July [5074], September [3971] and December [5231] contracts in the period 1989-1999 (number of observations for each contract in brackets). For each contract we estimate four models: *Black76*, *Fackler99*, *Bates91* and *New*. The three former models are constrained versions of the latter (see table (2.1) for parameter constraints for each model.) Sum of squared errors (SSE) are reported for each model, and t-values are in parentheses.

September contracts								
Parms.	Bl	ack76	Fackler99		Bates91		New	
σ	0.24	(330.8)			0.18	(1290)	0.34	(706.9)
γ			ļ .		0.11	(158.1)	0.14	(21.3)
ν					0.17	(60.8)	0.46	(636.3)
λ					0.56	(60.7)	0.14	(23.7)
δ		,					1.20	(173.2)
α_1							-0.15	(421.4)
β_1							-0.03	(169.8)
SSE	5	5913	-		53359		42426	
			Dece	mber con	tracts	5		
Parms.	Bl	ack76	Fac	kler99	Bates91		New	
σ	0.23	(805.3)	0.29	(477.0)	0.15	(156.5)	0.30	(24.5)
γ					0.01	(78.0)	0.05	(271.1)
ν					0.24	(61.3)	0.35	(402.1)
λ					0.65	(442.1)	0.22	(24.4)
δ			1.03	(268.1)			1.56	(21.7)
α_1			0.01	(4.7)			0.05	(5.7)
eta_1			-0.12	-(144.8)			-0.12	-(11.3)
SSE	47345		45480		43608		41732	

Table 2.2 cont. (see caption on previous page)

where H_0 is the null hypothesis and H_1 is the alternative hypothesis. The appropriate restrictions for each model are in table 2.1. The results, given in table 2.3, shows that we can reject the null hypothesis of a pure lognormal model assumptions against both the volatility time-dependent model and the jump-diffusion model with constant volatility. We also find that, for all contracts, *Bates91* is rejected in favour of the model *New* with both jumps and time-dependent volatility.

2.4.3 A closer look at the time-dependent volatility

Recall that the volatility dynamics is modelled in (2.5), (2.8) and (2.9). In the case of order one (symmetric) seasonality model the four parameters $\overline{\sigma}$, α_1, β_1 and δ are governing the volatility time-dependence. From table 2.2 we see that these parameters differ somewhat across contracts. There is little

Testing H_0 versus H_1	F-value	F-critical	Decision
March contracts			
Black76 vs. Fackler99	187.0	8.5	Reject H_0
Black76 vs. Bates91	202.1	8.5	Reject H_0
Bates91 vs. New	151.0	8.5	Reject H_0
May contracts			
Black76 vs. Fackler99	49.2	8.5	Reject H_0
Black76 vs. Bates91	105.5	8.5	Reject H_0
Bates91 vs. New	98.9	8.5	Reject H_0
July contracts			
Black76 vs. Fackler99	415.0	8.5	Reject H_0
Black76 vs. Bates91	67.2	8.5	Reject H_0
Bates91 vs. New	338.2	8.5	Reject H_0
September contracts			
Black76 vs. Fackler99	-	8.5	Reject H_0
Black76 vs. Bates91	63.3	8.5	Reject H_0
Bates91 vs. New	340.5	-	-
December contracts			
Black76 vs. Fackler99	71.4	8.5	Reject H_0
Black76 vs. Bates91	149.3	8.5	Reject H_0
Bates91 vs. New	78.3	8.5	Reject H_0

Table 2.3: Model specification tests. The table reports the results from several hypothesis tests. The null hypothesis of constant volatility ($H_0 = Black76$) is tested separately against time-dependent volatility ($H_1 = Fackler99$) and the presence of jumps ($H_1 = Bates91$). A pure jump model ($H_0 = Bates$) is tested against the full model ($H_1 = New$). The critical value of the F-tests are given for a confidence level of 95 per cent.

point in comparing each parameter against each other for different contracts, since, as mentioned above, different parameter values may cause quite similar option prices. Hence, we need to consider all the relevant parameters simultaneously when we investigate the volatility time-dependence. We have plotted the volatility time-dependence in figure 2.1, using the estimated parameters in table 2.2. For each contract the volatility term structure spans one year and ends as the futures contract expires.

We see that March, July and September contracts reveal the most profound maturity effect. The December contract combines high summer volatility and a maturity effect during autumn. It seems to be more volatile during the second half of the year. The July contract shows few signs of seasonality at all, but from table 2.2 we see that the seasonal parameters are significantly different from zero. In sum figure 2.1 illustrates the far bigger effect maturity has on volatility time-dependence, than seasonal variations over the year.

2.4.4 A closer look at the jump parameters

If wheat futures prices are characterised solely by deterministic time-dependent volatility, they are lognormally distributed. Furthermore, the implied volatility from option prices will be constant across strike prices.¹¹ However, if jumps are likely to occur, implied volatility will be not be constant across strike prices. As argued elsewhere, implied volatility curves reveal the effects of jumps on option prices. As an illustration of the effect of jumps on implied volatility, we computed theoretical option prices on American calls for different strikes using parameters from the New model of the May contract in table 2.2. The futures price is set to $F(t, T^*) = 300$, the maturity of the contract $T^* = 7/12$ months and the risk free rate r = 0.05. We backed out implied volatility curves using 5 strikes (K = 240, 270, 300, 330 and 360) for option maturities 2, 4 and 6 months from now (T = 2/12, 4/12 and 6/12). The results are given in figure 2.2.

We note that implied volatility is not constant across strike prices. This is known as the volatility "smile". This is caused by the possibility of both upward and downward jumps. It is also evident that this "smile" gets more pronounced as option expiration gets closer. If there is only a short time to

¹¹The fact that we are dealing with American options, means that implied volatility is not necessarily constant across strikes. However, prices on American and European futures option differ very little (Bates (2000)), hence implied volatility from American futures options are close to horizontal in a lognormal model.



Figure 2.1: Estimated time-dependence of volatility. The functional form of the volatility time-dependence is given in (2.5), (2.8) and (2.9). The parameters underlying the volatility functions $(\bar{\sigma}, \alpha_1, \beta_1, \delta)$ are implied from option prices, and they are given in table 2.2 in the column termed *New* for each of the contracts. Each contract is plotted during a one year cycle. For example the May contract is initialised in May, with maturity the following May $(T^* = 1)$.



Figure 2.2: Implied volatility smiles from wheat call options. Parameters for the May contract reported in table 2.2 are used in the computations. The futures price is set to 300 for a futures contract with maturity 7 months from now $((T^* = 7/12))$, and the risk free rate is 5%. Option prices are computed using the formula in (2.10) adjusting for the early exercise feature as in Bates (1991) for different strikes (K = 240, 270, 300, 330 and 260) and option maturities (T = 2/12, 4/12 and 6/12). To back out implied volatilities we use the Black (1976) model adjusted for early exercise premium of American options as described in Barone-Adesi and Whaley (1987).

maturity, far out-of-the-money (OTM) options in a lognormal model will be worth relatively little, since an extreme upward price swing is very unlikely. In a jump-diffusion model, these options may end up in-the-money (ITM) if a jump occurs, and consequently, these options will be relatively more valuable in a jump-diffusion than in a lognormal world. When there is long time to option maturity, the jump component plays a less prominent part when it comes to moving futures prices upwards or downwards. In the case of OTM options say, the diffusion term alone will be able to move the futures price so that the option will end up ITM.¹² We also note from figure 2.2 that the volatility curve shifts upwards when option maturity increases. This fact is mainly caused by the maturity effect captured by the volatility term structure. When the option maturity is close to the maturity of the futures contract, the maturity effect results in high average volatility during the life of the option. When the option matures long before the futures contract, the average volatility during the life of the option is lower, since the futures contract is less volatile far from maturity.

2.4.5 A numerical example

Finally, we provide a numerical example showing the economic significance of our findings. Assume that our model specification is correct; that both the volatility term structure and jumps are present in futures prices, and hence our option pricing formula calculates the true option price. What kind of mispricing will take place if we use the model of Black (1976) or Bates (1991) previously suggested in the literature? We set the futures price to $F(t, T^*) = 300$ and the maturity of the contract $T^* = 7/12$ months and the risk free rate r = 0.05. We computed prices for 3 strikes (K = 260, 300, 340) for option maturities 2, 4 and 6 months from now (T = 2/12, 4/12, 6/12). We compute American call option prices in the Black76, Bates91 and New model. The parameters are from the May contract in table 2.2. The results are given in table 2.4. The first three columns report the actual American

¹²In our case, there is roughly an equal chance for the jump to be either positive or negative under the EMM ($\bar{\kappa} \approx 0$). This means that as time to option expiration increases, multiple jumps will have a tendency to cancel each other out. This will enforce the flattening effect on the volatility smile as time to expiration increases. However, jump effects will in general be more visible in terms of implied volatility as time to expiration shortens (see Das and Sundaram (1999) for an investigation of term structure effects in a jump-diffusion model).

					%Diff.		
Maturity	Strike	Black76	Bates91	New	Black76 vs.	Bates91 vs.	
ļ					New	New	
T = 2/12	260	40.17	40.29	40.14	0.1~%	0.4 %	
$T^* = 7/12$	300	9.68	9.41	7.59	27.5~%	23.9 %	
	340	0.70	1.12	1.35	-48.1 %	-16.5 %	
T = 4/12	260	41.41	41.46	40.97	1.1 %	1.2~%	
$T^* = 7/12$	300	13.60	13.50	12.43	9.4~%	8.6 %	
1	340	2.55	3.10	3.19	-20.2 %	-2.9 %	
T = 6/12	260	43.01	43.20	43.61	-1.4 %	-0.9 %	
$T^* = 7/12$	300	16.71	16.86	18.14	-7.9 %	-7.1 %	
	340	4.61	5.31	6.55	-29.6 %	-18.9 %	

Table 2.4: Comparison of American wheat futures option pricing models. Option prices are computed using (2.10) and adjusting for the early exercise premium of American options as in Bates (1991). Parameters estimates used in the computations are those estimated for the May contract in table 2.2 are used in the computations. Prices are computed from the following pricing models *Black76*, *Bates91* and *New*. The futures price is set to $F(t, T^*) = 300$, the maturity of the contract $T^* = 7$ months and the risk free rate r = 0.05. Prices are computed for 3 strikes (K = 260, 300, and 340,) for option maturities 2 4 and 6 months from now (T = 2/12, 4/12 and 6/12). The two columns to the right report the relative pricing differences between *Black76* vs. New and *Bates91* vs. New respectively.

option prices for each of the models. The last two columns report percentage differences between the *New* model and *Black76* and *Bates91* respectively. Below we will comment on the pricing differences for each strike separately.

Prices for ITM options (K = 260) are more or less the same for all three models for all maturities. This is due to the fact that the intrinsic value dominates the value of an option when deep ITM, and hence most models would produce quite similar results. The at-the-money (ATM) (K = 300)price differences are basically influenced by the maturity effect of volatility. Both *Black76* and *Bates91* use an average volatility for the whole period as input. The fact that the volatility of futures contract increases as maturity approaches, means that using an average value for the volatility will produce too high option prices for short maturity options and too low prices for long maturity options. We note that the prices from *Black76* and *Bates91* are in quite good agreement with each other; however, they differ quite severely from the *New* model. For the shortest option maturity (T = 2/12) the prices of *Black76* and *Bates91* are roughly 25% higher than *New*. This number is

down to about 9% at maturity T = 4/12. At the maturity closest to the maturity of the futures contract (T = 6/12), we see that ATM option prices from Black 76 and Bates 91 produce prices 7% lower than New. Lastly, the two alternative models produce significantly lower price for OTM calls than New (K = 340). For the Black 76 model, this fact is not surprising since OTM calls will be more valuable in a jump-diffusion world. The results from the *Bates91* model deserve some explanation. We see that the parameters estimated for Bates 91 give a less pronounced smile effect than New. This is because, as the volatility term structure is restricted to be flat, the jump parameters will influence both the prices across strikes, and the overall price level. From the discussion on implied volatility, the jump parameters influence both the "smile" and the level of the implied volatility curve.¹³ In New, the term structure of volatility can take care of the level, and the jump parameters can "concentrate" on "smile" effects. Hence the parameters in Bates 91, through the estimation method, emerge as a compromise of the two effects. The results reported here might be important in other valuation contexts. For example, Hilliard and Reis (1999) argue that average based Asian options are popular in commodity over-the-counter (OTC) markets. They show that Asian option prices in the Black76 versus Bates91 differ even more than is the case for European/American options prices. Our results indicate, in addition to the jump effect, that Asian option prices will differ quite substantially depending on where in the life of the option the average is calculated. Especially, the relative strong maturity effect will give very different prices on Asian options depending on both the length of averaging period and how close the averaging period is to the maturity of the futures contract.

¹³This fact may partly explain the observation reported in Hilliard and Reis (1999) that parameter values are not stable over time. In their estimation procedure, they calibrate the model each day. Using their procedure, Bates91 will be able to replicate New as long as we are only considering options with one maturity date. When either the option or futures maturity change, the parameters in Bates91 must change to capture the volatility time-dependence. Hence we would expect unstable parameters in the analysis of Hilliard and Reis (1999) if, in fact, there exist volatility time-dependence effects in the underlying futures price dynamics.

2.5 Summary

In this paper we have developed an option pricing model that incorporates several stylised facts reported in the literature on commodity futures price dynamics. The volatility may depend on both calendar-time and time to maturity. Furthermore, futures prices are allowed to make sudden discontinuous jumps. We estimated the parameters of the futures price dynamics by fitting our model to eleven years of wheat options data using non-linear least squares. Several models suggested in the literature are nested within our model, and they all gave significantly poorer fit compared with the full model. In a numerical example we showed that ignoring volatility time-dependence and jump effects in futures prices might lead to severe mispricing of options.

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2.6 Appendix: Closed form futures call option

In this appendix we will give a formal proof to option pricing formula in (2.10). Our formula is a special case of the general futures option pricing formula provided in Björk and Landén (2002), but it is provided here for completeness. In their paper the change of measure technique, pioneered by Jamshidian (1989) and Geman et al. (1995), is used repeatedly in the derivation of the option pricing formula. We will sketch the proof of the pricing formula following Björk and Landén (2002) closely. First we present our model in a more formal way from a measure theoretic point of view. Then we sketch the proof along the lines of Björk and Landén (2002).

We consider a market where the uncertainty is characterised by the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where Ω is a set, \mathcal{F} is a σ -algebra of subsets of Ω and $\mathbb{Q}: \mathcal{F} \to [0, 1]$ is a probability measure. All economic activity is assumed to take place on a finite horizon $[0, T^*]$, with the filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T^*\}$ satisfying the usual conditions. The filtered probability space is assumed to carry a standard Brownian motion, and a homogenous Poisson counter process. We assume that \mathbb{F} is equal to the information structure generated by Wand a Poisson random measure. A homogenous Poisson process is a process with stationary (they do not depend on time) and independent increments such that q(t) is $Po(\lambda t)$, where $Po(\bullet)$ means Poisson distributed. Hence $P(q(t) = n) = e^{-\lambda t} \frac{\lambda t^k}{k!}$. The dynamics of the futures prices are assumed to be

$$\frac{dF(t,T^*)}{F(t,T^*)} = -\lambda \overline{\kappa} dt + \sigma(t,T^*) dB(t) + \kappa dq \qquad (2.13)$$

where B is standard Brownian motion under the EMM and κ is the random percentage jump conditional upon q occurring. We assume that $\ln(1+\kappa)$ is a normally distributed random variable with mean $(\gamma - \frac{1}{2}\nu^2)$ and variance ν^2 . Consequently, the expected percentage jump size is $E[\kappa] \equiv \overline{\kappa} = e^{\gamma} - 1$.

Now consider an equivalent martingale measure. The measure \mathbb{Q}^F is defined by

$$d\mathbb{Q}^F = L_t^F d\mathbb{Q} \tag{2.14}$$

where

$$L_t^F = \frac{F(T, T^*)}{F(t, T^*)}$$
(2.15)

The dynamics of L^F is obtained as

$$dL_t^F = L_t^F \left(\kappa dq - \lambda \overline{\kappa} dt\right) + L_t^F \sigma\left(t, T^*\right) dB(t)$$
(2.16)

Now the solution to (2.13) can be expressed under both measures. Conditional on that there has been n jumps $F(T, T^*)$ can be written

$$F(T,T^*) = F(t,T^*) \exp\left(\begin{array}{c} -\lambda \overline{\kappa} \left(T-t\right) - \frac{1}{2} \int_t^T \sigma\left(s,T^*\right)^2 ds \\ + \int_t^T \sigma\left(s,T^*\right) dB(s) \end{array}\right) \prod_{j=0}^n \left(1+\kappa_j\right)$$

$$(2.17)$$

where $\ln(1 + \kappa_j) \sim \mathcal{N}\left(\gamma - \frac{1}{2}\nu^2, \nu^2\right)$, κ_j , j = 1, ..., n are *i.i.d.* random variables under the measure \mathbb{Q} , and $X \sim \mathcal{N}(m, s)$ is a normally distributed random variable with mean m and variance s. The possibility of n number jumps occurring in the period [t, T] can be found from the Poisson distribution which is $P(n) = \frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}$ under \mathbb{Q} . Alternatively $F(T, T^*)$ can be written

$$F(T,T^*) = F(t,T^*) \exp\left(\begin{array}{c} -\lambda \overline{\kappa} \left(T-t\right) + \frac{1}{2} \int_t^T \sigma\left(s,T^*\right)^2 ds \\ + \int_t^T \sigma\left(s,T^*\right) dB^F(s) \end{array}\right) \prod_{n=0}^{\infty} \left(1+\kappa_j^F\right)$$
(2.18)

where $\ln\left(1+\kappa_{j}^{F}\right) \sim \mathcal{N}\left(\gamma+\frac{1}{2}\nu^{2},\nu^{2}\right)$ and κ_{j}^{F} , j=1,...,n are *i.i.d.* random variables under the measure \mathbb{Q}^{F} . Under this measure the probability of n jumps is $P^{F}(n) = \frac{e^{-\lambda^{F}(T-t)}\left(\lambda^{F}(T-t)\right)^{n}}{n!}$ where $\lambda^{F} = \lambda e^{\gamma}$ (see Björk and Landén (2002) for details).

Consider a European call option C with maturity T and strike K on a futures contract with maturity T^* , where $T < T^*$. Let $E_t^{\mathbb{Q}}[\bullet]$ denote the conditional expectation with respect to the measure \mathbb{Q} and let I_D be an indicator function with the set $D = \{F(T, T^*) > K\}$. The value at time t of this call can then be expressed as

$$c(F(t, T^*), t, T) = e^{-r(T-t)} E_t^{\mathbb{Q}} [F(T, T^*) - K]$$

= $e^{-r(T-t)} E_t^{\mathbb{Q}} [F(T, T^*) I_D] - e^{-r(T-t)} E_t^{\mathbb{Q}} [KI_D]$
= $e^{-r(T-t)} F(t, T^*) \mathbb{Q}^F [F(T, T^*) > K]$
 $-e^{-r(T-t)} K \mathbb{Q} [F(T, T^*) > K]$

The two probabilities, $\mathbb{Q}^{F}[\bullet]$ and $\mathbb{Q}[\bullet]$, can be expressed as follows

•

$$\mathbb{Q}^{F}[F(T,T^{*}) > K] = \sum_{n=0}^{\infty} \frac{e^{-\lambda^{F}(T-t)} \left(\lambda^{F}(T-t)\right)^{n}}{n!} \Phi(d_{1n})$$
(2.19)

$$\mathbb{Q}\left[F\left(T,T^{*}\right) > K\right] = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)} \left(\lambda(T-t)\right)^{n}}{n!} \Phi(d_{2n})$$
(2.20)

where $\Phi(\bullet)$ is the standard cumulative normal distribution and

$$d_{1n} = \frac{\ln\left(\frac{F(t,T^*)}{K}\right) + \frac{1}{2}\left(\omega^2 + n\nu^2\right) + b(n)(T-t)}{\sqrt{\omega^2 + n\nu^2}}$$
$$b(n) = -\lambda\overline{\kappa} + \frac{n\gamma}{T-t}$$
$$d_{2n} = d_{1n} - \sqrt{\omega^2 + n\nu^2}$$
$$\omega = \sqrt{\int_t^T \sigma\left(s, T^*\right)^2 ds}$$

Using the fact that $\lambda^F = \lambda e^{\gamma}$ and the relationship $\overline{\kappa} = e^{\gamma} - 1$ we find that

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda^{F}(T-t)} \left(\lambda^{F}(T-t)\right)^{n}}{n!} = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)} \left(\lambda(T-t)\right)^{n}}{n!} e^{-\overline{\kappa}\lambda(T-t)+n\ln(\overline{\kappa}+1)}$$
$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)} \left(\lambda(T-t)\right)^{n}}{n!} e^{-\overline{\kappa}\lambda(T-t)+n\gamma}$$

Setting it all together yields

$$c(F(t,T^{*}),t,T) = e^{-r(T-t)}F(t,T^{*})\sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\left(\lambda(T-t)\right)^{n}}{n!}e^{-\overline{\kappa}\lambda(T-t)+n\gamma}\Phi(d_{1n})$$
$$-e^{-r(T-t)}K\sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\left(\lambda(T-t)\right)^{n}}{n!}\Phi(d_{2n})$$
(2.21)

which is the formula given in (2.10).

Chapter 3

Forward curve dynamics in the Nordic electricity market

This paper is co-authored with Fridthjof Ollmar¹

ABSTRACT - The purpose of this paper is to investigate the forward curve dynamics in an electricity market. Six years of price data on futures and forward contracts traded in the Nordic electricity market are analysed. For the forward price function of electricity, we specify two different multi-factor term structure models in a Heath-Jarrow-Morton framework. Principal component analysis is used to reveal the volatility structure in the market. A two-factor model explains 75% of the price variation in our data, compared to approximately 95% in most other markets. Further investigations show that correlation between short- and long-term forward prices is lower than in other markets. We briefly discuss possible reasons why these special properties occur, and some consequences for hedging exposures in this market.

3.1 Introduction

With the rapid growth of derivative securities in deregulated electricity markets, the modelling and management of electricity price risk have become important topics for researchers and practitioners. In the case of electricity,

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contingent claims valuation and risk management were not considered important issues prior to market deregulation. Due to the special properties of this commodity volatility in deregulated electricity markets can reach extreme levels and a proper understanding of volatility dynamics is important for all participants in the market place.

There are two lines of research focusing of commodity contingent claims valuation and risk management. The traditional way has concentrated on modelling the stochastic process of the spot price and other state variables such as the convenience yield² (see for example Brennan and Schwartz 1985, Gibson and Schwartz 1990, Schwartz 1997 and Hilliard and Reis 1998). This approach has been adopted and modified in the recent electricity literature by, among others Deng (2000), Kamat and Ohren (2000), Pilipović (1998) and Lucia and Schwartz (2000). As far as we know Lucia and Schwartz (2000) represent the first thorough empirical work on electricity spot prices.

The main problem with spot price based models is that forward prices are given endogenously from the spot price dynamics. As a result, theoretical forward prices will in general not be consistent with market observed forward prices. As a response to this, a line of research has focused on modelling the evolution of the whole forward curve using only a few stochastic factors taking the initial term structure as given. Examples of this research building on the modelling framework of Heath et al. (1992), are Clewlow and Strickland (1999a) and (1999b), Miltersen and Schwartz (1998) and Bjerksund et al. (2000).

Empirical investigations of forward curve models in commodity markets have been conducted by, among others, Cortazar and Schwartz (1994) and Clewlow and Strickland (2000). Cortazar and Schwartz (1994) studied the term structure of copper futures prices using principal component analysis and found that three factors were able to explain 99% of the term structure movements. Clewlow and Strickland (2000) investigated the term structure of NYMEX oil futures and found that three factors explained 98.4% of the total price variation in the 1998-2000 period. The first factor (explained 91% of total variation) shifted the whole curve in one direction. They termed this

²This direction is rooted in the theory of storage developed by Kaldor (1939), Working (1948) and (1949), Telser (1958) and Brennan (1958) and (1991). According to the theory of storage, the futures and spot price differential is equal to the cost of storage (including interest) and an implicit benefit that producers and consumers receive by holding inventories of a commodity. This benefit is termed the convenience yield. The most obvious benefit from holding inventory is the possibility to sell at an occurring price peak.

3.1. INTRODUCTION

a "shifting" factor. The second factor, termed the "tilting" factor, influenced short and long-term contracts in opposite directions. The third factor, the "bending" factor, moved the short and long end in opposite direction of the mid-range of the term structure.³

In this paper we adopt the forward curve approach and perform an empirical examination of the dynamics of the forward curve in the Nordic electricity market during the 1995-2001 period. Following the work of Cortazar and Schwartz (1994) and Clewlow and Strickland (2000) we use principal component analysis to analyse the volatility factor structure of the forward curve. The forward price of electricity is the price today for a delivery of electricity at some point in time in the future. This forward price function is not directly observable in the market place. Power contracts trading on Nord Pool are all written on a future average; the delivery periods of the contracts. Instead of working directly with the different financial contracts with various delivery periods, we compute a continuous forward price function from each day's futures and forward prices. This data transformation process is similar to the process of extracting a forward interest rate curve from a set of fixed income products. We apply the principle of maximum smoothness described in Adams and van Deventer (1994) and Bjerksund et al. (2000) to compute daily electricity forward curves. We specify two different models for the evolution of the forward price of electricity in the framework of Heath et al. (1992); the geometric and the arithmetic Brownian motion. Two sets of data are constructed. For the arithmetic model forward price differences are analysed, and forward price returns are analysed in the case of the geometric model. The maturities for the contracts that constitute the data sets range from one week to two years. Following the work of Cortazar and Schwartz (1994) and Clewlow and Strickland (2000) we use principal component analysis to analyse the volatility factor structure of the forward curve. In the short end of the term structure, the volatility increases sharply as time to maturity decreases. In other commodity markets one typically find that a few factors are able to explain most of the variation in the forward prices. The portion of explained variance is lower in the electricity market. We find that

³The multi-factor forward approach by Heath et al. (1992) was originally developed for interest rate markets. Empirical work on factor dynamics in fixed income securities markets have been conducted by Steely (1990), Litterman and Scheinkman (1991) and Dybvig (1997), among others. The results in these studies are quite similar to the work reported from the commodity markets. Typically, three factors explain 95%-98% of the total variation in the forward curve.

a two-factor model explains 75% of the price variation in our data, compared to approximately 95% in most other markets. Pilipović (1998) conjecture that electricity prices exhibit "split personalities". By this she means that the correlation between short- and long term forward prices are lower in electricity markets than in other markets. We provide some empirical support of this claim. The most important factors driving the long end of the curve have very little impact on price changes in the short end. Furthermore we find some evidence of changing volatility dynamics both seasonally and from one year to another. Finally, we are unable to decide if an arithmetic or geometric model describes the data best.

This paper is organised as follows: We give a short description of the Nordic electricity market in section 3.2. Section 3.3 presents the multi-factor models and section 3.4 describes the data set. In section 3.5 we show how principal component analysis can be used in order to estimate the empirical volatility functions and section 3.6 reports the results. Section 3.7 concludes the paper.

3.2 The Nordic electricity market

3.2.1 History of the Nordic Power Exchange

From 1971 to 1993 a market called Samkjøringen co-ordinated the Norwegian electricity production. Every week Samkjøringen set the daily or part-ofthe-day price for electricity. This price was used to decide the Norwegian electricity production and the exchange with other countries. A new Energy Law was approved by the Norwegian Parliament in 1990 and came into effect in 1991. This law introduced market-based principles for production and consumption of electricity in Norway. After England and Wales in 1989, Norway was the second country to deregulate the electricity market.

In 1993 Samkjøringen merged with Statnett SF to create a new company called Statnett Marked AS. Statnett Marked AS organised the new Norwegian market place for electricity from 1993 to 1996. In 1996 the Swedish grid company, Svenska Kraftnät, bought 50% of Statnett Marked AS and became part of the power exchange area. At the same time Statnett Marked AS was renamed to Nord Pool ASA. Finland joined the power exchange area in 1998, western Denmark in 1999 and eastern Denmark in 2000. The Nordic electricity market is non-mandatory and a significant share of the physical power and financial contracts are traded bilaterally.

3.2.2 The physical market

Today Nord Pool organises and operates Elspot, Eltermin, Eloption, and Elclearing. Elspot is a spot market for physical delivery of electricity. Each day at noon, spot prices and volumes for each hour the following day are determined in an auction. The equilibrium price is termed the *system price*, which may be considered a one-day futures contract. The following day, the national system operators organise a *regulating*- or *balance* market, where short term up- or down regulation is handled. Since 1993 the turnover in Elspot market has increased steadily from 10.2 TWh in 1993 to 96.2 TWh in 2000. In 1999, more than one fifth of the total consumption of electric power in the Nordic countries was traded via Nord Pool.

3.2.3 The financial market

Eloption and Eltermin are Nord Pool's financial markets for price hedging and risk management. Financial contracts traded on Eltermin are written on the arithmetic average of the system price at a given time interval.⁴ This time interval is termed the delivery period. The time period prior to delivery is called the trading period. Both futures and forward contracts are traded at Eltermin. The contract types differ as to how settlement is carried out during the trading period. For futures contracts, the value is calculated daily, reflecting changes in the market price of the contracts. These changes are settled financially at each participant's margin account. For forward contracts there is no cash settlement until the start of the delivery period. European options written on underlying futures and forward contracts are traded on Eloption. Asian options written on the system price do no longer trade on Eloption. This is due to low liquidity.

The power contracts refer to a delivery rate of 1 MW during every hour for a given delivery period. Futures contracts feature daily market settlement in their trading and delivery periods. Forward contracts, on the other hand, do not have settlement of market price fluctuations during the trading period.

⁴We only give a brief description of the different products traded at Nord Pool here. For a detailed description see www.nordpool.no or Lucia and Schwartz (2000). Some contracts traded in the OTC market have a different underlying reference price than the system price. Such contracts are not considered in this study.

Daily settlement is made in the delivery period. None of the contracts traded at Nord Pool are traded during the delivery period.

The contracts with the shortest delivery periods are futures contracts. Daily futures contracts with delivery period of 24 hours are available for trading within the nearest week.⁵ Weekly futures contracts with delivery periods of 168 hours can be traded 4-8 weeks prior to delivery. Futures contracts with 4 weeks delivery period, are termed block contracts. The forward contracts have longer delivery periods. Each year is divided into three seasons: V1 - late winter (January 1- April 30), S0 - summer (May 1 - September 30) and V2 - early winter (October 1 - December 31). Seasonal contracts⁶ are written on each of these seasonal delivery periods. In January each year, seasonal contracts on S0 and V2 the coming year and all three seasonal contracts for the next two years are available. Furthermore, yearly forward contracts are available for the next three years. In other words, the (average based) term structure goes 3 to 4 years into the future, depending on current time of year.

In 1995 the total volume of financial contracts traded on Nord Pool and OTC was 40.9 TWh. In 2000, this number was 1611.6 TWh. The most heavily traded contracts are weekly contracts and the two nearest seasonal contracts. On average 20-30 weekly contracts and 30-80 seasonal contracts are traded each day.

3.3 Multi-factor forward curve models

Our model setting is similar to the forward interest rate model of Heath et al. (1992). The two models we investigate in this paper are special cases of the general multi-factor term structure model developed for commodity markets in Miltersen and Schwartz (1998). We consider a financial market where the uncertainty can be described by a K-dimensional Brownian motion $(W_1, ..., W_K)$ defined on an underlying probability space $(\Omega, \mathbb{F}, \mathbb{Q})$ with the filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T^*]\}$ satisfying the usual conditions and representing the revelation of information. The probability measure \mathbb{Q} represents

⁵These contracts have only a short (and illiquid) history, and will not be included in our data set when analysing the volatility structure in the market.

 $^{^{6}}$ From 1995 to the end of 1999 seasonal futures contract were traded. In our empirical analysis, all contracts traded in the 1995-2001 period are used in the estimation of the models.

the equivalent martingale measure. Throughout the paper we assume constant risk free interest rate, so that futures prices and forward prices with common maturity are identical (see Cox et al. (1981)). The two terms will be used interchangeably in the following sections.

Let the forward market be represented by a continuous forward price function, where f(t,T) denotes the forward price at date t for delivery of the commodity at time T, where $t < T < T^*$. Given constant interest rates the futures and forward prices are by construction martingales under the measure \mathbb{Q} .

• Model A: Deterministic volatility functions are independent of the forward price level

Consider a model where the dynamics of the forward price is

$$df(t,T) = \sum_{i=1}^{K} \sigma_i^A(t,T) dW_i(t)$$
(3.1)

where the $(W_1, ..., W_K)$ are independent Brownian motions, and $\sigma_i^A(t, T)$ are time dependent volatility functions.⁷ The solution to (4.1) is

$$f(t,T) = f(0,T) + \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}^{A}(s,T) dW_{i}(s)$$
(3.2)

This means that the forward prices are distributed

$$f(t,T) \sim \mathcal{N}\left(f(0,T), \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}^{A}(s,T)^{2} ds\right)$$
(3.3)

where $\mathcal{N}(s, v)$ denotes a normally distributed variable with mean s and variance v.

• Model B: Deterministic volatility functions are proportional to the forward price level

⁷Volatility is a term usually associated with the (time dependent) function of the diffusion term in a lognormal model (model B above). In this paper we use the term "volatility functions" for the time dependent functions in the diffusion term in both models.

Consider a model where the dynamics of the forward price is given by

$$\frac{df(t,T)}{f(t,T)} = \sum_{i=1}^{K} \sigma_i^B(t,T) dW_i(t)$$
(3.4)

with solution

$$f(t,T) = f(0,T) \exp\left(-\frac{1}{2}\sum_{i=1}^{K}\int_{0}^{t}\sigma_{i}^{B}(s,T)^{2}ds + \sum_{i=1}^{K}\int_{0}^{t}\sigma_{i}^{B}(s,T)dW_{i}(s)\right)$$
(3.5)

The distribution of the natural log of the forward price is given by

$$\ln f(t,T) \sim \mathcal{N}\left(\ln f(0,T) - \frac{1}{2}\sum_{i=1}^{K}\int_{0}^{t}\sigma_{i}^{B}(s,T)^{2}ds, \sum_{i=1}^{K}\int_{0}^{t}\sigma_{i}^{B}(s,T)^{2}ds\right)$$
(3.6)

where $\mathcal{N}(s, v)$ is defined as above.

Versions of both class A and B models have been proposed for the Nordic electricity market. Lucia and Schwartz (2000) propose a spot price model and derive analytical expressions for futures/forward prices. They consider mean reverting spot price models both in level and log form. It is easy to show that their models are consistent with forward price models with

$$\sigma_1^A(t,T) = \sigma e^{-\kappa(T-t)}$$

and

$$\sigma_1^B(t,T) = \sigma e^{-\kappa(T-t)}$$

respectively, where σ and κ are positive constants. This model produces a falling volatility curve in T, approaching zero as $T \to \infty$. Bjerksund et al. (2000) on the other hand, propose two different kinds of class B models. The one factor model is given by

$$\sigma_1^B(t,T) = \frac{a}{T-t+b} + c$$

where a, b and c are positive constants. With realistic parameter values, this specification produces a sharply falling volatility curve in T. As $T \to \infty$ the

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volatility converges to c. Bjerksund et. al. (2000) also propose a three factor model

$$\begin{split} \sigma_1^B(t,T) &= \frac{a}{T-t+b} \\ \sigma_2^B(t,T) &= \left(\frac{2ac}{T-t+b}\right)^{\frac{1}{2}} \\ \sigma_3^B(t,T) &= c \end{split}$$

with all parameters assumed positive. This three-factor model allows a richer structure of the forward price dynamics. They argue that the one factor model may be adequate for pricing contingent claims, while the three-factor model is better suited for risk management purposes. Note that in all the models above, given that all the parameters are positive, each individual Brownian motion will move forward prices of all maturities in the same direction. As we will see from the empirical analysis, this property of the proposed models is inconsistent with our empirical findings.

3.4 Descriptive analysis and data preparation

We are interested in the volatility dynamics of the forward price function described above. This forward price function, giving us today's price of a unit of electricity delivered at a specific instant in the future, is not directly observable in the market place. The power contracts trading on Nord Pool are all written on a future average; the delivery periods of the contracts. We need to pin down the relationship between the forward price function and the average based contracts. Let $F(t, T_1, T_2)$ be today's contract price of an average based futures contract delivering one unit of electricity at a rate of $\frac{1}{T_2-T_1}$ in the time period $[T_1, T_2]$, where T_1 and T_2 is the beginning and the end of the delivery period of the contract, and $t \leq T_1 < T_2$. Suppose that the contract price is paid as a constant cash flow during the delivery period. Then the expression for the average contract is (see Bjerksund et al. (2000)):

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} w(r, u) f(t, u) du$$
(3.7)

where

• . .

$$w(r,u) \equiv \frac{e^{-ru}}{\int_{T_1}^{T_2} e^{-ru} du}$$
(3.8)

Lucia and Schwartz (2000) note that $F(t, T_1, T_2) \approx \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, u) du$ is a very good approximation of (3.7) and (3.8) for reasonable levels of interest rates. We use this approximation in the empirical analysis.

3.4.1 Smoothed data

Instead of working directly with the different financial contracts with various delivery periods, we compute a continuous forward price function from each day's futures and forward prices. The smoothing procedure is based on the principle of maximum smoothness suggested by Adams and van Deventer (1994). The smoothness criterion they state for the forward rate function is the one that minimises the functional

$$\min \int_0^T f''(t,s)^2 ds$$
 (3.9)

while at the same time fitting observed market prices.⁸ They show, in an interest rate setting, that the yield curve with the smoothest possible forward rate function according to this criterion, is a quartic spline function, with the cubic term dropped, that is fitted between each knot point on the yield curve.⁹ We apply the quartic spline function as described in Adams and van Deventer (1994), and estimate the forward price function prices all traded assets within the bid/ask spread using (3.7).¹⁰ The result of this smoothing procedure on March 27, 2000 is illustrated in figure 3.1. The horizontal dotted lines are closing prices on weekly, block and seasonal contracts. We have computed the smoothed forward price function on each of the 1340 trading days in our sample using all the contracts available each day. In figure 3.2 we have plotted weekly forward curves during the 1995-2001 sample period. Note the clear annual seasonal variation with high winter and low summer prices. The contract with the longest time to maturity increases from 80 weeks in 1995 to 208 weeks in 2001.

⁸Here the derivatives are taken with respect to the second time index.

⁹For a comprehensive description of the maximum smoothness approach see Adams and van Deventer (1994), Bjerksund and Stensland (1996) and Forsgren (1998).

¹⁰A sinusoidal prior function is defined prior to estimation to pick up the strong seasonal pattern in this market. The smoothed forward price functions were computed using the software ELVIZ developed by Viz Risk Management Services AS. For more information of the ELVIZ software, see www.viz.no.

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Sampl	e period:	1995 - 2	2001
	···		
Maturity	W-01	W-52	W-104
Mean	145.51	159.54	163.47
Median	130.18	153.49	158.88
Min	45.25	99.91	101.24
Max	356.00	262.03	275.75
Std.dev	64.10	36.04	33.17
Skewness	1.21	0.76	0.63
Kurtosis	3.91	3.18	3.26
Nobs	1340	1340	1279

Table 3.1: Descriptive statistics for electricity forward prices. The table reports statistics from three points on the smoothed term structure, the one week forward price (W-01), the one year forward price (W-52) and the two year forward price (W-104).

Sample period: 1995-2001											
Maturity	W-01	W-52	W-104	W-01	W-52	W-104					
	Prie	Price differences Price returns									
Mean	-0.32	0.00	-0.04	-0.00	0.00	-0.00					
Median	-0.25	-0.02	-0.00	-0.00	-0.00	-0.00					
Min	-32.75	-17.42	-29.00	-0.39	-0.07	-0.25					
Max	37.25	21.36	26.80	0.22	0.09	0.23					
Std.dev	6.03	2.64	2.34	0.04	0.01	0.01					
Skewness	0.13	0.28	-1.06	-0.42	0.36	-1.01					
Kurtosis	9.44	12.56	45.07	11.23	8.11	116.14					
Nobs	1339	1339	1278	1339	1339	1278					

Table 3.2: Descriptive statistics for electricity forward price differences and returns. Descriptive statistics of daily forward price differences and forward price returns from the smoothed term structure of the total sample. The table reports statistics from three points on the smoothed term structure, the one week forward price (W-01), the one year forward price (W-52) and the two year forward price (W-104).



Figure 3.1: Power contracts and the smoothed forward curve on March 26, 2000. The dotted lines represent the actual market prices, and the length of the dotted lines corresponds to the delivery period on which the contracts are written. The weekly contracts (one dot) and block contracts (four dots) are futures contracts, and the seasonal contracts are forward contracts. The solid line is the smoothed term structure.

Table 3.1 shows descriptive statistics on three different points on the term structure; W-01 (one week to maturity), W-52 (one year to maturity) and W-104 (two years to maturity). We note that the mean forward price is increasing with maturity. This means that the market on average can be described by normal backwardation¹¹ (a positive risk premium). We note that the one-week forward price has fluctuated substantially during the sample period. The fluctuations decrease with time to maturity. To further examine the time series properties of the data, we have plotted the time series

¹¹Normal backwardation is used to describe the relationship $f(t, T_2) \ge f(t, T_1)$ when $T_2 > T_1$. We must be careful when using this relationship in markets with seasonal price variation. By choosing maturities exactly one year apart, forward prices on the same time of the year are compared and seasonal variation is no longer a problem.



Figure 3.2: Surface plots of smoothed forward curves. Weekly surface plots (each wednesday) for each of the years in our sample.



Figure 3.3: Time series of futures prices. The graphs are time series plots of the futures prices with one week (solid line), one year (dashed line) and two years (dotted line) to maturity.

of forward prices with the same three maturities in figure 3.3. It is obvious that the one-week contract is much more erratic than the one- and two-year contract. Note that the short-term price varies around the long-term price indicating some sort of mean reversion. Roughly speaking the market was in contango in 1996 and in normal backwardation in the 1997-2001 period.

3.4.2 Constructing two data sets

The forward price models in (4.2) and (3.5) describe the stochastic evolution under an equivalent martingale measure, and not under the real world measure where observations are made. Although there may be risk premia in the market that cause futures prices to exhibit non-zero drift terms, the diffusion terms are equal under both measures. So the volatility functions in (4.2) and (3.5) can be estimated from real world data. As noted by Cortazar and Schwartz (1994), this is only strictly correct when observations are sampled continuously. In our analysis we use daily observations as a proxy to a continuously sampled data set. Let $f(t_n, t_n + \tau_m)$ denote the forward price at date t with maturity at date $t_n + \tau_m$, where $\tau_m = T_m - t_n$ is time to maturity for the contract. Our discrete approximations of model A and B are

$$df(t_n, t_n + \tau_m) \approx f(t_n, t_n + \tau_m) - f(t_{n-1}, t_n + \tau_m) = x_{n,m}^A$$
(3.10)

 and

$$\frac{df(t_n, t_n + \tau_m)}{f(t_n, t_n + \tau_m)} \approx \frac{f(t_n, t_n + \tau_m) - f(t_{n-1}, t_n + \tau_m)}{f(t_{n-1}, t_n + \tau_m)} = x_{n,m}^B$$
(3.11)

where n = 1, ..., N. For a set of maturity dates $\{\tau_1, ..., \tau_M\}$, we construct 2 different data sets from the smoothened data, $\mathbf{X}^A_{(N \times M)}$ with forward price differences

$$\mathbf{X}_{(N\times M)}^{A} = \begin{bmatrix} x_{1,1}^{A} & x_{1,2}^{A} & \cdots & x_{1,M}^{A} \\ x_{2,1}^{A} & x_{2,2}^{A} & \cdots & x_{2,M}^{A} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N,1}^{A} & x_{N,2}^{A} & \cdots & x_{N,M}^{A} \end{bmatrix}$$
(3.12)

and $\mathbf{X}^{B}_{(N \times M)}$ with forward price returns

$$\mathbf{X}_{(N \times M)}^{B} = \begin{bmatrix} x_{1,1}^{B} & x_{1,2}^{B} & \cdots & x_{1,M}^{B} \\ x_{2,1}^{B} & x_{2,2}^{B} & \cdots & x_{2,M}^{B} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N,1}^{B} & x_{N,2}^{B} & \cdots & x_{N,M}^{B} \end{bmatrix}$$
(3.13)

The matrices above deserve a thorough description. We first compute daily forward price functions from the observed market prices. From these forward functions we compute 104 weekly midpoint prices (equidistant forward prices), one price for each week along a two years term structure. Within each week these maturities are held constant. Next we compute N = 1339 time series observations on price returns and price differences. The contracts are rolled over each Friday. Let us illustrate our approach using the contract with maturity in one week: The daily returns and differences from Monday to Friday are computed from the contracts with maturity the following week

 (τ_1) . On Friday we observe the price of the contact with maturity two weeks ahead (τ_2) . The return and difference on this contract is calculated from Friday to Monday. Reaching Monday, this contract has now become the new one-week contract. We use this approach of fixing the time to maturity to avoid problems of seasonality in prices over the year. Finally we pick M = 21price returns and differences with different maturities among the 104 weekly prices. If we scale τ_m in "weeks-to-maturity" the specific maturities chosen are $\tau_1, \ldots, \tau_M = [1, 2, 3, 4, 5, 6, 7, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 48]$ 52, 70, 88, 104]. The maturities are chosen in such a way that they reflect the actual traded contracts. In the shortest end we pick 7 maturities with weekly intervals, mimicking the weekly contracts. The next 11 maturities are 4 weeks apart. There are only three maturities in the last year of the term structure, representing seasonal contracts. In table 3.2 we report descriptive statistics on the one week-, one year- and two year forward price differences and forward price returns for the whole sample period. The standard deviation of both price returns and price differences is sharply falling with time to maturity. We also note that kurtosis is high, and that skewness is different from zero. The sign of the skewness changes along the term structure. In tables 2.3 and 2.4 we report descriptive statistics on semi-annual and seasonal sub-interval of forward price differences and forward price returns respectively. We note that the standard deviation of price differences is markedly higher in the 1995-1996 sub-period than in 1997-1998 and 1999-2001.

3.5 Principal component analysis and volatility functions

Principal component analysis (PCA) is concerned with the identification of structure within a set of interrelated variables. It establishes dimensions within the data, and serves as a data reduction technique. The aim is to determine factors (i.e. principal components) in order to explain as much of the total variation in the data as possible. In order to use principal component analysis to estimate the volatility functions in (4.2) or (3.5) we assume that these functions only depend on time to maturity (τ_m). Not allowing the volatility functions to depend explicitly on t precludes seasonal variation in the volatility functions. Assume that we have a total of N observations of M different variables contained in vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_M$ all of which dimension

is $(N \times 1)$.¹² Let the data matrix, **X**, be given by

$$\mathbf{X}_{(N \times M)} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_M \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}$$
(3.14)

The corresponding sample covariance matrix, of order M, is denoted Ψ . The orthogonal decomposition of the covariance matrix is

$$\Psi = \mathbf{P} \mathbf{\Lambda} \mathbf{P}' \tag{3.15}$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_M \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1M} \\ p_{21} & p_{22} & \cdots & p_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MM} \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & 0 & \cdots & 0 \\ 0 & \lambda_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{MM} \end{bmatrix}$$

 Λ is a diagonal matrix whose diagonal elements are the eigenvalues λ_{11} , $\lambda_{22}, ..., \lambda_{MM}$, and where **P** is an orthogonal matrix of order *M* whose *i*th column, \mathbf{p}_i , is the eigenvector corresponding to λ_{ii} . **P'** is the transpose of **P**. The matrix $\mathbf{Z} = \mathbf{XP}$ is called the matrix of principal components. Its columns, \mathbf{z}_i , are linear combinations of the columns of **X** with the weights given by the elements of \mathbf{p}_i . That is, the *i*th principal component is

$$\mathbf{z}_i = \mathbf{X}\mathbf{p}_i = \mathbf{x}_1 p_{1i} + \mathbf{x}_2 p_{2i} + \dots + \mathbf{x}_M p_{Mi}$$
(3.16)

¹²Throughout this section we write matrices in bold upper case letters, vectors in bold lower case letters and elements in plain text. The principal component analysis is conducted on both forward price differences (\mathbf{X}^A) and forward price returns (\mathbf{X}^B) . We suppress superscripts for notational convenience throughout this section.

where p_{ji} is the element in the *j*th row and *i*th column of **P**. The sample covariance matrix of **Z** is given by

$$Var(\mathbf{Z}) = \mathbf{P}' \Psi \mathbf{P} = \mathbf{P}' \mathbf{P} \mathbf{\Lambda} \mathbf{P}' \mathbf{P} = \mathbf{\Lambda}$$
(3.17)

since $\mathbf{PP'} = \mathbf{P'P} = \mathbf{I}$, where \mathbf{I} is the identity matrix, hence the \mathbf{Z} variates are uncorrelated, and the variance of $\mathbf{z}_i = \lambda_{ii}$. The eigenvectors on the diagonal of $\boldsymbol{\Lambda}$ are of convention ordered so that $\lambda_{11} \geq \lambda_{22} \geq ... \geq \lambda_{MM}$. To explain all the variation in \mathbf{X} , we need M principal components. Since the objective of our analysis is to explain the covariance structure with just a few factors, we approximate the theoretical covariance matrix using the first K < Meigenvalues in (3.15). Unfortunately we lack any solid statistical criterion to determine the number of factors that constitute the theoretical covariance matrix. Hair et al. (1995) discuss several criteria:

- 1. Eigenvalue criterion; only factors eigenvalues greater than 1 are considered significant.
- 2. Scree test criterion; the test conducted by plotting the eigenvalues against the number of factors in their order of extraction, and the shape of the curve is used to evaluate the cut-off point.
- 3. Percentage of variance criterion; additional factors are added until the cumulative percentage of the variance explained reach a pre-specified level.

We consider all of these criteria, but the latter criterion is the one frequently employed in the finance literature. The K factors should explain a "big" part of the total covariance of the underlying variables (typically around 95%). The proportion of total variance accounted for by the first Kfactors is

Cumulative contribution of first K factors =
$$\frac{\sum_{i=1}^{K} \lambda_i}{\sum_{i=1}^{M} \lambda_i}$$

Component loadings are often computed to facilitate interpretation of the results from a principal component analysis. Here, we instead plot the empirical volatility function, $\hat{\sigma}_i(.)$, directly from the eigenvalue decomposition as

$$\widehat{\sigma}_i\left(\tau_m\right) = \sqrt{\lambda_i} p_{mi} \tag{3.18}$$

where i = 1, ..., K. Here we have suppressed the time index, emphasising the fact that the volatility is independent on calendar time. We can use (3.18) to plot easy-to-interpret volatility functions can be graphed.

3.6 Empirical results

In table 3.3 we report the results from the PCA analysis conducted on the full sample. We note that a one-factor model is able to explain 68% and 70% of the variation of price returns and price differences respectively. The eigenvalue and scree test criteria both agree on a two-factor model for both returns and differences with a total of 75% and 78% variation explained respectively. This is considerably lower than in most other markets. Typically two or three factors explain more than 95% of total variations in forward prices. For example, Clewlow and Strickland (2000) investigate the term structure of NYMEX oil futures and find that three-factors explain 98.4% of the total price variation. The fact that as much as 25% of the variance in the electricity market is maturity specific, as far as we know, a feature unique to this market. If we increase the number of factors the percentage variations explained will naturally increase. We also note from table 3.3 that a target of say 95% explained variation requires more than 10 factors in our data. It is obvious that the 8 additional factors do not explain variation common to the whole term structure. We will examine this more closely below, but first we will investigate the shape of the first two factors.

We now want to take a closer look at the volatility dynamics represented by the first two factors that affect the whole term structure. From the eigenvalues and the corresponding eigenvectors for the two first factors, we use (3.18) to plot the corresponding volatility function in figure 3.4 along with the overall volatility. The scaling on the vertical axes are annualised volatilities. Data for the whole sample period is used in these calculations. We note that the overall volatility is very high in the short end of the term structure, and it falls rapidly with time to maturity. After approximately one year it stabilises. This pattern applies to both price differences and price returns. Turning to the individual volatility functions, we see that the first factor is positive for all maturities, shifting all forward prices in the same direction. It causes much bigger movements in the short end than in the long end. The second factor causes short and long term forward prices to move in opposite directions. Again this pattern applies to both price differences and price

Sample period: 1995 - 2001											
Data	Price	Price returns Price differences									
Sample		199	5-2001								
Factor	Ind.	Cum.	Ind.	Cum.							
Fn1	0.68	0.68	0.70	0.70							
Fn2	0.07	0.75	0.08	0.78							
Fn3	0.05	0.80	0.05	0.83							
Fn4	0.03	0.83	0.02	0.86							
Fn5	0.03	0.86	0.02	0.88							
Fn6	0.02	0.88	0.02	0.89							
Fn7	0.02	0.89	0.02	0.91							
Fn8	0.02	0.91	0.01	0.92							
Fn9	0.01	0.92	0.01	0.93							
Fn10	0.01	0.94	0.01	0.94							

Table 3.3: Principal component analysis of forward price differences and returns. The analysis is performed on the whole data set, 1339 observations from September 1995 to March 2001. The table reports the individual contribution (Ind.) of each factor (Fn.) of the total variance, and the cumulative effect (Cum.) of adding one additional factor.

returns. These two factors are qualitatively equal to the first two factors reported in Clewlow and Strickland (2000) for NYMEX oil futures, which they termed the *tilting* factor and *shifting* factor respectively.

Pilipović (1998) argued that the correlation between short-term and longterm forward prices seem to be lower in electricity markets than in other markets. If this is indeed the case, we would expect factors explaining a lot of variation in the long end of the term structure, being able to explain far less of the short term movements, and vice versa. We conducted the PCA analysis once again to take a closer look at this. First we computed 10 principal components capturing about 95% of variation of both price differences and price returns. Then all 10 factors were sorted according to size for each of the maturities. Hence for each of the 21 maturities, the 10 volatility functions resulting from the PCA analysis are sorted according to their ability to explain the overall variation for that particular maturity. The results are given in tables 3.4 and 3.5.¹³ The first column reports the variation explained

¹³The rest of the tables and figures are located in the appendix for space considerations.



Figure 3.4: Volatility functions and overall volatility in the full sample period 1995-2001. The volatility functions on the left hand side are computed from price returns and the volatility functions on the right hand side are computed from price differences. The functions are annualized using a factor of square root of 250 (number of trading days).

by the most important factor for that particular maturity. The number in superscript is the factor number. Hence factor number 1 is the most important factor for explaining overall volatility. The second column reports the cumulative variance explained by adding the second most important factor for that particular maturity. Again, the superscript indicates the importance of this factor in explaining total variation for all maturities. The results in tables 3.4 and 3.5 are very similar, and we comment only the latter. We note that factor number 1 is the factor explaining most of the variation for each maturity within the first year. Factors number 1 and 2 are among the 4 most important factors for all maturities. However, in the long end of the term structure, factors number 9 and 6 are the most important ones. In other words, the most important factors driving the long end of the curve have very little impact on price changes in the short end. On average, very little is gained in terms of percentage variation explained, by increasing the number of factors beyond 5. Combined, this evidence supports the conjecture made by Pilipović (1998) that electricity prices exhibit "split personalities". Why do we see this kind of forward curve behaviour in the electricity market? The answer possibly lies in the non-storable nature of electricity. For example, assume that the Swedish government makes a final decision to phase out their nuclear electricity production and decides to start cutting production two years from now. This would lower future supply, resulting in rising futures prices with more than 2 years to maturity. In a market where storage is possible, speculator would buy for storage (or producer would hold back production), as a reaction to the anticipated rise in electricity prices in the future. This would in turn result in a positive shift in spot and short-term futures prices as well as long term futures prices. Since buying for storage is impossible¹⁴ in electricity markets, the price on electricity will stay low until the date of reduced production. Consequently, only futures contracts with maturity after the production cut will react to this information.

Using the whole sample period in our calculations, we implicitly assume that volatility dynamics have been constant in the 1995-2001 period. Investigating the validity of this assumption, we plotted the volatility series from the shortest maturity for each of our models in figure 3.5. Annualised volatility of price differences and price returns of the one-week forward price is calculated using a 30 day moving window. Volatility of price differences is measured on the left vertical axis, and volatility of price returns is measured on the right vertical axis. The volatility of price differences was high in the period 1995-1997 and relatively much lower in the 1998-2001 period. We also note that the volatility is all but constant.

The volatility of price returns was not especially high the first years. In this model we see a relatively regular pattern; volatility peaks during summer. We want to investigate yearly and seasonal differences further. However, our methodology does not allow calendar time dependence in the volatility functions. As a second best alternative, we repeat our PCA analysis in different sub-samples. In table 3.6 we report the results from PCA analysis on two years sub-intervals and seasonal sub-intervals for model A and B. The two first volatility functions and overall volatility for each sub-sample are plotted in figures 3.6 and 3.7. From table 3.6 we see that the V1 and S0 sub-periods, fewer factors are needed to explain 95% of the variation in the data. Dividing into semi-yearly samples resulted in increased explanatory power of the 10 factors. This indicates that volatility dynamics changes both seasonally and from one year to the other¹⁵. Still, from the volatility function

¹⁴A large part of the electricity consumed the Nordic market is produced in hydropower based production units. Many of these units have reservoir facilities that, to some extent, enables them to move energy between periods. Such reservoir facilities provide a relatively high level of operating flexibility. Still, the reservoir capacity is not big enough for producers to shut down production for long periods of time without spilling water.

¹⁵We also computed the non-parametric Kolmogorov-Smirnoff test on equality of distributions across seasons and years. The test results, not reported here, showed rejections of



Figure 3.5: Estimated volatility using a moving window. The volatility series are computed from daily data of the one week ahead forward price returns (dashed line) and forward price differences (solid line). Both volatility series are simple arithmetic average of the last 30 trading days. We have annualized the series by the square root of 250 (number of trading days).

in figures 3.6 and 3.7 we recognise the shifting and tilting factor as the most important factors driving the forward curve.

Finally, we are interested in which of the two models, A or B, best resembles the data generating process. We know that model A assumes normally distributed price differences and model B assumes normally distributed price returns. In table 3.7 we report statistics on skewness, kurtosis and the combined effect of the two (the Jarque-Bera test) under the null hypothesis of normality of price differences and price returns respectively. The tests are conducted on 3 points on the term structure, with one week, one year and two years to maturity. We note that both price differences and price returns are positively skewed in the short end, and negatively skewed in the

equal distributions on 1% level in all cases.

long end. Excess kurtosis is substantially different from zero for both models and increases with maturity for both specifications. The high degree of kurtosis may indicate that jumps are present in the data. Not surprisingly, the Jarque-Bera tests reject the null hypothesis of normality for both models, and so further modifications and testing of the models are necessary to decide upon the winning candidate.

3.7 Concluding remarks

In this paper we have conducted an exploratory investigation of the volatility dynamics in the Nordic futures and forward market in the period 1995-2001. We have used smoothed data and performed a principal component analysis to reveal the factor structure of the forward price curve. We specified two different models in the framework of Heath et al. (1992); one model where the volatility was independent of the forward price level and one model where the volatility was proportional to the price level.

The main results are as follows: Two factors are common across all maturities. A two-factor model explains around 75% of total variation in the data. The first two factors governing the forward curve dynamics are comparable to other markets. The first factor is positive for all maturities, hence it shifts all forward prices in the same direction. The second factor causes short and long term forward prices to move in opposite directions. In contrast to other markets, more than 10 factors are needed to explain 95% of the term structure variation. Furthermore, the main sources of uncertainty affecting the movements in the long end of the forward curve, have virtually no influence on variation in the short end of the curve. We argue that this behaviour may occur because electricity is a non-storable commodity. Note that the maximum maturity in our analysis is 2 years. One might suspect that contracts sold in the OTC market with maturities further into the future are even less correlated with short term contracts. These results indicate that modelling the whole forward curve has less merit in this market than others. For example, hedging long-term commitments using short-term contracts may prove disastrous.

The results reported above apply to both models. Both models fail the normality test, and so neither of them are completely satisfactory. Results from semi-yearly and seasonal sub-intervals suggest that volatility is not constant through time. Hence extending the basic model to include stochastic volatility, possibly with a seasonally time-dependent component, may be fruitful.

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3.8 Appendix: Tables and figures

Relativ	Relative importance of factors across maturities for price differences												
Matu	rity		Ċı	ımulat	ive va	riance	explai	ned (?	%)				
	1^{th}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	8^{th}	9^{th}	10^{th}			
W-01	0.85^{1}	0.92^{2}	0.94^{4}	0.95^{3}	0.96^{8}	0.96 ⁹	0.967	0.97^{5}	0.97^{6}	0.97^{10}			
W-02	0.89^{1}	0.94^{2}	0.95^{4}	0.96^{3}	0.96^{9}	0.96^{8}	0.96^{6}	0.96^{7}	0.96^{10}	0.96^{5}			
W-03	0.91^{1}	0.94^{2}	0.95^{8}	0.96^{7}	0.96^{9}	0.96^{5}	0.96^{6}	0.96^{10}	0.96^{4}	0.96^{3}			
W-04	0.91^{1}	0.93^{2}	0.94^{4}	0.94^{5}	0.95^{8}	0.95^{3}	0.96^{7}	0.96^{9}	0.96^{10}	0.96^{6}			
W-05	0.91^{1}	0.93^{4}	0.95^{3}	0.96^{5}	0.96^{2}	0.96^{6}	0.96^{10}	0.96^{8}	0.96^{9}	0.96^{7}			
W-06	0.90^{1}	0.93^{4}	0.95^{3}	0.95^{8}	0.96^{5}	0.97^{7}	0.97^{6}	0.97^{9}	0.97^{10}	0.97^{2}			
W-07	0.88^{1}	0.91^{4}	0.92^{8}	0.94^{3}	0.94^{7}	0.95^{5}	0.95^{2}	0.95^{9}	0.96^{6}	0.96^{10}			
W-12	0.81^{1}	0.88^{5}	0.90^{9}	0.92^{2}	0.92^{3}	0.93^{4}	0.94^{8}	0.94^{7}	0.94^{10}	0.94^{6}			
W-16	0.82^{1}	0.89^{5}	0.92^{2}	0.93^{4}	0.93^{8}	0.94^{7}	0.94^{9}	0.94^{3}	0.94^{10}	0.94^{6}			
W-20	0.81^{1}	0.87^{5}	0.90^{2}	0.92^{9}	0.92^{4}	0.92^{10}	0.93^{6}	0.93^{8}	0.93^{3}	0.93^{7}			
W-24	0.79^{1}	0.85^{2}	0.89^{9}	0.91^{5}	0.92^{10}	0.92^{7}	0.93^{6}	0.94^{3}	0.94^{8}	0.94^{4}			
W-28	0.75^{1}	0.82^{2}	0.85^{7}	0.87^{3}	0.88^{6}	0.89^{9}	0.89^{10}	0.89^{4}	0.89^{5}	0.89^{8}			
W-32	0.66^{1}	0.82^{3}	0.91^{2}	0.93^{10}	0.93^{9}	0.94^{7}	0.94^{5}	0.94^{4}	0.95^{8}	0.95^{6}			
W-36	0.70^{1}	0.82^{3}	0.91^{2}	0.93^{5}	0.94^{7}	0.95^{9}	0.95^{10}	0.95^{8}	0.95^{4}	0.95^{6}			
W-40	0.72^{1}	0.82^{2}	0.88^{7}	0.90^{8}	0.91^{5}	0.92^{4}	0.93^{3}	0.93^{10}	0.94^{6}	0.94^{9}			
W-44	0.66^{1}	0.77^{2}	0.84^{4}	0.89^{3}	0.91^{7}	0.93^{8}	0.93^{6}	0.93^{5}	0.93^{9}	0.93^{10}			
W-48	0.59^{1}	0.74^{4}	0.85^{2}	0.94^{3}	0.95^{5}	0.95^{7}	0.95^{6}	0.96^{10}	0.96^{8}	0.96 ⁹			
W-52	0.58^{1}	0.72^{2}	0.81^{4}	0.86^{3}	0.90^{7}	0.92^{8}	0.93^{5}	0.93^{10}	0.94^{9}	0.94^{6}			
W-70	0.59^{10}	0.76^{1}	0.86^{8}	0.88^{2}	0.91 ⁹	0.92^{5}	0.94^{7}	0.94^{3}	0.94^{4}	0.94^{6}			
W-88	0.49^{6}	0.64^{1}	0.71^{9}	0.76^{2}	0.78^{4}	0.80^{10}	0.81^{3}	0.82^{8}	0.83^{5}	0.83^{7}			
W-104	0.72^{6}	0.81^{1}	0.85^{2}	0.87^{4}	0.89^{10}	0.90^{7}	0.90^{5}	0.90^{3}	0.90^{9}	0.90^{8}			
Avg.	0.76	0.84	0.89	0.91	0.92	0.93	0.93	0.94	0.94	0.94			

Table 3.4: Most important factors across maturities for price differences. we have first conducted a principal component analysis using 10 factors. Then the importance of each factor is sorted for each maturity. The table reports the cumulative variance explained when adding one additional factor. The factor number is in superscript. The bottom row reports the the average cumulative variance explained.

Relativ	Relative importance of factors across maturities for price returns $(\%)$											
Maturi	ty			Cumu	lative	varian	ice exp	olainec	1			
	1^{th}	2^{nd}	3 rd	4^{th}	5^{th}	6 th	7^{th}	8^{th}	9^{th}	10^{th}		
W-01	0.86^{1}	0.91 ²	0.95^{3}	0.96^{6}	0.967	0.97^{5}	0.97 ⁸	0.97 ⁹	0.97^{4}	0.97^{10}		
W-02	0.90^{1}	0.95^{2}	0.96^{3}	0.96^{6}	0.96^{7}	0.96^{5}	0.96 ⁸	0.96^{9}	0.96^{4}	0.96^{10}		
W-03	0.91^{1}	0.93^{2}	0.95^{6}	0.96^{7}	0.96^{5}	0.96^{3}	0.96^{8}	0.96^{9}	0.96^{10}	0.964		
W-04	0.91^{1}	0.93^{3}	0.94^{2}	0.95^{6}	0.95^{7}	0.96^{5}	0.96^{8}	0.96^{9}	0.96^{4}	0.96^{10}		
W-05	0.89^{1}	0.95^{3}	0.96^{2}	0.96^{6}	0.96^{5}	0.96^{9}	0.96^{7}	0.96^{8}	0.96^{10}	0.964		
W-06	0.88^{1}	0.94^{3}	0.96^{6}	0.97^{7}	0.97^{2}	0.97^{8}	0.97^{5}	0.97^{10}	0.97^{9}	0.97 ⁴		
W-07	0.85^{1}	0.90^{3}	0.93^{6}	0.94^{7}	0.95^{2}	0.95^{5}	0.95^{8}	0.95^{9}	0.95^{10}	0.95 ⁴		
W-12	0.76^{1}	0.81^{2}	0.86^{5}	0.88^{10}	0.89^{7}	0.91 ⁹	0.92^{8}	0.92^{6}	0.92^{4}	0.92 ³		
W-16	0.75^{1}	0.84^{2}	0.89^{5}	0.91^{7}	0.92^{8}	0.92^{4}	0.92^{9}	0.93^{6}	0.93^{10}	0.93 ³		
W-20	0.72^{1}	0.83^{2}	0.87^{5}	0.88^{4}	0.89^{9}	0.90^{10}	0.90^{7}	0.91 ³	0.91^{6}	0.91 ⁸		
W-24	0.70^{1}	0.82^{2}	0.86^{9}	0.89^{8}	0.90^{10}	0.91^{5}	0.92^{4}	0.93 ³	0.93^{6}	0.93 ⁷		
W-28	0.67^{1}	0.80^{2}	0.85^{8}	0.87^{4}	0.88^{7}	0.89^{3}	0.89^{9}	0.89^{10}	0.89^{6}	0.89^{5}		
W-32	0.61^{1}	0.77^{2}	0.85^{4}	0.88^{10}	0.90^{5}	0.92^{3}	0.93^{9}	0.94^{7}	0.94^{8}	0.94^{6}		
W-36	0.63^{1}	0.78^{2}	0.85^{5}	0.89^{4}	0.92^{8}	0.93^{3}	0.94^{7}	0.94^{9}	0.94^{6}	0.94^{10}		
W-40	0.63^{1}	0.77^{2}	0.85^{8}	0.88^{5}	0.90^{9}	0.91 ⁴	0.92^{10}	0.93^{3}	0.93^{6}	0.947		
W-44	0.59^{1}	0.77^{4}	0.88^{2}	0.90^{8}	0.91 ⁹	0.92^{3}	0.92^{5}	0.92^{6}	0.92^{7}	0.93^{10}		
W-48	0.61^{4}	0.83^{1}	0.93^{2}	0.94^{8}	0.94^{3}	0.95^{7}	0.95^{10}	0.96^{6}	0.96^{9}	0.96^{5}		
W-52	0.57^{1}	0.75^{4}	0.86^{2}	0.89^{8}	0.91 ⁹	0.92^{10}	0.92^{7}	0.93^{3}	0.93^{6}	0.93^{5}		
W-70	0.55^{9}	0.76^{10}	0.89^{1}	0.93^{2}	0.94^{8}	0.95^{6}	0.95^{7}	0.95^{4}	0.95^{5}	0.95 ³		
W-88	0.38^{6}	0.53^1	0.64^{7}	0.72^{2}	0.76^{5}	0.79^{3}	0.80^{8}	0.81^{10}	0.81 ⁹	0.814		
W-104	0.53^{6}	0.73^{7}	0.79^{1}	0.83^{2}	0.85^{3}	0.86^{5}	0.87^{10}	0.89^{4}	0.89 ⁹	0.89 ⁸		
Avg.	0.71	0.83	0.88	0.90	0.92	0.92	0.93	0.93	0.93	0.93		

Table 3.5: Most important factors across maturities for price returns. Relative importance of factors across maturities for price returns. We have first conducted a principal component analysis using 10 factors. Then the importance of each factor is sorted for each maturity. The table reports the cumulative variance explained when adding one additional factor. The factor number is in superscript. The bottom row reports the the average cumulative variance explained.

Sample	mple V2		V1		S0		1995-1996		1997-1998		1999-2001	
Factor	Ind.	Cum.	Ind.	Cum.	Ind.	Cum.	Ind.	Cum.	Ind.	Cum.	Ind.	Cum.
Fn1	0.60	0.60	0.72	0.72	0.76	0.76	0.73	0.73	0.53	0.53	0.73	0.73
Fn2	0.08	0.67	0.07	0.79	0.07	0.84	0.08	0.80	0.11	0.63	0.07	0.81
Fn3	0.05	0.73	0.06	0.85	0.04	0.88	0.05	0.85	0.07	0.70	0.05	0.85
Fn4	0.04	0.77	0.04	0.89	0.03	0.91	0.03	0.89	0.06	0.76	0.02	0.88
Fn5	0.03	0.80	0.03	0.92	0.01	0.93	0.03	0.91	0.05	0.81	0.02	0.90
Fn6	0.03	0.83	0.02	0.94	0.01	0.94	0.01	0.93	0.04	0.85	0.02	0.91
Fn7	0.02	0.86	0.02	0.95	0.01	0.95	0.01	0.94	0.03	0.87	0.01	0.93
Fn8	0.02	0.88	0.01	0.96	0.01	0.96	0.01	0.95	0.02	0.90	0.01	0.94
Fn9	0.02	0.90	0.01	0.97	0.01	0.97	0.01	0.96	0.02	0.92	0.01	0.95
Fn10	0.02	0.92	0.01	0.97	0.01	0.97	0.01	0.97	0.02	0.93	0.01	0.96

Panel A: Analysis of forward price differences

Panel	B:	Analysis	OI	forward	price	returns	

Sample	۱ ۱	/2	N	/1	5	30	1995-1996		1995-1996 1997-1998		995-1996 1997-1998 1999-20		-2001
Factor	Ind.	Cum.	Ind.	Cum.	Ind.	Cum.	Ind.	Cum.	Ind.	Cum.	Ind.	Cum.	
Fn1	0.59	0.59	0.70	0.70	0.80	0.80	0.70	0.70	0.58	0.58	0.73	0.73	
Fn2	0.09	0.68	0.08	0.78	0.06	0.87	0.08	0.78	0.09	0.67	0.08	0.81	
Fn3	0.05	0.73	0.05	0.83	0.04	0.91	0.06	0.83	0.07	0.74	0.05	0.86	
Fn4	0.04	0.77	0.03	0.86	0.02	0.93	0.03	0.86	0.06	0.80	0.02	0.88	
Fn5	0.04	0.81	0.03	0.89	0.01	0.95	0.03	0.89	0.04	0.83	0.02	0.90	
Fn6	0.03	0.84	0.02	0.91	0.01	0.95	0.02	0.91	0.04	0.87	0.02	0.92	
Fn7	0.02	0.86	0.02	0.93	0.01	0.96	0.02	0.92	0.02	0.89	0.01	0.93	
Fn8	0.02	0.89	0.01	0.94	0.01	0.97	0.01	0.94	0.02	0.91	0.01	0.94	
Fn9	0.02	0.90	0.01	0.95	0.01	0.98	0.01	0.95	0.02	0.93	0.01	0.95	
Fn10	0.02	0.92	0.01	0.96	0.00	0.98	0.01	0.96	0.01	0.94	0.01	0.96	

Table 3.6: Principal component analysis of forward price differences and price returns. In panel A the analysis is performed on each two year sub-interval of the total sample. In panel B the data set is reshuffled, and the analysis is performed on 3 seasonal subintervals, V2 (early winter), V1 (late winter) and S0 (summer) (see the text for exact period specifications). The table reports the individual contribution (Ind.) of each factor (Fn.) of the total variance, and the cumulative effect (Cum.) of adding an additional factor.



Volatility functions: Model A

Figure 3.6: Volatility functions and overall volatility in subperiods for price differences. The two first volatility functions and overall volatility. The volatility functions on the left hand side are computed from different seasons corresponding to seasonal contracts traded at Nord Pool and the functions on the right hand side are computed from the time periods 1995-1996, 1997-1998 and 1999-2001. The functions are annualized using a factor of square root of 250 (number of trading days).



Volatility functions: Model B

Figure 3.7: Volatility functions and overall volatility in subperiods for price returns. The two first volatility functions and overall volatility. The volatility functions on the left hand side are computed from different seasons corresponding to seasonal contracts traded at Nord Pool and the functions on the right hand side are computed from the time periods 1995-1996, 1997-1998 and 1999-2001. The functions are annualized using a factor of square root of 250 (number of trading days).

Sample period 1995 - 2001											
Data Price differences Price returns											
Maturity	W-01	W-52	W-104	W-01	W-52	W-104					
Std. Skew	0.15	0.25	-1.06	0.08	0.33	-1.01					
Sign.	(0.04)	(0.00)	(0.00)	(0.26)	(0.00)	(0.00)					
Std. Kurt.	6.53	9.49	42.07	2.65	5.07	113.14					
Sign.	(0.00)	(0.00)	(0.00)	(0.20)	(0.00)	(0.00)					
Jarque-Bera	2277.17	4809.96	94483.59	374.69	1394.68	681867					
Sign.	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)					
Nobs	1339	1339	1278	1339	1339	1278					

Table 3.7: Results from normality tests. Std.skew and Std.kurt are calculated as $\frac{(skewness)}{(std.deviation)^3}$ and $\frac{(kurtosis)}{(std.deviation)^4} - 3$, respectively. The test statistics are both normally distributed, and the p-value for a two sided test for the null hypotesis of zero Std. skew and zero Std.kurtosis is reported in parantheses. The Jarque-Bera test statistics is calculated as $\frac{T}{6}(std.skewn)^2 + \frac{T}{24}(std.kurt)^2$. This statistic is chi-squred distributed. The significance level at which the null hypothesis of normality can be rejected using a 2-sided test is reported in parantheses.

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Chapter 4

A multi-factor forward curve model for electricity derivatives

ABSTRACT - In this paper we develop a general framework for valuation and hedging electricity derivatives. We propose a multi-factor forward curve model consistent with market prices. The electricity forward price is modelled as arithmetic Brownian motion. The main advantage of our model compared to the geometric Brownian forward curve models suggested previously in the literature is that closed form solution to average based derivatives can be easily computed. This is important in the electricity industry, since most contingent claims in this market are derived from (arithmetic) price averages. The dynamic properties of two different average based forward contracts are investigated. Furthermore, closed form solutions to both European and Asian options and corresponding hedge ratios are calculated. Finally we implement the model and provide some numerical examples using data from the Nordic electricity market.

4.1 Introduction

In this paper we develop a simple model that provides easy valuation, hedging and risk management of electricity contingent claims. There are currently two different approaches to commodity contingent claims valuation. The traditional line of research starts with a stochastic specification of the underlying asset or some state variables, such as the convenience yield or in-

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terest rates (see Brennan and Schwartz (1985), Gibson and Schwartz (1990), Schwartz (1997) and Hilliard and Reis (1998))). These models are usually referred to as spot price models. Deng (2000) and Kamat and Ohren (2000) have investigated different spot price models for electricity contingent claims. They modify the standard lognormal spot price assumption by adjusting for jumps, regime switching and/or stochastic volatility in the log of the spot price. Knittel and Roberts (2001) claim that the assumption of lognormal electricity prices is inappropriate for several reasons, with the appearance of negative prices in electricity markets being one. Their starting point is a mean reverting model, where the price level is conditional Gaussian. More sophisticated models that include seasonality, jumps and stochastic volatility are also investigated. Lucia and Schwartz (2000) estimated both lognormally and normally distributed spot price models using data from the Nordic electricity market. However, they do not reach any conclusion on which model best describes the data.

Spot price models have some disadvantages. First, some or all of the state variables, like the convenience yield, are typically unobserved. Second, the forward and futures prices are endogenous in these models, and in general the endogenous prices are typically inconsistent with prices observed in the market place. To cope with these drawbacks, a second line of research has concentrated on the evolution of the whole forward curve. The idea is to model the entire forward curve using multiple (few) sources of risk, and all observed futures and forward prices are taken as initial values of the forward curve. Cortazar and Schwartz (1994) and Clewlow and Strickland (1999a) have used this model to analyse copper index notes and energies respectively. Bjerksund et al. (2000) apply the model in the Nordic electricity market. These studies have the assumption of lognormally distributed forward prices in common.

In this paper we develop a multi-factor *arithmetic* forward curve model consistent with observed market prices. Electricity, once produced, cannot be stored. Production and consumption have to balance in a power network. This property makes electricity unique compared to other commodities, and often electricity is described as a *flow commodity*. Consequently, contracts traded in the electricity industry are typically specified with a future time *period* for delivery, not a future time *point*. The value of such a contract depends on the arithmetic average of the electricity spot price in the delivery period. In a lognormal electricity forward price model, simple closed form solutions to such derivatives does not exist, since the distribution of the sum of lognormal random variables is unknown. Hence, in a lognormal model, approximations are needed even for simple European contingent claims. Our model, being a forward price model, provides an important generalisation of the Gaussian spot price model proposed by Lucia and Schwartz (2000) and Knittel and Roberts (2001), since it is consistent with observed market prices. But the most important property our model is that it provides simple closed form pricing formulae for arithmetic average based contingent claims. We investigate the dynamic properties of two different average based forward contracts. Furthermore, closed form solutions to both European and Asian options and corresponding hedge ratios are calculated.

The rest of the paper is organised as follows: Section 4.2 describes the multi-factor model and section 4.3 investigates the distributional properties of two average based forward contracts. Section 4.4 provides closed form expressions for European forward and Asian spot price options. Hedging ratios are calculated and hedging in a multi-factor model is discussed. In section 4.5 we implement the model and provide some numerical examples using data from the Nordic electricity market. Section 4.6 concludes the paper.

4.2 The multi-factor model

We consider a financial market where the uncertainty is characterised by the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where \mathcal{F} is a σ -algebra of subsets of Ω and $\mathbb{Q} : \mathcal{F} \to [0, 1]$ is a probability measure. The probability measure \mathbb{Q} is the equivalent martingale measure by assumption. All economic activity is assumed to take place on a finite horizon $[0, T^*]$. We consider a K-dimensional Brownian motion $(W_1, ..., W_K)$ defined on this probability space. We fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T^*]\}$ with \mathcal{F}_t defined as the sigma algebra representing available information at time t (for technical details see e.g. Duffie (1996)). Let the forward market be represented by a continuous forward price function, where f(t, T) denotes the forward price at date t for delivery of one unit electricity at time T, where $t < T < T^*$. We consider a model where the dynamics of the forward prices are given by

$$df(t,T) = \sum_{i=1}^{K} \sigma_i(t,T) dW_i(t), \ f(0,T) \ \forall \ T$$
(4.1)

and where the $(W_1, ..., W_K)$ are independent Brownian motions, and $\sigma_i(t, T)$ are time dependent volatility functions associated with each source of uncertainty.¹ The integral form of (4.1) is

$$f(t,T) = f(0,T) + \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(s,T) dW_{i}(s)$$
(4.2)

This means that the forward prices are distributed

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$$f(t,T) \sim \mathcal{N}\left(f(0,T), \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(s,T)^{2} ds\right)$$
(4.3)

where $X \sim \mathcal{N}(a, b)$ means that the random variable X is normally distributed with mean a and variance b. The process of the spot price process can be found by setting S(t) = f(t, t)

$$S(t) = f(0,t) + \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(s,t) dW_{i}(s)$$
(4.4)

The spot price is normally distributed with

$$S(t) \sim \mathcal{N}\left(f(0,t), \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(s,t)^{2} ds\right)$$

The spot price process can be written as^2

$$S(t) = S(0) + \int_0^t \zeta(u) du + \sum_{i=1}^K \int_0^t \sigma_i(u, u) dW_i(u)$$
(4.5)

where ζ is the following process

$$\zeta(t) = \frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^{K} \int_{0}^{t} \frac{\partial \sigma_{i}(u,t)}{\partial t} dW_{i}(u)$$
(4.6)

¹Note that volatility is a term usually associated with the diffusion term in a lognormal model. In this paper we use the term "volatility function" for the time dependent function in the diffusion term in the arithmetic model presented above.

 ^{2}A proof is given in appendix A.

4.2. THE MULTI-FACTOR MODEL

The differential form of S is

$$dS(t) = \left(\frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^{K} \int_{0}^{t} \frac{\partial \sigma_{i}(s,t)}{\partial t} dW_{i}(s)\right) dt + \sum_{i=1}^{K} \sigma_{i}(t,t) dW_{i}(t) \quad (4.7)$$

Since the last term in $\zeta(t)$ integrates over Brownian motions, the spot price will in general depend upon its past evolution. In other words the spot price process is, in general, non-Markovian (see Carverhill (1994) and Ritchken and Sanakarasubramanian (1995) for discussions on the Markov-property of spot rates in Heath-Jarrow-Morton models).

Lucia and Schwartz (2000) propose a spot price model and they derive analytical expressions for futures/forward prices. They consider a mean reverting spot price model of the form

$$dS(t) = \kappa (\theta(t) - S(t)) dt + \sigma dW(t), \ S(0) = S_0$$
(4.8)

with solution

$$S(t) = e^{-\kappa t}S_0 + \int_0^t e^{-\kappa(t-s)}\theta(s)ds + \int_0^t e^{-\kappa(t-s)}\sigma dW(s)$$
(4.9)

where $\theta(t)$ is a time dependent function that can capture seasonal variation in the spot price and κ is a positive constant that pulls the spot price back to the normal time-dependent mean.³ At first glance, the interpretation of the of eq. (4.8) seems to be that, under the equivalent martingale measure, the spot price reverts toward a time dependent mean, $\theta(t)$, with speed of mean reversion equal to κ . This is not correct. When θ is time dependent, it is no longer identical to the long run mean of the process. To see this, consider the expectation of the spot price, m(t), given by

$$m(t) = e^{-\kappa t}S_0 + \int_0^t e^{-\kappa(t-s)}\theta(s)ds$$

and note that

$$\begin{aligned} \frac{\partial m(t)}{\partial t} &= -\kappa e^{-\kappa t} S_0 + \kappa \theta(t) - \kappa \int_0^t e^{-\kappa (t-s)} \theta(s) ds \\ &= \kappa \theta(t) + \kappa m(t) \end{aligned}$$

³Note that we are presenting the model under the risk neutral measure, which means that $\theta(t)$ is adjusted for risk. If we follow Lucia and Schwartz (2000) and assume constant market price of risk, we have the following relationship $\theta(t) = \theta^*(t) - \frac{\sigma\lambda}{\kappa}$, where $\theta^*(t)$ is the corresponding time dependent function under the objective probability measure and λ is the market price of risk.

Inserting $\theta(t) = \frac{1}{\kappa} \frac{\partial m(t)}{\partial t} + m(t)$ in (4.8) expresses the SDE in terms of the mean of the process

$$dS(t) = \left(\frac{\partial m(t)}{\partial t} + \kappa \left(m(t) - S(t)\right)\right) dt + \sigma dW(t)$$
(4.10)

We see from (4.9) that the term structure of forward price volatility in this single factor model is given by

$$\sigma_1(t,T) = \sigma e^{-\kappa(T-t)} \tag{4.11}$$

Substituting (4.11) into (4.7) we get

$$dS(t) = \left(\frac{\partial f(0,t)}{\partial t} - \kappa \int_0^t \sigma e^{-\kappa(t-s)} dW_i(s)\right) dt + \sigma dW_1(t)$$

= $\left(\frac{\partial f(0,t)}{\partial t} - \kappa \left(S(t) - f(0,t)\right)\right) dt + \sigma dW_1(t)$ (4.12)

where S(0) = f(0,0). The last equality follows from (4.4). Lucia and Schwartz (2000) show that forward prices in their spot-based model are given analytically as

$$f(0,T) = e^{-\kappa T} \left(S(0) - m(0) \right) + m(T)$$
(4.13)

Substituting (4.13) into (4.12) and making the proper differentiation leads to (4.8). Our model is a generalisation of the Gaussian spot price model. In the case of a one-factor model, using the spot price specification in (4.12)ensures that the spot price model is consistent with the initial forward curve. This is an important generalisation when the model is used for option pricing and hedging, since observed market prices are allowed as model input. In the following all results will be given for the general model in (4.1) of which the spot representation in (4.12) is a special case.

4.3 Average based contracts

We will study two arithmetic based forward contracts in the next subsections: a contract with settlement at maturity, and a contract with continuous settlement. The dynamic properties of these contracts provide us with the tools necessary to value European and Asian options in section 4.4.

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4.3.1 A forward contract with settlement at maturity

Assume that there exists an arithmetic based forward contract that will give the owner $\frac{1}{T_2-T_1}$ unit of electricity each instant during a delivery period $[T_1, T_2]$ where $T_1 < T_2$. Let the price at time t of such a contract be $A(t, T_1, T_2)$ where t is the present time and T_1 and T_2 is the start and end of the delivery period respectively. Let $R(t, T_1, T_2)$ be the value at time t of entering into such a contract. The owner of the contract will receive, at maturity, the difference between the average electricity price during the period $[T_1, T_2]$ and the contract price. Hence, we have the following relationship at time T_2 :

$$R(T_2, T_1, T_2) = \left[\int_{T_1}^{T_2} \frac{f(u, u)}{T_2 - T_1} du - A(T_2, T_1, T_2) \right]$$
(4.14)

Since there is no costs involved in entering into a forward contract, we set $R(t, T_1, T_2) = 0$, and find that the contract price can be expressed as

$$A(t, T_1, T_2) = E_t^{\mathbb{Q}} \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(u, u) du \right]$$

= $\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, u) du$ (4.15)

Hence the value of such a contract equals the average of the forward prices for each instant in the delivery period. Define $\Theta_i(s,a) \equiv \int_s^a \sigma_i(s,u) du$. We show in appendix B that the SDE of (6.25) can be written as

$$dA(t, T_1, T_2) = \begin{cases} \frac{1}{T_2 - T_1} \sum_{i=1}^{K} \left(\Theta_i(t, T_2) - \Theta_i(t, T_1)\right) dW_i(t) & \text{for } t \le T_1 \\ \frac{1}{T_2 - T_1} \sum_{i=1}^{K} \Theta_i(t, T_2) dW_i(t) & \text{for } t > T_1 \end{cases}$$

$$(4.16)$$

The distribution of A at some future time point $T_0 \leq T_2$ conditioning on information at time $t < T_0$ is

$$A(T_0, T_1, T_2) \sim \mathcal{N}(A(t, T_1, T_2), \operatorname{Var}_A(t, T_0, T_1, T_2))$$
 (4.17)

where

$$\operatorname{Var}_{A}(t, T_{0}, T_{1}, T_{2}) = \begin{pmatrix} \sum_{i=1}^{K} \int_{\min(t, T_{1})}^{\min(T_{1}, T_{0})} \left(\Theta_{i}(s, T_{2}) - \Theta_{i}(s, T_{1})\right)^{2} ds \\ + \sum_{i=1}^{K} \int_{\max(t, T_{1})}^{\max(T_{1}, T_{0})} \Theta_{i}(s, T_{2})^{2} ds \end{pmatrix}$$
(4.18)

and $X \sim \mathcal{N}(a, b)$ denotes a random variable X with mean a and variance b.

4.3.2 A forward contract with continuous settlement

Now consider another forward contract which is defined as follows: During a delivery period $[T_1, T_2]$, $T_1 < T_2$, the owner of the contract receives or pays the difference between the price of $\frac{1}{T_2-T_1}$ unit of electricity and the contract price each instant. Let the contract price agreed upon today be $F(t, T_1, T_2)$ where t is the present time and T_1 and T_2 is the start and end of the delivery period respectively and let $t \leq T_1 < T_2$. Denote the value today of entering into such a contract by $V(t, T_1, T_2)$, and assume continuous settlements in the delivery period. Then we have the following relationship at time T_2 :

$$V(T_2, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} e^{-r(u-t)} \left(f(u, u) - F(t, T_1, T_2) \right) du \qquad (4.19)$$

The value at time t of entering such a contract is

$$V(t, T_{1}, T_{2}) = E_{t}^{\mathbb{Q}} \left[\frac{1}{T_{2} - T_{1}} \int_{T_{1}}^{T_{2}} e^{-r(u-t)} \left(f(u, u) - F(t, T_{1}, T_{2}) \right) du \right]$$

$$= E_{t}^{\mathbb{Q}} \frac{1}{T_{2} - T_{1}} \left[\int_{T_{1}}^{T_{2}} e^{-r(u-t)} f(u, u) du \right]$$

$$- \frac{F(t, T_{1}, T_{2})}{T_{2} - T_{1}} \int_{T_{1}}^{T_{2}} e^{-r(u-t)} du$$

$$= \frac{1}{T_{2} - T_{1}} \int_{T_{1}}^{T} e^{-r(u-t)} f(t, u) du$$

$$- \frac{F(t, T_{1}, T_{2})}{T_{2} - T_{1}} \int_{T_{1}}^{T_{2}} e^{-r(u-t)} du$$
(4.20)

Again, since there is no initial cost of entering a forward contract, its value must be zero at time t. Setting $V(t, T_1, T_2) = 0$ and rearranging gives the following expression

$$F(t, T_1, T_2) = \int_{T_1}^{T_2} w(u; r) f(t, u) du$$
(4.21)
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where

$$w(u;r) = \frac{e^{-ru}}{\int_{T_1}^{T_2} e^{-ru} du}$$
(4.22)

Note that setting r = 0 implies $w(u; 0) = \frac{1}{T_2 - T_1}$ and consequently $A(t, T_1, T_2) = F(t, T_1, T_2)$. Define $\Psi_i(s, a) \equiv \int_s^a w(u; r) \sigma_i(s, u) du$. In appendix A it is shown that the SDE of (4.21) is given by

$$dF(t, T_1, T_2) = \begin{cases} \sum_{i=1}^{K} \left(\Psi_i(t, T_2) - \Psi_i(t, T_1) \right) dW_i(t) & \text{for } t \le T_1 \\ \sum_{i=1}^{K} \Psi_i(t, T_2) dW_i(t) & \text{for } t > T_1 \end{cases}$$
(4.23)

with initial condition $F(0, T_1, T_2)$ given in (4.21). The distribution of F at some future time point $T_0 \leq T_2$ condition on information on time $t < T_0$ is

$$F(T_0, T_1, T_2) \sim \mathcal{N}(F(t, T_1, T_2), \operatorname{Var}_G(t, T_0, T_1, T_2))$$
 (4.24)

with

$$\operatorname{Var}_{F}(t, T_{0}, T_{1}, T_{2}) = \begin{pmatrix} \sum_{i=1}^{K} \int_{\min(t, T_{1})}^{\min(T_{1}, T_{0})} \left(\Psi_{i}(s, T_{2}) - \Psi_{i}(s, T_{1}) \right)^{2} ds \\ + \sum_{i=1}^{K} \int_{\max(t, T_{1})}^{\max(T_{1}, T_{0})} \Psi_{i}(s, T_{2})^{2} ds \end{pmatrix}$$
(4.25)

Note that since the contracts described in (6.25) and (4.21) only differs in the way the contracts are settled, the net cash flow from the contracts are equal. This also means that if we have zero interest rates, the contract prices are the same. It is easy to see that (4.21) collapses to (6.25) when r = 0.

4.4 Option pricing and hedging

We want to find the values of a European style options. We consider both European forward options and Asian spot price options.

4.4.1 European forward options

Consider a European call option, FC, with maturity $T_0 \leq T_1$ written on a forward price with continuous settlements in the delivery period (denoted $F(\bullet)$ in the section above). At time T_0 the owner of the call receives the

positive difference, if any, between the value of the forward price and the option strike price, K. The value is given by

$$FC(t) = e^{-r(T_0 - t)} E_t^{\mathbb{Q}} \left[\left(F(T_0, T_1, T_2) - K \right)^+ \right]$$
(4.26)

This expectation can be calculated explicitly as

$$FC(t) = e^{-r(T_0-t)} \sqrt{\frac{\operatorname{Var}_F(t, T_0, T_1, T_2)}{2\pi}} e^{-\frac{1}{2}(d_F)^2} + e^{-r(T_0-t)} \left(F(t, T_1, T_2) - K\right) \Phi(d_F)$$
(4.27)

where

$$d_F = \frac{F(t, T_1, T_2) - K}{\sqrt{\operatorname{Var}_F(t, T_0, T_1, T_2)}}$$
(4.28)

 $\Phi(\bullet)$ is cumulative distribution function of a standard normally distributed random variable and $\operatorname{Var}_F(t, T_1, T_0)$ is the conditional variance of the forward contract. Proof of the formula is given in appendix C.

The corresponding European forward put option, FP(t), can be found from the well known put-call parity

$$FP(t) - FC(t) = e^{-r(T_0 - t)} \left(K - F(t, T_1, T_2) \right)$$
(4.29)

Noting the property $\Phi(a) = 1 - \Phi(-a)$ by the symmetry of the normal distribution, we can write the valuation expression for the European put option explicitly as:

$$FP(t) = e^{-r(T_0-t)} \left(K - F(t, T_1, T_2)\right) \Phi(-d_F) + e^{-r(T_0-t)} \sqrt{\frac{\operatorname{Var}_F(t, T_0, T_1, T_2)}{2\pi}} e^{-\frac{1}{2}(d_F)^2}$$
(4.30)

with d_F given above.

4.4.2 Asian spot price options

Now consider an Asian spot price option. An Asian call option, AC, delivers at T_0 the owner of the option the positive difference, if any, between the

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realised average spot price during a time period $[T_1, T_2]$ where $T_1 < T_2 = T_0$, and the fixed strike price K. It is obvious that the Asian option is identical to a European forward option written on an arithmetic based forward contract with delivery period $[T_1, T_2]$ and settlement at maturity (denoted $A(\bullet)$ in the section above) where the maturity of the option is equal to the maturity of the forward contract $(T_0 = T_2)$. The value is given by

$$AC(t) = e^{-r(T_2-t)} E_t^{\mathbb{Q}} \left[\left(\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(u, u) du - K \right)^+ \right]$$

= $e^{-r(T_2-t)} E_t^{\mathbb{Q}} \left[(A(T_2, T_1, T_2) - K)^+ \right]$

The conditional expectation can be computed explicitly as

$$AC(t) = e^{-r(T_2-t)} \sqrt{\frac{\operatorname{Var}_A(t, T_0, T_1, T_2)}{2\pi}} e^{-\frac{1}{2}(d_A)^2} + e^{-r(T_2-t)} \left(A(t, T_1, T_2) - K\right) \Phi(d_A)$$
(4.31)

where

$$d_A = \frac{A(t, T_1, T_2) - K}{\sqrt{\operatorname{Var}_A(t, T_0, T_1, T_2)}}$$
(4.32)

and $\operatorname{Var}_A(t, T_0, T_1, T_2)$ is given in (4.18). Proof of the formula is given in appendix C. The corresponding Asian put option, AP(t), can again be found from the put-call parity

$$AP(t) - AC(t) = e^{-r(T_2 - t)} \left(K - A(t, T_1, T_2) \right)$$
(4.33)

or explicitly as

$$AP(t) = e^{-r(T_2-t)} \left(K - A(t, T_1, T_2)\right) \Phi(-d_A) + e^{-r(T_2-t)} \sqrt{\frac{\operatorname{Var}_A(t, T_0, T_1, T_2)}{2\pi}} e^{-\frac{1}{2}(d_A)^2}$$
(4.34)



Table 4.1: Comparative statistics for futures options. The statistics are partial derivatives of the closed form expressions for European forward call and put options given in (4.27) and (4.30) respectively.

4.4.3 Hedging a single forward option

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In table 4.1 we have computed comparative statistics for European put and call options. These expressions apply to Asian options as well by replacing F with A and d_F with d_A everywhere. We see that the derivative of the call with respect to the underlying forward contract, usually termed the delta of the option, is always positive. The logic behind this result is straightforward. An increase in the underlying asset price will increase the probability of a positive terminal payoff, resulting in a higher option value. The reverse argument explains the negative delta of a put. The delta of the option tells us the number of units of the underlying forward contract to hold if we want the value of the forward position to change by the same amount as an option, when the underlying forward contract change by a little amount.

4.4.4 Hedging a portfolio of contingent claims

In the case of a single forward option, we can use standard delta-hedging techniques. If we want to hedge a portfolio of derivatives, and there are several types of derivatives and several maturities involved, delta-hedging each exposure becomes impractical. Alternatively we can immunise the portfolio against possible changes in the forward curve caused by each of the K factors. This is called factor hedging and proceeds in two steps: First, compute the changes in portfolio value when the forward curve is "shocked" by each source of uncertainty separately. Next, find positions in some hedging instruments that exactly offsets these portfolio changes.

We can represent discrete shocks to forward contract for each of the volatility functions as

$$\Delta f_i(t,T) = \sigma_i(t,T) \Delta W_i \qquad i = 1, ..., K \tag{4.35}$$

where the size of the shock ΔW_i depends on the hedging period. Now let

P(f(t,T)) denote the value of the portfolio P of some contingent claims as a function of the forward price. Compute the changes in this portfolio when the forward curve shifts up or down. For each factor we have

$$\Delta P_{i} = P(f_{i,u}(t,T)) - P(f_{i,d}(t,T)) \qquad i = 1, ..., K$$
(4.36)

where ΔP_i is the change in the portfolio value and $f_{i,u}(t,T)$ and $f_{i,d}(t,T)$ are the forward curve after a positive and negative shock respectively. In theory, K hedging instruments are needed to hedge this portfolio. Suppose we in addition to our portfolio P want to take positions in K different forward contracts $(\delta_1 \Delta f_1(t,T_1) + \delta_2 \Delta f_1(t,T_2) + ... + \delta_K \Delta f_i(t,T_K))$ in such a way that forward curve changes are immunised. The problem consists of determining the positions in the forward contracts (the $\delta'_i s$). If we denote the hedged portfolio H, our job is to solve the following set of equations

$$\Delta H_1 = \Delta P_1 + \delta_1 \Delta f_1(t, T_1) + \delta_2 \Delta f_2(t, T_2) + \dots + \delta_K \Delta f_K(t, T_K) = 0 \Delta H_2 = \Delta P_2 + \delta_1 \Delta f_2(t, T_1) + \delta_2 \Delta f_2(t, T_2) + \dots + \delta_K \Delta f_K(t, T_K) = 0 \dots \Delta H_K = \Delta P_K + \delta_1 \Delta f_K(t, T_1) + \delta_2 \Delta f_K(t, T_2) + \dots + \delta_K \Delta f_K(t, T_K) = 0 (4.37)$$

This gives us K linear equations in K unknowns which can easily be solved.

4.5 Application to the Nordic electricity market

In this section we illustrate an implementation of our model in the Nordic electricity market. We only provide some simple illustrations. For a detailed analysis of the forward curve dynamics in this market, see Koekebakker and Ollmar (2001).

The Nordic electricity market consists of Norway, Sweden, Denmark and Finland. The power exchange is called Nord Pool. It is the world's first multinational commodity exchange for electric power. Financial contracts traded on Nord Pool are written on the arithmetic average of the spot price (called the system price) during a delivery period. The period may be a given week, season or year. During the delivery period, the difference between the contract price and the spot price is paid/received by the holder of the contract. This corresponds approximately to the specification of the average based forward contract with continuous settlement during the delivery period as discussed in section 4.2.

4.5.1 The forward price function

Instead of working directly with the different financial contracts with various delivery periods, we can compute a continuous forward price function from the contract prices observed in the market place. A forward price function consistent with observed market prices should be able to reproduce market prices using (4.21). In Koekebakker and Ollmar (2001) the daily smoothed forward price functions were computed using software with the name ELVIZ developed by Viz Risk Management Services AS. Yearly cyclical price differences are accounted for by a sinusoidal prior function. The methodology underlying the forward market representation in ELVIZ is called *maximum smoothness.*⁴ An example of a forward price function is provided in figure 4.1. The yearly cycle, with high winter and low summer prices, is evident from the plot.

4.5.2 One- and two-factor models

Feeding the computer with observed market prices on power contracts, we can compute such forward price functions each day. We compute daily forward price functions from January 1, 1999 to March 15, 2001 following Koekebakker and Ollmar (2001). Each day we observe the forward price for the following week, the week after, and so on for next two years; a total of 104 weekly prices collected from the forward price function each day. Hence we obtain a term structure of 104 prices each day during our sample period. Now we can compute price differences for all our 104 fixed maturities. We now have the data to take a closer look at the volatility functions. In figure 4.2 we have plotted the annualised historical volatility term structure of during the period 1999-2001.

This is simply the standard deviation of each of the 104 columns in our data matrix of forward price differences. The standard deviations are transformed into annual volatility by multiplying with the square root of 250 (average number of trading days). Note that volatility falls rapidly in this

⁴For a comprehensive description of the maximum smoothness approach see Adams and van Deventer (1994), Bjerksund and Stensland (1996) and Forsgren (1998). For more information of the ELVIZ software, see www.viz.no.



Figure 4.1: The forward price function on January 12, 2001.

market, and then it stabilises for contracts with maturity further into the future than 13 weeks.

A theoretical factor model should be able the reproduce this volatility term structure. In figure 4.2 we have also given two examples the negative exponential one-factor model, $\sigma_1(t,T) = \sigma e^{-\kappa(T-t)}$, suggested by Lucia and Schwartz (2001). The graph Volatility 1 appears from the parameters $\kappa =$ 0.38 and $\sigma = 37.49$. The parameters result from a mean square error fit between historical and model volatility of all 104 maturities. Note that the sharp increase of volatility in the short end is completely ignored. Focusing instead on the short end, the parameters $\kappa = 4.21$ and $\sigma = 71.56$ produce Volatility 2. The good fit in the short end is accomplished by raising the value of κ . This implies strong mean reversion in the underlying spot price model, and as a consequence, sharply falling volatility as time to maturity increases. An implication of Volatility 2 is that long term forward prices should hardly move at all. This is not in accordance with historical data.

The negative exponential one-factor model is not able to fit both the short and long end of the volatility term structure. To cope with the sharply falling volatility Bjerksund et al. (2000) suggested a different functional form



Figure 4.2: Fitting a negative exponential one-factor model. The solid line is the annualised volatility of forward price differences along a 104 weeks term structure during the period 1999 - 2001. Volatility 1 and 2 are the best fit (in mean square error sense) of the negative exponential one-factor model using data from all 104 weeks and the first 15 weeks, respectively.



Figure 4.3: Fitting the BSR one-factor model. The solid line is the annualised volatility of forward price differences along a 104 weeks term structure during the period 1999 - 2001. Vol BSR is the best fit (in mean square error sense) of the one-factor model suggested by Bjerksund et al. (2000).

of the volatility functions. A one-factor version of their model is: $\sigma_1(t,T) = \frac{a}{T-t+b} + c$. We will term this the BSR-model.⁵ A mean square error fit of the whole term structure resulted in the following parameter values: a = 2.62, b = 0.027 and c = 21.02. The fitted model together with historical volatilities are given in figure 4.3. It is evident that the BSR model gives a remarkable good fit both in the short and the long end of the term structure. However, a direct comparison to the negative exponential model is hardly fair, since a three-parameter model allows more flexibility than a two-factor model.

A one factor model illustrated in figure 4.3 shifts all forward prices in the same direction, hence a price increase in the short end dictates price increase in the long end as well. A two-factor model may provide more realistic forward price dynamics. In figure 4.4 the solid lines represent the

⁵Note that Bjerksund et al. (2000) modelled the forward price as geometric Brownian motion, hence in their study $\sigma_1(t,T) = \frac{a}{T-t+b} + c$ represent the volatility term structure of forward price returns, not forward price differences as in this study.



Figure 4.4: Historic and fitted volatility in a two-factor model. The solid lines are historical factors from principal component analysis of forward price differences during the period 1999 - 2001. FIT F1 and FIT F2 are functions of a two-factor model fitted using mean square error.

two most important factors in the 1999-2001 period computed by principal component analysis.⁶ Factor one is most important, and it is positive for all maturities. A shock to this factor (called the shifting factor) moves all the forward prices in one direction. The second factor moves forward prices with short and long maturities in opposite directions. This factor is called the twisting factor. Inspired by nice fit of the one-factor model, we suggest the following two-factor extension of the BSR model $\sigma_1(t,T) = \frac{a_1}{T-t+b_1} + c_1$ and $\sigma_2(t,T) = \frac{a_2}{T-t+b_2} + c_2$. Mean square error fits between theoretical and historical factors resulted in the following parameter values; $a_1 = 4.29$, $b_1 = 0.052$, $c_1 = 10.54$, and $a_2 = -1.00$, $b_2 = 0.01$, $c_2 = 10.15$. We see from

figure 4.4 that the theoretical volatility factors closely mimics the empirical counterparts.

⁶See Koekebakker and Ollmar (2001) for a description of the principal component analysis applied to electricity forward price data.



Figure 4.5: Volatility time-dependence for an average based forward contract. The graph shows volatility of the arithmetic average based forward contract with settlement at maturity. The specific model used is: $\sigma(t,T) = \frac{a}{T-t+b} + c$ with the following parameter values a = 2.62, b = 0.027, c = 21.02, $T_1 = 1$ and $T_2 = 2$.

4.5.3 Volatility of an average based contract

The forward price function gives us the price today for the delivery of one unit of electricity at time T in the future. We also see from figure 4.2 and 4.3 that the volatility increases sharply when such a (theoretical) contract approaches maturity. The power contract traded in the market place are based on the average electricity price during a delivery period. The term structure of volatility for an average based contract is quite different from those derived from the forward price function. Consider the BSR-model with the same parameters as in figure 4.3. Furthermore, set $T_1 = 1$ and $T_1 = 2$, hence we are considering a forward contract with one year delivery period starting one year from now. For simplicity, we assume that the forward contract has settlement at maturity. Then we can use the $\Theta_i(\bullet)$ -functions in (4.16) to compute the term structure of volatility for such a contract. This is illustrated in figure 4.5. We see that the volatility term structure is rather



Figure 4.6: Delta profile of European forward call options. The graph shows the delta profile of three different call options written on arithmetic average based forward contracts with continuous settlement during the delivery period. The volatility of the forward price function is given by a two-factor BSR model (see text for parameter values). In all three cases considered the strike price is (K = 160), the maturities of the calls concide with the start of the delivery periods $(T_0 = T_1)$ and the delivery period is set to one year $(T_2 = T_1 + 1)$ for all contracts. Maturities of the options are one month $(T_0 = 1/12)$, six months $(T_0 = 6/12)$ and a year $(T_0 = 1)$.

flat at first. When approaching the delivery period, volatility rises and peaks just when the contract is entering the delivery period. Volatility then rapidly decreases and collapses at T_2 . This is intuitive, as more and more information concerning the average price in the $[T_1, T_2]$ time interval becomes known as $t > T_1$.

4.5.4 Delta hedging a forward option

The idea of delta hedging an option involves dynamic trading a position in the underlying power contract in a way so that a change in the option price during a small time interval is offset by an identical change in value of the underlying contract in opposite direction. We illustrate delta hedging of call options

written on three different forward contracts with continuous settlement in the delivery period. A two-factor BSR model gives the volatility of the forward price function. The parameters values are $a_1 = 4.29$, $b_1 = 0.052$, $c_1 = 10.54$, and $a_2 = -1.00$, $b_2 = 0.01$, $c_2 = 10.15$. All contracts are specified with delivery periods of one year, and option maturities coincide with the start of the delivery periods $(T_0 = T_1)$. In all three cases considered the strike price is K = 160, and the maturities of the options are one month, six months and one year. We can now use the analytical expression in table 4.1 to calculate the delta for different forward prices. The delta profile for the three options are plotted in figure 4.6. When the forward price is low relative to the strike, the delta (the position in the underlying asset) is low, reflecting the low probability for the option to end in the money. An at-the-money option has a delta of about 0.5. The delta approaches one for very high forward prices, reflecting the high probability of ending in the money. We also note that the delta profile is steeper the closer the maturity of the option. The reason for this is that the probability of the option ending in-the-money becomes more sensitive to small changes in the forward price, when this is close to the strike price.

4.5.5 Factor hedging a portfolio of options

In this section we will illustrate the factor hedging approach. Consider a portfolio of two short forward call options. The first option has strike price 130. The call option, denoted $FC_{(K=130)}$, is written on a forward contract with one year delivery period starting one month from now $(T_1 = 1/12 \text{ and }$ $T_2 = 1/12 + 1$). We denote this contract $F\left(0, \frac{1}{12}, 1\frac{1}{12}\right)$. The maturity of the option is equal to the start of the delivery period for the forward contract $(T_0 = T_1)$. The second call option has strike price 140. This call option, denoted $FC_{(K=140)}$, is written on a forward contract with one year delivery period starting one year from now (F(0,1,2)). Both forward contracts are continuously settled during the delivery period. The dynamics describing the evolution of the price processes for these contracts are given in (4.23). We calculate the forward prices of the specified contracts from the forward price function calibrated on 12. January 2001 (see figure 4.1, and we get $F\left(0,\frac{1}{12},1\frac{1}{12}\right) = 129.36$ and F(0,1,2) = 139.55 hence both call options are slightly out of the money. Furthermore, we assume that the forward price function can adequately be described by a two-factor BSR-model with parameter values $a_1 = 4.29$, $b_1 = 0.052$, $c_1 = 10.54$, and $a_2 = -1.00$, $b_2 = 0.01$,



Figure 4.7: Shifts in a two-factor forward curve BSR-model. The parameters in the model are $a_1 = 4.29$, $b_1 = 0.052$, $c_1 = 10.54$ and $a_2 = -1.00$, $b_2 = 0.01$, $c_2 = 10.15$. The curve is given a positive/negative one standard deviation shock of each factor for a one-week hedging period.

 $c_2 = 10.15$. In a real world situation, the market would deliver the call prices, but in this application we simply compute the options premium setting r = 0.07, and using (4.27) inserting the forward prices and the volatility parameters from the two-factor BSR-model given above. The premiums are $FC_{(K=130)} = 2.35$ and $FC_{(K=140)} = 7.03$.

Our task now is to immunise a short position in each of the call options using factor hedging. Hence our unhedged portfolio, P, is given by $P = -FC_{(K=130)} - FC_{(K=140)} = -9.38$. Since our model consists of two factors we need to determine hedging positions in two different underlying forward contracts. We use the two average based forward contracts described above as hedging instruments. Thus our hedged portfolio is given by $H = P + \delta_1 F\left(0, \frac{1}{12}, 1\frac{1}{12}\right) + \delta_2 F(0, 1, 2)$. We want to determine δ_1 and δ_2 in such a way that the hedged portfolio value stays the same, when the forward curve is "shocked" by each source of uncertainty separately. Hence we want $\Delta H = 0$. The size of the shock (ΔW_i) is chosen to give a typical movement of the curve over the hedging period. We have chosen one week hedging period

	Today	$\overline{F1_U}$	$F1_D$	$\overline{F2}_U$	$F2_D$	
$F\left(0,\frac{1}{12},1\frac{1}{12}\right)$	129.36	132.10	126.62	130.43	128.29	
F(0, 1, 2)	139.55	141.40	137.70	140.87	138.23	
$FC_{(K=130)}$	2.35	3.84	1.31	2.88	1.90	
$FC_{(K=140)}$	7.03	7.91	6.21	7.65	6.44	
						Weights
ΔP		-4.22		-2.19		
$\Delta F\left(\frac{1}{12}, \frac{1}{12}, 1\frac{1}{12}\right)$		5.48		2.14		0.46
$\Delta F(1,1,2)$		3.71		2.63		0.46
ΔH		0.00		0.00		

Table 4.2: Factor hedging a portfolio of options. This table illustrates factor hedging in a two-factor model. The model specification is given in the caption of figure 4.7. The unhedged portfolio consists of two short forward call option positions $FC_{(K=130)}$ and $FC_{(K=140)}$ (strike prices in subscript) written on the contracts $F(0, \frac{1}{12}, 1\frac{1}{12})$ and F(0, 1, 2). The options mature as the forwards enter their respective delivery periods. In the upper columns, the prices today of the derivatives are given along with the prices in case of one standard deviations shocks during one week hedging period. Changes in the unhedged portfolio (ΔP) and the forward contracts ($\Delta F(.)$) are calculated using (4.36) for a shock to either factor 1 (F1) or 2 (F2). The hedging positions that immunise the hedged portfolio (ΔH) are calculated from (4.37) and given in the column termed "weights".

 $(\Delta t = 1/52)$ and one standard deviation shifts in each factor. Figure 4.7 illustrates the shifts in the forward price function, by simulating positive and negative shocks to each of the two volatility functions in our two-factor BSR model.

Now we can calculate the change in value of the options and forward contracts conditional on shocks in factor 1 and 2. The values of the derivative securities are given in the upper rows of table 4.2. Next we solve for δ_1 and δ_2 using (4.36) and (4.37). The numerical values are given in the lower part of table 4.2. We see that long positions of 0.46 units in each average based forward contract offsets the risk in our portfolio. Note that the advantages of factor hedging are greater the bigger the portfolio is. In theory we only need two underlying assets in a two factor model to completely offset the changes in value of a portfolio of any number of different contingent claims. Of course, this again depends crucially on correct model specification.

4.6 Summary and conclusions

In this paper we have developed a multi-factor term structure model for the electricity forward curve. This modelling approach has the advantage of being consistent with power contracts observed in the market place. Previously Clewlow and Strickland (1999a) and Bjerksund et al. (2000) have modelled the electricity forward price as multi-factor geometric Brownian motion. These authors provide a link between electricity contingent claims and the pricing of derivatives in mainstream finance, since the assumption of lognormal asset prices is most widely used. One clear advantage of this approach is that valuation methodologies developed previously for commodities and financial assets can be applied in the electricity market as well. This paper represents an alternative approach. We suggest a multi-factor arithmetic Brownian motion for the forward price dynamics of electricity. Our approach can be seen as a generalisation of several electricity spot price models suggested by Lucia and Schwartz (2000) and Knittel and Roberts (2001).

An electricity contingent claim will typically dependent on the arithmetic average price during a time interval. In our set up, which implies normally distributed spot prices, the arithmetic average spot price is normally distributed. Hence, closed form solutions to average based contingent claims are available. This is the most salient feature of our proposed model. We investigate several contingent claims. First we define two different average based financial forward contracts differing only in the way they are settled. The specification of the first contract is such that the owner will receive, at maturity, the difference between the average electricity price during the delivery period and the contract price. The second contract specification assumes continuous settlement during the delivery period. Both contracts are normally distributed, and the appropriate dynamic representations (in the form of stochastic differential equations) are derived. The specification of the latter contract is equal to those traded at the Nordic power exchange. Closed form solution to both European forward options and Asian spot price options are derived. Both option types are traded in the Nordic market. In the last part of the paper we discuss different volatility specifications and hedging issues using data from this market.

This research can be extended in both empirical and theoretical directions. Here we briefly sketch some possibilities. Our model rest upon an assumption of arithmetic based forward prices. This again implies normally distributed spot prices. Whether or not this is a reasonable stochastic specification is an empirical issue that is yet to be resolved. One nice feature of the closed form representations of the average based forward contracts provided in this paper is that it makes possible a more direct empirical testing of different volatility models. The conditional distribution of the average based forward contracts can be utilised in maximum likelihood estimation of the volatility dynamics using market prices of traded power contracts in stead of smoothed prices. One possible theoretical extension of the model would be to include a jump process to the forward price dynamics. Several authors have argued that electricity prices exhibit jumps (see e.g. Clewlow and Strickland (1999b) and Deng (2000)). Whether or not closed form solutions are available in an arithmetic forward curve jump-diffusion model is left for future research.

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4.7 Dynamic properties of spot prices in an arithmetic forward price model

In this appendix we will give show that an arithmetic forward price model in (4.2) implies a spot price process given in (4.5)-(4.6). A similar proof in a more general forward price model is given in Musial and Rutkowski (1997). First recall that the spot price can be written as

$$S(t) = f(t,t) = f(0,t) + \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(u,t) dW_{i}(u)$$
(4.38)

Applying the stochastic Fubini theorem to the Itô integral, we obtain

$$\sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(u,t) dW_{i}(u) = \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(u,u) dW_{i}(u) + \sum_{i=1}^{K} \int_{0}^{t} (\sigma_{i}(u,t) - \sigma_{i}(u,u)) dW_{i}(u) \quad (4.39)$$

$$= \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(u,u) dW_{i}(u) + \sum_{i=1}^{K} \int_{0}^{t} \int_{u}^{t} \frac{\partial \sigma_{i}(u,s)}{\partial t} ds dW_{i}(u) \quad (4.40)$$

$$= \sum_{i=1}^{K} \int_{0}^{t} \sigma_{i}(u,u) dW_{i}(u) + \dots \quad (4.41)$$

$$= \sum_{i=1}^{K} \int_{0}^{\infty} \sigma_{i}(u, u) dW_{i}(u) + \qquad (4.41)$$

$$\sum_{i=1}^{K} \int_{0}^{t} \int_{0}^{s} \frac{\partial \sigma_{i}(u,s)}{\partial t} dW_{i}(u) ds \qquad (4.42)$$

Finally we have

$$f(0,t) = S(0) + \int_0^t \frac{\partial f(0,u)}{\partial t} du$$
(4.43)

Combining (4.38), (4.41) and (4.43) gives (4.5)-(4.6).

4.8 Appendix B: The dynamic properties of average based forward contracts

In this appendix we derive the expressions (4.16) and (4.23). First consider

the average based forward contract with settlement at maturity. We remember the following relationship $A(T_2, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(u, u) du$ from (4.14). Applying the stochastic Fubini theorem to the Itô integral, we can write $A(T_2, T_1, T_2)$ as a stochastic process conditional on information on time $t < T_2$ as

$$\begin{split} A(T_2,T_1,T_2) &= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(u,u) du \\ &= \frac{1}{T_2 - T_1} \left(\int_{T_1}^{T_2} f(t,u) du + \int_{T_1}^{T_2} \sum_{i=1}^{K} \int_{t}^{u} \sigma_i(s,u) dW_i(s) du \right) \\ &= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t,u) du \\ &+ \frac{1}{T_2 - T_1} \sum_{i=1}^{K} \int_{\max(t,T_1)}^{T_2} \int_{s}^{T_2} \sigma_i(s,u) du dW_i(s) \\ &- \frac{1}{T_2 - T_1} \sum_{i=1}^{K} \int_{\min(t,T_1)}^{T_1} \int_{s}^{T_1} \sigma_i(s,u) du dW_i(s) \\ &= A(t,T_1,T_2) + \frac{1}{T_2 - T_1} \sum_{i=1}^{K} \int_{\max(t,T_1)}^{T_2} \Theta_i(s,T_2) dW_i(s) \\ &- \frac{1}{T_2 - T_1} \sum_{i=1}^{K} \int_{\min(t,T_1)}^{T_1} \Theta_i(s,T_1) dW_i(s) \end{split}$$
(4.44)

where $\Theta_i(s, a) \equiv \int_s^a \sigma_i(s, u) du$. Note that the last term disappears when $t \geq T_1$. The differential form of (4.44) is given in (4.16).

Now consider the average based forward contract with continuous settlement in the delivery period $[T_1, T_2]$. We have the following relationship from (4.19) $F(T_2, T_1, T_2) = \int_{T_1}^{T_2} w(u; r) f(u, u) du$ where $w(u; r) = \frac{e^{-ru}}{\int_{T_1}^{T_2} e^{-ru} du}$. Applying the stochastic Fubini theorem to the Itô integral, we can write $F\left(T_{2},T_{1},T_{2}\right)$ as a stochastic process conditional on information on time t $< T_{2}$ as

$$\begin{split} F\left(T_{2},T_{1},T_{2}\right) &= \frac{\int_{T_{1}}^{T_{2}} e^{-r(u-t)}f(u,u)du}{\int_{T_{1}}^{T_{2}} e^{-r(u-t)}du} \\ &= \frac{\int_{T_{1}}^{T_{2}} e^{-r(u-t)}f(t,u)du + \int_{T_{1}}^{T_{2}} \sum_{i=1}^{K} \int_{t}^{u} e^{-r(u-t)}\sigma_{i}(s,u)dW_{i}(s)du}{\int_{T_{1}}^{T_{2}} e^{-r(u-t)}du} \\ &= \frac{\int_{T_{1}}^{T_{2}} e^{-ru}f(t,u)du}{\int_{T_{1}}^{T_{2}} e^{-r(u-t)}du} + \\ &= \frac{\sum_{i=1}^{K} \int_{\max(t,T_{1})}^{T_{2}} e^{-r(u-t)}\sigma_{i}(s,u)dudW_{i}(s)}{\int_{T_{1}}^{T_{2}} e^{-r(u-t)}du} \\ &- \frac{\sum_{i=1}^{K} \int_{\min(t,T_{1})}^{T_{1}} \int_{s}^{T_{1}} e^{-r(u-t)}\sigma_{i}(s,u)dudW_{i}(s)}{\int_{T_{1}}^{T_{2}} e^{-r(u-t)}du} \\ &= F\left(t,T_{1},T_{2}\right) + \sum_{i=1}^{K} \int_{\max(t,T_{1})}^{T_{2}} \Psi_{i}(s,T_{2})dW_{i}(s) \\ &- \sum_{i=1}^{K} \int_{\min(t,T_{1})}^{T_{1}} \Psi_{i}(s,T_{1})dW_{i}(s) \end{split}$$
(4.45)

where $\Psi_i(s,a) \equiv \int_s^a w(u;r)\sigma_i(s,u)du$. Note that the last term disappears when $t \geq T_1$. The differential form of (4.45) is given in (4.23).

4.9 Appendix C: European-style call option

This appendix provides the general European-style call option pricing formula when the underlying asset is normally distributed. This a modified version of the very first option pricing formula developed by Bachelier (1900). The formulas in (4.27) and (4.31) are both special cases.

Proposition 1 The value of a European style call option, C(t), where the underlying asset is Gaussian with mean, μ and variance v, under the equivalent martingale measure, is

$$C(t) = e^{-r(T-t)} \sqrt{\frac{v}{2\pi}} e^{-\frac{1}{2}(d)^2} + e^{-r(T-t)} \left(\mu - K\right) \Phi(d)$$

where

$$d = \frac{\mu - K}{\sqrt{v}}$$

and Φ is the cumulative standard normal distribution function.

Proof. Consider the Gaussian variable, X with conditional mean μ and variance σ^2 . The distribution on time T given the information at time t can be represented as

$$X_T | \mathcal{F}_t = \mu + \sqrt{v} Z$$

where Z is standard normal random variable. Define the pay-off of a European style call option by

$$C(T) = (X(T) - K)^{+} = X_{T}1_{D} - K1_{D}$$

where 1 is an indicator function and

$$D = \{X_T > K\}$$

is the exercise set. Furthermore, assume a constant risk free interest rate. The standard risk neutral valuation formula gives us

$$C_{t} = E_{t}^{\mathbb{Q}} \left[e^{-r(T-t)} \left(X_{T} - K \right)^{+} \right] = e^{-r(T-t)} E_{t}^{\mathbb{Q}} \left[X_{T} \mathbf{1}_{D} \right] - e^{-r(T-t)} E_{t}^{\mathbb{Q}} \left[K \mathbf{1}_{D} \right]$$

Taking the latter expectation is straight forward:

$$e^{-r(T-t)}E_t^{\mathbb{Q}}[K1_D] = e^{-r(T-t)}K\mathbb{Q}[D]$$

= $e^{-r(T-t)}K\mathbb{Q}[X_T > K]$
= $e^{-r(T-t)}K\mathbb{Q}[\mu + \sqrt{vZ} > K]$
= $e^{-r(T-t)}K\mathbb{Q}\left[-Z < \frac{\mu - K}{\sqrt{v}}\right]$
= $e^{-r(T-t)}K\Phi(d)$

where Φ denotes the standard cumulative normal distribution and $d = \frac{\mu - K}{\sqrt{v}}$. Solving the expectation $E_t^{\mathbb{Q}}[X_T \mathbf{1}_D]$ requires explicit integration of the normal density:

$$e^{-r(T-t)} E_t^{\mathbb{Q}} [X_T 1_D] = e^{-r(T-t)} E_t^{\mathbb{Q}} [X_T 1_D]$$

= $e^{-r(T-t)} \int_{X_T = K}^{\infty} X_T dX_T$
= $e^{-r(T-t)} \int_{X_T = K}^{\infty} (\mu + \sqrt{v}Z) df(Z)$

Now, change the variables to express the result as the integral of a $\mathcal{N}(0,1)$ rather than the expectation of a function of a $\mathcal{N}(0,1)$,

$$e^{-r(T-t)}E_t^{\mathbb{Q}}[X_T \mathbf{1}_D] = e^{-r(T-t)} \int_{X_T=K}^{\infty} \left(\mu + \sqrt{v}Z\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ$$
$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{X_T=K}^{\infty} \mu e^{-\frac{1}{2}Z^2} dZ + e^{-r(T-t)} \sqrt{\frac{v}{2\pi}} \int_{X_T=K}^{\infty} Z e^{-\frac{1}{2}Z^2} dZ$$

The lower bound $X_T = K$ in terms of Z is

$$X_T = K = \mu + \sqrt{v}Z$$
$$Z = \frac{K - \mu}{\sqrt{v}}$$

so that

$$e^{-r(T-t)} E_t^{\mathbb{Q}} [X_T 1_D] = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{\frac{K-\mu}{\sqrt{v}}}^{\infty} \mu e^{-\frac{1}{2}Z^2} dZ + e^{-r(T-t)} \sqrt{\frac{v}{2\pi}} \int_{\frac{K-\mu}{\sqrt{v}}}^{\infty} Z e^{-\frac{1}{2}Z^2} dZ$$

Finally we use the symmetry

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-\frac{1}{2}Z^{2}} dZ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} e^{-\frac{1}{2}Z^{2}} dZ = \Phi(-a)$$

to get

$$e^{-r(T-t)}E_t^{\mathbb{Q}}\left[X_T \mathbf{1}_D\right] = e^{-r(T-t)}\mu\Phi\left(d\right) + e^{-r(T-t)}\sqrt{\frac{v}{2\pi}}e^{-\frac{1}{2}(d)^2}$$

 \mathbf{where}

.

$$d=\frac{\mu-K}{\sqrt{v}}$$

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Putting all together yields the general formula

$$C_t = e^{-r(T-t)} \mu \Phi(d) + e^{-r(T-t)} \sqrt{\frac{v}{2\pi}} e^{-\frac{1}{2}(d)^2} - e^{-r(T-t)} K \Phi(d)$$

= $e^{-r(T-t)} \sqrt{\frac{v}{2\pi}} e^{-\frac{1}{2}(d)^2} + e^{-r(T-t)} (\mu - K) \Phi(d)$

which completes the proof.

BIBLIOGRAPHY

Chapter 5

Approximate Asian option pricing in the Black '76 framework

ABSTRACT - In this paper we present an approximate lognormal valuation model for Asian average rate options when the underlying asset is geometric Brownian motion. Our model is a modification of the Black (1976) formula. We analyse a futures contract written on the arithmetic average of the underlying asset. This contract is lognormally distributed prior to the average period, and approximately lognormally distributed inside the averaging period. We propose a new Asian option pricing formula based on our analysis. In a Monte Carlo study, we find that our formula is more accurate than the lognormal approximation proposed by Levy (1992).

5.1 Introduction

The purpose of this paper is to present a new approximate lognormal valuation model for Asian options. An Asian option is a path dependent contingent claim, with payoff based on an average of some underlying variable. Average based options have become popular in several markets for different reasons. Asian options are often used in thinly traded commodity markets to avoid problems with price manipulation of the underlying asset near or at maturity. For this purpose, the average period will typically be rather

short, and the main trading period will occur before the averaging period (so-called forward starting Asian option). Asian options are natural hedging instrument for domestic firms with continuous sales and payments in a foreign currency. In some commodity markets, the nature of the commodity naturally promotes average based contracts. For example in electricity or natural gas markets limited possibility of storage leads to continuous purchases for energy consumers, and Asian options are natural instruments for risk management purposes (see Levy (1997) for several other examples).

As shown by Kemna and Vorst (1990), an Asian option with geometric averaging has a closed form solution in a standard geometric Brownian asset-pricing framework. Arithmetic average options are the most commonly traded ones, but they are also the more difficult to value. Exact closed form option formulas for these options do not exist, since the distribution of the arithmetic average of a lognormal process is unknown. This fact has resulted in a rather voluminous literature on different valuation approaches.¹ Albeit unjustifiable from a mathematical point of view, the preferred valuation model among practitioners to value arithmetic average rate Asian options seem to by the lognormal approximation proposed by Levy (1992).

Some reasons for its popularity may be that the valuation formula scores high on performance criteria important in practise, such as *speed*, *accuracy* and *familiarity*. We comment on these three criteria below.

In a hectic trading environment, a valuation model based on Monte Carlo simulation might be considered to slow. Furthermore, simulation based methods or other numerical approaches require specialised software. A lognormal approximation is readily implemented in a spreadsheet package or even on an advanced calculator.

Secondly, and most importantly, the formula has to be *fairly accurate*. Exact closed form solutions are satisfactorily from a mathematical (and intellectually) point of view. Still, a formula is exact conditional on a specific mathematical model. This mathematical model is at best a good approximation of the real world. From this perspective, fairly accurate is all you can

¹Kemna and Vorst (1990) is the seminal paper on Asian option valuation. They used Monte Carlo simulation to price the options. Numerical solutions to the partial differential equation which characterises the price of an Asian option have been the focus of work by Rogers and Shi (1995), Dewynne and Wilmott (1995), Alziary, Decamps and Koehl (1997) and Zhang (2000). Yor (1993) and Geman and Yor (1993) develop analytical solutions to the Asian option problem, but non-standard numerical integration techniques are needed to compute explicit prices (see Geman and Eydeland (1995) for a numerical application).

5.1. INTRODUCTION

hope for.

Lastly it might be regarded beneficial if the formula is of a *familiar form*. If the sole purpose of an option model is to deliver the correct price, we would not accord the pricing method much importance. But an option pricing formula is frequently used as a link between the mathematical model and real world data generating process through the calculation of implied volatility from market option prices. This world wide practise of reversed financial engineering has led Taleb (1997) to conclude that the Black and Scholes (1973) formula, or modifications thereof, is used more or less non-parametrically by market participants. The model simply filters real world prices into implied volatility. The implied volatility across strike prices, the so-called "volatility smile", is a picture well known to the trading community after years of use. A lognormal formula enables us to use this established technique when analysing price data.

In this paper we will derive an approximate lognormal valuation model for Asian options. Our model is a modification of the Black (1976) formula. Fischer Black published this modification of the original stock option model to value options on commodity forward and futures contracts. As the use of futures contracts has penetrated all major financial markets, the Black (1976) model is perhaps the most frequently used option pricing formula there is. Thus our method is fast and familiar, and as we will argue, more accurate than the existing lognormal approximation of Levy (1992).

We need two inputs for our model that are not readily available in the marketplace; a futures price and a plug-in volatility. The first step in our analysis is to calculate today's "price" of the arithmetic average. This is simply a conditional expectation, which is very easy to compute. We then interpret this price as a financial futures contract, which delivers the value of the arithmetic average of the underlying asset price at maturity. This means that an Asian option can be reinterpreted as a European futures option. We show that this contract is actually lognormally distributed prior to the averaging period. After entering the averaging period, the arithmetic average contract is no longer lognormally distributed. We then propose a lognormal approximation of the contract inside the averaging period. Based on the analysis above, we calculate a plug-in volatility for the futures option model. In a Monte Carlo exercise, we show that our model has some advantage over the Levy (1992) model in terms of accuracy. We finally study the implicit volatility of the average rate options. We calculate "exact" market prices by Monte Carlo simulation and use the Black (1976) formula to back out implicit

volatilities. An Asian call option in a standard Black-Scholes economy has an upward sloping implied volatility "smile" across maturities due to the deviation from lognormality of the arithmetic average. This smile cannot be captured by a lognormal approximation.

The paper is organised as follows: In the following section we give a description of the economy and we state the valuation problem. In section 5.3 the average based futures contract approach is explained. In section 5.4 our closed form approximation to Asian average rate option is presented and compared to the Levy (1992) formula. Comparisons to Monte Carlo prices are made, and we investigate the implicit volatility smile graphically. Section 5.5 concludes the paper.

5.2 The economy

Our setting is a standard continuous time Black-Scholes economy where the uncertainty is characterised by the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where \mathcal{F} is a σ -algebra of subsets of Ω and $\mathbb{Q} : \mathcal{F} \to [0, 1]$ is a probability measure. The probability measure \mathbb{Q} is the equivalent martingale measure by assumption. All economic activity is assumed to take place on a finite horizon [0, T]. The financial market consists of one traded financial asset. Let X(t) be the market price of this risky asset at time t. We fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ with \mathcal{F}_t defined as the sigma algebra representing available information at time t (for technical details see e.g. Duffie (1996)). Frictionless borrowing and lending is possible at the constant riskless rate r. Furthermore the dynamics of the asset X is governed by the stochastic differential equation (SDE)

$$dX(t) = rX(t)dt + \sigma X(t)dW(t), \ X(0) = X_0$$
(5.1)

with solution

$$X(t) = X_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}$$
(5.2)

where σ is constant and W(t) is a one dimensional Brownian motion under the measure \mathbb{Q} . The asset does not pay any dividends. In the following we use $E_t^{\mathbb{Q}}(\cdot)$ to denote expectation conditional on \mathcal{F}_t .

5.2.1 The valuation problem

In this paper we will concentrate on average rate options. Furthermore we will focus on continuous averaging. The payoff at maturity of an average rate Asian call option, AC, is

$$(A(T, t_1, T) - K)^+$$
(5.3)

where $A(T, t_1, T)$ is the average value of the underlying asset during the time period $[t_1, T]$. We consider valuation at time t. If $t < t_1$ the option is forward starting. It is well known from financial theory that the value of a contingent claim is given by the expectation with respect to the equivalent martingale measure discounted by the risk free rate. The market value of the Asian average rate call at time t is

$$AC(t) = e^{-r(T-t)} E_t^{\mathbb{Q}} \left[(A(T, t_1, T) - K)^+ \right]$$
(5.4)

So far nothing has been said about the specific form of $A(T, t_1, T)$. When the average is geometric, the valuation problem in (6.6) has an exact closed form solution (see Kemna and Vorst (1990)). However, Asian options, which are traded in various financial markets, are usually based on the arithmetic average.² If the average is recorded continuously, $A(T, t_1, T)$ is given by:

$$A(T, t_1, T) = \frac{1}{T - t_1} \int_{t_1}^T X(u) du$$
(5.5)

In the case of an arithmetic average, there is no longer a closed form solution available. Since the geometric average is less than or equal to the arithmetic average, the corresponding geometric average call will in general be worth less (and in some cases considerably less) than its arithmetic counterpart. We will focus on the arithmetic average call option in this paper.

The main idea of this paper is to replace the arithmetic average value, a number unknown to us prior to maturity, with a futures contract written on the arithmetic average. A standard futures contract can be interpreted as the price we are willing to set today to receive the underlying asset at some future time point. In many cases there is no actually delivery of the underlying asset. Only the monetary difference between the futures price

 $^{^{2}}$ There are other possible weighting schemes which will not be considered here, see Zhang (1998) for a discussion.

and the underlying asset is exchanged between the holder and the issuer of the contract. This is particular convenient if the underlying asset is not a tradable asset, like electricity or weather. We will investigate a futures contract written on the arithmetic average of an underlying asset during a future time period. In our model the underlying asset is a tradable asset. Denote the time t value of an arithmetic average based futures contract as $F(t, t_1, T)$, where the averaging period is $[t_1, T]$ and $t \leq T$. This contract can be interpreted as a the price set today to deliver at time T, the monetary value of the arithmetic average of the underlying asset during the period $[t_1, T]$. Market values of futures contracts are found by taking expectations under the equivalent martingale measure (see Duffie (1996), p. 169). Hence, for t < T the market value of the arithmetic based futures contract is

$$F(t, t_1, T) = E_t^{\mathbb{Q}} \left[\frac{1}{T - t_1} \int_{t_1}^T X(u) du \right]$$
(5.6)

If t_1 is close to T, this contract resembles a standard futures contract, and indeed $\lim_{t_1\to T} (F_t(t_1,T)) = e^{r(T-t)}X_t$, which is the expression for a standard futures contract in the Black-Scholes economy. Standard futures contract converges to the underlying asset price at maturity. This is also the case for the average based futures contracts. Standard arbitrage arguments imply that the price at time T, for the average asset price during the interval $[t_1, T]$ is equal to the realised average, hence $F(T, t_1, T) = \frac{1}{T-t_1} \int_{t_1}^T X(u) du$. In other words, the average based futures contract can replace the arithmetic average in expression (6.6). Hence, the payoff of an Asian average rate option can equivalently be stated as

$$AC(t) = e^{-r(T-t)} E_t^{\mathbb{Q}} \left[(F(T, t_1, T) - K)^+ \right]$$
(5.7)

In the next section we will study the dynamic properties of this arithmetic average rate contract in order to find an explicit expression for the conditional expectation in expression (5.7).

5.3 Dynamic properties of arithmetic average rate futures contracts

Even though average rate futures contracts rarely trade in real life³, from a theoretical point of view they can be treated as tradable assets, because they can easily be created synthetically by the risk free and the underlying asset. In the next section we analyse the dynamic properties of the average based futures contracts.

5.3.1 The dynamics of $F(t, t_1, T)$ before the averaging period

Let first $t < t_1 < T$, that is, the contract is forward starting. From (6.7) we calculate the expectation to get

$$F(t, t_1, T) = \frac{e^{r(T-t)} - e^{r(t_1-t)}}{(T-t_1)r} X_t$$
(5.8)

To arrive at the stochastic differential equation for $F(t, t_1, T)$, we apply Itô's lemma to the expression above and arrive at:

$$dF(t, t_1, T) = \left(\frac{e^{r(T-t)} - e^{r(t_1-t)}}{(T-t_1)r} X_t\right) \sigma dW_t = F(t, t_1, T) \sigma dW_t$$
(5.9)

From (6.8) we see that the arithmetic futures contract is governed by the same SDE as a standard futures contract in a Black-Scholes economy, hence it is lognormally distributed when $t < t_1$.

5.3.2 The dynamics of $F(t, t_1, T)$ inside the averaging period

We now value the average futures contract within the averaging period, that is when $t_1 < t < T$. Define the running average as $a(t) \equiv \frac{1}{t-t_1} \int_{t_1}^t X(u) du$. Again we can calculate the average futures contract as:

³The futures/forward electricity contracts traded in various power markets are often arithmetic average based.

$$F(t, t_1, T) = E_t^{\mathbb{Q}} \left[\int_{t_1}^T \frac{X(u)}{T - t_1} du \right]$$

= $E_t^{\mathbb{Q}} \left[\frac{1}{T - t_1} \left(\int_{t_1}^t X(u) du + \int_t^T X(u) du \right) \right]$
= $\frac{t - t_1}{T - t_1} a(t) + \frac{1}{T - t_1} \int_t^T E_t^{\mathbb{Q}} [X(u)] du$
= $\frac{t - t_1}{T - t_1} a(t) + \frac{e^{r(T - t)} - 1}{(T - t_1) r} X_t$ (5.10)

We see from (6.7) that the contract now can be divided into two components; the first is non-stochastic and second is stochastic. We also see from the expression above that the last term vanishes as we approach maturity; then there is no uncertainty about the average anymore, and $F(T, t_1, T) = A(T, t_1, T)$.

Recalling Leibniz' differential rule $\frac{d}{dt}\left(\frac{t-t_1}{T-t_1}a(t)\right) = \frac{X(t)}{T-t_1}$, we can express the SDE for $F(t, t_1, T)$ when $t_1 < t < T$ by Itô's formula

$$dF(t, t_1, T) = \left(\frac{e^{r(T-t)} - 1}{(T-t_1)r}X(t)\right)\sigma dW(t)$$
(5.11)

From (5.11) we see that $F(t, t_1, T)$ is no longer lognormally distributed when $t > t_1$. Dividing each side of (5.11) with $F(t, t_1, T)$ we get a representation of the instantaneous return of the contract inside the averaging period:

$$\frac{dF(t,t_1,T)}{F(t,t_1,T)} = \frac{\left(\frac{e^{r(T-t)}-1}{(T-t_1)r}X_t\right)}{F(t,t_1,T)}\sigma dW(t)$$
(5.12)

We will return to this expression when we derive a lognormal approximation for the dynamics of the average rate contract.

5.4 Lognormal approximations of the Asian option
We start this section by examining the formula of Levy (1992), and we show that it can be considered a modified Black (1976) formula. We then propose a new lognormal approximating formula, using a plug-in volatility based on the analysis of the previous section. The accuracy of both models are then compared with Monte Carlo prices. Finally, we back out implicit volatility curves using Monte Carlo prices and the Black (1976) formula to give a graphical understanding of the limitation of the lognormal approximation.

5.4.1 Levy's approximation

Levy (1992) suggests to approximate the true unknown distribution of the arithmetic average, $A(T, t_1, T)$, with a lognormal distribution where the first two moments match (termed the Wilkinson's approximation). Let random variable, Z, be normally distributed with expectation and variance μ_p and σ_p^2 , respectively. The k'th moments of the lognormal random variable, $Y = e^Z$, is given by $E[Y^k] = e^{k\mu_p + \frac{1}{2}k^2\sigma_p^2}$. The Wilkinson approximation instructs us to determine μ_p and σ_p^2 by the following equation:

$$e^{\mu_{p} + \frac{1}{2}\sigma_{p}^{2}} = E_{t}^{\mathbb{Q}} \left[A\left(T, t_{1}, T\right) \right]$$

$$e^{2\mu_{p} + 2\sigma_{p}^{2}} = E_{t}^{\mathbb{Q}} \left[A\left(T, t_{1}, T\right)^{2} \right]$$
(5.13)

Solving yields:

$$\sigma_{p}^{2} = \ln E_{t}^{\mathbb{Q}} \left[A(T, t_{1}, T)^{2} \right] - 2 \ln E_{t}^{\mathbb{Q}} \left[A(T, t_{1}, T) \right]$$

$$\mu_{p} = 2 \ln E_{t}^{\mathbb{Q}} \left[A(T, t_{1}, T) \right] - \frac{1}{2} \ln E_{t}^{\mathbb{Q}} \left[A(T, t_{1}, T)^{2} \right]$$
(5.14)

The expression for the second moment of the arithmetic average is given explicitly in Levy (1992) and it is not repeated here. Levy's approximating formula is simply

$$AC(t) \approx e^{-r(T-t)} \left(e^{\mu_p + \frac{1}{2}\sigma_p^2} \Phi(d_1) - K\Phi(d_2) \right)$$
 (5.15)

with

$$d_1 = \frac{\mu_p - \ln K + \sigma_p^2}{\sigma_p}$$

 and

$$d_2 = d_1 - \sigma_p$$

We claim that the Levy formula can be interpreted as a modified Black (1976) futures option formula. To see this note that

$$\mu_p + \frac{1}{2}\sigma_p^2 = \ln E_t^{\mathbb{Q}}\left[A\left(T, t_1, T\right)\right] = \ln F\left(t, t_1, T\right)$$
(5.16)

where the last equation follows from the definition of the average based futures contract. This means that Levy's approximating formula can be expressed as a futures option. Thus formula (5.15) can be written as

$$AC(t) \approx e^{-r(T-t)} \left(F(t, t_1, T) \Phi(d_1) - K \Phi(d_2) \right)$$
(5.17)

with

$$d_{1} = \frac{\mu_{p} - \ln K + \frac{1}{2}\sigma_{p}^{2} + \frac{1}{2}\sigma_{p}^{2}}{\sigma_{p}}$$

$$= \frac{\ln F(t, t_{1}, T) - \ln K + \frac{1}{2}\sigma_{p}^{2}}{\sigma_{p}}$$

$$= \frac{\ln \left(\frac{F(t, t_{1}, T)}{K}\right) + \frac{1}{2}\sigma_{p}^{2}}{\sigma_{p}}$$
(5.18)

and

$$d_2 = d_1 - \sigma_p \tag{5.19}$$

Turnbull and Wakeman (1992) match the first four moments of the lognormal distribution with the corresponding moments of the arithmetic average (termed Edgeworth approximation). In a comparative study of different approximating methods, Levy and Turnbull (1993) claim that annual volatility of the underlying asset less than 20% the Levy formula yields satisfying results. Volatility between 20%-30% requires an adjustment for higher moments, and hence the Turnbull and Wakeman (1992) procedure is recommended. They conclude that neither of the models perform satisfactory for higher volatility. They furthermore note that when the option is forward starting, approximating the first two moments is the key to accurate prices. We now know, from the analysis of section 5.3 that the average future contract is lognormally distributed prior to averaging, hence the matching procedure described in Levy (1992) is in fact exact. That is the reason why Levy's formula performs better when the Asian option is forward starting. A lognormal approximation will perform better the longer the period before averaging, i.e. the bigger $t_1 - t$ is, and the shorter the averaging period, i.e. the shorter $T - t_1$ is.

5.4.2 A new lognormal approximation

Instead of matching the first two moments of the arithmetic average and the lognormal distribution, we use expression (6.9) as a starting point. The fraction $\frac{\left(\frac{e^{r(T-t)}-1}{(T-t_1)r}X_t\right)}{F(t,t_1,T)}$ in the diffusion term is worrisome, since both the nominator and the denominator is stochastic. We compute our approximation by simply replacing the stochastic elements in both the nominator and the denominator by their expected values. Needless to say this approximation is somewhat ad hoc, but so is the Wilkinson approximation.

Now assume that $t = t_1$. The conditional expected value of the nominator at time s, where $t_1 < s < T$ is

$$E_{t_1}^{\mathbb{Q}}\left[\left(\frac{e^{r(T-s)}-1}{(T-t_1)r}X_s\right)\right] = \left(\frac{e^{r(T-s)}-1}{(T-t_1)r}X_{t_1}e^{r(s-t_1)}\right) \\ = \left(\frac{e^{r(T-t_1)}-e^{r(s-t_1)}}{(T-t_1)r}X_{t_1}\right)$$
(5.20)

For the denominator we have

$$E_{t_1}^{\mathbb{Q}}[F(s,t_1,T)] = F(t_1,t_1,T) \\ = \left(\frac{e^{r(T-t_1)}-1}{(T-t_1)r}X_{t_1}\right)$$
(5.21)

And so we get

$$\frac{E_{t_1}^{\mathbb{Q}}\left[\left(\frac{e^{r(T-t_1)}-1}{(T-t_1)r}X_s\right)\right]}{E_{t_1}^{\mathbb{Q}}\left[F\left(s,t_1,T\right)\right]} = \frac{e^{r(T-t_1)}-e^{r(s-t_1)}}{e^{r(T-t_1)}-1}$$
(5.22)

The approximate SDE governing the arithmetic average rate contract is now

$$dF(t, t_1, T) \cong F(t, t_1, T) V(t) \sigma dW(t)$$
(5.23)

where

$$V(t) = \begin{cases} 1 & \text{for } t \le t_1 \\ \frac{e^{r(T-t_1)} - e^{r(t-t_1)}}{e^{r(T-t_1)} - 1} & \text{for } t_1 < t < T \end{cases}$$

Furthermore, $\ln F(T, t_1, T)$ is approximately normal distributed with

$$\ln F\left(T, t_{1}, T\right) \sim \mathcal{N}\left(\ln F\left(t, t_{1}, T\right) - \frac{1}{2} \int_{t}^{T} \left(V(s)\sigma\right)^{2} ds, \int_{t}^{T} \left(V(s)\sigma\right)^{2} ds\right)$$
(5.24)

where $Z \sim \mathcal{N}(\mu, \sigma^2)$ denotes a normally distributed random variable, Z, with expectation μ and variance σ^2 .

The approximate input for the variance in the Black (1976) model in (5.17) for valuing a forward starting Asian call then becomes:

$$\begin{split} \sigma_F^2 &= \int_t^T \left(V(s)\sigma \right)^2 ds \\ &= \int_t^{t_1} \left(V(s)\sigma \right)^2 ds + \int_{t_1}^T \left(V(s)\sigma \right)^2 ds \\ &= \sigma^2 \left(t_1 - t \right) + \int_{t_1}^T \left(\frac{e^{r(T-t_1)} - e^{r(s-t_1)}}{e^{r(T-t_1)} - 1} \sigma \right)^2 ds \\ &= \sigma^2 \left(t_1 - t \right) + \frac{\sigma^2}{\left(e^{(T-t_1)r} - 1 \right)^2} \int_{t_1}^T \left(e^{r(T-t_1)} - e^{r(s-t_1)} \right)^2 ds \\ &= \sigma^2 \left(t_1 - t \right) + \sigma^2 \left(\frac{2re^{2r(T-t_1)} \left(T - t_1 \right) - 3e^{2r(T-t_1)} + 4e^{r(T-t_1)} - 1}{2r \left(e^{r(T-t_1)} - 1 \right)^2} \right) \end{split}$$

This is the volatility plug-in we use in (5.17). In the section below, we compare our model with the one suggested by Levy (1992) and Monte Carlo prices.

5.4.3 A Monte Carlo comparison

In this section we conduct a simulation study to check the accuracy of the lognormal approximations. In the simulations, we use the price of a geometric Asian option as a control variate, a technique proposed by Kemna and Vorst (1990) to reduce the variance of simulated prices. The two competing lognormal approximations are based on a continuous average, hence they provide approximate prices of *continuous* arithmetic Asian options. In a recent paper, Fu et. al. (1997) consider Monte Carlo valuation of Asian options with continuous averaging. They argue that the judicious choice of *biased* control variates can, in addition to reducing variance, correct the discretisation bias inherent in simulation, if the object of interest is the continuous time limit. In accordance with the result in the above-mentioned study, we use the continuous geometric Asian option as (a biased) control.

In table 5.1 we present numerical comparisons between Monte Carlo estimates (termed "MC") and the two different lognormal approximations

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Volatility	Strike	MC	Std.	New	Levy	
0.30	K = 90	15.23720	0.0052	15.30758	15.32306	
	K = 100	9.051251	0.0055	9.090714	9.113903	
	K = 110	4.856152	0.0056	4.838257	4.862787	
0.50	K = 90	18.36302	0.0151	18.53306	18.62493	
	K = 100	13.21493	0.0162	13.28350	13.39332	
	K = 110	9.291660	0.0164	9.258294	9.373827	
0.90	K = 90	25.65053	0.0698	25.96211	26.54387	
	K = 100	21.41393	0.0637	21.68586	22.32281	
	K = 110	18.09593	0.0683	18.08589	18.75454	

Table 5.1: Comparison of different valuation approaches. The valuation results refer to an Asian average rate option with continuous averaging. The averaging period starts this instant and lasts one year $(t_1 = 0 \text{ and } T = 1)$. The initial stock price is set to $X_0 = 100$ and r = 0.1. The result from our proposed method is in the column New, and the method in Levy (1992) is given in the column Levy. Monte Carlo estimates are conducted using a fine discretization of the stock price of 4000 observations per year. The number of Monte Carlo runs are 10000 for K = 90,20000 for K = 100 and 30000 for K = 110.

(termed "New" and "Levy"). We have chosen a fine discretisation of the stock price of 4000 observations a year. Since the two approaches yield similar results prior to averaging, we have set $t = t_1 = 0$ and T = 1, hence we are pricing an average rate Asian call with maturity in one year at the moment averaging starts. Furthermore the initial stock price is set to $X_0 = 100$ and r = 0.1. In general the accuracy of the Monte Carlo prices is higher for lower volatility of the underlying asset. We investigate prices for annualised volatility of 30%, 50% and 90%, with strike prices set at K = 90, 100 and 110. The number of Monte Carlo runs are 10 000 for K = 90, 20 000 for K = 100 and 30 000 for K = 110. We increase the numbers of runs along with moneyness to get approximately equal levels of precision of the computed prices for any given level of volatility. The estimated standard deviations are given in the column termed "Std.".

The main results are as follows. The new approximation overprices at K = 90 and K = 100 and underprices at K = 110. This pattern is consistent for all the levels of volatility. Levy's formula consistently overprices the options. The pricing errors increase with increasing volatility of the underlying asset. This is the main drawback of the formula of Levy. The variance obtained by matching moments of lognormal and the arithmetic average overshoots the real variance. The overshooting gets worse when volatility increases.

5.4.4 Implicit volatility in the Black (1976) framework

This section is a first attempt to learn more about average rate arithmetic options using implied volatility. In liquid option markets, market participants calculate volatility implied from the Black-Scholes option pricing formula from market prices, to better understand the limitation of the formula. Implied volatility is typically not constant across strike prices. This is the so-called "moneyness bias" or "volatility smile". This effort of financial engineering has in turn led to the theoretical development of option pricing models with more sophisticated asset price dynamics, such as jumps and stochastic volatility. In our setup we know that the average rate futures contract is not lognormally distributed within the averaging period. We will now investigate the "volatility smile" resulting from this deviation from lognormality. First we compute prices of the average rate futures contracts. Furthermore, we do a Monte Carlo experiment to compute the "correct" option prices. Finally, we use the Black (1976) formula to back out the implicit volatility. It is well known that lognormality produce constant implied volatility across strikes. Hence, a lognormal approximation will produce a horizontal volatility "smile".

We use the same initial parameters as described above, and we calculate prices for average rate call options with K = 80, 90, ..., 120 for the following volatility levels of the underlying asset; $\sigma = 0.3$, 0.5 and 0.7. We then back out the implicit volatility from the Black (1976) formula. Plotting the implicit volatilities (termed "MC") against the strike prices produced the volatility curves in figures 5.1 - 5.3.⁴ We also plotted the volatility using Wilkinson's approximation and our new lognormal approximation (termed "Levy" and "New" respectively).

We see that the implied volatility is increasing in the strike. This is consistent for all levels of volatility of the basic underlying asset. Furthermore, the difference between our new approximation and the Wilkinson's approximation is increasing in the basic underlying asset volatility. Our new approximation seems to be better except for call option far out of the money.

The fact that the smile is upward sloping is an indication of positive skewness for the natural log of the arithmetic average. Out of the money call options are relatively more expensive than in the money options, since positive skewness makes the right tail of the distribution heavier than the

⁴Note that the underlying asset volatility in the captions refers to the basic underlying asset, not the volatility of the average rate contract.



Figure 5.1: Implied volatility when underlying asset volatility is 30%. "Correct" prices are estimated using Monte Carlo methods for an average rate Asian call option with continuous averaging starting today with maturity one year from now $(t_1 = 0 \text{ and } T = 1)$. Todays stock price is $X_0 = 100$ and r = 0.1 and underlying asset volatility is $\sigma = 0.3$. The implied volatility is backed out using the Black (1976) formula. "MC", "New" and "Levy" are the implied volatility from Monte Carlo prices, our new formula and the formula presented in Levy (1992) respectively.



Figure 5.2: Implied volatility when underlying asset volatility is 50%. (See caption on figure 5.1)



Figure 5.3: Implied volatility when underlying asset volatility is 70%. (See caption on figure 5.1)

lognormal. It is not clear from the figures whether this moneyness effect increases or decreases, when the volatility of the basic underlying asset increases. In table 5.2 we have computed the implied volatility (termed "imp") in percent of the basic underlying asset for K = 80 and K = 120.

We see no conclusive result. From $\sigma = 0.3$ to $\sigma = 0.5$ the difference between implied volatility at K = 80 and K = 120 increases. But the difference decreases from $\sigma = 0.5$ to $\sigma = 0.7$. It is important to note that the Monte Carlo prices that produce the implied volatility inhabit measurement errors. Table 5.2 reports a fairly stable moneyness effect in percentage terms.

	K = 80	K = 120		
$\frac{imp}{0.3}$	57.0%	59.3%		
$\frac{imp}{0.5}$	56.6%	59.9%		
$\frac{imp}{0.7}$	56.9%	58.9%		

Table 5.2: Volatility of the arithmetic average relative to the volatility of the underlying asset. The stock price today is $X_0 = 100$ and r = 0.1. Monte Carlo prices are calculated for an average rate Asian call option with continuous averaging starting this instant with maturity one year from now $(t_1 = 0 \text{ and } T = 1)$. Prices are calculated for different strikes (K = 80 and K = 120) and different levels of underlying asset volatility. Implied volatility for these Monte Carlo prices are backed out from the Black (1976) formula. The table reports the ratio of the implied Monte Carlo prices and the underlying asset volatility.

Some further testing not reported here, indicate that the implicit smile curve is unaffected by maturity. For example setting T = 0.5 or T = 2 produce the same smile curve as for T = 1. This means that an appropriate correction for moneyness may further improve the accuracy of the formula.

5.5 Conclusions and suggestions for future research

In this paper we develop a new lognormal approximation for the Asian option. We start our analysis by creating a synthetic asset; a futures contract written on the arithmetic average asset price, which equals the arithmetic average at maturity. This way the Asian option can be interpreted as a European futures option. The average rate futures contract is lognormally distributed prior to the averaging period. Within the averaging period, the contract is no longer lognormally distributed, and we propose a new lognormal approximation. We then value the average rate Asian call in the Black (1976) framework. In a Monte Carlo experiment we show that our formula gives quite accurate prices as long as the strike is not too far in or out-ofthe-money. The mispricing of the lognormal approximation formula by Levy (1992) becomes increasingly severe for higher levels of volatility of the underlying asset. Finally we shoved, that Asian options interpreted as European futures option produce implicit volatility curves increasing with strike price. This analysis using implied volatility in the Black (1976) framework might be extended to a more realistic model for the underlying asset price dynamics (for example allowing for jumps and stochastic volatility) in the Monte Carlo exercise. In such an environment, a lognormal approximation may even perform better in terms of accuracy than in a standard Black-Scholes economy, since pricing errors due to underlying asset price dynamics versus arithmetic averaging may cancel each other out. This is left for future research.

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Chapter 6

Valuation of Asian options by matching moments

ABSTRACT - Arithmetic Asian options are difficult to value since the distribution of the arithmetic average of lognormal random variables is unknown. Turnbull and Wakeman (1991) suggested matching the moments of the unknown distribution by the lognormal distribution using the Edgeworth expansion. It is well known that their method is inaccurate when the volatility of the underlying asset is high. In this paper we first investigate a certain futures contract that equals the arithmetic average of the underlying asset at maturity. We approximate the dynamics of this average based contract with a standard lognormal futures contract with mean reverting square-root variance. By adjusting parameter values of this approximate process, we are able to simultaneously match all four moments of the arithmetic average. Valuation is achieved by Fourier inversion techniques. The suggested method is shown to give very accurate option prices, even at high levels of volatility.

6.1 Introduction

In this paper we propose a new method for valuation of Asian options. Using the standard lognormal model for the underlying asset, arithmetic average options are difficult to value. Closed form option formulas for these options do not exist, since the distribution of the arithmetic average of lognormal

random variables is unknown. This fact has resulted in a abundant literature on different valuation approaches, falling into four broad categories:

- Monte Carlo simulation. The seminal paper on Asian options is the work by Kemna and Vorst (1990). They provide closed form solutions on geometric average options, and they develop a Monte Carlo pricing approach, using the geometric average option as control variate. Recently Grant, Vora and Weeks (1997) have developed a simulation scheme that allows for early exercise features of Asian options.
- **PDE-methods.** Kemna and Vorst (1990) derive a partial differential equation, which characterises the price of an Asian option. Numerical solution of this PDE has been the focus of work by Rogers and Shi (1995), Alziary et al. (1997) and Zhang (2002).
- Analytical solutions. Yor (1993) and Geman and Yor (1993) develop analytical solutions to the Asian option problem, but non-standard numerical integration techniques are needed to compute prices (see Geman and Eydeland (1995) for a numerical application).
- Approximate closed form solutions. Closed form solutions are useful since they provide prices and hedge ratios very quickly. In the seminal Black Scholes (1973) analysis, the assumption of lognormally distributed asset prices provides the closed form solution. Vorst (1992) presents a formula based on an ad hoc adjustment of the strike price and then replacing the arithmetic average by its geometric counterpart. Turnbull and Wakeman (1991) use the Edgeworth expansion to approximate the unknown distribution of the arithmetic average to the lognormal. Levy (1992) shows that matching only the first two moments yields a closed form solution that provide sufficient accuracy for low levels of volatility. Koekebakker (2002) suggests an alternative lognormal approximation. Levy and Turnbull (1992) compare different methods and conclude that the approximating formulas are problematic when the volatility is high. Milevsky and Posner (1998) repeat the analysis of Levy (1992), using the reciprocal gamma distribution as the approximating distribution of the arithmetic average. Curran (1994) derives a closed form approximation by conditioning the Asian payoff on the geometric average.

This paper falls into the last category and it is linked to the work of Turnbull and Wakeman (1991). They apply the Edgeworth expansion using the lognormal density as an approximate distribution. In their work the series expansion is truncated after the fourth term, hence information from the first four moments of the arithmetic average is utilised in their valuation approach. It is well known that this method is inaccurate when the volatility of the underlying asset is high. In this paper we take a somewhat different approach to the method of matching moments. We start by defining a futures contract written on the arithmetic average. At maturity this futures contract equals the arithmetic average. This means that a European option on this average rate futures contract is equivalent to an Asian option. We then investigate the dynamics of this futures contract and we give a stochastic volatility interpretation of the dynamics. Unfortunately the resulting stochastic differential equation is unfamiliar and a closed form pricing formula cannot be reached. Instead we choose a lognormal futures model with stochastic variance. The variance is modelled as a mean reverting square-root process. Valuation is done in the following steps: 1) Calculate the variance, skewness and excess kurtosis of the arithmetic average. 2) Use an optimising routine to pick parameters of the approximate model that produce variance, skewness and excess kurtosis close to (by minimising mean square error) the arithmetic average. 3) Use the Fourier inversion technique to calculate the price of a European option on the average rate contract. This procedure allows us to simultaneously match the first four moments of the arithmetic average. Our method produces very accurate option prices, also when the underlying asset volatility is high. From our analyses we can conclude that the first four moments contain enough information about the density of the arithmetic average of the geometric Brownian motion to facilitate accurate option pricing.

The paper is organised as follows: In the following section we describe our modelling framework and state the valuation problem. Section 6.3 introduces the arithmetic average rate futures contract. Section 6.4 explains the valuation method suggested by Turnbull and Wakeman (1992) and our new stochastic variance approach. We present some numerical results in section 6.5, and section 6.6 concludes.

6.2 The valuation problem

We work in a standard continuous time Black-Scholes economy where the uncertainty is characterised by the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where \mathcal{F} is a σ -algebra of subsets of Ω and $\mathbb{Q} : \mathcal{F} \to [0, 1]$ is a probability measure. The probability measure \mathbb{Q} is the equivalent martingale measure (EMM) by assumption. All economic activity is assumed to take place on a finite horizon $[0, T^*]$. The financial market consists of one traded financial asset. Let X(t)be the market price of this risky asset at time t. We fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ with \mathcal{F}_t defined as the sigma algebra representing available information at time t (for technical details see e.g. Duffie (1996)). Frictionless borrowing and lending is possible at the constant riskless rate r. Furthermore the dynamics of the asset X is governed by the stochastic differential equation (SDE)

$$dX(t) = rX(t)dt + \sigma X(t)dW(t), \ X(0) = X_0$$
(6.1)

with solution

$$X(t) = X_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}$$
(6.2)

where σ is constant and W(t) is a one dimensional Brownian motion under the measure \mathbb{Q} . The asset does not pay any dividends. In the following $E_t^{\mathbb{Q}}(\cdot)$ denotes expectation under EMM conditional on \mathcal{F}_t .

This paper investigates the valuation of arithmetic average rate options. The value of such an options is defined as the positive difference (if any) between the observed difference between the average of observed prices prior to maturity and a fixed strike, K. Let the tick times at which the underlying asset is sampled be given by $\{t_1, t_2, ..., t_n\}$. We furthermore assume that the maturity of the option coincides with the last observation, hence $t_n = T$. The arithmetic average at time T, A(T), is given by

$$A(T) = \sum_{i=1}^{n} X_i$$
 (6.3)

At time $t < t_n$ this average is unknown. In the case of continuous sampling A(T) becomes

$$A(T) = \frac{1}{T - t_1} \int_{t_1}^T X(u) du$$
(6.4)

The payoff of an average rate Asian call option, AC, is

$$(A(T) - K)^+ (6.5)$$

and the time t value is

$$AC(t) = e^{-r(T-t)} E_t^{\mathbb{Q}} \left(A(T) - K \right)^+$$
(6.6)

It is not easy to calculate the expectation in (6.6) explicitly, since the distribution of A(T) is unknown for t < T. We will instead proceed by investigating an average based futures contract that equals A(T) at time T. We study this contract and its dynamic properties in the next section.

6.3 Introducing the arithmetic average rate futures contract

Throughout this section we will assume continuous averaging since this simplifies the mathematical exposition. Let $F(t, t_1, T)$ denote the time t value of the arithmetic average over some future time period $[t_1, T]$. Here the subscript t is the current time and t_1 and T denote the start and end of the averaging period, respectively. $F(t, t_1, T)$ can be interpreted as a financial futures contract which delivers, at time T, the average asset price during the period $[t_1, T]$. From a theoretical point of view such a contract is a traded assets, meaning that it can easily be created synthetically by a dynamic strategy of the risk free and the underlying asset. The value (or contract price) of $F(t, t_1, T)$ can be found by calculating the conditional expectation under the equivalent martingale measure (see Duffie (1996), p. 169) as

$$F(t,t_{1},T) = E_{t}^{\mathbb{Q}} \left[\int_{t_{1}}^{T} \frac{X(u)}{T-t_{1}} du \right] = \begin{cases} \frac{e^{r(T-t)} - e^{r(t_{1}-t)}}{(T-t_{1})r} X_{t}, & \text{when } t \leq t_{1} \\ \int_{t_{1}}^{t} \frac{X_{u}}{T-t_{1}} du + \frac{e^{r(T-t)} - 1}{(T-t_{1})r} X_{t}, & \text{when } t > t_{1} \end{cases}$$

$$(6.7)$$

To arrive at the SDE for $F(t, t_1, T)$, we apply Ito's lemma to the expression above to get (see Koekebakker (2002)):

$$dF(t, t_1, T) = \begin{cases} F(t, t_1, T) \, \sigma dW(t) & \text{when } t \leq t_1 \\ \frac{e^{r(T-t)} - 1}{(T-t_1)r} X(t) \sigma dW(t) & \text{when } t > t_1 \end{cases}$$
(6.8)

From (6.8) we see that the arithmetic futures contract is governed by the same SDE as a standard futures contract in a Black-Scholes economy when $t \leq t_1$, hence the futures contract is lognormally distributed prior to t_1 . At time T it is obvious that $F(T, t_1, T) = \int_{t_1}^T \frac{X(u)}{T-t_1} du = A(T)$ and consequently $F(T, t_1, T)$ can replace A(T) in (6.6). In the next sub-section we will further investigate the dynamics of the return on $F(t, t_1, T)$ inside the averaging period.

6.3.1 A stochastic volatility representation of the average futures contract

Dividing each side of (6.8) with $F(t, t_1, T)$ we get a representation of the instantaneous return of the contract:

$$\frac{dF(t,t_1,T)}{F(t,t_1,T)} = V(t)\sigma dW(t)$$
(6.9)

where

$$V(t) \equiv \begin{cases} 1, & \text{when } t \le t_1 \\ \frac{\left(\frac{e^{r(T-t)}-1}{(T-t_1)r}X_t\right)}{F(t,t_1,T)}, & \text{when } t > t_1 \end{cases}$$
(6.10)

Applying Ito's formula on V(t) we arrive at the following SDE for $t > t_1$:

$$dV(t) = \left(\frac{-r}{e^{r(T-t)} - 1} - \sigma V(t)\right) V(t)dt + \sigma V(t) \left(1 - V(t)\right) dW(t) \quad (6.11)$$

From (6.9) and (6.11) we see that the instantaneous return of the futures contract can be given a stochastic volatility interpretation when $t > t_1$. We note that $V(t_1) = 1$ and V(T) = 0 (since $\frac{e^{r(T-T)}-1}{(T-t_1)r}X_T = 0$). These facts can easily be explained. When entering the averaging period the uncertainty of the final contract price (at maturity) decreases, since price information is constantly revealed to the investor. At maturity all information about the average is known, and the uncertainty vanishes. We also note that the diffusion term contains the volatility parameter of the underlying asset. This means that the higher the volatility of the underlying asset, the bigger the uncertainty regarding the volatility (the volatility of volatility) of the futures contract. Unfortunately the SDE in (6.11) is rather complicated. In the next sub-section we will instead consider an approximate representation that allows closed form option pricing.

6.3.2 An approximate SDE of the average rate futures contract

In the continuous time finance literature, stochastic volatility is typically modelled using an additional Brownian motion correlated with the Brownian motion driving the underlying asset. Early contributors to this literature are Wiggins (1987), Scott (1987), Chesney and Scott (1989), Hull and White (1987), Stein and Stein (1991) and Heston (1993). These studies differ in the specification of the stochastic volatility (or variance) and in the different numerical techniques employed to price options. The square root variance specification in Heston (1993) has become the most widely used in later research. One important reason for this is that it allows fast numerical calculations of options prices. This is also our motivation for considering the square root variance process. A stochastic volatility model typically gives rise to the problem of market incompleteness, and the specification of market price of volatility risk. In our model we only have one source of randomness, thus market incompleteness is not an issue in our setting. To be more specific we introduce a futures price process, \overline{F} with variance v as proxies for the system (6.9) and (6.11). The differential form of the approximate model is by assumption:

$$d\overline{F}(t) = \sqrt{v(t)}\overline{F}(t)dZ(t) \tag{6.12}$$

 and

$$dv(t) = \kappa \left(\theta - v(t)\right) dt + \sigma_v \sqrt{v(t)} dZ(t)$$
(6.13)

where Z(t) is a standard Brownian motions under the equivalent martingale measure. The initial values the two processes are \overline{F}_t and v_t respectively. The variance is modelled as a square-root process with κ controlling the rate of reversion to the mean of the variance θ , and σ_v representing the diffusion term of the variance. All parameters are assumed non-negative. There is only one source of randomness driving the system above, as is also the case in the economy defined in section 6.2. In the standard stochastic volatility model set up, the underlying asset and the stochastic variance are driven by separate, possible correlated Brownian motions. Still, the system described in (6.12) and (6.13) is equivalent to the risk neutral processes of Heston (1993), with the two Brownian motions therein being perfectly correlated. In the next section we will apply the Fourier inversion technique to find approximate prices of Asian options.

6.4 Pricing Asian options by matching moments

Our Asian option pricing approach can be described in the following manner: Adjust the parameters in the system (6.12) and (6.13) so that the moments of $\overline{F}(T)$ match those of A(T). Then use the price of a European option on $\overline{F}(T)$ as an approximation of the Asian option price. Before we detail our approach, a brief review of the moment matching approach suggested by Turnbull and Wakeman (1991) is given. Following Jarrow and Rudd (1982) they suggested to use the lognormal distribution as an initial guess on the distribution of A(T) and the Edgeworth expansion adjusting for differences in third and fourth moments.

6.4.1 The Turnbull and Wakeman (1991) approach

The Edgeworth expansion is usually expressed in terms of cumulants instead of moments. There is a close relationship between the cumulants and the moments of a distribution. Let $\phi(s; Y) = E\left[e^{sY}\right]$ denote the moment generating function of a random variable Y. The *j*'th cumulant, κ_j , is defined as

$$\left. \frac{\partial^j \phi(s;Y)}{\partial s^j} \right|_{s=0} = \kappa_j$$

Let the unknown distribution of A(T) be given by $\psi(A)$. Furthermore, let $\kappa_j(A; \psi)$ and $m_j(A; \psi)$ denote, respectively, the j'th cumulant and central moments of A(T). Then we have the following relationships:

$$\begin{split} \kappa_{1}\left(A;\psi\right) &= m_{1}\left(A;\psi\right) \\ \kappa_{2}\left(A;\psi\right) &= m_{2}\left(A;\psi\right) \\ \kappa_{3}\left(A;\psi\right) &= m_{3}\left(A;\psi\right) \\ \kappa_{4}\left(A;\psi\right) &= m_{4}\left(A;\psi\right) - 3\left(m_{2}\left(A;\psi\right)\right)^{2} \end{split}$$

Skewness and excess kurtosis can be expressed in terms of cumulants by

$$\gamma_{1}\left(A;\psi\right) = \frac{\kappa_{3}\left(A;\psi\right)}{\left(\kappa_{2}\left(A;\psi\right)\right)^{\frac{3}{2}}}$$

and

$$\gamma_{2}(A;\psi) = rac{\kappa_{4}(A;\psi)}{\left(\kappa_{2}(A;\psi)
ight)^{2}}$$

respectively. All moments of A(T) can be calculated both in the case of discrete and continuous sampling of the average (see appendix A). But the distribution is unknown. The Edgeworth expansion is one way of approximating this distribution using the cumulants. The Edgeworth expansion consists of applying a series expansion for $\psi(A)$ to adjust for higher moments effects. Let $\varphi(A)$ be an approximate distribution of A(T), then we can expand $\psi(A)$ as (see e.g. Jarrow and Rudd (1982))

$$\psi(A) = \varphi(A) - Q_1 \varphi^1(A) + \frac{Q_2}{2!} \varphi^2(A) - \frac{Q_3}{3!} \varphi^3(A) + \frac{Q_4}{4!} \varphi^4(A) - \dots \quad (6.14)$$

where $\varphi^{j}(A)$, j = 1, 2, ..., is the *j*th derivative of the approximate distribution and Q_{j} are terms involving the differences of the cumulants between the exact and approximate distribution. Define $\varepsilon_{j} \equiv \kappa_{j}(A; \psi) - \kappa_{j}(A; \varphi)$. Then the first four coefficients Q_{j} are given by:

$$\begin{split} Q_1 &= \varepsilon_1 \\ Q_2 &= \varepsilon_1^2 + \varepsilon_2 \\ Q_3 &= \varepsilon_1^3 + 3\varepsilon_1\varepsilon_2 + \varepsilon_3 \\ Q_4 &= \varepsilon_1^4 + 3\varepsilon_2^2 + 4\varepsilon_1\varepsilon_3 + 6\varepsilon_1^2\varepsilon_2 + \varepsilon_4 \end{split}$$

Following Jarrow and Rudd (1982) we truncate (6.14) after the fifth term. The Asian option pricing problem now becomes

$$AC(t) \approx e^{-r(T-t)} \int_{K}^{\infty} \left(A(T) - K\right) \psi(A) dA \tag{6.15}$$

where

$$\psi(A) = \varphi(A) - Q_1 \varphi^1(A) + \frac{Q_2}{2!} \varphi^2(A) - \frac{Q_3}{3!} \varphi^3(A) + \frac{Q_4}{4!} \varphi^4(A)$$
(6.16)

The question of choosing the approximate distribution still remains. Turnbull and Wakeman (1991) suggest the lognormal distribution as a candidate

for the approximate distribution. This distribution is completely characterised by its mean and variance, which have to be determined somehow. Turnbull and Wakeman (1991) suggest that we set the first two moments equal to each other (known as the Wilkinson approximation). If the random variable W is normally distributed with mean μ and variance s^2 , then the kth moment of the lognormal random variable, $Y = e^W$, is given by $E[Y^k] = e^{k\mu + \frac{1}{2}k^2s^2}$. The Wilkinson approximation instructs us to determine μ and s^2 by the following equation:

$$e^{\mu + \frac{1}{2}s^2} = E_t \left[A(T) \right]$$
$$e^{2\mu + 2s^2} = E_t \left[(A(T))^2 \right]$$

Solving yields:

$$s^{2} = \ln E_{t} \left[\left(A(T) \right)^{2} \right] - 2 \ln F(t, t_{1}, T)$$
(6.17)

$$\mu = 2\ln F(t, t_1, T) - \frac{1}{2}\ln E_t \left[(A(T))^2 \right]$$
(6.18)

The Wilkinson approximation implies equality between the first two cumulants of the real and the approximate distributions, and consequently that $Q_1 = Q_2 = 0$. The approximate Asian option call price according to Turnbull and Wakeman (1991), AC^{TW} , becomes

$$AC(t)^{TW} = e^{-r(T-t)} \left(F(t, t_1, T) \Phi(d_1) - K \Phi(d_1) \right) - M_1 + M_2$$
(6.19)

where $\Phi(.)$ is the standard cumulative normal distribution, $M_1 = e^{-r(T-t)} \left(\frac{Q_3}{3!}\varphi^1(K)\right)$ and $M_2 = e^{-r(T-t)} \left(\frac{Q_4}{4!}\varphi^2(K)\right)$ are correction terms that picks up differences between $\psi(A)$ and $\varphi(A)$ of the third and fourth moments, respectively. Finally, $\varphi^j(K)$ is the *j*th derivative of the lognormal distribution evaluated at K and $d_1 = \frac{\ln\left(\frac{F(t,t_1,T)}{K}\right) - \frac{1}{2}s^2}{s}$, $d_2 = d_1 - s$. Levy (1992) suggested matching only the first to moments of the distribution, resulting in a formula involving only the first term in (6.19).

6.4.2 The stochastic volatility approach

We suggest the following approximation: Use the process \overline{F} in (6.12) - (6.13) as an approximating process for A(T). An Asian option price can be approximated by the price of a European option written on \overline{F} . A European

call option, $\overline{FC}(t)$, written on the underlying asset \overline{F} with dynamics given in (6.12) - (6.13) and strike price K can be valued as

$$\overline{F}C(t) = e^{-r(T-t)} E_t^{\mathbb{Q}} \left[\left(\overline{F}(T) - K \right)^+ \right]$$
$$= e^{-r(T-t)} \left(\overline{F}_t G_1^{\overline{\mathbb{Q}}}(D) - K G_2^{\mathbb{Q}}(D) \right)$$
(6.20)

with the Radon-Nikodym derivative

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} = e^{-r(T-t)} \frac{\overline{F}(T)}{\overline{F}(t)}$$
(6.21)

where $\overline{\mathbb{Q}}$ is a martingale measure equivalent to \mathbb{Q} and $D = \{\ln \overline{F}(T) > \ln K\}$. The functions $G_1^{\overline{\mathbb{Q}}}$ and $G_2^{\mathbb{Q}}$ can be interpreted as probability distribution functions under two different probability measures (see Zhu (1999) for a nice exposition). Dropping superscript for convenience and setting v = T - t and $x_t \equiv \ln(\overline{F}(t))$ we can write G_1 and G_2 in terms of the characteristic functions, $g_1(.)$ and $g_2(.)$, using Fourier inversion

$$G_{j} = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \operatorname{Re}\left(g_{j}\left(x_{t}, v_{t}; \phi\right) \frac{e^{-i\phi \ln(K)}}{i\phi}\right) d\phi, \ j = 1, 2$$
(6.22)

where $\text{Re}(\bullet)$ returns the real part of the expression in paranthesis. The explicit expressions for g_i can be shown to be (see appendix B)

$$g_j(i\phi; x_t, v_t) = \exp\left(i\phi x_t - p_j\left(v_t + \kappa\theta\right) + A\left(v; s_j, p_j\right)v_t + C\left(v; s_j, p_j\right)\right)$$
(6.23)

with

$$s_{j} = -(1+i\phi)\left(-\frac{1}{2} + \frac{\kappa}{\sigma_{v}} + \frac{1}{2}(2-j+\phi i)\right) \text{ and } p_{j} = \frac{(2-j+i\phi)}{\sigma_{v}}$$
(6.24)

$$A(v; s_j, p_j) = \frac{(1 - e^{-\gamma \tau}) (2s_j + \kappa p_j) - \gamma p_j (1 + e^{-\gamma \tau})}{-2\gamma e^{-\gamma \tau} - (\kappa + \gamma - \sigma_v^2 p_j) (1 - e^{-\gamma \tau})}$$
(6.25)

$$C(\upsilon; s_j, p_j) = \frac{2\kappa\theta}{\sigma_v^2} \ln\left[\frac{2\gamma e^{\frac{1}{2}(\kappa-\gamma)\tau} + \frac{1}{2}\left(\sigma_S k - \kappa - \gamma\right)}{2\gamma e^{-\gamma\tau} + \left(\kappa + \gamma - \sigma_v^2 p_j\right)\left(1 - e^{-\gamma\tau}\right)}\right]$$
(6.26)

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$$\gamma = \sqrt{\kappa^2 + 2\sigma_v^2 s_j} \tag{6.27}$$

The integrals in (6.22) can be evaluated efficiently using Gaussian quadrature. The formula in (6.20) gives us the price of a European option given parameter values of κ , θ and σ_v and the initial values \overline{F}_t and v_t . We want to select these values so that the moments of $\overline{F}(T)$ and A(T) match each other. We have the following relationship between the moment generating function $M(\phi; x_t, v_t)$ for $x(T) = \ln \overline{F}(T)$ and the moments of $\overline{F}(T)$ under the equivalent martingale:

$$M(\phi; x_t, v_t) \equiv E_t^{\mathbb{Q}} \left[e^{\ln \overline{F}(T)\phi} \right] = g_1(\phi; x_t, v_t) = E_t^{\mathbb{Q}} \left[\overline{F}(T)^\phi \right]$$
(6.28)

In other words, we can find the moments of $\overline{F}(T)$ using the expressions in (6.23) - (6.27) setting j = 2 and replacing i with 1 everywhere. This provides us with an analytical expression for calculating moments of $\overline{F}(T)$. Now define the following scalars

$$w_{1} = \left(\text{Variance } (A(T)) - \text{Variance } (\overline{F}(T))\right)^{2}$$

$$w_{2} = \left(\text{Skewness } (A(T)) - \text{Skewness } (\overline{F}(T))\right)^{2}$$

$$w_{3} = \left(\text{Excess Kurtosis } (A(T)) - \text{Excess Kurtosis } (\overline{F}(T))\right)^{2}$$

Our hope is that variance, skewness and excess kurtosis of $\overline{F}(T)$ and A(T) are close to each other. If so, then w_1, w_2 and w_3 will be close to zero. Finally we define a new scalar, W, given by

$$W = \frac{1}{3} \sum_{l=1}^{3} w_l \tag{6.29}$$

This scalar is our measure of fit between the random variables $\overline{F}(T)$ and A(T). Our task is to choose parameters and initial values governing the dynamics of to minimize the value of W. Technically this can be expressed as

$$\min_{\left\{\overline{F}_t, v_t\right\}, \Psi} W$$

where

$$\Psi = (\kappa, \theta, \sigma_v)$$

We set $\overline{F}_t = F(t, t_1, T)$ to ensure that mean of the two processes are equal. Furthermore we set $v_t = \frac{s^2}{(T-t)}$ where s^2 is given in (6.17). This means that

in the special case of $\kappa = \theta = \sigma_v = 0$, the approximation model collapses to the model of Levy (1992). ¹ Let $\overline{F}(t; \Psi^*)$ be the process of $\overline{F}(t)$ with parameters Ψ^* that minimises W. Then our approximate formula the Asian call option is

$$AC(t)^{NEW} = e^{-r(T-t)} E_t^{\mathbb{Q}} [A(T) - K]^+ \approx e^{-r(T-t)} (F(t, t_1, T)G_1(D) - KG_2(D))$$
(6.30)

with $D = \left\{ \ln \overline{F}(T; \Psi^*) > \ln K \right\}.$

6.5 Numerical results

In this section we present some numerical results of the different valuation approaches. In table 6.1 we have reported the results for an Asian call option with continuous averaging. The price of the underlying asset is 100. The averaging period starts instantly, and maturity is one year from now, hence $t = t_1 = 0$ and T = 1. The continuously compounded interest rate is 9%, and the strike prices are set so that we can compare with the results presented previously in the literature. R-S lower is the lower bound of the option given in Rogers and Shi (1995) and T upper is the upper bound derived in Thompson (2000). The column termed Zhang reports prices given in Zhang (2002). He derives an analytical approximate expression and shows that the resulting error can be expressed explicitly as a partial differential equation (PDE) that can be solved easily by standard numerical methods. This method, he claims, is accurate down to five decimals. The last columns include prices from three moment matching approaches: The Wilkinson approximation proposed by Levy (1992), the Edgeworth expansion by Turnbull and Wakeman (1991) and our new stochastic volatility based method. They are termed Levy, TW and New, respectively.

¹We used the minimization routine QNewton in the $GAUSS^{\textcircled{o}}$ programming language to minimize W. Transformed versions of the parameters (using the exponential function) were employed in the routine to ensure non-negativity. To ensure non-complex moments of $\overline{F}(T)$ further restrictions have to be imposed on κ , θ , and σ_v . This can be implemented using an optimising routine with restrictions on the parameters. Trying different starting values in the unconstrained search will typically result in a non-complex minimum after a couple of trials. In all the numerical examples reported, the routine picked parameters resulting in $W \leq 0.00001$. This means an essentially identical match between the first four moments of A(T) and $\overline{F}(T)$.

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Vol	Strike	R-S lower	T upper	Zhang	Levy	TW	New
0.05	95	8.8088	8.8089	8.8088	8.8089	8.8088	8.8088
	100	4.3082	4.3084	4.3082	4.3097	4.3082	4.3082
	105	0.9583	0.9585	0.9584	0.9582	0.9584	0.9584
0.1	95	8.9118	8.9130	8.9118	8.9172	8.9119	8.9118
	100	4.9150	4.9155	4.9151	4.9231	4.9146	4.9152
	105	2.0699	2.0704	2.0701	2.0705	2.0700	2.0700
0.3	90	14.9827	14.9929	14.9840	15.0670	14.925	14.9847
	100	8.8275	8.8333	8.8288	8.8858	8.8019	8.8290
	110	4.6949	4.7027	4.6967	4.6951	4.7174	4.6963
0.5	90	18.1829	18.2208	N/A	18.4370	17.6999	18.1931
	95	N/A	N/A	15.4427	15.6649	15.1355	15.4449
	100	13.0225	13.0569	13.0282	13.2120	12.8994	13.0301
	105	N/A	N/A	10.9296	11.0675	10.9540	10.9303
	110	9.1179	9.1561	N/A	9.2132	9.2655	9.1238

Table 6.1: Asian option prices and bounds. The table reports Asian call option prices with continuous averaging using various methods. The current price of the underlying asset is 100. The averaging period starts instantly, and maturity is one year from now $(t = t_1 = 0 \text{ and } T = 1)$. and the interest rate is 9%. R-S lower is the lower bound of the option given in Rogers and Shi (1995), and T upper is the upper bound derived in Thompson (2000). The column termed Zhang reports prices given in Zhang (2002). The last columns report prices from the approximation proposed by Levy (1992) (termed Levy), the Edgeworth expansion by Turnbull and Wakeman (1991) (termed TW). Prices from our new method are in the column New.

We see that all three moment matching methods give very similar results when the volatility of the underlying asset is 5% or 10%. When raising underlying asset volatility to 30% we note that for K = 90, the *Levy* price is above the upper bound, and the *TW* price is below the lower bound. This pattern repeats itself when the underlying volatility is set to 50%. Now the *TW* price is rather far from the lower-upper bound. The prices in the column *New* is between the upper and lower bounds and practically indistinguishable from the prices reported in Zhang (2002). We can conclude that our method is very accurate. This degree of accuracy results from simultaneously matching the first four moments.²

To further compare the three moment matching approaches we have computed the implied volatility across strike prices. This needs some explanation. A standard option pricing model is based on an assumption that the underlying asset is lognormally distributed. Observed option prices from real markets for instance, can then be inverted into implied volatility; the volatility that reproduces the observed price. If observed prices coincide with theoretical prices for all strikes, then implied volatility will be constant across strikes. We compute the arithmetic average rate futures price, and use the Black (1976) futures option pricing model to back out the implied volatility from option prices generated by each of three moment matching methods. In figure 6.1 we have computed implied volatilities for strikes $K = \{80, 90, 100, 110, 120\}$ and 30% volatility of the underlying asset. All other parameters are unchanged. The horisontal implied volatility resulting from Levy prices is due to the fact that this method assumes lognormality of the arithmetic average (and consequently lognormality of the arithmetic average futures contract). This approach clearly produces too high prices for options that are in the money compared to our new accurate method. For out of the money calls it produces too low prices. The Turnbull and Wakeman (1991) differs from Levy (1992) by the adjustment terms M_1 and M_2 in (6.19). We see that TWprices are closer to *New* prices; the value of in the money call options are cheaper than out of the money options relative to prices from a lognormal model.

In figure 6.2 we repeat the exercise from figure 6.1, but the volatility of the underlying asset is now raised to 50%. The *Levy* prices are too high

²We might also be tempted to conclude that moments higher than the fourth have no impact on option prices. But our approximate model might possibly match higher moments of the aritmetic average as well as the first four and this might be the reason for the good performance.



Figure 6.1: Implied volatility when the underlying asset volatility is 30%. Asian option call prices are computed using three valuation approaches; the lognormal approximation of Levy (1992), the Edgeworth expansion method by Turnbull and Wakeman (1991) and the stochastic volatility approach suggested in this paper. The volatilities are backed out using the Black (1976) option pricing formula, where the input futures price is calculated from the first moment of the arithmetic average. Parameters used for option pricing are: $X_0 = 100$, r = 0.09, $\sigma = 0.3$, $t_1 = 0$, T = 1 and $n = \infty$ (continuously sampled average).



Figure 6.2: Implied volatility when the underlying asset volatility is 50%. (See caption on figure 6.1).

for all strikes. But surprisingly, the TW prices are even worse. This may seem strange since Levy prices are computed from a simplified version of the formula producing TW prices. Accounting for higher moments worsens the performance of the pricing model! The bad performance of the truncated Edgeworth expansion applied to Asian options contrasts the results reported in Jarrow and Rudd and more recently in Backus et al. (1997). Why is it that Edgeworth expansion performs well in some cases and bad in others? The answer lies in the specification of the approximate distribution. In the Edgeworth series expansion, setting the first cumulant of the approximating distribution equal to the cumulant of the true distribution is dictated by the argument of no arbitrage. Choosing the lognormal distribution as the approximate distribution, we only need a second parameter to completely describe the approximating distribution. Jarrow and Rudd (1982) provide a brief discussion on how to choose this second parameter and they do admit that the approach is somewhat arbitrary. They mention three distinct methods: (1) Equate the second cumulant of the approximating distribution with the true distribution (i.e. setting variances equal). (2) Equate the second cumulant of the natural log of the underlying variable with the normal distribution. (3) Equate the instantaneous variance of the true and approximating distribution. In this case both the instantaneous return variance and the instantaneous variance are equal. Jarrow and Rudd (1982) chose the second approach, which produced accurate results. The second approach also has the advantage of being directly in compliance with empirical data. This fact has lead Backus et al. (1997) to restate the formula directly in terms of skewness and excess kurtosis of the natural log of the underlying asset.³ In the case of Asian option pricing we have to use method number (1) since methods (2) and (3) involves the natural log of the arithmetic average, and we do not know the moments of the natural log of the arithmetic average.

Our method, although motivated by the pricing of options written on the continuous arithmetic average, can be applied to Asian option where the average is discretely sampled. The method proposed by Zhang (2002) is limited to options written on a continuously sampled average. Of course, all real life Asian options are discretely sampled. Sometimes continuously based formulas are used as approximations for discrete option pricing. In figure 6.3

³To be precise, Backus et al. (1997) use the Gram-Chalier expansion in stead of the Edgeworth expansion. Both expansions converge to the same density in the limit but differ slightly when truncated after a finite number of lags.



Figure 6.3: Implied volatility for different sampling intervals. Asian option call prices are computed by the stochastic volatility approach suggested in this paper. The volatilities are backed out using the Black (1976) option pricing formula. The input futures price is calculated from the first moment of the arithmetic average. Parameters used for option pricing are: $X_0 = 100$, r = 0.1, $\sigma = 0.5$, $t_1 = 0$, T = 1.

we have computed the implied volatility for an Asian call option using our new method. The first observation in the average is observed today, and the option matures in one year. The other parameters are: $X_0 = 100$, r = 0.08, K = 100 and $\sigma = 0.5$. We note that the fewer observations in the sample the cheaper is the option. The reason for this is quite obvious; when the first observation is known, this means that a greater part of the total average is known when the average consists of only of few observations. Less uncertainty about the final average means lower option price and lower implied volatility in figure 6.3. We also see that an weekly sampling produce implied volatility quite close to continuously sampling. For monthly or quarterly sampling, a continuously approximation is rather inaccurate.

Finally, in figure 6.4, we compute implied volatility for Asian call options



Figure 6.4: Implied volatility for different maturities. Asian option call prices are computed by the stochastic volatility approach suggested in this paper. The volatilities are backed out using the Black (1976) option pricing formula. The input futures price is calculated from the first moment of the arithmetic average. Parameters used for option pricing are: $X_0 = 100, r = 0.1, \sigma = 0.5, t_1 = 0, T = 1$ and $n = \infty$.

when the maturity of the options are varied. Here we use continuously sampled averages and the parameters from figure 6.3, and maturities were set to T = 0.2, 0.4, 0.6 and 0.8. We see that the maturity has very little effect on the implied volatility. This indicates that the distributional properties of the arithmetic average remain quite stable independent of the length of the averaging period.

6.6 Concluding remarks

In this paper we have developed a new approximation for the price of an arithmetic Asian option. We start our analysis by creating a synthetic asset - a futures contract written on the arithmetic average. This financial contract equals the arithmetic average at maturity, hence the Asian option can be interpreted as a European futures option. Unfortunately the dynamics of the futures contract is too complex for us to reach a closed form European option pricing formula. Instead, we define a new futures model approximating the average based contract. The instantaneous return on our contract is specified with stochastic mean-reverting square-root variance. This representation leads to a closed form solution to European options. The parameters of this specification are chosen by matching the variance, skewness and kurtosis of the arithmetic average and the approximate futures model. Numerical comparisons are made to the relevant literature and in particular to the moment matching approaches presented in Turnbull and Wakeman (1991) and Levy (1992). We find that our method produce accurate option prices, even when volatility of the underlying asset is high. Hence the information contained in the first four moments can be utilised to produce far more accurate results than the previous mentioned studies. The futures based approach allows us to back out implied volatility from computed option prices. When comparing the Edgeworth expansion method of Turnbull and Wakeman (1991) and the simplified Wilkinson approximation of Levy (1992) to our method, we find that when the volatility of the underlying asset is high the Edgeworth approximation actually perform worse than the Wilkinson approximation.

Our futures based approach to Asian option pricing can be generalised to a situation where the underlying asset exhibits jumps. In some markets the underlying asset will typically experience sudden, discontinuous jumps. The recursive algorithm described in appendix A can be modified to allow for the possibility of jumps, as long as the jumps are independent of the asset price.

Such an extension is left for future research.

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6.7 Appendix A: Moments of the arithmetic average of geometric Brownian motion

In this appendix we demonstrate how to compute the moments of the arithmetic average of geometric Brownian motion both with continuous and discrete sampling. Throughout this appendix assume that the dynamics of X(t) is given by (6.1) with solution (6.2).

6.7.1 Continuous sampling

First consider continuous sampling, hence the average is given by $A(T) = \frac{1}{T-t_1} \int_{t_1}^{T} X(u) du$. The k'th moment of A(T) at $t = t_n$ is given by:

$$E\left[A(T)^{k}\right] = X_{t}^{k} \frac{k!}{\sigma^{2k}} \left\{ \sum_{j=0}^{k} d_{j}^{(\beta)} \exp\left[\left(\frac{\sigma^{2} j^{2}}{2} + \sigma j \left(\frac{r}{\sigma} - \frac{1}{2}\sigma\right)\right) (T-t)\right] \right\}$$

$$(6.31)$$

where

$$d_{j}^{(\beta)} = 2^{k} \prod_{\substack{h \neq j \\ 0 \leq h \leq n}} \frac{1}{\left((\beta + j)^{2} - (\beta + h)^{2} \right)}$$

 and

$$\beta = \left(\frac{r}{\sigma^2} - \frac{1}{2}\right)$$

See Geman and Yor (1993) for proof.

6.7.2 Discrete sampling

Now consider the discrete average $A(T) = \frac{1}{n} \sum_{i=1}^{n} X(t_i)$ where the *n* tick times $\{t_1 < ... < t_n = T\}$ are equally spaced. As shown in Turnbull and Wakeman (1991) it is possible to set up a recursive relationship to compute the first *k* moments of A(T). First we write A(T) as

$$A(T) = \frac{X(t_1) + \dots + X(t_n)}{n}$$

= $\frac{X(t_1)}{n} \left[1 + \frac{X(t_2)}{X(t_1)} + \frac{X(t_3)}{X(t_1)} \dots + \frac{X(t_n)}{X(t_1)} \right]$
= $\frac{X(t_1)}{n} \left[1 + Z(t_1, t_2) + \dots + Z(t_1, t_n) \right]$
= $\frac{X(t_1)}{n} [1 + Z(t_1, t_2) [1 + Z(t_2, t_3) \times [1 + Z(t_3, t_4)] \dots + Z(t_{n-2}, t_{n-1})] + Z(t_{n-1}, t_n)]]]]$ (6.32)

It is obvious that

$$Z(t_{n-2}, t_{n-1})Z(t_{n-1}, t_n) = \frac{X(t_{n-1})}{X(t_{n-2})} \frac{X(t_n)}{X(t_{n-1})} = \frac{X(t_n)}{X(t_{n-2})}$$

and that $Z(t_{n-2}, t_{n-1})$ and $Z(t_{n-1}, t_n)$ are independent due the independence of increments of Brownian motion. Define the following

$$Y_{1} \equiv 1 + Z(t_{n-1}, t_{n})$$
$$Y_{l} \equiv 1 + Z(t_{n-l-1}, t_{n-l})Y_{l-1}$$

We can find the first four moments of Y_1 in the following way

$$\begin{split} E\left[Y_{1}\right] &= E\left[1 + Z(t_{n-1}, t_{n})\right] \\ &= 1 + E\left[Z(t_{n-1}, t_{n})\right] \\ E\left[Y_{1}^{2}\right] &= E\left[(1 + Z(t_{n-1}, t_{n}))^{2}\right] \\ &= 1 + 2E\left[Z(t_{n-1}, t_{n})\right] + E\left[Z(t_{n-1}, t_{n})^{2}\right] \\ E\left[Y_{1}^{3}\right] &= E\left[(1 + Z(t_{n-1}, t_{n}))^{3}\right] \\ &= 1 + 3E\left[Z(t_{n-1}, t_{n})\right] + 3E\left[Z(t_{n-1}, t_{n})^{2}\right] + E\left[Z(t_{n-1}, t_{n})^{3}\right] \\ E\left[Y_{1}^{4}\right] &= E\left[(1 + Z(t_{n-1}, t_{n}))^{4}\right] \\ &= 1 + 4E\left[Z(t_{n-1}, t_{n})\right] + 6E\left[Z(t_{n-1}, t_{n})^{2}\right] \\ &+ 4E\left[Z(t_{n-1}, t_{n})^{3}\right] + 4E\left[Z(t_{n-1}, t_{n})^{4}\right] \end{split}$$

From the lognormal property of geometric Brownian motion, the kth moment of $Z(t_{n-1}, t_n)$ is explicitly given as

6.8. APPENDIX B: CHARACTERISTIC FUNCTIONS

$$E\left[Z(t_{n-1},t_n)^k\right] = e^{k(r-\frac{1}{2}\sigma^2)\Delta + k^2\frac{1}{2}\sigma^2\Delta}$$

where $\Delta \equiv \frac{t_n-t_1}{n-1}$ is time between ticks. The independence of the Z(.)'s implies independence of the Y's. Working our way backwards from time t_n we can set up the following recursion to compute the moments of Y_l :

$$\begin{split} E\left[Y_{l}\right] &= E\left[1 + Z(t_{n-l-1}, t_{n-l})Y_{l-1}\right] \\ 1 + E\left[Z(t_{n-l-1}, t_{n-l})\right] E\left[Y_{l-1}\right] \\ E\left[Y_{l}^{2}\right] &= E\left[\left(1 + Z(t_{n-l-1}, t_{n-l})Y_{l-1}\right)^{2}\right] = \\ 1 + 2E\left[Z(t_{n-l-1}, t_{n-l})\right] E\left[Y_{l-1}\right] + E\left[Z(t_{n-l-1}, t_{n-l})^{2}\right] E\left[Y_{l-1}^{2}\right] \\ E\left[Y_{l}^{3}\right] &= E\left[\left(1 + Z(t_{n-l-1}, t_{n-l})Y_{l-1}\right)^{3}\right] = \\ 1 + 3E\left[Z(t_{n-l-1}, t_{n-l})\right] + 3E\left[Z(t_{n-l-1}, t_{n-l})^{2}\right] E\left[Y_{l-1}^{2}\right] \\ &+ E\left[Z(t_{n-l-1}, t_{n-l})^{3}\right] E\left[Y_{l-1}^{3}\right] \\ E\left[Y_{l}^{4}\right] &= E\left[\left(1 + Z(t_{n-l-1}, t_{n-l})Y_{l-1}\right)^{3}\right] = \\ 1 + 4E\left[Z(t_{n-l-1}, t_{n-l})\right] E\left[Y_{l-1}\right] + 6E\left[Z(t_{n-l-1}, t_{n-l})^{2}\right] E\left[Y_{l-1}^{2}\right] \\ &+ 4E\left[Z(t_{n-l-1}, t_{n-l})\right] E\left[Y_{l-1}^{3}\right] + 4E\left[Z(t_{n-l-1}, t_{n-l})^{4}\right] E\left[Y_{l-1}^{4}\right] \end{split}$$

Finally the moments of the discrete average condition on time $t \leq t_1$ can be written as:

$$E_t \left[A(T)^k \right] = E_t \left[\left(\frac{X(t_1)}{n} \left[1 + \frac{X(t_2)}{X(t_1)} + \frac{X(t_3)}{X(t_1)} \dots + \frac{X(t_n)}{X(t_1)} \right] \right)^k \right]$$
$$= \frac{X_t e^{k(r - \frac{1}{2}\sigma^2)(t_1 - t) + k^2 \frac{1}{2}\sigma^2(t_1 - t)}}{n} E \left[Y_{n-1}^k \right]$$

for k = 1, 2, 3 and 4.

6.8 Appendix B: Characteristic functions of the distributions $G_1^{\overline{\mathbb{Q}}}$ and $G_2^{\mathbb{Q}}$

In this appendix we will sketch the solution to the explicit expressions of $g_1(i\phi; x_t, v_t)$ and $g_2(i\phi; x_t, v_t)$. In doing so we follow the exposition in Zhu

(1999) closely. Let the dynamics of \overline{F} be given by (6.12) and (6.13). Define $f(t) \equiv \ln \overline{F}(t)$. It is easy to verify that f follows the stochastic process

$$df(t) = -\frac{1}{2}v(t)dt + \sqrt{v(t)}dW(t)$$
(6.33)

$$dv(t) = \kappa \left(\theta - v(t)\right) dt + \sigma_v \sqrt{v(t)} dW(t)$$
(6.34)

Expressing (6.34) in integral form and rearrange gives

$$\int_{t}^{T} \sqrt{v(t)} dW(s) = -\frac{1}{\sigma_{v}} \left(v_{0} + \kappa \theta(T-t) - \kappa \int_{t}^{T} v(t) ds + v(T) \right) \quad (6.35)$$

The characteristic function, $g_1(.)$ for x(T) under the measure $\overline{\mathbb{Q}}$ is given by

$$g_1(i\phi; x_t, v_t) \equiv E_t^{\overline{\mathbb{Q}}} \left[\exp\left(i\phi x(T)\right) \right] = E_t^{\mathbb{Q}} \left[e^{-r(T-t)} \frac{\overline{F}(T)}{\overline{F}(t)} \exp\left(i\phi x(T)\right) \right]$$
(6.36)

with $\overline{f}(T-t)$ defined in (6.21). The last expression in (6.36) can be further manipulated

$$\begin{split} E_t^{\mathbb{Q}} \left[e^{-r(T-t)} \frac{\overline{F}(T)}{\overline{F}(t)} \exp\left(i\phi x(T)\right) \right] \\ &= E_t^{\mathbb{Q}} \left[\exp\left(\left(1+i\phi\right) x(T) - r(T-t) - x(t)\right) \right] \\ &= E_t^{\mathbb{Q}} \left[\exp\left(\left(x(t) + r(T-t)\right) i\phi - \frac{(1+i\phi)}{2} \int_t^T v(s) ds \\ &+ (1+i\phi) \int_t^T \sqrt{v(s)} dW(s) \end{array} \right) \right] \end{split}$$

Inserting for $\int_t^T \sqrt{v(s)} dW(s)$ in (6.35) yields

$$= E_t^{\mathbb{Q}} \left[\exp \left(\begin{array}{c} (x(t) + r(T-t)) i\phi - \frac{(1+i\phi)}{2} \int_t^T v(s) ds \\ -\frac{(1+i\phi)}{\sigma_v} \left(v_0 + \kappa\theta(T-t) - \kappa \int_t^T v(t) ds + v(T) \right) \end{array} \right) \right] \\ = \exp \left(\begin{array}{c} (x(t) + r(T-t)) i\phi - \\ \frac{(1+i\phi)}{\sigma_v} \left(v_0 + \kappa\theta(T-t) \right) \end{array} \right) E_t^{\mathbb{Q}} \left[\exp \left(\begin{array}{c} (1+i\phi) \left(\frac{\kappa}{\sigma_v} - \frac{1}{2} \right) \int_t^T v(s) ds \\ -\frac{(1+i\phi)}{\sigma_v} v(T) \end{array} \right) \right] \\ = \exp \left(\begin{array}{c} (x(t) + r(T-t)) i\phi - \\ p_1 \left(v_0 + \kappa\theta(T-t) \right) \end{array} \right) E_t^{\mathbb{Q}} \left[\exp \left(s_1 \int_t^T v(s) ds - p_1 v(T) \right) \right] \right]$$

where in the last equality we have defined $s_1 \equiv -(1+i\phi)\left(-\frac{1}{2}+\frac{\kappa}{\sigma_v}+\frac{1}{2}(1+\phi i)\right)$ and $p_1 \equiv \frac{(1+i\phi)}{\sigma_v}$. The expectation value $y = E_t^{\mathbb{Q}}\left[\exp\left(s_1\int_t^T v(s)ds - p_1v(T)\right)\right]$ can be found by the Feynman-Kac formula. Let v be time to maturity. When v(t) follows the SDE in (6.34) we obtain the following partial differential equation

$$rac{\partial y}{\partial \upsilon}\left(v,t
ight)=-s_{1}vy\left(v,t
ight)+\kappa\left(heta-v
ight)rac{\partial y}{\partial \upsilon}\left(v,t
ight)+rac{1}{2}\sigma^{2}vrac{\partial^{2}y}{\partial \upsilon^{2}}\left(v,t
ight)$$

with boundary condition

$$y\left(v_t,t\right) = \exp\left(p_1 v(t)\right)$$

This PDE can easily be solved and has the solution

$$y\left(v,\upsilon\right)=e^{A\left(\upsilon;s_{1},p_{1}\right)+C\left(\upsilon;s_{1},p_{1}\right)}$$

where

$$A(\upsilon; s_1, p_1) = \frac{(1 - e^{-\gamma\tau})(2s_1 + \kappa p_1) - \gamma p_1(1 + e^{-\gamma\tau})}{-2\gamma e^{-\gamma\tau} - (\kappa + \gamma - \sigma_v^2 p_1)(1 - e^{-\gamma\tau})}$$
$$C(\upsilon; s_1, p_1) = \frac{2\kappa\theta}{\sigma_v^2} \ln\left[\frac{2\gamma e^{\frac{1}{2}(\kappa-\gamma)\tau} + \frac{1}{2}(\sigma_S k - \kappa - \gamma)}{2\gamma e^{-\gamma\tau} + (\kappa + \gamma - \sigma_v^2 p_j)(1 - e^{-\gamma\tau})}\right]$$
$$\gamma = \sqrt{\kappa^2 + 2\sigma_v^2 s_1}$$

The function $g_2(i\phi; x_t, v_t) \equiv E_t^{\mathbb{Q}} \left[\exp(i\phi x(T)) \right]$ can be found in a similar way, leading to the general expressions (6.23) - (6.27).