Asymmetric Information and Irreversible Investments: Real Option Valuation and Strategies

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Chapter 1

Introduction

In this dissertation we combine finance/real option theory with principal-agent theory and auction theory, using stochastic calculus. The topic is asymmetric information, valuation of uncertain and irreversible investments, and optimal strategies. A decision maker has a real option consisting of a right to implement some project by paying some investment cost. One or more agents have private information about some state variable affecting the investment profitability. The project owning principal and the privately informed agent enter into a mutually beneficial contractual relationship. We show that asymmetric information causes an additional wedge affecting the critical price of project implementation, with the inverse hazard rate being a key component. The owner of the project constructs an optimal contract where the compensation to the agent depends on only observable variables at the time the investment decision is made. The decision to be made is formulated as an optimal stopping problem.

In this introductory chapter the topic of the thesis is motivated, an outline of the thesis is given, the theoretical background is briefly presented, and some related literature is discussed.

1.1 Motivation

The uncertainty considered in the real option literature is usually uncertainty that is common knowledge. However, in many investment projects some uncertainty is privately revealed, and this may result in incentive problems. This aspect is studied in this dissertation. By introducing asymmetric information we extend previous results on valuation and strategies of uncertain investments, using a principal-agent framework and an auction setting, respectively.

The benchmark model is a classic real option problem. An investor owns a right to invest in a project, and his optimization problem is to find the optimal time to invest in the project. The incentive problem arises because an agent has private information: the owner of the investment needs an expert to manage the investment project, as the owner does not possess the technical knowledge of the investment. In chapters 2-4 we assume that the expert privately observes the investment cost, whereas in the last chapter the expert privately receives signals about the output value of the project. The investor and the expert enter into a contractual relationship. We find optimal compensation functions, that induces the expert to make the decisions preferred by the investor. Due to the incentive problem, the optimal investment strategies found are inefficient compared to the benchmark case of no private information. In our framework incentive problems lead to under-investment when the private information is constant, whereas the effect on the optimal investment strategy is ambiguous when the private information is driven by a stochastic variable.

Note that the investor cannot do better than to enter into a contractual relationship as specified in the principal-agent models (chapters 2, 3 and 5) and the auction model (chapter 4)[1]. This means that the investor is better off by entering into a contractual relationship, than to sell the option to invest. The reason is market failure because of asymmetric information. If the investor ex ante wants to sell the investment project at a price based on his expectation of the investment cost, the investor knows that if the agent accepts the price, the investor can do better by entering into a contract with the agent. If the investor's price is too high, then the agent will not buy the investment project. This market failure only occurs when the information is asymmetric at the time the parties contract. If the information is symmetric at the time of contracting, and becomes asymmetric afterwards, the investor is indifferent between selling the option to

[1]This effect was shown in a classic paper by Akerlof (1970). The paper describes a model where the informed party has no way to signal the quality of the good it is selling, which may hinder the functioning of the market.

invest ex ante, or entering into a contractual relationship.

The problem applies to all types of real options where there is private information. Applications can be found both within corporate valuation and government regulation.

One situation where there may be conflicts between option valuation and incentives is the case where a government owns natural resources. Production of natural resources involves large and (partly) irreversible investments, and uncertainty due to future output prices. A feature of production of natural resources is that uncertainty in output prices usually is common knowledge, whereas investment and production costs may be private information for those investing in and operating such projects. To exploit the resources, the government delegates the production of the resources to companies. The privately informed companies may have incentives to signal a higher cost than the true cost in order to obtain a larger profit. Or they may have incentives to increase slack in the organization, or to increase the organization, thereby realizing larger and more expensive projects than necessary.

In a different setting, a company owning the real option delegates some of the investment decisions to suppliers. The same conflicts as described in the situation above, may arise if the suppliers have private information about the costs of the supplies.

Moreover, within corporate valuation, incentive problems resulting from private information may occur between management at different levels in the organization, and between shareholders and managers. Managers will typically know more about their projects than the shareholders do[2].

Contractual relationships as found in the models presented in this dissertation can be applied on such investment problems as described above. Using the models we will study how the private information affect the values and strategies of investment projects managed by privately informed agents.

[2]The contingent claims theory has more recently been applied to evaluation of R&D projects, for example developing new drugs (Schwartz and Moon (2000)), and to projects within information technology (Schwartz and Zozaya-Gorostiza (2000)). In such projects the incentive problems may be large due to the high uncertainty and the specialized learning that occurs.

1.2 Outline of the thesis

In chapters 2 and 3 the optimal stopping models are analyzed within a principal-agent framework. We assume that an agent has private information about the cost of investing in some project. Optimal contracts are found where the investment decisions are delegated to the agent, and where the agent is induced to make the investment decisions preferred by the principal. We find optimal compensation functions that are concavely increasing in the observable, stochastic output value. The optimal compensations are not dependent on the unobservable investment cost, and the compensations are paid at the time the options to invest are exercised. As will be shown from the numerical examples in chapters 2 and 3, the value of private information may be considerable.

In the model of chapter 2 we assume that the agent's private information is constant. This assumption results in a (second-best) optimal investment strategy where the critical price of investment is higher under asymmetric information than under full information. The reason is that the principal's cost of exercising the option is higher under asymmetric information than in a situation of full information. Hence, the contract results in under-investment as long as the stochastic output value is lower than the entry threshold under asymmetric information. When the output value is higher than the critical price under asymmetric information, the optimal investment strategy leads to the same decision whether the information is symmetric or asymmetric. Thus, in this interval the optimal contract just implements a rule of sharing the project value between the principal and the agent, without having any inefficiency effects. Numerical examples in the case where the output value is driven by a geometric Brownian motion show that the agent's value of the contract decreases in volatility, whereas the the effect of volatility on the principal's value is ambiguous. These results are in contrast to the value of the contract under full information, corresponding to the value of an American call option, where the value increases in volatility.

The model formulated in chapter 3 is richer, as the agent's private information is allowed to change stochastically, and new information is continuously obtained. In this case we do not find closed form solutions. Thus, values and strategies are found by a numerical approximations. When the output value and the privately observed investment cost are driven by geometric Brownian motions, we find that both over- and under-investments may occur, depending on the parameter values. The reason is that two effects pull in opposite directions. As in chapter 2 the principal's exercise cost of the contract is higher under under asymmetric information than under full information, tending to an increase in the critical price of investment. However, the higher exercise cost decreases the option value of the project, which tends to reduce the entry threshold. As the volatility increases, the over-investment effect increases in our numerical examples. Moreover, in the examples the agent's value of the contract decreases in volatility, whereas the principal's value increases in this parameter.

In chapter 4 the optimal stopping problems of chapters 2 and 3 are extended to incorporate competition. We assume that n agents compete about winning the contract of managing the investment project. The problem is analyzed within an auction model. We assume that each agent participating in the auction has (perfect) private information about his own costs, but that he does not observe the competitors' cost levels. We find that the optimal (second-best) investment strategies are identical under the cases of competition and no competition. The optimal compensation, however, is lower for the winning agent under competition. than in the case of only one agent.

In the case where the private information is constant, the contract is assigned to the agent who (truthfully) reports the lowest investment cost. The investment decision is delegated to the winner of the auction. On the other hand. when the private information changes stochastically, the investment decision cannot be delegated to an agent. The reason is that under this assumption the winner of the auction is not chosen prior to the time when the investment is exercised. The agent having the lowest cost level at the time the auction starts, does not necessarily have the lowest cost at the time of the investment. Therefore all the agents participate in the auction until the option to invest is exercised. or the investment option expires. Thus, the winner of the auction in the case of a stochastically changing private information is the agent reporting the lowest cost at the time of investment.

In chapter 5 we formulate an investment project where investment decisions are made sequentially. This implies that there is a possibility to temporarily stop the project if the value of the completed project falls[3]. Incentive problems are formulated within a principal-agent framework, where the agent of the project obtains private information about the value of the completed project as new investment phases are completed. The source of private information is different in this model, compared to the models in earlier chapters: the private information is related to the gross project value from exercising the investment option rather than being related to the cost variables. Under this assumption also, we find a delegation based contract.

In chapter 5 we assume that the agent obtains private information only when new investment phases are finished: each time an investment phase is finished the agent obtains a signal about the output value of the completed investment project. We find that the optimal contract only depends on the most current private information, not on earlier reported signals. The reason is that the signals follow a Markov process.

Chapter 6 gives a short summary and points out some simplifying assumptions we have made in the models of the dissertation.

1.3 Related literature

How private information affects the decision strategies and values of real options is a subject that is not treated in many papers. Related literature includes Bjerksund and Stensland (2000), Moel and Tufano (2000), Antle, Bogetoft, and Stark (1998) and MacKie-Mason (1985).

MacKie-Mason (1985) motivates the problems of option values combined with asymmetric information by stating that "making decisions incrementally allows parties to use newly-arriving information, but interested parties are likely to have differential access to new information. Thus, spreading decisions over time creates opportunities to exploit informational asymmetries. Dynamic information generates costs." He models a sequential investment problem, where an agent privately obtains a signal about the value of the project when an investment phase is realized. If the signal is favorable, the next investment phase is started.

[3] I.e., the investment project is analogous to compound options.

An unfavorable signal leads to abandonment of the investment project, and the privately informed agent is given an "abandonment compensation" in order to induce him to abandon the project. MacKie-Mason finds a contract consisting of constant payments, dependent on the agent's private information when it comes to decision making, whereas the contracted payments are independent of the private information. The model formulated in chapter 5 is closely related to MacKie-Mason (1985).

The interaction between options and diverging incentives between a principal and an agent is also analyzed in Antle et al. (1998). They assume that an agent privately observes an investment cost that changes stochastically from one period to the next in a discrete two-period model. The principal makes an investment in one of the two periods, or no investment at all. The only uncertainty in the model is the cost of the investment project, which changes stochastically from one period to another. Thus, the problem can be interpreted as compound options of the European type. The agent observes the current cost at each period in time, but the knowledge about future costs is the same for the principal and the agent. The respective investment triggers at each of the two periods are given by two constants. Antle *et al.* find that the incentive effects from private information tend to defer investment because the investment is done at a higher cost under asymmetric information than under full information. On the other hand, increased volatility by postponing investment tend to reduce the value of waiting, thereby leading to earlier investment. The reason is an inefficient investment trigger in the last period, reduces the principal's advantage of delaying the investment to the last period. The results Antle et al. (1998) with respect to the investment triggers are consistent to the results in chapter 3. This is no surprise as in both models the privately observed investment cost change stochastically over time. The main difference is that Antle et al. (1998) is a discrete two-period model, whereas the model in chapter 3 is formulated in continuous time.

The model formulation in chapter 2 is inspired by Bjerksund and Stensland (2000). In their article it is assumed that an owner of some resource may exploit the resource in two ways: (*i*) Sell the resource in a competitive spot market at a constant price, or (*ii*) ship the resource to an agent for processing and sell the processed resource in a competitive market where the price of the processed resource is stochastic. Bjerksund and Stensland assume that the processing may be switched on and off at no cost[4]. In alternative (*ii*), the owner of the resource ("the regulator") must compensate the agent for the cost of processing the resource. The cost of processing is perfect, private information to the agent, whereas the regulator knows the probability distribution of the costs. The stochastic variable used in Bjerksund and Stensland (2000) is more general than the diffusion process presented in the model in chapter 2. However, the results are consistent: In both models it is found that the agent's private information yields higher costs for the principal, thus creating a wider interval of inaction.

Moel and Tufano (2000) analyze the effects of an actual contract offered by a government, where a copper mine were offered for sale. Moel and Tufano point out that the government did not achieve its stated objectives of the sale, because it failed to recognize the combined effects of options and incentives. Each bid in the auction were required to specify the price of the copper mine, in addition to the amount the winner would spend on investments. The last requirement may induce the winner to make uneconomical investments. Moel and Tufano (2000) discuss the value of the winner's default option, the possibility of renegotiating the contract, and the type of company which is most likely to bid aggressively.

1.4 Theoretical background

Valuation of projects using real option theory has been a field of much research during the last two decades. Real option analysis originates from the insight that uncertain and partly controllable project cash-flows can be reduced to contingent claims on traded assets. Hence, the term "real option valuation" is commonly used when the *contingent-claims* framework is applied to project evaluation.

The theory of contingent claims (or risk-neutral pricing) comes from the literature of financial options. The foundation of contingent claims analysis is the work of Black and Scholes (1973) and Merton (1973), and later extended and formalized by Cox and Ross (1976), Harrison and Kreps (1979) and Harrison and Pliska (1981). Among the first to employ the contingent claims theory on real investments were Brennan and Schwartz (1985) and McDonald and Siegel

[4]Thus, Bjerksund and Stensland (2000) formulate a "switching option", similar to Brennan and Schwartz (1985).

(1986)[5].

In some real investment contexts it may be difficult to reduce the controllable cash-flows to claims on traded assets. The reason is that real assets, much more often than is the case for financial assets, are sold in imperfect markets (if there exists a market at all), see Williams (1995). In the problems we study in this thesis, this problem is avoided: for simplification we assume that uncertain values either can be spanned (in complete markets), or that the risk from the uncertain variables consists of unsystematic risk, only. These assumptions are unrealistic in many cases, but makes it possible to study the problems with respect to asymmetric information within the simple framework of contingent claims analysis in complete markets. Thus, we can analyze the models using the technique of risk-neutral (equivalent martingale) pricing, as described in the books by Huang and Litzenberger (1988) and Duffie (1996), among others.

The incentive problems are modelled based on *adverse selection* models. Much of the adverse selection literature is adapted from the pioneering work of Akerlof (1970). Salanié (1997) defines on page 4 adverse selection models as models where "the uninformed party is imperfectly informed of the characteristics of the informed party; the uninformed party moves first." The adverse selection model is a Stackelberg game in which the principal (the uninformed party) is the leader, and the agent (the informed party) is the follower. The term "adverse selection" comes from insurance: if an insurance company offers insurance based on average risk only, then only the "higher risk" part of the population is attracted. The term is now also used more generally about incentive problems caused by private information. Baron and Myerson (1982) wrote a seminal paper in the context of regulating a firm, and since then much work is done on the case where the firm is better informed of its costs than the regulator. Much of the work is collected in the book by Laffont and Tirole (1993). The principal-agent models of chapters

[5]Other early, and often cited, contributions to the real option theory are (among others) given by Brennan and Schwartz (1985), Majd and Pindyck (1987), Majd and Pindyck (1987), Paddock, Siegel, and Smith (1988), Ekern (1988), Dixit (1989) and Gibson and Schwartz (1990). Some later applications of real option theory are given in Pindyck (1993), Leahy (1993), Quigg (1993), Ekern (1993), Smit and Ankum (1993), Lambrecht and Perraudin (1996), Schwartz (1997), Antle et al. (1998) and Grenadier (1999). Collections on real options have been edited by Lund and Øksendal (1991), Trigeorgis (1995) and Brennan and Trigeorgis (2000), with Schwartz and Trigeorgis (2001) containing early classical readings as well as more recent contributions. Textbooks on real options are Dixit and Pindyck (1994) and Trigeorgis (1996).

2 and 3 are modelled with basis in this tradition.

An important concept for solving adverse selection models is the *revelation principle.* For definition and explanation of the concept, see for example Salanié (1997), Mas-Colell, Whinston, and Green (1995) or Laffont and Tirole (1993). Our definition is based on Salanié (1997), page 16-18: The revelation principle is based on the observation: for each set of implementable *incentive mechanisms*, a contract with the same outcome can also be implemented through a *direct truthful mechanism* where the agent reveals his private information. By an incentive mechanism, we mean the tools the principal employ in order to induce the agent to behave in a certain way. A *direct* mechanism is a mechanism where the agent reports to the principal, and in a *truthful* mechanism the agent finds it optimal to announce the true value of his private information.

Thus, under a revelation mechanism the agent truthfully reports his private information to the principal, and the decision in question is then made according to a decision rule to which the principal has committed himself. The revelation principle simplifies the principal's optimization problem as it reduces the problem to optimizing over the set of truthful mechanisms.

In the principal-agent models of this thesis, we aim at finding compensation functions based on observable variables only, i.e., where there is no need for the agent to report his private information to the principal, and where the decision to be made is delegated to the agent. Hence, the revelation principle is just a device to find such compensation functions. We show that under our assumptions we indeed find optimal contracts where the investment decisions are delegated to the privately informed agent, and where there is no communication between the parties. Moreover, the outcomes of the "delegation contracts" are identical to the contracts where the agent reports his private information. This is consistent with the results of Melumad and Reichelstein (1987) and Melumad and Reichelstein (1989), who find that communication between the principal and the agent is of no value when the private information is perfect or can be spanned.

We often say that a contract is *incentive compatible* when the contract for each privately observed value level gives incentive to implement the decisions preferred by the principal.

In chapter 4 we assume that more than one agent has private information, and we formulate the valuation and strategy problems as auctions. We apply a *second-price sealed-bid private-values auction*, also called a *Vickrey auction*. In such an auction, each bidder simultaneously submits a bid, without seeing others' bids, and the contract is given to the bidder who makes the best bid. However, the contract is priced according to the second-best bidder.

In a Vickrey auction it is a dominant strategy for the bidder to bid according to his true value. Hence, we see a correspondence to the situation of no competition: truth telling is an optimal strategy for the agents participating in a Vickrey auction, as well as in a principal-agent relationship. This resemblance is emphasized in Laffont and Tirole (1987).

Although we will follow the approach of Laffont and Tirole (1987) in chapter 4, it can be shown by the *revenue equivalence theorem* that the results do not depend on the organization of the auction. The revenue equivalence theorem was developed by Vickrey (1961) for some special cases. Myerson (1981) and Riley and Samuelson (1981) were the first to show that Vickrey's results about the equivalence in expected revenue of different auctions apply very generally. One source where the revenue equivalence theorem is presented is Klemperer (1999), and the same version of the theorem is reproduced here: The revenue equivalence theorem says that by any auction mechanism in which (*i*) the contract always goes to the buyer with the best bid, and (*ii*) any bidder with the worst bid expects zero surplus, yields the same expected revenue, and results in each bidder being given the same compensation as a function of his report. Thus, when the revenue equivalence theorem is satisfied, the expected outcome from the auction is the same no matter how the auction is organized. The main assumptions for the revenue equivalence theorem to hold, are that each player has private information about signals, or know their own private value parameters, and that their respective information is independent of others' private information. These assumptions are satisfied in the auction models of chapter 4.

Chapter 2

Asymmetric Information about a Constant Investment Cost

This chapter introduces a base case for examination of dynamic investment decisions when there is an agency problem. A principal delegates to an agent the decision of when to make an investment, given uncertainty about future values. The agent has private information about a deterministic investment cost, whereas the principal only knows the probability distribution of the cost. The principal's problem is how to compensate the agent in order to optimize the value of the principal's investment opportunity.

An optimal compensation function dependent on the observable output from the investment is found. We show that the incentive problem results in under-investment. Furthermore, we find that the agent's option value decreases in volatility, whereas there are two opposing effects on the principal's option value as a function of volatility. These results form a contrast to the full information case, where the increased volatility increases the option value.

2.1 Introduction

We assume that an investor owns an opportunity to invest in a project. The value of the investment project can be formulated as an American call option. At any time the investor has an opportunity to make an irreversible investment at a deterministic cost. Upon the investment, the investor obtains a stochastic output value. We introduce an incentive problem by assuming that the investor delegates the investment strategy of the project to an agent, and the agent has private information about the exact investment cost. The investor (also called the principal) only knows the probability distribution of the cost. A reason for an owner of an investment possibility to delegate the management of a project to an agent, may be that the management requires expertise that the principal does not possess, or that is too costly for him to obtain.

The principal's problem is to find a compensation function that optimizes his value of the contract, given delegation of the investment strategy, and given the agent's preferences. The problem is solved using the revelation principle, which is a direct, truthful mechanism that exploits the fact that to each contract that leads to misreporting of the private information, there exists a contract with the same outcome and with no incentive to misreport.

Under the model assumptions made in this chapter, we show that the principal's and the agent's respective values of the contract are the same whether the investment decision is delegated or not. The reasons are that there are no costs of reporting private information, and the investment trigger is a one-to-one function of the privately observed investment cost.

We find an optimal (second-best) contract that depends on the observable stochastic output value (the value of the "asset in place"), and not on the unobservable cost. However, the agent's compensation depends on the value of the agent's private information.

The optimal contract results in under-investment for some output values as compared to the full information case. If the optimal investment decision under asymmetric information differs from the optimal decision under full information, the total value of the investment is lower than the project value under full information. When the optimal investment decision is the same under asymmetric and full information, the contract only results in a rule of sharing the value between the principal and the agent, without having inefficiency effects.

Furthermore, in numerical examples we find that the agent's option value decreases in the volatility, whereas there are two opposing effects on the principal's option value as a function of volatility. These results contradict the full information case, where the increased volatility increases the option value.

The model in this chapter applies to situations where the production from a project is sold at the market price, and where there are asymmetric information with respect to the investment costs of the project. More generally, the model applies to all types of real options where an agent has private information about the (constant) costs of exercising the option.

In the problem presented here we assume that the agent has private information prior to the date when the parties enter into a contract. Then the principal cannot do better than to enter into a contractual relationship. The reason is that the asymmetric information leads to market failure.

Bjerksund and Stensland (2000) have formulated a somewhat similar model to the one in this chapter. In both models an agent has private information about a constant cost, and the project to be evaluated is affected by a stochastic market value. Furthermore, the projects in both models are evaluated as contingent claims. Whereas our model focuses on an option to invest, the problem in Bjerksund and Stensland (2000) models a production process that can be switched between two modes (on and off), and the processed output is sold at stochastic market prices. The processing cost is private information to an agent. Analogously to the model presented in this chapter, Bjerksund and Stensland (2000) find that private information results in an increase in the principal's cost. The optimal (second-best) compensation function found in Bjerksund and Stensland (2000) is linear in the market price. This result differs from ours as we find that the compensation function is concavely increasing in the market price. The explanation of the difference is that in Bjerksund and Stensland (2000) a compensation *stream* is found, payable whenever processing occurs, whereas the compensation in our model is based on a stochastic market *value*. Thus, in our model, the optimal compensation paid today takes account of future market values, leading to a concavely increasing compensation as a function of the output value.

This chapter is organized as follows: Section 2.2 formulates the problem, and states the model assumptions. Market uncertainty is evaluated assuming dynamically complete markets in section 2.3. In section 2.4 we find the value of the real option, and the corresponding optimal investment strategy under full information. The full information case is our benchmark when analyzing the efficiency of the asymmetric information case. The agent's optimization problem and his value of private information is presented in section 2.5. The optimal investment strategy under asymmetric information is derived in section 2.6, and the optimal compensation function is found in section 2.7. In section 2.8 the results are examined for the special case of a uniformly distributed investment cost, and an output value driven by a geometric Brownian motion. This special case is illustrated numerically in 2.9. Section 2.10 concludes the chapter.

2.2 Model assumptions

A principal owns an option to invest in a project. The investment decision of the project is undertaken by an agent, and the principal compensates the agent based on the output from the project. The output is observable by both parties, whereas the agent has private information about the investment cost. In order to keep a larger part of the profit from the project, the agent may have incentives to report a higher investment cost than the true cost. Thus, the problem for the principal is how to compensate the agent to maximize the value of the principal's investment opportunity, given the agent's incentives.

The agent has perfect knowledge of the true investment cost K of the project, whereas the principal knows only the probability density, $f(\cdot)$, of the investment cost. The cumulative distribution is denoted by $F(\cdot)$, and upper and lower levels of the investment cost are denoted by $\overline{K}$ and $\underline{K}$, respectively. The distribution function $F(\cdot)$ is assumed to be absolutely continuous.

We assume that the option to invest is perpetual, and that the value of the output follows a stochastic process where the uncertainty is common knowledge. Upon investment the owner of the project obtains the stochastic value S_t, which is a function of future cash-flow streams. For short, we refer to S_t as the "output value", or the value of the "asset in place"[1]. The stochastic process is defined on

[1]An example of an interpretation of S_t is given in Bjerksund and Ekern (1990). In their model the value of the "asset in place", S_t, is defined by $S_t = \pi_t q_t$, where π_t is the spot price of oil, following the risk-adjusted price process, $d\pi_t = r\pi_t dt + \sigma\pi_t dB_t^S$, and q_t is a time-adjusted quantity of oil, found by discounting the output $a(t)$ by the convenience yield rate δ, i.e.,

a complete probability space $(\Omega, \mathcal{F}, P)$, and the state space $(0, \infty)$. Furthermore, the stochastic process is adapted to a filtration $\{\mathcal{F}_t^S\}_{0 \leq t \leq T}$, satisfying the usual conditions[2]. The parameter T is the time horizon. The output value is, under the equivalent martingale measure (also called the Q-measure, or the risk-neutral measure)[3] driven by a linear diffusion process[4]

$$dS_t = (rS_t - \delta(S_t))dt + \sigma(S_t)dB_t^S, \quad s \equiv S_0, \tag{2.1}$$

where r is a constant risk free rate, $\delta(S_t) > 0$ reduces the drift in the stochastic process because of the convenience yield, and B_t^S is a standard Brownian motion with respect to the equivalent martingale measure. It is assumed that $\delta(S_t) > 0$ and $\sigma(S_t) > 0$ are bounded, continuous functions.

As the agent privately observes K, in addition to the stochastic process S_t, we denote his information at time t by $\mathcal{F}_t^{S,K}$, whereas the principal's information at time t is given by $\mathcal{F}_t^S$.

It is assumed that the principal's evaluation of the uncertainty in the investment cost is identical under the P and the Q measures. The stopping time with respect to $\mathcal{F}_t^{S,K}$ is denoted by τ_K, where the footnote indicates that the stopping time is a function of the privately observed K. The agent's compensation is denoted by X, and is payable at the investment time.

Below the principal's optimization problem is formulated, with the expectations

$q_t = \int_t^{T-t} e^{-\delta\xi} a(\xi)d\xi$. Their model's investment cost K is defined by $K = \int_t^{T-t} e^{-r\xi} k(\xi)d\xi$, where $k(\xi)$ is the combined investment and production cost rate at time $\xi \geq t$.

[2]We may interpret Ω as the set of all given states, $\mathcal{F}$ is a collection of subsets of Ω, where the elements of $\mathcal{F}$ may be interpreted as events, and P is a probability measure assigning to any event in $\mathcal{F}$. References on this subject are Øksendal (1998), Borodin and Salminen (1996), or Duffie (1996), among many others. By the term "usual conditions" we mean that $\mathcal{F}_t^S$ is *right continuous* and *complete*.

[3]The probability measure P is the measure an investor uses to model his beliefs about future values, and is referred to as the *subjective* probability measure (also called the *true* measure). The measure Q is equivalent to P, but under the Q-measure, the stochastic processes are adjusted so that future values can be evaluated by a risk-free discounting rate. By *equivalent* probability measures we mean that they assign positive probabilities to the same domains, such that with appropriate transformations it is always possible to recover one measure from the other. Useful references to valuation using the equivalent martingale measure are, among others, Duffie (1996), ch. 6.H), Huang and Litzenberger (1988), and Neftci (1996).

[4]A linear diffusion is a one-dimensional, strong Markov process with continuous values paths taking values on an interval. Furthermore, a linear diffusion is a Feller process. See Borodin and Salminen (1996), ch. II, for definition of a linear diffusion.

given under the Q-measure:

The principal optimizes his value function with respect to the compensation function $X(\cdot)$,

$$V^P(s; X(\cdot)) = \sup_{X(\cdot)} E\left[e^{-r\tau_K}\left(S_{\tau_K} - X(S_{\tau_K})\right)^+ \middle| \mathcal{F}_0^S\right] \tag{2.2}$$

subject to the agent's optimization problem,

$$V^A(s, K) = \sup_{\tau_K} E\left[e^{-r\tau_K}\left(X(S_{\tau_K}) - K\right)^+ \middle| \mathcal{F}_0^{S,K}\right]. \tag{2.3}$$

The agent will never reject the contract, as the formulation of the optimization problems implies that the agent's value of participating is always positive. Later we shall see that it is never optimal for the principal to reject the contract for any cost levels $K \in [\underline{K}, \overline{K}]$.

In the formulation of equations (2.2)-(2.3) the principal's problem is to find a compensation function $X(\cdot)$ such that the agent is induced to follow the investment strategy preferred by the principal. In order to ensure that the agent is willing to enter into a contractual agreement with the principal, the value of the contract must be positive.

We want to find a compensation function that is based on observable variables only. In the model, it is assumed that the value of the output upon investment, S_t, is common knowledge. To avoid the agent from behaving opportunistically, the value of the compensation is not paid before the time of investment.

We may note that we assume that the option to invest is perpetual, whereas the equivalent martingale measure is well defined when $T < \infty$. Thus, by the term "perpetual option" we here mean that the expiring date of the option approaches infinity[5]. Formally, if the date the option expires is represented by $\overline{\tau}$, then $\tau_K \in [0, \overline{\tau}]$, where $\overline{\tau} = T \in [0, \infty)$, and $T \to \infty$.

[5]See Aase (2000) for some further remarks on this.

2.3 Valuation of future cash flows

With a perpetual option to invest, and a stationary stochastic process, the optimal investment strategy is time homogeneous. Thus, the "trigger value" of the output is independent of time. This means that the optimal stopping time τ_K^* is given by

$$\tau_K^* = \inf\{t \in [0,T] | S_t \geq S^*(K)\},$$

where $S^*(K)$ is the optimal critical price.

Denote the "unoptimized" value functions of the principal and the agent as $v^P(\cdot)$ and $v^A(\cdot)$, respectively. The initial condition on the stochastic process is given by $s = S(0)$. Because of the time homogeneity we rewrite the principal's and the agent's value functions as,

$$v^P(s; X(\cdot)) = E[e^{-r\tau_K} | \mathcal{F}_0^S] E\left[(S_{\tau_K} - X(S_{\tau_K}))^+ | \mathcal{F}_0^S\right],$$

and,

$$v^A(s, K) = E[e^{-r\tau_K} | \mathcal{F}_0^{S,K}] E\left[(X(S_{\tau_K}) - K)^+ | \mathcal{F}_0^{S,K}\right],$$

respectively. Note that the expected value of the discounting factor is written independently of the output value and the compensation function. This independence simplifies the problem of finding the optimal investment strategy, since we will be able to optimize the option to invest with respect to a "deterministic" trigger level $\hat{S}(K)$, instead of the stochastic trigger S_{τ_K}.

Using results from the theory of linear diffusions[6], the expected value of the discounting factor is formulated as a function of the (arbitrary) trigger level $\hat{S}(K)$, and the time 0 value of the output, s,

$$E[e^{-r\tau_K} | \mathcal{F}_0^{S,K}] = \begin{cases} \frac{\phi(s)}{\phi(\hat{S}(K))} & \text{if } s \leq \hat{S}(K) \\ 1 & \text{if } s > \hat{S}(K). \end{cases} \tag{2.4}$$

Define $u(s) = E[e^{-r\tau_K} | \mathcal{F}_0^{S,K}]$. The function $\phi(\cdot)$ is the strictly positive and increasing, unique solution[7] to the ordinary differential equation,

$$\frac{1}{2}(\sigma(s))^2 u_{ss}(s) + (rs - \delta(s)) u_s(s) - ru(s) = 0, \tag{2.5}$$

[6]Confer Itô and McKean (1965), sect. 4.6. and Borodin and Salminen (1996), ch. II.10.

[7]Given some boundary conditions, the differential equation in (2.5) has two linearly independent and unique solutions, one increasing and the other one decreasing.

with boundaries $\lim_{s \uparrow \hat{S}(K)} u(s) = 1$ and $\lim_{s \downarrow 0} u(s) = 0$. The proof of (2.4)-(2.5) is given in Itô and McKean (1965), section 4.6.

The fraction $\frac{\phi(s)}{\phi(\hat{S}(K))}$ is interpreted as the present value of a claim that pays the value 1 when the investment trigger is represented by $\hat{S}(K)$.[8]

By equation (2.4) the principal's and the agent's value functions can be reformulated. The principal's value function is written as,

$$\begin{aligned} v^P(s; X(\cdot)) &= E\left[\frac{\phi(s)}{\phi(\hat{S}(K))}\left(\hat{S}(K) - X(\hat{S}(K))\right) \mathrm{I}_{\{s \le \hat{S}(K)\}} \right. \\ &\quad \left. +(s - X(s))\mathrm{I}_{\{s > \hat{S}(K)\}} \right| \mathcal{F}_0^S \Big], \end{aligned}$$

where $\mathrm{I}_{\{A\}}$ is the indicator function of the event A. Equivalently, the principal's value function is formulated as

$$\begin{aligned} v^P(s; X(\cdot)) &= \int_{\underline{K}}^{\overline{K}} \left\{\frac{\phi(s)}{\phi(\hat{S}(K))}\left(\hat{S}(K) - X(\hat{S}(K))\right) \mathrm{I}_{\{s \le \hat{S}(K)\}} \right. \\ &\quad \left. +(s - X(s))\mathrm{I}_{\{s > \hat{S}(K)\}} \right\} f(K)dK \end{aligned} \tag{2.6}$$

The agent's value function is given by

$$v^A(s, K) = \begin{cases} \frac{\phi(s)}{\phi(\hat{S}(K))}\left(X(\hat{S}(K)) - K\right) & \text{if } s \le \hat{S}(K) \\ X(s) - K & \text{if } s > \hat{S}(K). \end{cases} \tag{2.7}$$

Note that the value functions are no longer stochastic, but are functions of the "deterministic" trigger level $\hat{S}(K)$ and the value of the output $s = S(0)$. In equations (2.6) and (2.7) the term $\phi(s)/\phi(\hat{S}(K))$ could have been replaced by the simpler notation $u(s)$, where $u(s) \equiv \phi(s)/\phi(\hat{S}(K))$, as $\phi(\hat{S}(K))$ is a constant. However, the term $\phi(s)/\phi(\hat{S}(K))$ is applied as this is the form on which the value of the discounting function is given in the references Borodin and Salminen (1996) and Itô and McKean (1965). In Alvarez and Stenbacka (2001) a similar approach is used.

[8] Similar interpretations are made in Goldstein, Ju, and Leland (2001).

2.4 Benchmark: The principal observes the deterministic costs

As a benchmark, we first study the case where the principal observes the agent's deterministic investment cost K. When the agent has no private information, there is no need for the principal to compensate the agent with more than his true cost, as the principal observes the agent's investment strategy, and can punish the agent if he does not act according to the principal's preferences. Thus, the agent is compensated for his capital cost only, i.e., $X(\cdot) = K$ at the time of investment, and $X(\cdot) = 0$ as long as it is optimal to postpone the investment. Inserting a compensation equal to the true investment cost into the agent's value function in equation (2.7), we find that the agent's optimal value is given by

$$V^A_{sym}(s, K) = 0, \tag{2.8}$$

where the subscript *sym* indicates that this is the value under symmetric information[9].

This means that the principal's optimization problem is formulated identical to the perpetual American call option,

$$V^P_{sym}(s, K) = \sup_{\tau} E\left[e^{-r\tau}\left(S_\tau - K\right)^+ |\mathcal{F}_0^{S,K}\right], \tag{2.9}$$

with the sup being taken over all stopping times τ. The optimal stopping time is given by $\tau^* = \inf\left\{t \geq 0; S_t \geq S^*_{sym}(K)\right\}$, and $S^*_{sym}(K)$ is the optimal investment trigger when the principal has full information. Equation (2.9) is identical to the formulation of an American call option, with τ representing the optimal time of exercising the option, S_t denoting the underlying asset value, and K being the exercise price.

Similarly to the result in equation (2.4), we know that we can express

$$E\left[e^{-r\tau^*}|\mathcal{F}_0^{S,K}\right] = \begin{cases} \frac{\phi(s)}{\phi(S^*_{sym}(K))} & \text{if } s \leq S^*_{sym}(K) \\ 1 & \text{if } s > S^*_{sym}(K) \end{cases}, \tag{2.10}$$

[9]By "symmetric information" we here mean that both parties observe the investment cost parameter. Another interpretation could be that none of the parties have full information, but the probability density of K is common knowledge. In this case the agent will still have a value of zero, as he will have no private information.

where the trigger function now is the optimal investment trigger under full information. Deterministic K and substitution of $X(\cdot)$ for K into the principal's value function in equation (2.6), observing that the upper and lower limits of the integration coincide, leads to

$$V^P_{sym}(s,K) = \begin{cases} \frac{\phi(s)}{\phi(S^*_{sym}(K))}\left(S^*_{sym}(K) - K\right) & \text{if } s \leq S^*_{sym}(K) \\ s - K & \text{if } s > S^*_{sym}(K), \end{cases} \tag{2.11}$$

where the optimal entry threshold $S^*_{sym}(K)$ satisfies the first-order condition,

$$\frac{\partial V^P_{sym}(s,K;S^*_{sym}(K))}{\partial S^*_{sym}(K)} = \frac{\phi(s)}{\phi(S^*_{sym}(K))}\left[1 - \frac{\phi'(S^*_{sym}(K))}{\phi(S^*_{sym}(K))}\left(S^*_{sym}(K) - K\right)\right] = 0. \tag{2.12}$$

For the trigger value in equation (2.12) to be optimal, the second-order condition has to be non-positive, i.e., $\frac{\partial^2 V^P_{sym}(s,K;S^*_{sym}(K))}{\partial S^*_{sym}(K)^2} \leq 0$, (see appendix A.1 for derivation of the second-order condition), yielding the restriction

$$\phi''(S^*_{sym}(K)) \geq 0. \tag{2.13}$$

Note that $\phi(s) = V^P_{sym}(s,K)$ when $s \leq S^*_{sym}(K)$. Thus, the restriction in (2.13) means that the value of a perpetual American call option always increases convexly in the stochastic price, when the stochastic price is driven by a linear diffusion as given in (2.1).

The first-order condition (2.12) can be formulated as

$$S^*_{sym}(K) - K = \frac{\phi(S^*_{sym}(K))}{\phi'(S^*_{sym}(K))}, \tag{2.14}$$

where $S^*_{sym}(K)$ is the optimal critical value for investment. The term on the right-hand side can be interpreted as the opportunity cost of exercising the option with immediate payoff $S^*_{sym}(K) - K$. The fraction captures the wedge between the critical value $S^*_{sym}(K)$ and the investment cost K.

2.4.1 A special case: Geometric Brownian motion

The special case where the stochastic process is given by a geometric Brownian motion gives a well-known closed form solution, as introduced in the economics literature by McKean (1965).

The geometric Brownian motion process of the value of "the asset in place" is given by

$$dS_t = (r - \delta_S)S_t dt + \sigma_S S_t dB_t^S, \quad s = S_0, \tag{2.15}$$

under the equivalent martingale measure Q. The parameter δ_S represents the proportional convenience yield rate, whereas σ_S is the volatility parameter. The function $\phi(s)$ is the strictly positive and increasing solution to the ordinary differential equation (compare equations (2.4) and (2.5)),

$$\frac{1}{2}\sigma_S^2 s^2 u_{ss}(s) + (r - \delta_S) s u_s(s) - r u(s) = 0 \tag{2.16}$$

is then found to equal $\phi(s) = As^\beta$, for some constant A, where

$$\beta = \frac{1}{\sigma_S^2}\left[\frac{1}{2}\sigma_S^2 - (r - \delta_S) + \sqrt{\left((r - \delta_S) - \frac{1}{2}\sigma_S^2\right)^2 + 2r\sigma_S^2}\right] > 1. \tag{2.17}$$

Hence, the solution to the expected value of the discounting factor is (using equation (2.4)),

$$E[e^{-r\tau}|\mathcal{F}_0^{S,K}] = \begin{cases} \left(\frac{s}{S_{sym}^*(K)}\right)^\beta & \text{if } s \leq S_{sym}^*(K) \\ 1 & \text{if } s > S_{sym}^*(K). \end{cases} \tag{2.18}$$

Note that, because $\beta > 1$, the expected value of the discounting factor is a strictly positive, strictly increasing, and strictly convex function of the current price s as long as it is below the trigger price $S_{sym}^*(K)$. Moreover, the term $\left(s/S_{sym}^*(K)\right)^\beta$ is always lower than, or equal to, 1 in the interval $s \leq S_{sym}^*(K)$.

For the benchmark symmetric information case, the right-hand side of equation (2.14) becomes $S_{sym}^*(K)/\beta$, and hence the optimal critical value for investment is

$$S_{sym}^*(K) = K\frac{\beta}{\beta - 1} > K, \tag{2.19}$$

as $\beta > 1$. From equation (2.11), the corresponding value of the investment opportunity is

$$V_{sym}^P(s, K) = \begin{cases} \left(\frac{s}{S_{sym}^*(K)}\right)^\beta K\frac{1}{\beta-1} & \text{if } s \leq S_{sym}^*(K) \\ s - K & \text{if } s > S_{sym}^*(K) \end{cases} \tag{2.20}$$

In appendix A.2 it is shown that the optimality conditions, leading to the values in equations (2.19) and (2.20), are satisfied.

The value in equation (2.19) equals the value of a real option when the asset price follows a geometric Brownian motion, as analyzed in the papers by McDonald and Siegel (1986), Dixit (1989) and Bjerksund and Ekern (1990), among others. The traditional method of finding the optimal value in (2.11) is to exploit the *value matching* and *high contact* conditions[10].

The full information case in equation (2.19) represents the benchmark for the illustrations in sections 2.8-2.9.

2.5 The agent's value of private information

In this section we find an expression of the agent's value of private information. This quantity mirrors how much the principal has to compensate the agent in order to induce the agent to follow the investment strategy preferred by the principal.

The agent's optimal investment strategy under asymmetric information, given a pre-specified compensation $X(\cdot)$, has to satisfy the first-order condition of equa-

[10]For instance, this method is used in Dixit and Pindyck (1994). Applied on the problem in equation (2.9) the value-matching/high-contact approach works as follows: When immediate investment is optimal, the value of the option is given by the payoff $s - K$. In the interval postponing the investment is optimal, the option value v^P_{sym} needs to satisfy the ordinary differential equation

$$(r - \delta_S)\frac{\partial v^P_{sym}}{\partial s} + \frac{1}{2}\sigma_S^2 s^2 \frac{\partial^2 v^P_{sym}}{\partial s^2} - r v^P_{sym} = 0,$$

which has a positive and increasing solution equal to As^β, where A is a constant, and β is given in (2.17). Denote $\hat{S}$ as an arbitrary investment trigger. The value matching condition says that the option value As^β must be equal to the payoff from immediate investment, $s - K$, at the trigger price, i.e.,

$$A\hat{S}^\beta = \hat{S} - K.$$

Furthermore, by the high contact condition (also called the *smooth pasting* condition) the first-order derivative of the two functions is equal at the trigger,

$$A\beta\hat{S}^{\beta-1} = 1.$$

Solving the two equations with respect to the unknown constants, A and $\hat{S}$, lead to $v^P_{sym}(s,K) = V^P_{sym}(s,K)$ and $\hat{S} = S^*_{sym}(K)$ as given by equations (2.19) and (2.20).

tion (2.7) with respect to the investment strategy,

$$\frac{\partial v^A(s,K;\hat{S}(K))}{\partial \hat{S}(K)} = \frac{\phi(s)}{\phi(\hat{S}(K)}\left[X'(\hat{S}(K)) - \frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\Big(X(\hat{S}(K)) - K\Big)\right] = 0. \tag{2.21}$$

For the investment strategy $\hat{S}(K)$ to be optimal, the second-order condition must be non-positive, i.e., $\frac{\partial^2 v^A(s,K;\hat{S}(K))}{\partial \hat{S}(K)^2} \leq 0$. In appendix A.3 it is shown that this requirement yields the sufficient conditions, $X''(\hat{S}(K)) \leq 0$ and $\phi''(\hat{S}(K)) \geq 0$.

Equation (2.21) leads to the agent's optimal investment strategy, given a compensation $X(\cdot)$. However, we do not yet know the optimal compensation function.

It may seem to be a difficult task to optimize the principal's problem with respect to a compensation function. However, the *revelation principle* helps us at the task of capturing the set of possible compensation functions[11]. By the revelation principle, the agent's value of private information can be found.

As described in chapter 1, the revelation principle is a direct mechanism, which means that the principal makes the decision, and compensates the agent, based on the agent's report of his private information. Therefore we temporarily reformulate the model to the situation where the agent reports his private information, and the principal makes the investment decision based on the report. The report is denoted $\hat{K}$. When the agent is to report $\hat{K}$ to the principal, his value function is given by

$$v^A(s,K;\hat{K}) = \begin{cases} \frac{\phi(s)}{\phi(\hat{S}(\hat{K}))}\Big(X(\hat{S}(\hat{K}),\hat{K}) - K\Big) & \text{if } s \leq \hat{S}(\hat{K}) \\ X(s,\hat{K}) - K & \text{if } s > \hat{S}(\hat{K}) \end{cases} \tag{2.22}$$

Note that the investment trigger $\hat{S}(K)$ now is a function of the report $\hat{K}$. The reason is that the report must be consistent with the investment strategy $\hat{S}(\cdot)$, otherwise the principal will detect that the agent lies, and can punish him accordingly. Furthermore, under a direct mechanism the compensation $X(\cdot)$ may depend on the report $\hat{K}$. However, when we have found an optimal investment

[11]See definition in chapter 1, section 1.4. References are the classical articles of Baron and Myerson (1982) and Laffont and Tirole (1993), or the textbooks Laffont and Tirole (1993), Mas-Colell et al. (1995), and Salanié (1997), among others.

strategy using a direct truthful mechanism, we find a delegation based contract that implements this investment strategy[12].

Applied to our model, we show in the appendix, section A.4, that the revelation principle does indeed lead to an optimal contract.

The agent's first-order condition with respect to the report is

$$\frac{\partial v^A(s,K;\hat{K})}{\partial \hat{K}} = 0. \tag{2.23}$$

The agent is induced to report truthfully if his first-order condition is satisfied in the point where $\hat{K} = K$. By the envelope theorem, we find that the first-order condition for optimization with respect to the reported cost parameter[13] is,

$$\frac{dv^A(s,K)}{dK} = \begin{cases} -\frac{\phi(s)}{\phi(\hat{S}(K))} & \text{if } s \leq \hat{S}(K) \\ -1 & \text{if } s > \hat{S}(K). \end{cases} \tag{2.24}$$

In the appendix, section A.5, we find the second-order condition for incentive compatibility. For a contract to satisfy the incentive compatibility constraint we

[12] Melumad and Reichelstein (1987) have shown that in the case where the agent has perfect information about the privately observed quantity, delegation of the decision gives an outcome identical to the situation where the principal makes the decision himself. Denote the optimal entry threshold by $S^*(K)$. Applying the result in Melumad and Reichelstein (1987) we know that a compensation function $X(S^*(K), K)$ under a communication-based centralized contract, by the revelation principle is compatible with the compensation function $X(S^*(K))$ under a direct delegation contract, if for all $K \in [\underline{K}, \overline{K}]$, $X(S^*(K), K) = X(S^*(K))$. This restriction is satisfied when the function $S^*(K)$ is one-to-one. The investment trigger $S^*(K)$ is a one-to-one function as long as it is continuous and strictly increasing in the interval $S^*(K) \in [S^*(\underline{K}), S^*(\overline{K})]$. Later (by equation (2.32)) we will see that the optimal investment trigger actually is strictly increasing in K.

Note that in our model delegation of the investment decision never gives a better outcome than the setting where the principal makes the investment decision himself based on the agent's report. The reason is that in our model communication between the parties is costless.

[13] The report $\hat{K}$ is dependent upon the true investment cost K, i.e., $\hat{K} = \hat{K}(K)$. The first-order condition is

$$\frac{dv^A(s,K;\hat{K}(K))}{dK} = \frac{\partial v^A(s,K;\hat{K}(K))}{\partial \hat{K}(K)} \frac{d\hat{K}(K)}{dK} + \frac{\partial v^A(s,K;\hat{K}(K))}{\partial K}.$$

When the first-order condition in (2.23) is satisfied, the first term on the right-hand side is zero. Incentive compatibility implies $v^A(s,K;\hat{K}) = v^A(s,K;K)$. Thus, the notation can be simplified such that $v^A(s,K) = v^A(s,K;K)$.

here mean that for any privately observed cost level, the agent has incentives to report truthfully. For a delegation based contract we say that the contract is incentive compatible when the agent for any cost level is induced to choose the investment strategy preferred by the principal. In appendix A.5 it is shown that the second-order condition requires that the critical price $\hat{S}(K)$ must be increasing in K for incentive compatibility to be satisfied.

Denote the "inverse trigger function"[14] by $\vartheta(s)$, interpreted as follows: If $K \geq \vartheta(s)$, the option to invest is postponed, whereas it is optimal to invest immediately when $K < \vartheta(s)$. Integrating the condition in (2.24) on both sides of the equality, results in an equivalent condition on the agent's value function (the computations are shown in appendix A.6):

$$v^A(s,K) = \begin{cases} \int_K^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))} du + v^A(s,\overline{K}) & \text{if } s \leq \hat{S}(K) \\ \vartheta(s) - K + \int_{\vartheta(s)}^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))} du + v^A(s,\overline{K}) & \text{if } s > \hat{S}(K). \end{cases} \tag{2.25}$$

Equation (2.25) represents the agent's value of accepting the contract, or in other words, his value of private information.

The term $v^A(s,\overline{K})$, is the agent's value when his true cost equals $\overline{K}$. Since the principal knows that the agent does not have higher cost than $\overline{K}$, the principal does not compensate the agent with more than $\overline{K}$, i.e., $v^A(s,\overline{K}) = 0$ (i.e., we assume that the agent's reservation utility equals zero). However, so far we have not explicitly expressed the investment strategy function $\hat{S}(K)$, which means that we do not yet know what the agent's value of private information amounts to. The optimal investment strategy is found below.

2.6 Asymmetric information: The optimal exercise strategy

In this section we solve the problem of finding the optimal investment strategy, given the agent's private information.

[14]The trigger function $\hat{S}(K)$ has an inverse as long as $\hat{S}(K)$ is strictly increasing for all $K \in [\underline{K}, \overline{K}]$.

In order to simplify the problem of finding an optimal strategy, we substitute the unknown function $X(\cdot)$ in the principal's value function in equation (2.6), with an expression of the agent's private information. Using equations (2.7) and (2.25), remembering that $v^A(s,\overline{K}) = 0$, the compensation function is written as the sum of the value of the true investment cost and the value of the agent's private information,

$$\frac{\phi(s)}{\phi(\hat{S}(K))}X(\hat{S}(K),K) = \frac{\phi(s)}{\phi(\hat{S}(K))}K + \int_K^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))}du, \tag{2.26}$$

when $s \leq \hat{S}(K)$.

Substituting the expression for $\frac{\phi(s)}{\phi(\hat{S}(K))}X(\hat{S}(K))$ in equation (2.26) into the principal's optimization problem in equation (2.6) leads to

$$\begin{aligned} v^P(s) &= \textstyle\int_{\underline{K}}^{\overline{K}} \Big\{ \Big[\frac{\phi(s)}{\phi(\hat{S}(K))} \Big(\hat{S}(K) - K \Big) - \int_K^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))} du \Big] \mathrm{I}\Big\{ s \leq \hat{S}(K) \Big\} \\ &\quad + (s - X(s,K))\mathrm{I}_{\{s > \hat{S}(K)\}} \Big\} f(K)dK. \end{aligned} \tag{2.27}$$

Note that the principal's value function in equation (2.27) no longer consists of the unknown function $X(\cdot)$. The only unknown function is the entry threshold $\hat{S}(K)$. Thus, the optimal threshold is found by finding the first-order condition with respect to the trigger $\hat{S}(K)$. In order to simplify this optimization, we partially integrate the term $\int_{\underline{K}}^{\overline{K}} \int_K^{\overline{K}} \Big[\phi(s)/\phi(\hat{S}(u))\Big] du f(K)dK$, leading to (see the appendix, section A.7),

$$\begin{aligned} v^P(s;\hat{S}(K)) &= \textstyle\int_{\underline{K}}^{\overline{K}} \Big\{ \Big[\frac{\phi(s)}{\phi(\hat{S}(K))} \Big(\hat{S}(K) - K - \frac{F(K)}{f(K)} \Big) \Big] \mathrm{I}\Big\{ s \leq \hat{S}(K) \Big\} \\ &\quad + (s - X(s))\mathrm{I}_{\{s > \hat{S}(K)\}} \Big\} f(K)dK. \end{aligned} \tag{2.28}$$

Equation (2.28) shows that the principal's optimization problem is now identical to the problem of optimally exercising an American call option, with optimal exercise price $K + F(K)/f(K)$. Thus, the principal's value of the contract is positive for any report of $K \in [\underline{K}, \overline{K}]$, and the principal therefore do not reject the contract for any admissible reports.

The term $F(K)/f(K)$ is called the *inverse hazard rate*[15]. The ratio represents the probability that the investment cost is lower than or equal to K, divided by the probability density of K. In our problem $F(K)/f(K)$ can be interpreted as an "inefficiency cost" due to the agent's private information.

In order to ensure a separating optimum, we assume that the fraction $F(K)/f(K)$ is increasing in K.[16] Hence, if we increase K by one unit, the "exercise price" of the option, $K + F(K)/f(K)$, is increased by more than one unit. The reason is that the incentive problem implies that an increase in the compensation for a certain level of K, means that the compensation must be increased for all values of K, as well[17].

The inverse hazard rate often turns up in adverse selection problems: in the classical papers of Baron and Myerson (1982) and Laffont and Tirole (1986) the ratio contributes to "overstate" the costs because of an agent's private information. Such a result is found in our model as well: the principal's "exercise price" of the option to invest increases from K to $K + \frac{F(K)}{f(K)}$.

Optimization of equation (2.28), i.e.,

$$V^P(s) = \sup_{\hat{S}(K)} v^P(s; \hat{S}(K)), \tag{2.29}$$

results in the following optimal investment strategy (see appendix, section A.8)

$$S^*(K) - K - \frac{F(K)}{f(K)} = \frac{\phi(S^*(K))}{\phi'(S^*(K))} > 0. \tag{2.30}$$

Given the compensation function (to be evaluated in the next section), the trigger value in equation (2.30) is also the optimal exercise strategy for the agent. Equation (2.30) shows that the trigger value is based on the principal's total

[15]The term hazard rate (also called the *failure rate*), defined as $\frac{f(a)}{1-F(a)}$, comes from the statistical literature: if the distribution $F(a)$ is the probability of dying before age a, then $\frac{f(a)}{1-F(a)}$ represents the instantaneous probability of dying at age a provided one has survived until then. The term is used in reliability theory and life insurance.

[16]This condition is often called the monotone likelihood ratio property in the principal-agent literature. The condition is satisfied by usual distributions such as the uniform, normal and exponential ones, among others. See Laffont and Tirole (1993), pp 66-67.

[17]Similar interpretations are made in the textbook by Laffont and Tirole (1993), p. 65, and in a paper by Antle et al. (1998).

cost of exercising the investment option, i.e., it is based on $K + F(K)/f(K)$. The fraction $F(K)/f(K)$ is positive, and increasing in K. Thus, it tends to increase the investment trigger $S^*(K)$. As $\phi(\cdot)$ is strictly increasing and positive, $S^*(K) > K + \frac{F(K)}{f(K)}$.

As in equation (2.14), the right-hand side of equation (2.30) represents the opportunity cost of exercising the option. If we rearrange the respective investment triggers of the full information and the asymmetric information cases, we find that

$$S^*_{sym}(K) = K + \frac{\phi(S^*_{sym}(K))}{\phi'(S^*_{sym}(K))}$$

and

$$S^*(K) = K + \frac{F(K)}{f(K)} + \frac{\phi(S^*(K))}{\phi'(S^*(K))}$$

The incentive problem may imply under-investment if $S^*_{sym}(K) < S^*(K)$: In the interval where $S^*(K) < s \leq S^*_{sym}(K)$ it is optimal to invest immediately if the principal has full information, whereas it is not optimal to invest in the case where we assume that the agent has private information. We cannot from the difference

$$S^*(K) - S^*_{sym}(K) = \frac{F(K)}{f(K)} + \frac{\phi(S^*(K))}{\phi'(S^*(K))} - \frac{\phi(S^*_{sym}(K))}{\phi'(S^*_{sym}(K))},$$

say whether asymmetric information leads to under- or over-investment, as we cannot determine the sign of the difference. However, the difference above show that for $K = \underline{K}$ we obtain $S^*(K) = S^*_{sym}(K)$. In section 2.7 we show that because the agent's value of private information is always positive, we need to have $S^*_{sym}(K) \leq S^*(K)$.

Implicit differentiation of the entry thresholds tells us that they always are increasing in the investment cost (derived in the appendix, section A.9):

$$(S^*)'_{sym}(K) = \frac{\phi'(S^*_{sym}(K))^2}{\phi(S^*_{sym}(K))\phi''(S^*_{sym}(K))} \tag{2.31}$$

and

$$(S^*)'(K) = \left[1 + \frac{\partial(F(K)/f(K))}{\partial K}\right] \frac{\phi'(S^*(K))^2}{\phi(S^*(K))\phi''(S^*(K))}. \tag{2.32}$$

By assumption we know that $\frac{\partial(F(K)/f(K))}{\partial K} > 0$ and $\phi''(\cdot) > 0$. In addition, we have $\phi(\cdot) > 0$, $\phi'(\cdot) > 0$. This means that the right-hand sides of (2.31) and (2.32) are positive. Thus, by (2.32) the requirement for incentive compatibility, given by the second-order condition in appendix A.5, is satisfied.

2.7 Implementation of the optimal compensation function

We are now left with the problem of finding an implementable compensation function that leads to the optimal investment strategy. By equations (2.4), (2.7), (2.25), (2.26) and (2.30), the time zero value of the optimal compensation function is given by

$$E\left[e^{-r\tau_K^*}X(S_{\tau_K},K)|\mathcal{F}_0^{S,K}\right]$$
$$= \begin{cases} \frac{\phi(s)}{\phi(S^*(K))}K + \int_K^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))}du & \text{if } s \leq S^*(K) \\ \vartheta(s) + \int_{\vartheta(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))}du & \text{if } s > S^*(K). \end{cases} \tag{2.33}$$

By observing that $\vartheta \in [\underline{K}, \overline{K}]$, equation (2.33) shows that the value of the compensation cannot be higher than $\overline{K}$. This result is in accordance with intuition: as the principal knows that the agent does not have a cost higher than $\overline{K}$, the principal will not compensate the agent by more than this quantity. Hence, we obtain the following optimal compensation function at the time the investment is made, i.e., when $s > S^*(K)$, or equivalently, when $\vartheta(s) > K$,

$$X(s) = \begin{cases} \vartheta(s) + \int_{\vartheta(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))}du & \text{if } s \leq S^*(\overline{K}) \\ \overline{K} & \text{if } s > S^*(\overline{K}). \end{cases} \tag{2.34}$$

Observe that at the time the investment is made, the optimal compensation $X(\cdot)$ does not depend on the unobservable investment cost K. Thus, we have found a compensation that gives the same outcome whether the agent's privately observed investment cost parameter is reported or not, i.e., $X(s) = X(s, K)$. This means that if the investor offers the agent a contract where the agent is paid according to the compensation in equation (2.34), and the compensation is paid at the time the investment is made, the agent will accept the contract, and it is optimal for him to follow the investment strategy given by equation (2.30). Hence, this contract results in an optimal investment strategy, and optimal values to both parties.

We have found a result that is consistent to the findings in Melumad and Reichelstein (1987). Recall that Melumad and Reichelstein find that a compensation function under a communication-based centralized contract (by the revelation principle) is compatible with the compensation function under a direct delegation contract if the decision to be made is a one-to-one function of the unobservable variable. In our model this implies that $\frac{\phi(s)}{\phi(S^*(K))}X(S^*(K)) = \frac{\phi(s)}{\phi(S^*(K))}X(S^*(K),K)$ when $s \leq S^*(K)$, whereas $X(s) = X(s,K)$ when $s > S^*(K)$. In appendix A.10 we show how to derive $\frac{\phi(s)}{\phi(S^*(K))}X(S^*(K)) = \frac{\phi(s)}{\phi(S^*(K))}X(S^*(K),K)$.

The optimal contract in equation (2.34) represents an implementable compensation function dependent upon the observable quantity s, only. Recall that although the investment cost K is unobservable to the principal, he observes the date the investment is made, implying that he observes at which level of the output value, s, the investment is made. Thus, he may be able to derive the true investment cost ex post. However, since the principal is committed to the contracted compensation he cannot exploit this ex post information.

The compensation in equation (2.34) satisfies the incentive compatibility constraint, and it is therefore optimal for the agent to follow the investment strategy found by (2.30). When $s \leq S^*(K)$, the agent postpones the option to invest until the point in time where the value of the output, s, reaches $S^*(K)$. It is optimal for him for invest at once if $s > S^*(K)$.

By equations (2.25) and (2.30) we find that the agent's optimized value is represented by

$$V^A(s,K) = \begin{cases} \int_K^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))} du & \text{if } s \leq S^*(K) \\ \vartheta(s) + \int_{\vartheta(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))} du - K & \text{if } S^*(K) < s \leq S^*(\overline{K}) \\ \overline{K} - K & \text{if } s > S^*(\overline{K}). \end{cases} \tag{2.35}$$

In section 2.4 it was shown that the agent's value from the investment is zero under symmetric information about the investment cost. Equation (2.35) states that the agent's value from the investment when he has private information about the cost, is strictly positive as long as his investment cost is below $\overline{K}$. If the agent's cost parameter equals $\overline{K}$, we find by inserting $K = \overline{K}$ into the agent's value in (2.35) that $V^A(s, \overline{K}) = 0$. The agent's value from the project will never exceed $\overline{K} - K$.

The principal's optimized value is, using equations (2.6), (2.27), (2.30) and (2.34),

$$\begin{aligned} V^P(s) \;=\; & \int_{\underline{K}}^{\overline{K}} \Big\{ \Big[\tfrac{\phi(s)}{\phi(S^*(K))} \left(S^*(K) - K\right) - \int_K^{\overline{K}} \tfrac{\phi(s)}{\phi(S^*(u))} du \Big] \mathrm{I}_{\{s \le S^*(K)\}} \\ & + \Big[s - \vartheta(s) - \int_{\vartheta(s)}^{\overline{K}} \tfrac{\phi(s)}{\phi(S^*(u))} du \Big] \mathrm{I}_{\{s > S^*(K)\}} \Big\} f(K) dK. \end{aligned} \tag{2.36}$$

In order to study the loss in value due to asymmetric information, we want to examine the principal's value for each cost level $K \in [\underline{K}, \overline{K}]$. For this reason we find the principal's value for a *given* cost level, and denote the value $\tilde{V}^P(s, K)$,

$$\tilde{V}^P(s, K) = \begin{cases} \frac{\phi(s)}{\phi(S^*(K))} \left(S^*(K) - K\right) - \int_K^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))} du & \text{if } s \le S^*(K) \\ s - \vartheta(s) - \int_{\vartheta(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))} du & \text{if } s > S^*(K). \end{cases} \tag{2.37}$$

Thus, $V^P(s) = \int_{\underline{K}}^{\overline{K}} \tilde{V}^P(s, K) f(K) dK$.

We can interpret the value in equation (2.37) as the principal's value just after the contract is entered into: The principal is committed to the contracted compensation in equation (2.34), and is informed about the investment cost.

Equation (2.37) shows that the principal's value for a given cost level is reduced by the agent's value of private information, where the value of private information is represented by the term $\int_K^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))} du$ when $s \le S^*(K)$. Note that even when $K = \overline{K}$, implying that the value of information equals zero, is the principal's value in (2.37) lower than the principal's value under full information. The reason is that the optimal investment strategy under asymmetric information is a second-best strategy.

In order to examine the efficiency of the optimal contract, we define deadweight-losses as follows. The value of the loss given that the principal does not know the value of the investment cost K, is defined as

$$L(s) \equiv \int_{\underline{K}}^{\overline{K}} \left[V^P_{sym}(s,K) + V^A_{sym}(s,K) - \left(\tilde{V}^P(s,K) + V^A(s,K) \right) \right] f(K)dK,$$

where $V^P_{sym}(s,K)$ is given by equation (2.11), $V^A_{sym}(s,K) = 0$ by equation (2.8), $\tilde{V}^P(s,K)$ is formulated in equation (2.37) and $V^A(s,K)$ is found in equation (2.35). For a *given* cost level, the loss is defined as

$$\tilde{L}(s,K) \equiv V^P_{sym}(s,K) + V^A_{sym}(s,K) - \left(\tilde{V}^P(s) + V^A(s,K) \right).$$

Observe that the loss $\tilde{L}(s,K)$ must be positive as the value of the project is at least as high under full information as the sum of the principal's and the agent's respective project values under asymmetric information. We can show that $\tilde{L}(s,K) \geq 0$ implies that the critical price under asymmetric information is larger than, or equal to, the critical price under full information, i.e., $S^*(K) \geq S^*_{sym}(K)$. This is found by the following arguments. As a contradiction, suppose that $S^*(K) < S^*_{sym}(K)$. Then, if $S^*(K) < s \leq S^*_{sym}(K)$, we find

$$\begin{aligned} &\tilde{L}(s,K)\mathrm{I}_{\{S^*(K)<s\leq S^*_{sym}(K)\}} \\ &= \left(V^P_{sym}(s,K) + V^A_{sym}(s,K) - \tilde{V}^P(s,K) - V^A(s,K) \right) \mathrm{I}_{\{S^*(K)<s\leq S^*_{sym}(K)\}} \\ &= \tfrac{\phi(s)}{\phi(S^*_{sym}(K))} \left(S^*_{sym}(K) - K \right) - (s-K). \end{aligned}$$

From equation (2.11) we know that $\frac{\phi(s)}{\phi(S^*_{sym}(K))} \left(S^*_{sym}(K) - K \right) - (s-K) \leq 0$ when $s \leq S^*_{sym}(K)$, with strict inequality for $s < S^*_{sym}(K)$. Thus, when $s < S^*_{sym}(K)$, we obtain $\tilde{L}(s,K) < 0$, which contradicts the requirement $\tilde{L}(s,K) \geq 0$. This implies that we will always have $S^*(K) \geq S^*_{sym}(K)$.

Below it is shown that the dead-weight loss is found as (in appendix A.11 the component value functions are presented),

$$\tilde{L}(s,K) = \begin{cases} \frac{\phi(s)}{\phi(S^*_{sym}(K))} \left(S^*_{sym}(K) - K \right) \\ -\frac{\phi(s)}{\phi(S^*(K))} \left(S^*(K) - K \right) & \text{if } s \leq S^*_{sym}(K) \\ s - K - \frac{\phi(s)}{\phi(S^*(K))} \left(S^*(K) - K \right) & \text{if } S^*_{sym}(K) < s \leq S^*(K) \\ 0 & \text{if } s > S^*(K). \end{cases} \tag{2.38}$$

The total dead-weight loss is positive when $s \leq S^*(K)$ and equal to zero when $s > S^*(K)$. The explanation is that when $s \leq S^*(K)$, the contract may lead to an inefficient investment strategy, whereas we have identical, optimal investment strategies under the full and asymmetric information cases when $s > S^*(K)$. Thus, when $s > S^*(K)$ the contracted compensation function only gives a sharing rule between the principal and the agent, i.e., the agent's gain exactly equals the principal's loss because of the asymmetric information.

The principal's expected loss, $L(s) = \int_{\underline{K}}^{\overline{K}} \tilde{L}(s, K) f(K) dK$, is represented by,

$$\begin{aligned} L(s) \quad = \quad & \int_{\underline{K}}^{\overline{K}} \left\{ \left[\frac{\phi(s)}{\phi(S^*_{sym}(K))} \left(S^*_{sym}(K) - K\right) - \frac{\phi(s)}{\phi(S^*(K))} \left(S^*(K) - K\right) \right] \mathrm{I}_{\{s \leq S^*_{sym}(K)\}} \right. \\ & \left. + \left[s - K - \frac{\phi(s)}{\phi(S^*(K))} \left(S^*(K) - K\right) \right] \mathrm{I}_{\{S^*_{sym}(K) < s \leq S^*(K)\}} \right\} f(K) dK. \end{aligned} \tag{2.39}$$

2.8 A special case: Geometric Brownian process and uniform distribution

The preceding sections used a time-homogeneous Ito diffusion (equation (2.1)) for the output process S_t, and an unspecified probability density $f(\cdot)$ for the assessed investment cost K. In order to illustrate our results, we will now examine the special case where the value of the "asset in place", S_t, follows a geometric Brownian motion, and the investment cost, K, is uniformly distributed. By these assumptions we obtain closed form solutions. We compare the optimal contract under asymmetric information to the case of full information in section 2.4.1.

A uniform distribution over the interval $[\underline{K}, \overline{K}]$ implies that $f(K) = \frac{1}{\overline{K} - \underline{K}}$ and $F(K) = \frac{K - \underline{K}}{\overline{K} - \underline{K}}$, and thus the inverse hazard rate $F(K)/f(K) = K - \underline{K}$. From the assumption of geometric Brownian motion, we find that $\phi(s) = As^{\beta}$, which yields $\frac{\phi(S^*(K))}{\phi'(S^*(K))} = \frac{S^*(K)}{\beta}$. By plugging this expression into the right-hand side of (2.30), we find that $S^*(K) = \left(K + \frac{F(K)}{f(K)}\right) \frac{\beta}{\beta - 1}$. Hence, the optimal trigger value is solved to yield

$$S^*(K) = (2K - \underline{K}) \frac{\beta}{\beta - 1}. \tag{2.40}$$

We find that, when $K > \underline{K}$, the entry threshold in (2.40) is higher than the trigger under symmetric information in (2.19), $S^*_{sym}(K) = K\beta/(\beta - 1)$. The fraction $\beta/(\beta - 1) > 1$ causes a wedge between the critical value for exercising the investment opportunity and the principal's cost of the investment, even in the case of symmetric information. The positive difference $S^*(K) - S^*_{sym}(K) = (K - \underline{K})\beta/(\beta - 1)$ is the increase in the investment trigger caused by asymmetric information in the special case.

Note that even though the principal knows that the agent does not have a higher investment cost than $\overline{K}$, the investment strategies in equation (2.40) and (2.19) are not equal when $K = \overline{K}$. The reason is that the investment trigger needs to be increasing in the report $\hat{K}$ in order to ensure incentive compatibility for all types of K, as shown by the second-order condition for incentive compatibility in equation (A.7) in the appendix. If the investment trigger were lower for a report of $\overline{K}$ than for a cost level lower than $\overline{K}$, then the agent always would report $\overline{K}$.

The inverse investment trigger $\vartheta(S^*(K)) \equiv K$, equals by equation (2.40),

$$\vartheta(S^*(K)) = \frac{1}{2}\left(S^*(K)\frac{\beta - 1}{\beta} + \underline{K}\right).$$

Thus, the inverse investment strategy is to invest when $\vartheta(s) > K$, where $\vartheta(s) = \frac{1}{2}\left(s\frac{\beta-1}{\beta} + \underline{K}\right)$.

In order to find the compensation function $X(s)$, we insert the above expression of the inverse entry threshold into the compensation function in (2.34). By some calculation we find the following expression for the optimal compensation function when $s > S^*(K)$,

$$X(s) = \begin{cases} \frac{1}{2}\left[s + \underline{K} - \left(\frac{s}{S^*(\overline{K})}\right)^{\beta} \frac{2\overline{K} - \underline{K}}{\beta - 1}\right] & \text{if } s \leq S^*(\overline{K}) \\ \overline{K} & \text{if } s > S^*(\overline{K}). \end{cases} \tag{2.41}$$

When $s \leq S^*(K)$, $X(s) = 0$.

The value functions of the agent and the principal (equations (2.35) and (2.37)), are then found to be equal to

$$V^A(s,K) = \begin{cases} \left(\frac{s}{S^*(K))}\right)^\beta \frac{1}{2}\left[\frac{2K-\underline{K}}{\beta-1} - \left(\frac{S^*(K)}{S^*(\overline{K})}\right)^\beta \frac{2\overline{K}-\underline{K}}{\beta-1}\right] & \text{if } s \leq S^*(K) \\ \frac{1}{2}\left[s - (2K-\underline{K})\left(\frac{s}{S^*(\overline{K})}\right)^\beta \frac{2\overline{K}-\underline{K}}{\beta-1}\right] & \text{if } S^*(K) < s \leq S^*(\overline{K}) \\ \overline{K} - K & \text{if } s > S^*(\overline{K}), \end{cases} \tag{2.42}$$

and

$$\tilde{V}^P(s,K) = \begin{cases} \left(\frac{s}{S^*(K)}\right)^\beta \frac{1}{2}\left[S^*(K) - \underline{K} + \left(\frac{S^*(K)}{S^*(\overline{K})}\right)^\beta \frac{2\overline{K}-\underline{K}}{\beta-1}\right] & \text{if } s \leq S^*(K) \\ \frac{1}{2}\left[s - \underline{K} + \left(\frac{s}{S^*(\overline{K})}\right)^\beta \frac{2\overline{K}-\underline{K}}{\beta-1}\right] & \text{if } S^*(K) < s \leq S^*(\overline{K}) \\ s - \overline{K} & \text{if } s > S^*(\overline{K}), \end{cases} \tag{2.43}$$

respectively.

Observe that the total combined value for the principal and the agent is

$$\tilde{V}^P(s,K) + V^A(s,K) = \begin{cases} \left(\frac{s}{S^*(K)}\right)^\beta (S^*(K) - K) & \text{if } s \leq S^*(K) \\ s - K & \text{if } s > S^*(K) \end{cases} \tag{2.44}$$

in the case of asymmetric information.

Consistent with (2.38), the deadweight loss $\tilde{L}(s,K)$ is, in the case the assumptions of a geometric Brownian motion and a uniform density, equal to

$$\tilde{L}(s,K) = \begin{cases} \left(\frac{s}{S^*_{sym}(K)}\right)^\beta (S^*_{sym}(K) - K) \\ -\left(\frac{s}{S^*(K)}\right)^\beta (S^*(K) - K)) & \text{if } s < S^*_{sym}(K) \\ s - K - \left(\frac{s}{S^*(K)}\right)^\beta (S^*(K) - K) & \text{if } S^*_{sym}(K) < s \leq S^*(K) \\ 0 & \text{if } s > S^*(K). \end{cases} \tag{2.45}$$

Below the results are illustrated numerically.

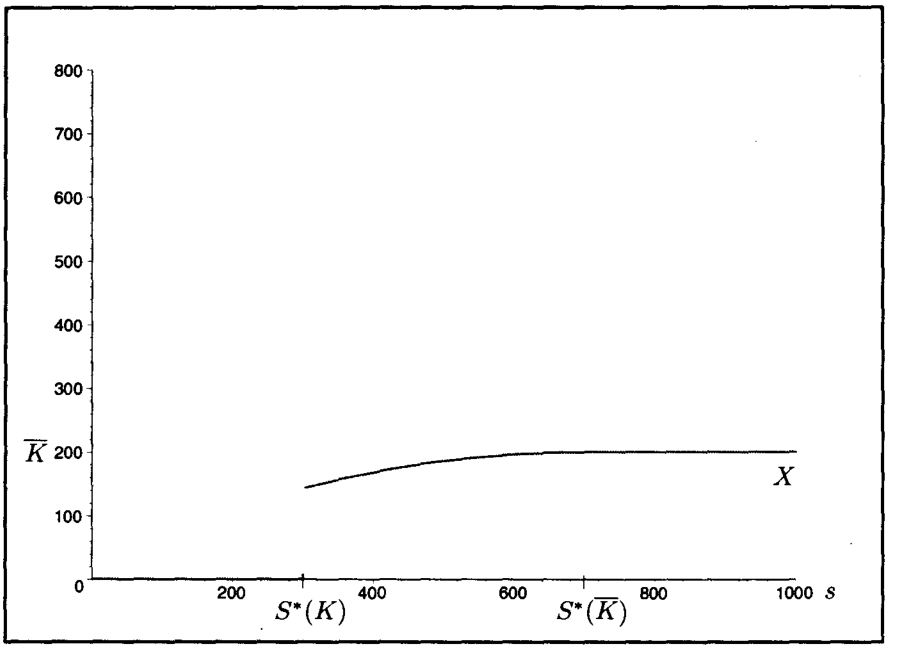

Figure 2.1: The compensation X as a function of the output value s.

2.9 Illustration of special case results

The results for the special case discussed in the previous section will now be illustrated numerically.

The parameter values are given in the table:

Base case:	The investment cost:	$K = 100$
	The lower limit of the investment cost:	$\underline{K} = 50$
	The upper limit of the investment cost:	$\overline{K} = 200$
	The risk-free rate:	$r = 0.05$
	The proportional convenience yield:	$\delta = 0.03$
	Volatility of asset in place:	$\sigma_S = 0.10$

These parameter values lead to the following pre-computed constants in the base case:

The probability density, $\underline{K} \le K \le \overline{K}$:	$f(K) = \frac{1}{150}$
The distribution:	$F(K) = \frac{50}{150}$
The inverse hazard rate:	$\frac{F(K)}{f(K)} = 50$
The positive root satisfying the ODE:	$\beta = 2$
The entry threshold, full information, eq. (2.19):	$S^*_{sym}(K) = 200$
The entry threshold, asymmetric information, eq. (2.40):	$S^*(K) = 300$
The entry threshold, asymmetric info., $K = \overline{K}$, eq. (2.40):	$S^*(\overline{K}) = 700$

In Figure 2.1 the compensation, specified in equation (2.41), is plotted as a function of the output value s. The compensation is zero when s is lower than or equal to the critical value of investment, $S^*(K) = 300$, as the compensation is not paid prior to the investment time. The compensation is increasing in the interval where $S^*(K) < s \le S^*(\overline{K})$. For $s > 700$ the compensation is constant at its maximum level $\overline{K} = 200$. In the case where the value of the output is driven by a geometric Brownian motion the compensation increases concavely (shown in appendix A.13).

Because of the agent's private information, the principal can never do better than to enter into a contractual relationship. However, in the appendix, section A.12, it is shown that when $s > S^*(\overline{K})$, the principal can do just as well by selling the project ex ante.

In the numerical example the compensation function is concave. The reason is that the upper level for the cost has a significant effect. If the upper level cost had been very high, the compensation would have approached a linear function of s. This can be seen by examination of equation (2.41): If $\overline{K}$ gets very large, then the trigger $S^*(\overline{K})$ becomes very large too, making the value of the discounting factor $\left(s/S^*(\overline{K})\right)^\beta$ close to zero. Thus, the term that makes the compensation function $X(s)$ concave goes to zero, and we are left with a compensation function that is linear in the stochastic output value s.

In Figure 2.2 the principal's and the agent's value functions (given by equations (2.20), (2.42) and (2.43)) are shown as functions of s. In addition, the sum of the principal's and the agent's value functions are drawn. The sum of $\tilde{V}^P$ and V^A is

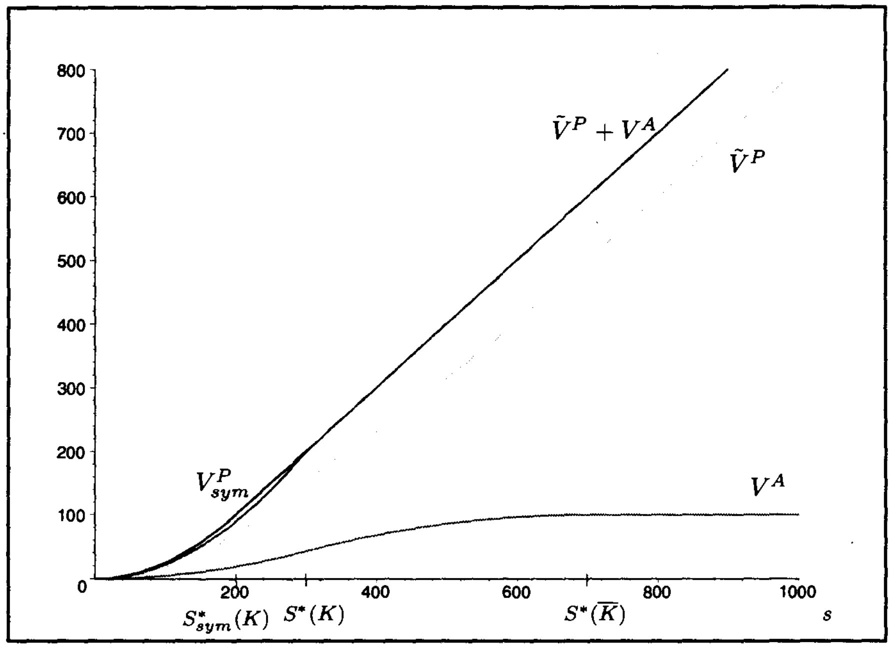

Figure 2.2: The values V^P_{sym}, $\tilde{V}^P + V^A$, $\tilde{V}^P$ and V^A, as functions of s.

identical to the principal's value under full information in the interval $s > S^*(K)$, and is otherwise lower than the principal's value under full information. Thus, here we illustrate that only when $s > S^*(K)$ does the the contract between the principal and the agent result in a sharing rule without having a dead-weight loss. In the interval $(0,\ S(K))$, $\tilde{V}^P(s, K) + V(s, K)$ is lower than $V^P_{sym}(s, K)$ because of a second-best investment strategy.

The principal's value under full information increases convexly when $s \leq S^*_{sym}(K) = 200$, and is linearly increasing in the interval where the optimal decision is to invest immediately. This corresponds to the value of a "standard" real option as a function of the output price. Under asymmetric information, it is also the case that the principal's and the agent's respective values increases convexly in the interval where it is ex ante profitable to postpone the investment, i.e., when

$s \leq S^*(K)$. This is for the same reason as under symmetric information: a volatility higher than zero implies a possibility of higher profitability in the future.

In the interval $S^*(K) < s \leq S^*(\overline{K})$ the agent's value is concavely increasing for the same reason as for the concavity in the compensation function: the upside potential for future profit is limited. For $s > S^*(\overline{K})$ the principal alone benefits from higher s, and the agent's value of the contract is constant at $\overline{K} - K = 100$.

Since the agent's value of information leaves less profit to the principal, and the agent's value function increases concavely in the interval $(S^*(K), S^*(\overline{K})]$, the principal's value increases convexly in the same interval. When $s > S^*(\overline{K})$, the principal's value under asymmetric information increases linearly, as the agent's value of information is zero in this interval.

By taking the first- and second-order conditions of the value functions $V^P_{sym}(s, K)$, $V^A(s, K)$ and $\tilde{V}^P(s, K)$ (given by equations (2.20), (2.42) and (2.43), respectively) with respect to s, we find that the concavity and convexity properties hold for all admissible parameter values. This is shown in appendix A.13.

In Figure 2.3 the relative dead-weight loss is plotted as a function of s in the lower curve. The relative dead-weight loss is defined as $(V^P_{sym} + V^A_{sym} - \tilde{V}^P - V^A)/V^P_{sym})$. The figure shows that the relative dead-weight loss is positive when $s \leq S^*(K) = 300$.

Furthermore, in Figure 2.3 the principal's relative loss, $(V^P_{sym} - \tilde{V}^P)/V^P_{sym}$, is plotted in the upper curve. Both the principal's relative loss, and the relative dead-weight loss are constant as long as the best decision under both asymmetric and symmetric information is to postpone the investment, i.e., when $s \leq S^*_{sym}(K) = 200$. When $s \leq 200$, $\tilde{L}/V^P_{sym}(K) = \frac{1}{9}$ and $(V^P_{sym} - \tilde{V}^P)/V^P_{sym} = \frac{19}{63}$ in our example.

The losses are decreasing in the interval $(S^*_{sym}(K), S^*(K)]$, since the inefficiency in the second-best investment strategy is decreasing as s approaches $S^*(K) = 300$. For all s higher than this point the investment decision is the same for the symmetric and the asymmetric information case, i.e., there is no dead-weight loss.

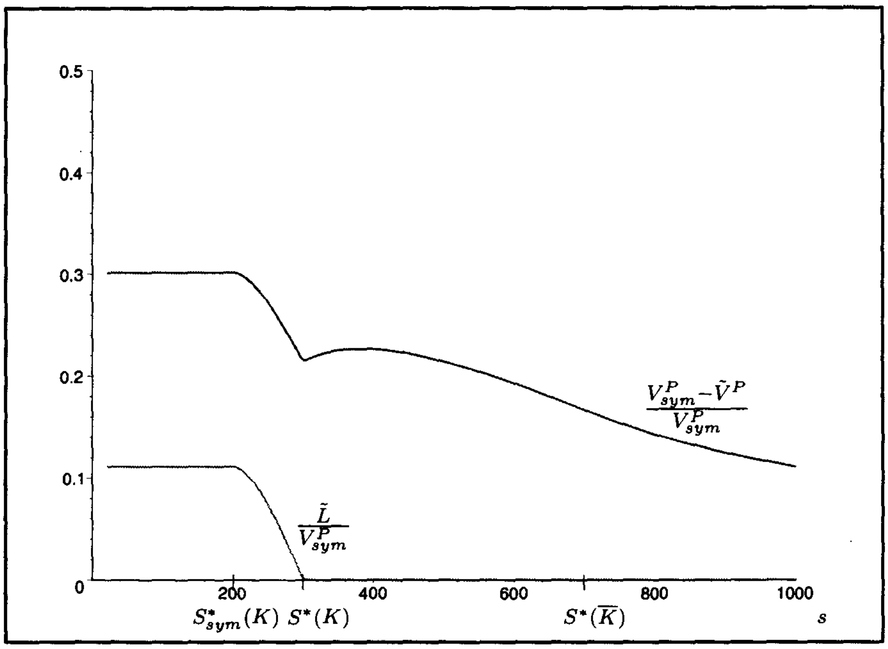

Figure 2.3: The principal's relative loss and the relative dead-weight loss as functions of the output value s.

In the interval $(S^*(K),S^*(\overline{K})]$, the principal's relative loss first increases and then decreases. The reason is that two effects pull in opposite directions: higher s leads to higher difference between the principal's values under symmetric and asymmetric information, which increases the relative loss, whereas an upper limit for the investment cost tends to decrease the agent's value of information as s gets closer to $S^*(\overline{K})$.

Figure 2.4 illustrates how the principal's value function $\tilde{V}^P(s,K)$ changes for different values of the asset in place s and the investment cost K, whereas Figure 2.5 shows the principal's value function $V^P(s)$ under the assumption that the principal does not observe K. As is to be expected, the value $\tilde{V}^P(s,K)$ is increasing in s and decreasing in K. With respect to s, the principal's value increases convexly in the area where it is optimal to postpone the investment, and

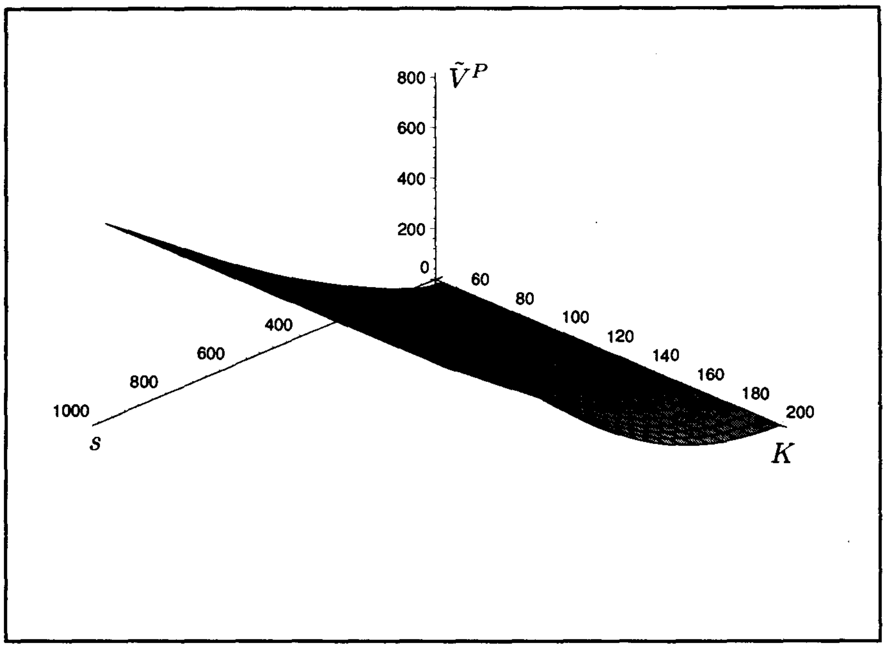

Figure 2.4: The principal's value function, $\tilde{V}^P$ as functions of s and K.

is linear when immediate investment is optimal. The value $\tilde{V}^P(s,K)$ is convexly decreasing in K as long as it is not optimal to invest. When the investment is made immediately, the value is constant with respect to K. The reason is that the compensation, which is the principal's "exercise price" of exercising the option, is independent of K. When we take the expectation of $\tilde{V}^P(s,K)$ with respect to K, we obtain the curve in Figure 2.5. The curve increases convexly as long as $s \leq S^*(\overline{K})$, and is linear for $s > S^*(\overline{K})$. Only when $s > S^*(\overline{K})$ does the principal know that the investment will be made immediately, resulting in a linear curve in this interval.

Figure 2.6 plots the parties' contract values as functions of the volatility parameter σ_S when the output value s equals 300, which in the base case equals the optimal critical price $S^*(K)$. In the "standard" real option problem, correspond-

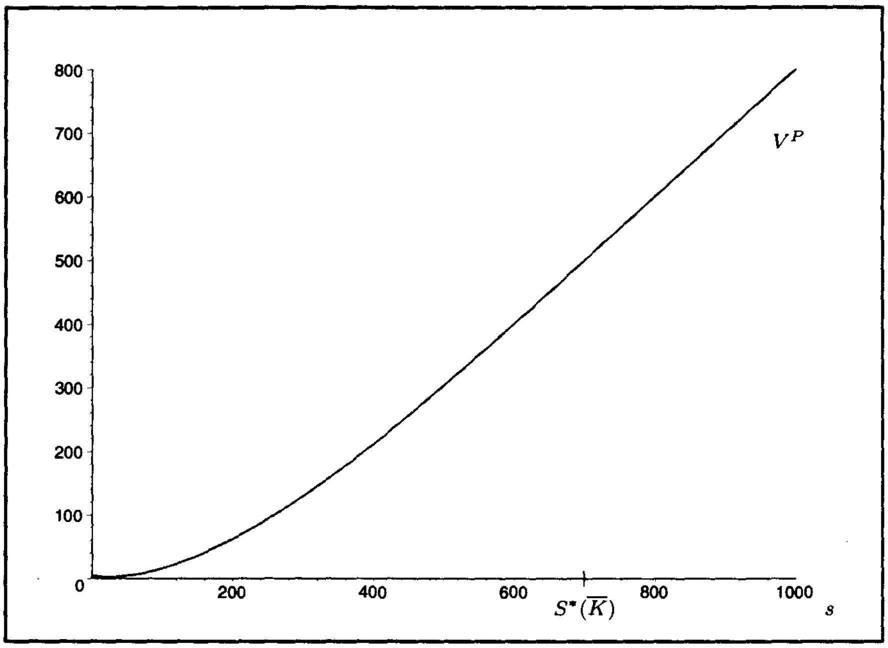

Figure 2.5: The principal's value, V^P as a function of the output value s.

ing to the value of $V^P_{sym}(s, K)$, the value is increasing with respect to σ_S in the interval where the best decision is to postpone the investment. The reason is that as long as the option is not exercised, higher volatility increases the possibility of a higher future profit.

The principal's value function under asymmetric information depends on the volatility σ_S also in the interval where the optimal decision is to invest immediately, i.e., the interval $s > S^*(K)$, in the figure corresponding to $\sigma_S < 0.1$. The reason is connected to the agent's value of information: as σ_S increases, the agent's value of information decreases, and therefore the share of the profit left to the principal is increasing. The agent's value is decreasing in σ_S because of the upper limit on the agent's compensation.

For $s \leq S^*(K)$, corresponding to $\sigma_S \geq 0.1$, there is an additional effect on

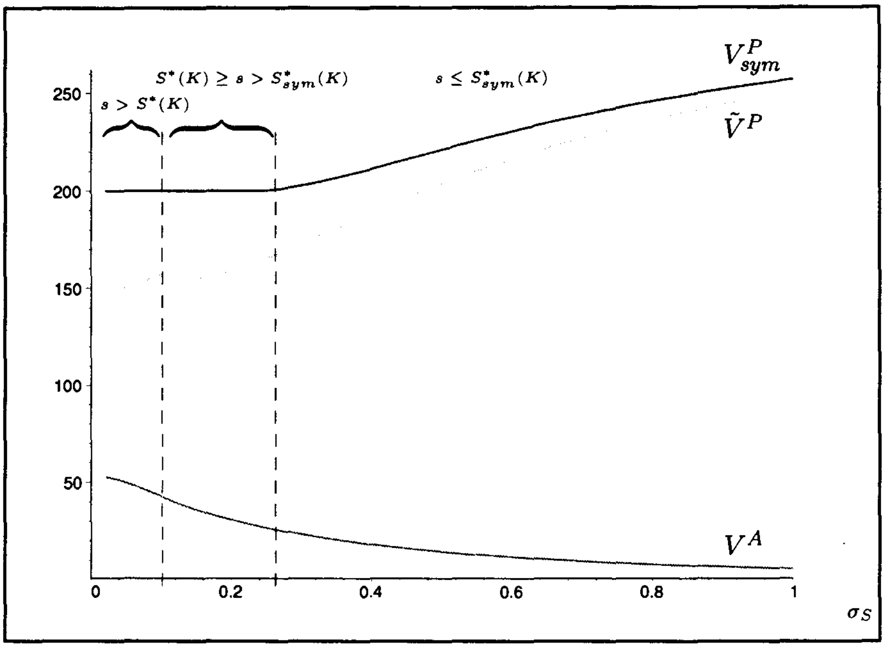

Figure 2.6: The values V^P_{sym}, $\tilde{V}^P$, and V^A as functions of the volatility σ_S, $s = 300$.

the principal's value under asymmetric information, which tends to depress the principal's value: the loss in value because of an inefficient investment strategy. We see that his effect is dominating when σ_S is between 0.1 and 0.15, where the output value s is close to the critical price $S^*(K)$. As the volatility parameter gets larger, the effect from the option value, which is increasing in volatility, dominates.

The same effects are reflected in the loss curves of figure 2.7. At $\sigma_S = 0.1$, corresponding to $S^*(K) = 300$, the relative dead-weight loss gets positive, because then it reaches the interval $s < S^*(K)$, in which we know that the loss is positive. Both the relative dead-weight loss and the principal's relative loss increase in this interval as long as the effect of a second-best investment strategy dominates the effects from the agent's value of information being decreasing in volatility, and

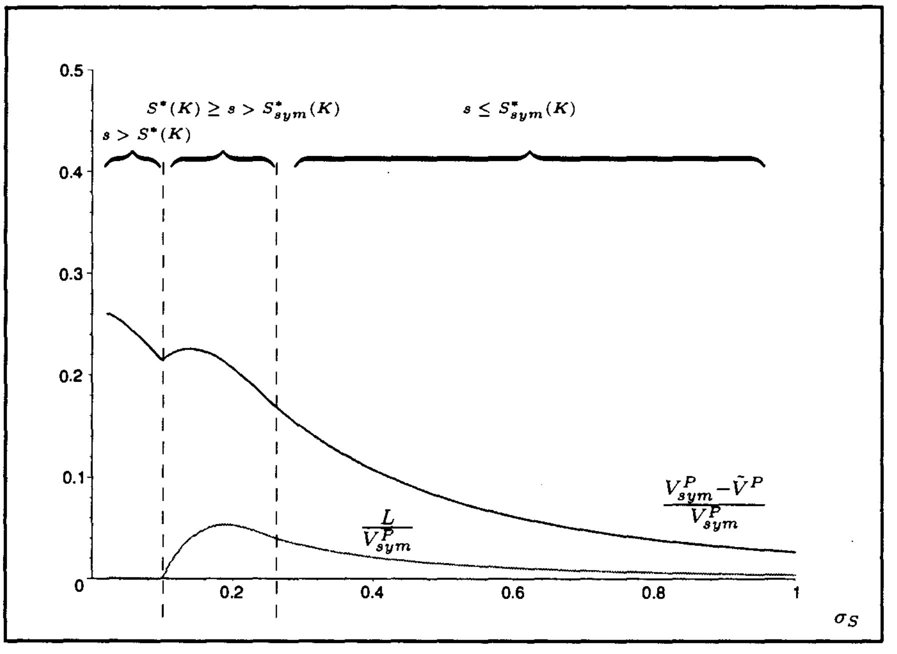

Figure 2.7: The principal's relative loss and the relative dead-weight loss as functions of the volatility parameter σ_S, the output value $s = 300$.

the option value being increasing in the same parameter. The two last-mentioned effects dominate when $s > S^*(K)$.

Figure 2.8 plots the principal's and the agent's values as functions of the investment cost, K. Both the principal's and the agent's value functions are non-increasing with respect to K, as a higher cost lowers the value of the investment for both. For $K < 100$, corresponding to $s > S^*(K)$, the principal's value under asymmetric information is independent of the agent's investment cost. The reason is that the compensation paid to the agent is not a function of the unobservable variable K.

Figure 2.9 shows that the relative dead-weight loss is increasing in K. The explanation is that higher costs lead to higher critical values for exercising the

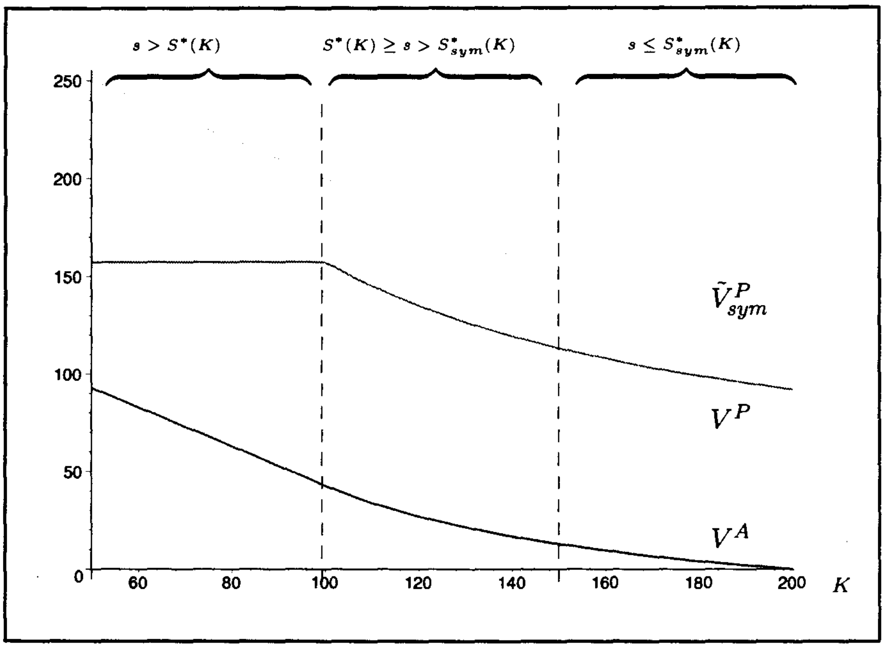

Figure 2.8: The values V_{sym}^{P}, $\tilde{V}^{P}$, and V^{A} as functions of the cost K, $s = 300$.

option, and thereby larger inefficiency in the investment decision.

The principal's relative loss is decreasing in K for K lower than or equal to 100, corresponding to $s > S^{*}(K)$. The reason is connected to the fact that when $s > S^{*}(K)$, the principal's value $\tilde{V}^{P}$ is independent of K, and therefore an increase in K results in a corresponding decrease in the principal's relative loss. For K corresponding to $s \leq S^{*}(K)$, the dominating effect is the same as in the dead-weight loss as long as K is lower than 130. For Ks higher than 100, the dominating effect is the agent's value of information getting lower the closer to the upper level cost the true investment cost is. This tends to decrease the loss.

At $K = \overline{K}$ the principal's loss and the dead-weight loss coincide, as the value of the agent's information is zero at this point.

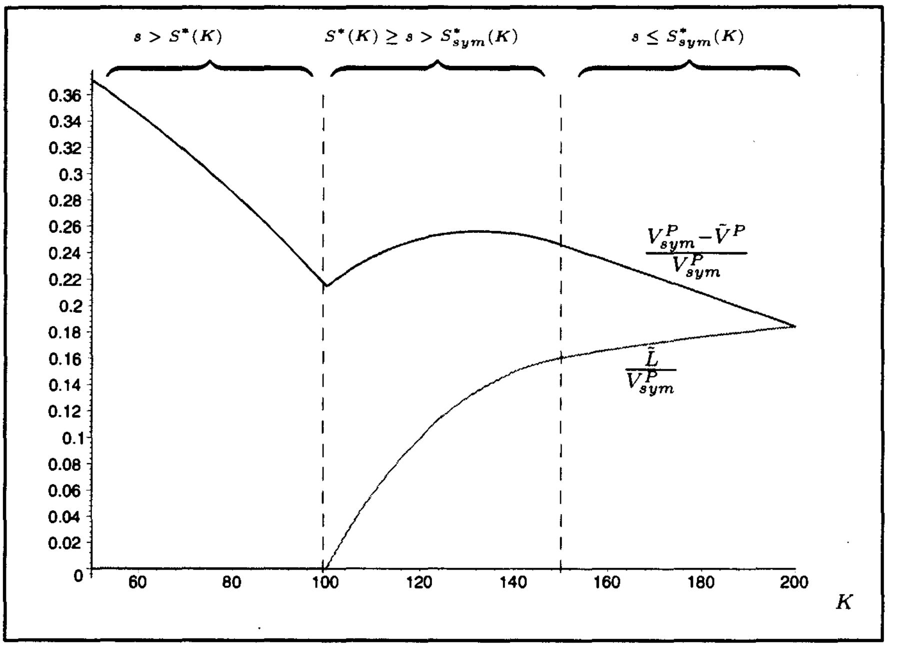

Figure 2.9: The principal's relative loss and relative dead-weight loss as functions of the investment cost K, the output value $s = 300$.

2.10 Conclusion

In this chapter, we study effects of asymmetric information on an optimal stopping problem. A principal owns an investment opportunity and delegates the investment strategy of the project to an agent. The agent has private information about the investment cost, whereas the stochastic output value is common knowledge.

This setting may apply to a number of "real option" situations, both within regulation and corporate finance.

The agent's private information about the cost implies that it is optimal for the principal to compensate the agent according to his value of private information.

Thus, the compensation will be higher than the true investment cost in most cases, thereby increasing the principal's cost of his investment opportunity. A higher cost leads to a higher critical value for investment. Hence, it is found that the agent's private information about the investment cost may lead to under-investment.

The agent's value of private information will, however, not always lead to an inefficient investment strategy. Inefficient decisions will occur only in the interval where the critical value of investment, given asymmetric information, is higher than the value of the output from the investment. If the value of the asset in place is higher than the critical value of investment, the compensation function just gives a rule for sharing the profit between the principal and the agent, without having any inefficiency effects.

In the numerical examples we find that the agent's value of the project decreases in volatility because of the upper limit on the agent's compensation. When the effect of an inefficient investment strategy dominates, the principal's value decreases in the volatility parameter, whereas his value increases when the effect of the possibility of a higher future profit dominates.

Acknowledgements
I would like to thank Petter Bjerksund, Kjell-Bjørn Nordal, an anonymous referee, and participants at the 3rd Annual International Conference on Real Options, at Wassenaar/Leiden in June 1999, for valuable comments.

Chapter 3

Asymmetric Information about a Stochastic Investment Cost

As in chapter 2 we assume that an investor delegates the investment decision of a project to an agent. Now, however, we extend the model in chapter 2 to the case where the agent's private information is driven by a stochastic process. This means that the full information case of the optimal stopping problem takes the form of an exchange option of American type.

Analogously to the case of constant private information of chapter 2, we find a second-best optimal compensation function that is concavely increasing in the output value. Numerical examples show that, depending on the parameter values, the factors leading to inefficiency can result in over-investment as well as under-investment. The last result is in contrast to the result in chapter 2, where the inefficiency always leads to under-investment.

3.1 Introduction

This chapter studies how the value of the contract changes when the private information is stochastic. The starting point of the analyzes is a standard real option problem: an investor owns a right to invest in a project that generates positive net cash flows when the investment is undertaken. Both the net present value of the future cash flows and the investment cost follow stochastic processes.

To maximize the value of the investment project, the investor aims to find the optimal time to exercise the option to invest.

The problem presented here is an extension of the problem in chapter 2. The main difference in assumptions between the two models, is that the agent's private information is constant in chapter 2, whereas it is driven by a stochastic process in the model presented here. A conclusion when the privately observed investment cost is constant, is that the optimal investment trigger is higher in the case of asymmetric information than under full information, and therefore may lead to under-investment. Furthermore, in chapter 2 we find an optimal (second-best) compensation function that is increasing and concave in the stochastic value of the future cash flows. In the numerical examples of the model in chapter 3, presented in section 3.7, we arrive at the same result with respect to the optimal compensation function, whereas we find that the optimal investment strategy, depending on the parameter values, may lead to over-investment, as well as under-investment.

An example of an application of the model in chapter 2 and of the model to be presented here, is the case where a principal owns natural resources (say, a petroleum resource), and needs an agent to manage the investment strategy of the project. The assumption of a constant investment cost is realistic in cases where the investment consists of standard technology. In other cases the investment project has the character of being more like a development project, with new technical solutions, or frequent changes in the design of these. For such investment projects it is more realistic to assume that the agent's private information changes as time passes, as is assumed in the model to be presented below. Moreover, the extension to stochastically changing private information gives us the opportunity to analyze the problem more extensively.

The optimization problem is solved by finding an optimal compensation function. In order to optimize the compensation function for all possible functions, we apply the *revelation principle*. For a definition of this concept, see for example Salanié (1997), or chapter 1 of this thesis. Other references are given in footnote 11.

The papers by Antle et al. (1998) and MacKie-Mason (1985) are related to the model below as they treat the problem of uncertain, irreversible investments in combination with changing private information. In their models new private information is obtained by an agent at certain points in time. Antle et al. (1998) find that the incentive effects from private information tend to defer investment because the investment is made at a higher cost under asymmetric information than under full information. On the other hand, increased volatility by postponing the investment tend to reduce the value of waiting, thereby leading to earlier investment. The reason is that an inefficient investment trigger in the last period, reduces the principal's advantage of delaying the investment to the last period. The result is ambiguous: the dominating effect depends on the parameters. In the numerical illustration of the results in this chapter, we arrive at similar results.

MacKie-Mason (1985) models a sequential decision problem in presence of private information and hence incentive problems. He finds that incentive problems lead to under-investment. However, in MacKie-Mason (1985) the private information concerns the output value, instead of the investment cost, and the problem is formulated as a sequential model. The article is therefore more closely related to the model to be presented in chapter 5.

This chapter is outlined as follows. In section 3.2 the problem is formulated and the assumptions are stated. The benchmark case of full information is presented in section 3.3. The agent's optimization problem is examined in section 3.4, whereas section 3.5 solves the principal's optimization problem. The implementable and optimal compensation function is presented in section in section 3.6. Numerical illustrations of the contract value, and of the corresponding optimal investment strategies, are given in section 3.7.

3.2 Problem formulation

We assume that an investor owns a possibility to invest in a project (for example production of petroleum), and that he needs an agent to manage the investment of the project. We can split the risk of the project into two main parts. The first one is *market uncertainty*, which is uncertainty related to the activity in the economy. The second is *technical uncertainty*. An example of technical uncertainty is uncertainty due to new technical solutions of the investment of the project. We assume that the agent has private information about the technical uncertainty.

The investment cost K_t of the project is a function of an observable variable C_t and an unobservable variable θ_t. The variable C_t represents the part of the cost that is due to market uncertainty, and θ_t is the part of the cost due to technical uncertainty. This approach is similar to the cost assumptions in Pindyck (1993), where the cost uncertainty consists of technical uncertainty defined as uncertainty of the physical difficulty of completing a project, and input cost uncertainty, which covers the uncertainty that is external to the agent.

In Pindyck (1993) it is assumed that the technical uncertainty changes only when investment occurs. Our model is, however, not directly comparable to Pindyck (1993), as we have not taken into consideration that investment takes time. Thus, in our model, the technical uncertainty changes even if no investment occurs. We assume that the agent obtains private information about the investment cost from other sources than the investment project in the model. For example, the agent may manage other, similar investment projects as well, continuously receiving private information from these. Another example is that the agent obtains private information about technical innovations.

Formally, the part of the investment cost that is private information to the agent is given by the geometric Brownian motion,

$$d\theta_t = \alpha\theta_t dt + \sigma_\theta \theta_t dB_t^\theta, \tag{3.1}$$

where α is the drift parameter of the cost due to technical uncertainty, σ_θ is the volatility parameter, and B_t^θ is a standard Brownian motion. As θ_t is a measure of technical uncertainty only, we assume that θ_t is independent of market uncertainty.

The observable part of the investment cost may be correlated with capital markets. The risk adjusted process is given by,

$$dC_t = rC_t dt + \sigma_C C_t dB_t^C. \tag{3.2}$$

The parameter r denotes the risk-free rate, the volatility parameter is given by σ_C, and B_t^C is a standard Brownian motion, under the Q measure, and may be correlated with capital markets. We assume that B_t^C and B_t^θ are independent Brownian motions.

The cost variables θ_t and C_t are both log-normal processes, and the product of the variables leads to a new log-normal process. We assume that the true cost, K_t, is given by the function $K_t = C_t\theta_t$. To justify this function we think of θ_t and C_t as suitably normalized "indexes" representing the market uncertainty and the technical uncertainty, respectively, and where the product of the indexes leads to the true investment cost.

By Ito's Lemma we find that the stochastic process of the true investment cost is given by

$$dK_t = (r + \alpha)K_t dt + \sigma_C K_t dB_t^C + \sigma_\theta K_t dB_t^\theta. \tag{3.3}$$

The stochastic variable S_t represents a present value of a completed project, commonly referred to as the "value of the asset in place"[1]. For short, we also refer to S_t as the output value. Information about S_t is common knowledge. Thus, when the option to invest is exercised, the agent has no more private information. The principal obtains all the cash flows when the investment is exercised, and the contracted compensation is transferred to the agent. The stochastic process of S_t is given by

$$dS_t = (r - \delta_S)S_t dt + \sigma_S S_t dB_t^S, \tag{3.4}$$

where δ_S is the proportional convenience yield parameter of S_t, σ_S is its volatility parameter, and B_t^S is a standard Brownian motion that may be correlated with B_t^C.

It is assumed that the parties are well diversified. As the variables S_t and C_t are assumed to be spanned by capital markets, all risk can be hedged against. The variable θ_t consists only of technical uncertainty privately known to the agent, and is uncorrelated to capital markets. Thus, the uncertainty in θ_t is fully diversifiable.

The probability space corresponding to the Brownian motions B_t^S, B_t^C and B_t^θ, is defined by a complete probability space $(\Omega, \mathcal{F}, Q)$. The agent's information at time t is given by $\mathcal{F}_t^{S,K}$, generated by $\{S_\xi, K_\xi, \xi \leq t\}$, i.e., the stochastic processes S and K are adapted to the filtration $\{\mathcal{F}_t^{S,K}\}_{0\leq t\leq T}$. The "twin assets" S_t and C_t are priced in complete markets, and hence the respective stochastic processes are

[1]McDonald and Siegel (1986) interpret this value as representing "the market value of a claim on the stream of net cash flows that arise from installing the investment project".

measured by the risk adjusted measure Q. Recall that the cost component θ_t is uncorrelated with capital markets. As long as the parties are well diversified, as assumed above, the stochastic process of θ_t is the same under the (true) P and the (risk adjusted) Q measures.

The principal's information at time t is formalized by the $\mathcal{F}_t^{S,C}$, which is generated by $\{S_\xi, C_\xi, \xi \leq t\}$. Thus, in this case the processes S and C are adapted to the filtration $\{\mathcal{F}_t^{S,C}\}_{0 \leq t \leq T}$. The principal knows the distribution of θ_t at any time t, but does not observe the variable θ_t.

Although we restrict our analysis to the case of geometric Brownian motions, we do not need to make this restriction. In equation (3.20) below we present the principal's optimization problem where the agent's value of private information is incorporated. The result presented in equation (3.20) is reached also when we assume that the stochastic processes are given by time-homogeneous Itô diffusions, and the true investment cost K_t is represented by a function $K_t = h(C_t, \theta_t)$. However, in order to find separating optima, we need to assume that the fraction $F(K_t)/f(K_t)$ is increasing in K_t, where $F(\cdot)$ is the cumulative distribution function of K_t, and $f(\cdot)$ is the corresponding density function. These assumptions are made when we, in section 4.3, extend the principal-agent model in this chapter to a situation where more than one agent has private information. The reason we assume that the stochastic processes are geometric Brownian motions in the computations below, is that the assumption makes it easier, and more informative, to compare our results to the well-known case of no private information. When the agent does not have private information, the option problem is identical to an exchange option of American type, presented in section 3.3.

We assume that the optimal stopping problem is delegated to the agent. For example, applied within the framework of corporate finance, this means that the owners delegate a certain investment decision to the privately informed management. In the case of regulation theory, delegation may have the interpretation that a government has given a firm the right to make an investment decision on the government's behalf. Note that, although we assume that the investment decision is delegated, the same outcome occurs if the principal makes the investment decision himself, and the agent only reports the costs. The reason is that we apply a truth telling mechanism in solving for the optimal investment strategy.

We want to find a compensation function that induces the agent to behave in the way preferred by the principal, and at the least cost to the him.

As in chapter 2, the compensation $X(\cdot)$ is transferred at the time when the investment is made. It depends on the observable variables S_t and C_t as well as a time variable. It may also be based on the agent's report at the investment time. The agent's cost report is denoted $\hat{K}$, where $\hat{K} \in [0, \infty)$.

Assume that the option to invest, and thereby the contract, expires at time $\bar{\tau}$, where $\bar{\tau} \leq T$, and T is a given time horizon. Furthermore, let $(s, c, k) = (S_t, C_t, K_t)$.[2] Starting at time t, with initial conditions (s, c, k), the principal's optimization problem is given by

$$V^P(s,c,t) = \sup_{X(\cdot)} E\left[e^{-r\tau_{\hat{K}}} \left(S_{\tau_{\hat{K}}} - X\left(S_{\tau_{\hat{K}}}, C_{\tau_{\hat{K}}}, \tau; \hat{K}\right)\right)^+ \middle| \mathcal{F}_t^{S,C} \right], \tag{3.5}$$

subject to the agent's optimization function

$$V^A(s,c,k,t) = \sup_{\hat{K}, \tau_{\hat{K}}} E\left[e^{-r\tau_{\hat{K}}} \left(X\left(S_{\tau_{\hat{K}}}, C_{\tau_{\hat{K}}}, \tau_{\hat{K}}; \hat{K}\right) - K_{\tau_{\hat{K}}}\right)^+ \middle| \mathcal{F}_t^{S,K} \right]. \tag{3.6}$$

The principal's problem in equation (3.5) is to find an optimal compensation function, subject to the agent's optimization problem in equation (3.6). The compensation function $X(\cdot)$ must be specified at the time the parties enter into a contract. The agent's optimization problem is dynamic: At any time during the contract period the agent must decide on whether to invest or not, and what report he is to give to the principal at the investment time. The stopping times $\tau_{\hat{K}}$ is based on the report given at the stopping time, $\hat{K}$. Thus, the optimal stopping time is defined by

$$\tau_{\hat{K}}^* = \inf\left\{ t \in [0,T] | V^A(S_t, C_t, K_t, t) = X(S_t, C_t, t; \hat{K}) - K_t \right\}.$$

The optimal stopping time may now be time dependent. The reasons are that we take into consideration that the compensation function $X(\cdot)$ may be dependent on time, and that the option to invest is not necessarily perpetual.

[2] Note that we now refer to s as the output value at time t, whereas we in chapter 2 defined s as the output value at time 0. In the time-inhomogeneous optimization problems in chapter 3 and section 4.3, we need a time variable and assume that the initial time is given by t, whereas in the time-homogeneous problems in chapter 2 and section 4.2 we assume that the initial time is zero.

As the agent continuously obtains new information, the agent correspondingly reports continuously to the principal. However, in the formulation of the problem the compensation function is not based on earlier reports. As long as the agent reports costs higher than the costs at which the parties find it optimal to exercise the option to invest, the value of the agent's compensation will not be dependent on the reports.

In solving the model we assume that the agent decides on the optimal investment strategy, and nevertheless reports the investment cost to the principal. However, the reports from the agent to the principal is just a device for finding the optimal investment strategy, and hence finding the optimal contract. Moreover, given truthful reports, it does not matter which party decides on the investment strategy, as the same outcome will occur. Our aim is to find a compensation function where communication between the principal and the agent is not necessary, and that is as good as any contract in which the agent communicates the private information to the principal.

The optimization problem (3.5)-(3.6) is similar to the problem in (2.2)-(2.3). The main difference is that in this chapter we assume that the private information changes stochastically, instead of being a constant as in the problem formulated in (2.2)-(2.3) of chapter 2. This means that in the model in chapter 2 the agent is committed to an earlier given report, whereas in the model presented in this chapter he is only committed to the report at the investment time. However, in spite of this difference, we shall see that the results in both chapters are similar.

In the next section we present the optimization problem in equations (3.5)-(3.6) in the case where the agent has no private information. This case of full information is used as a benchmark when we analyze the effects of private information.

3.3 Full information

If the principal observes the stochastic process K_t, the agent's value of the contract is zero, as the agent has no private information. Thus, the principal can design the contract in such a way as to punish the agent if he does not act in the way preferred by the principal. Hence the optimal transfer function under full information is given by $X(S_t, K_t, t) = K_t$ if the investment is made at time t, and zero otherwise.

The principal's optimization problem is given by

$$V^P_{sym}(s,k) = \sup_{\tau \in [t,\bar{\tau}]} E\left[e^{-r\tau}\left(S_\tau - K_\tau\right)^+ \middle| \mathcal{F}^{S,K}_t\right], \tag{3.7}$$

where τ denotes a stopping time. The subscript *sym* indicates that this is the principal's value of the contract when information is symmetric. The optimization problem has the form of an exchange option[3].

The optimization problem in (3.7) is analogous to the full information problem in the case where we assume that the private information is constant, represented by equation (2.9). There are two differences between the benchmarks: In this chapter the investment cost given by a stochastic variable, and the option to invest is not necessarily perpetual.

In the case where both S_t and K_t are geometric Brownian options, the optimization problem in equation (3.7) is discussed in Broadie and Detemple (1997). It is shown that the optimal exercise strategy is given by a linear relationship between S_t and K_t.

When the option has an infinite horizon, we are able to find a closed form solution, solved by, among others, McDonald and Siegel (1986), Gerber and Shiu (1996) and Øksendal and Hu (1998). Below the closed-form solution is presented.

Similarly to the optimal trigger under full information in chapter 2, equation (2.10), the trigger price $S^*_{sym}(t)$ is proportional to the investment cost K_t, $S^*_{sym}(t) = \frac{\beta}{\beta-1}K_t$, where β is a constant larger than 1. The solution to (3.7) is given by

$$V^P_{sym}(s,k) = \begin{cases} As^\beta k^{1-\beta} & \text{when } s \le \frac{\beta}{\beta-1}k \\ \\ s-k & \text{when } s > \frac{\beta}{\beta-1}k, \end{cases} \tag{3.8}$$

where

$$A = \frac{1}{\beta}\left(\frac{\beta}{\beta-1}\right)^{1-\beta}.$$

[3]An option to exchange one asset for another is sometimes called a Margrabe option, as Margrabe (1978) analyzed European options to exchange one asset for another.

Furthermore, we have

$$\beta = \begin{cases} \frac{1}{a}\left[\frac{1}{2}a + \delta_S - (r-\alpha) + \sqrt{\left(\frac{1}{2}a + \delta_S - (r-\alpha)\right)^2 + 2a(r-\alpha)}\right] & \text{if } a > 0 \\ \frac{r-\alpha}{r-\alpha-\delta_S} & \text{if } a = 0, \end{cases}$$

where $a = \sigma_S^2 - 2\rho_{SC}\sigma_S\sigma_C + (\sigma_C^2 + \sigma_\theta^2)$, and ρ_{SC} is the correlation coefficient between B_t^S and B_t^C. The parameter values must satisfy $\beta > 1$.

In addition to the conditions stated by McDonald and Siegel (1986), Øksendal and Hu (1998) find that the following restrictions are necessary in order to ensure that $\tau < \infty$,

$$\begin{cases} r - \alpha + \frac{1}{2}(\sigma_C^2 + \sigma_\theta^2) \geq \delta_S + \frac{1}{2}\sigma_S^2 & \text{if } a > 0 \\ r - \alpha > \delta_s & \text{if } a = 0. \end{cases}$$

The value function in (3.8) equals the principal's value of the contract when he observes the investment cost K_t, and when the option to invest is perpetual.

3.4 The agent's optimization problem

In this section we analyze the agent's optimization problem given by equation (3.6), and characterize his value of private information.

Following Salanié (1994), section 2.1.2, the revelation principle implies that we can "confine attention to mechanisms that are both *direct* (where the agent reports his information) and *truthful* (so that the agent finds it optimal to announce the true value of his information)". The principal's set of tools to induce the agent to behave in a certain way is given by the incentive mechanisms $(X(\hat{K}), \tau_{\hat{K}})$. If this set of mechanisms can be implemented, then we can also implement these incentive mechanisms through a direct truthful mechanism, $(X(\hat{K}), \tau_{\hat{K}}, \hat{K})$, where the agent reveals his private information. Thus, we can find a direct truthful mechanism $(X(\hat{K}), \tau_{\hat{K}}, \hat{K})$ with the same outcome as for any incentive mechanisms $(X(\hat{K}), \tau_{\hat{K}})$.

Define a trigger function $\psi(c, t; \hat{K})$ such that the option to invest is exercised immediately when $s > \psi(c, t; \hat{K})$, whereas it the option is postponed when $s \leq$

$\psi(c,t;\hat{K})$. Note that the investment strategy is based on the agent's report $\hat{K}$. The investment trigger ψ corresponds to the critical price $\hat{S}(K)$ in chapter 2.

Analogously to the approach in chapter 2 we now proceed by finding the agent's value of private information, also called the information rent. Our aim is to find an expression of the value of private information, and incorporate this value in the principal's optimization problem. We express the agent's value function as

$$v^A(s,c,k,t;\hat{K}) = \begin{cases} w^A(s,c,k,t;\hat{K}) & \text{if } s \leq \psi(c,t;\hat{K}) \\ e^{-rt}\left(X(s,c,t;\hat{K}) - k\right) & \text{if } s > \psi(c,t;\hat{K}), \end{cases} \tag{3.9}$$

where v^A represents an arbitrary value function of the agent, and w^A denotes the agent's value function when the investment decision is postponed.

At any time t the agent's truth telling condition is given by the first-order condition

$$\left.\frac{\partial v^A(s,c,k,t;\hat{K})}{\partial \hat{K}}\right|_{\hat{K}=k} = 0. \tag{3.10}$$

We now emphasize that the report $\hat{K}$ is dependent on the true investment cost k, i.e., $\hat{K} = \hat{K}(k)$. By the envelope theorem we find that[4]

$$\frac{dv^A(s,c,k,t;\hat{K}(k))}{dk} = \begin{cases} w_k^A(s,c,k,t;\hat{K}(k)) & \text{if } s \leq \psi(c,t;\hat{K}(k)) \\ -e^{-rt} & \text{if } s > \psi(c,t;\hat{K}(k)), \end{cases} \tag{3.11}$$

where $w_k^A(s,c,k,t;\hat{K})$ is defined as the derivative of $w^A(s,c,k,t;\hat{K})$ with respect to k.

From (3.11), we know that no contract that depends on the report $\hat{K}$ dominates a contract that is independent of the report. Thus, we find that $X(s,c,t;\hat{K}) = X(s,c,t)$.

Although the compensation is not dependent on the report $\hat{K}$, the agent's private information still is of value. To induce the agent to choose the principal's

[4] $\frac{dv^A(s,c,k,t;\hat{K}(k))}{dk} = \frac{\partial v^A(s,k,c,t;\hat{K}(k))}{\partial \hat{K}(k)}\frac{d\hat{K}(k)}{dk} + \frac{\partial v^A(s,c,k,t;\hat{K}(k))}{\partial k}$. The first term on the right-hand side is zero when $\hat{K}(k)$ is optimal.

preferred investment strategy, the agent must be compensated according to the value of his private information.

Let $\psi(c,k,t)$ be the entry threshold when truth telling is optimal. Define $w^A(s,c,k,t)$ as the agent's value function when $s \leq \psi(c,k,t)$ and when truth telling is the optimal strategy. Then the agent's value function, given truth telling, can be written in the form

$$v^A(s,c,k,t) = \begin{cases} w^A(s,c,k,t) & \text{if } s \leq \psi(c,k,t) \\ e^{-rt}\left(X(s,c,t) - k\right) & \text{if } s > \psi(c,k,t). \end{cases} \tag{3.12}$$

Let $\vartheta(s,c,t)$ be the inverse trigger function, i.e., it is optimal to invest when $k < \vartheta(s,c,t)$, and optimal to wait when $k \geq \vartheta(s,c,t)$. This function corresponds to the inverse entry threshold $\vartheta(\cdot)$ in chapter 2. We find the agent's value of private information by integrating both sides of the first-order condition in (3.11) with respect to the privately observed investment cost k. The following expression of the agent's value function is derived in appendix B.1,

$$v^A(s,c,k,t) = \begin{cases} -\int_k^\infty w_u^A(s,c,u,t)du & \text{if } s \leq \psi(c,k,t) \\ e^{-rt}\left(\vartheta(s,c,t) - k\right) - \int_{\vartheta(s,c,t)}^\infty w_u^A(s,c,u,t)du & \text{if } s > \psi(c,k,t). \end{cases} \tag{3.13}$$

The reason we express the value in (3.13) as unevaluated integrals, is that this is a convenient form when we are to incorporate the agent's value of private information into the principal's optimization problem. In the special case where the option to invest is perpetual, and the investment cost K_t is constant, equation (3.13) corresponds to the agent's value of private information in equation (2.25) in chapter 2.

For $s > \psi(c,k,t)$, put the right-hand sides of equations (3.12) and (3.13) equal to each other. Multiplying through by e^{rt} gives an expression for the compensation function,

$$X(s,c,t) = \vartheta(s,c,t) - e^{rt}\int_{\vartheta(s,c,t)}^\infty w_u^A(s,c,u,t)du. \tag{3.14}$$

Equation (3.14) shows that the compensation function is dependent on the trigger function $\vartheta(s,c,t)$, and thus only on observable variables, as the investment cost k has been cancelled out. However, so far we do not know the optimal investment strategy. This problem is analyzed in the section below.

3.5 The principal's optimization problem

The principal's optimization problem is given in (3.5), and rewritten here as,

$$V^P(s,c,t) = \sup_{X(\cdot)} E\left[g^P(S_\tau, C_\tau, \tau)^+ \middle| \mathcal{F}_t^{S,C} \right], \tag{3.15}$$

where

$$g^P(s,c,t) = e^{-rt}\left(s - X(s,c,t)\right). \tag{3.16}$$

The only difference in equations (3.15)-(3.16) compared to equation (3.5) is that the compensation X now is not dependent on the report $\hat{K}$. For simplicity we now have suppressed the subscript $\hat{K}$ in the notation of the stopping times. As long as the agent's truth telling constraint is satisfied, the stopping times depend on the true investment cost.

The principal's problem is to implement an optimal compensation function. In order to optimize the principal's value with respect to an optimal stopping time, we need to replace the unknown compensation function with known functions. Thus, we replace the unknown function $X(\cdot)$ by the expression in (3.14). Furthermore, in order to find a convenient expression of $g^P(s,c,t)$ we note that the function in (3.16) is equivalent to

$$g^P(s,c,t) = \int_0^\infty e^{-rt}\left(s - X(s,c,t)\right) f(k|c,t)dk, \tag{3.17}$$

where the function $f(k|c,t)$ is the probability density of k given the principal's information about c and t. Now, suppose that the option to invest is exercised at time t, and that $k = \psi(s,c,t)$ at time t. Then replace $X(s,c,t)$ in the principal's value of the payoff, by the right-hand side of (3.14), which by some calculations lead to (derived in the appendix, section B.2)

$$g^P(s,c,t) = \int_0^\infty \left\{ e^{-rt}(s-k) + w_k^A(s,c,k,t)\frac{F(k|c,t)}{f(k|c,t)} \right\} f(k|c,t)dk. \tag{3.18}$$

The function $F(k|c,t)$ is defined as the distribution function of the investment cost k, conditional on the observed cost component c and the time t.

A condition in the agent's optimal stopping problem is that the first-order derivative of $v^A(\cdot)$ must be continuous for all the variables included in the problem[5]. In

[5]This condition for optimum is called *smooth pasting* or *high contact*, and is, for instance, stated in McDonald and Siegel (1986) and Øksendal (1998), Theorem 10.4.1.

particular, we need to check that the value function is continuous at the trigger where the investment is exercised. The first-order derivative of the value function in (3.12) at the trigger where $k = \vartheta(s, c, t)$ leads to the following condition

$$w_k^A(s, c, \vartheta(s, c, t), t) = -e^{-rt}.$$

Under the assumption that the investment is made at time t, we use the smooth pasting condition to replace $w_k^A(s, c, k, t)$ in equation (3.18) by $-e^{-rt}$, leading to

$$g^P(s, c, t) = \int_0^\infty e^{-rt} \left(s - k - \frac{F(k|c, t)}{f(k|c, t)} \right) f(k|c, t) dk.$$

Thus, we have reformulated the principal's payoff value to an expression consisting of known functions only, and where the agent's truth telling condition is incorporated.

We can rewrite the principal's value at the investment time as

$$\begin{aligned} g^P(s, c, t) &= E\left[e^{-rt} \left(s - k - \tfrac{F(k|c,t)}{f(k|c,t)} \right) |\mathcal{F}_t^{S,C} \right] \\ &= E\left[E\left[e^{-rt} \left(s - k - \tfrac{F(k|c,t)}{f(k|c,t)} \right) |\mathcal{F}_t^{S,K} \right] |\mathcal{F}_t^{S,C} \right]. \end{aligned} \tag{3.19}$$

If we compare the result in (3.19) to the case where the principal has full information, given by (3.7), we see that the principal's payoff value is reduced by the fraction $\frac{F(k|c,t)}{f(k|c,t)}$.

The principal finds the optimal investment strategy by solving the following optimal stopping problem,

$$V^P(s, c, t) = \sup_\tau E\left[E\left[e^{-r\tau} \left(S_\tau - K_\tau - \frac{F(K_\tau|C_\tau, \tau)}{f(K_\tau|C_\tau, \tau)} \right) \middle| \mathcal{F}_t^{S,K} \right] \middle| \mathcal{F}_t^{S,C} \right], \tag{3.20}$$

as if he knows the unobservable variable k.

Recall that in the case of full information, presented in section 3.3, the optimal investment strategy involves a linear relationship between output value s and the investment cost k. However, the two variables are not linearly related in the case of asymmetric information because of the fraction $\frac{F(\cdot)}{f(\cdot)}$.

The principal's unconstrained optimization problem in (3.20) has the same form as the principal's optimization problem when the private information is constant (compare to equation (2.28) in chapter 2): the principal's expected payoff consists in both cases of the value of the asset in place minus the true investment cost and the fraction $\frac{F(\cdot)}{f(\cdot)}$. The only difference is that in the case of a constant private information, the fraction $\frac{F(\cdot)}{f(\cdot)}$ is a constant, too.

It may be surprising that the principal's expected value of the payoff is of the same type, because there seems to be an important difference between the case where the agent's private information is stochastic, and the case where it is constant. As the private information does not change in the first case, the agent is committed to the same report during the contracting time. However, when the private information changes stochastically, the agent continuously submits new reports without committing to earlier reports (given that the contract does not depend on reports earlier than the one at the investment time). Intuitively, one may therefore be led to believe that this gives the agent a higher value of his private information compared to the case where the private information is constant. The principal's optimization problem above (equation (3.20)) shows that this is not the case, as in both cases the principal's "exercise cost" is increased by $\frac{F(\cdot)}{f(\cdot)}$.

The explanation of the agent's value of private information being of the same form whether it is constant or stochastically changing, is that in both cases we find a contract where the investment strategy is delegated to the agent. Thus, communication has no value. For a discussion of the value of communication versus delegation, see Melumad and Reichelstein (1987) and (1989).

Let $\tilde{v}^P(s,c,k,t)$ be the principal's value function when he is informed about the agent's true investment cost, but is committed to the contract, i.e., when the principal's information is given by $\mathcal{F}_t^{S,K}$,

$$\tilde{v}^P(s,c,t) \quad = \quad E\left[e^{-r\tau}\left(S_\tau - K_\tau - \tfrac{F(K_\tau|C_\tau,\tau)}{f(K_\tau|C_\tau,\tau)}\right)\middle|\, \mathcal{F}_t^{S,K}\right]. \tag{3.21}$$

We find the optimal investment strategy by optimizing equation (3.21) with respect to the optimal stopping time. Moreover, define

$$\tilde{g}^P(s,c,k,t) = e^{-rt}\left(s - k - \frac{F(k|c,t)}{f(k|c,t)}\right). \tag{3.22}$$

The function $\tilde{g}^P(s,c,k,t)$ can be understood as the principal's value at the time the investment is made, given that he is committed to the second-best contract, and given that he is informed about the investment cost k. The optimal value function $\tilde{v}^P(s,c,k,t)$ must satisfy the variational inequalities

$$\tilde{v}^P(s,c,k,t) \geq \tilde{g}^P(s,c,k,t) \quad (3.23)$$

$$L\tilde{v}^P(s,c,k,t) - r\tilde{v}^P(s,c,k,t) \leq 0 \quad (3.24)$$

$$\max\left\{L\tilde{v}^P(s,c,k,t) - r\tilde{v}^P(s,c,k,t), \tilde{g}^P(s,c,k,t) - \tilde{v}^P(s,c,k,t)\right\} = 0. \quad (3.25)$$

The partial differential operator L that coincides with the generator A of the system $\{S_t, C_t, K_t, t\}$,[6] is given by

$$\begin{aligned} L\tilde{v}^P \;=\; & \tfrac{\partial \tilde{v}^P}{\partial t} + (r-\delta_S)s\tfrac{\partial \tilde{v}^P}{\partial s} + \tfrac{1}{2}\sigma_S^2 s^2 \tfrac{\partial^2 \tilde{v}^P}{\partial s^2} + (r+\alpha)k\tfrac{\partial \tilde{v}^P}{\partial k} + \tfrac{1}{2}(\sigma_C^2+\sigma_\theta^2)k^2\tfrac{\partial^2 \tilde{v}^P}{\partial k^2} \\ & + rc\tfrac{\partial \tilde{v}^P}{\partial c} + \tfrac{1}{2}\sigma_C^2 c^2 \tfrac{\partial^2 \tilde{v}^P}{\partial c^2} + \rho_{S,C}\sigma_S\sigma_C sc\tfrac{\partial \tilde{v}^P}{\partial s \partial c}, \end{aligned} \quad (3.26)$$

where $\rho_{S,C}$ is the correlation coefficient between the standard Brownian motions B_t^S and B_t^C. The expression $L\tilde{v}^P - r\tilde{v}^P = 0$ corresponds to the partial differential equation for arbitrage-free contingent claims[7], as described in, for example, Duffie (1996), ch. 5.G, or Merton (1992), ch. 13.2. The value function $\tilde{v}^P$ in (3.23) is always larger than, or equal to, the "payoff" $\tilde{g}^P$ as it consists of the decision flexibility (i.e., the option to wait) in addition to the value at the time the investment is made. As long as $\tilde{v}^P(s,c,k,t)$ is strictly larger than $\tilde{g}^P(s,c,k,t)$

[6]Denote $Y_t = \{S_t, C_t, K_t, t\}$, $y = Y_t$. Following Øksendal (1998), Definition 7.3.1, we define the generator A of the process Y_t by

$$A\tilde{v}^P(y) = \lim_{\xi \downarrow 0} \frac{E[\tilde{v}^P(Y_{t+\xi})|\mathcal{F}_t^{S,K}] - \tilde{v}^P(y)}{\xi}.$$

Thus, the generator A denotes the expected rate of change of $\tilde{v}^P(y)$. By Øksendal (1998), Theorem 7.3.3 we know that A and L coincide when the function $\tilde{v}^P$ has continuous derivatives up to the second order.

[7]The expression $L\tilde{v}^P - r\tilde{v}^P$ is lower or equal to zero for the following reason: The definition of $\tilde{v}^P$ implies that

$$\tilde{v}^P(s,c,k,t) \geq E\left[\tilde{v}^P(S_\tau, C_\tau, K_\tau, \tau)|\mathcal{F}_t^{S,K}\right]. \quad (*)$$

By Dynkin's formula we know that

$$E\left[\tilde{v}^P(S_\tau, C_\tau, K_\tau, \tau)|\mathcal{F}_t^{S,K}\right] = \tilde{v}^P(s,c,k,t) + E\left[\int_t^\tau A\tilde{v}^P(S_\xi, C_\xi, K_\xi, \xi)d\xi|\mathcal{F}_t^{S,K}\right].$$

Hence we need $A\tilde{v}^P \leq 0$ for the inequality in $(*)$ to be satisfied. A reference for Dynkin's formula is Theorem 7.4.1 in Øksendal (1998).

we are in the *continuation region*, defined by

$$D = \left\{(s,c,k,t) : \tilde{v}^P(s,c,k,t) > \tilde{g}^P(s,c,k,t)\right\}.$$

Moreover, in the continuation region we need $L\tilde{v}^P - r\tilde{v}^P = 0$, as required in equation (3.25). When $\tilde{v}^P(s,c,k,t) > \tilde{g}^P(s,c,k,t)$ the investment option is exercised.

There is no analytical solution to the problem in (3.21), and the optimal investment strategy must be solved numerically. However, as we know that $\frac{\partial(F(k|c,t)/f(k|c,t))}{\partial k} > 0$, we guess that the optimal investment trigger $\psi^*(c,k,t)$ is strictly increasing and convex in k, i.e., $\frac{\partial \psi^*}{\partial k} > 0$ and $\frac{\partial^2 \psi^*}{\partial k^2} > 0$.

Observe that in the case where the stochastic processes are given by geometric Brownian motions, we can reformulate the principal's value function to

$$V^P(s,c,t) = \sup_{\tau} E\left[e^{-r\tau}\left(S_\tau - C_\tau\left(\frac{F(\theta_\tau,\tau)}{f(\theta_\tau,\tau)} + \theta_\tau\right)\right)\middle| \mathcal{F}_t^{S,C}\right]. \qquad (3.27)$$

This result is derived in appendix B.3.

Thus, we see that there is a linear relationship between s and c. This result is consistent with the linear symmetric information case in subsection 3.3. The reason is that both s and c are observable to the principal. However, the part of the investment cost that is not observable to the principal, is not linearly dependent on s and c.

3.6 Implementation of the optimal investment strategy

Let $\psi^*(c,k,t)$ represent the optimal investment strategy found from equation (3.20). The optimal investment strategy is to invest immediately if $s > \psi^*(c,k,t)$ and postpone the investment if $s \leq \psi^*(c,k,t)$. This investment strategy must be implemented into the compensation function given by (3.14). Let $\vartheta^*(s,c,t)$ be the optimal inverse trigger function. From (3.14) we find by partial integration that the optimal compensation function is given by

$$X^*(s,c,t) = \vartheta^*(s,c,t) + e^{rt}w^A(s,c,\vartheta^*(s,c,t),t) \qquad (3.28)$$

if the investment is made at time t, and $X^*(s,c,t) = 0$ otherwise. Hence, if the principal compensates the agent according to the function in (3.28) at the time the investment is made, the agent is induced to follow the optimal investment strategy.

Above we have guessed that the critical price ψ^* is increasing in the investment cost k, since $\frac{F(k)}{f(k)}$ is increasing in k. As the critical cost, $\vartheta^*(s,c,t)$, is the inverse function of the critical price, $\psi^*(c,k,t)$, (keeping c and t fixed), we guess that the critical cost $\vartheta^*(s_0)$ is concavely increasing in s, i.e., $\frac{\partial \vartheta^*}{\partial s} > 0$ and $\frac{\partial^2 \vartheta^*}{\partial s^2} < 0$. Hence, we guess that the first term of the optimal compensation function in equation (3.28) is increasing and concave in the output value s. With respect to the second term in (3.28) we do not in general know whether the increase is concave or convex. However, in the numerical examples below, we will study these properties for some special cases.

3.7 Numerical illustration

The implementable compensation function in equation (3.28), the corresponding optimal investment strategies, and the resulting values of the principal and the agent, are in this section illustrated numerically.

For simplicity, we assume that $C_t = 1$ for all t. However, this simplification does not alter the results qualitatively, as we, corresponding to (3.27), can formulate the payoff as follows, $c\left[\frac{s}{c} - \left(\frac{F(\theta,t)}{f(\theta,t)} + \theta\right)\right]$. Hence, we may think of the option as consisting of c options on an asset with price s/c and investment cost equal to $\frac{F(\theta,t)}{f(\theta,t)} + \theta$. A similar interpretation of American exchange options is pointed out by Broadie and Detemple (1997).

In order to solve the optimization problem in equation (3.27) numerically, we use an implicit finite difference method for the two state variables, S_t and K_t. In the appendix, section B.4, our approach is described in more details.

In the example we assume that the option has a finite horizon and that $t = 0$. The parameter values for the base case are tabulated below.

Base case:	Expiration date of the contract:	$\bar{\tau} = 5$
	Investment cost, market uncertainty:	$C_t = 1$ for all $t \geq 0$
	Risk-free rate:	$r = 0.05$
	Convenience yield:	$\delta_S = 0.03$
	Drift, investment cost:	$\alpha = 0.04$
	Volatility, output value:	$\sigma_S = 0.1$
	Volatility, cost, technical uncertainty:	$\sigma_\theta = 0.2$
	Volatility, cost, market uncertainty:	$\sigma_C = 0$
	Initial investment cost:	$k = 100$
	Initial expectation of investment cost:	$E[k] = 100$
	Initial variance of investment cost:	$\text{Var}[k] = 0.16$

In Figures 3.1 and 3.2, the investor's values of the investment project are illustrated under full information and asymmetric information, respectively.

Figure 3.1 shows the investor's value of the project in the case of no private information, corresponding to the value of an exchange option of American type. As is typical for an American exchange option, the option value is convexly increasing in s, and convexly decreasing in k. When the combination of s and k leads to immediate investment, the option value is linear in both s and k.

In Figure 3.2 the investor's value function is illustrated in the case of asymmetric information, $\tilde{V}^P(s,k)$. Note that the value function illustrated here is the value for *given levels* of the partially observed k, such that $V^P(s) = \int_0^\infty \tilde{V}^P(s,k)f(k)dk$. Thus, the value function $\tilde{V}^P(s,k)$ is interpreted as the principal's value after he is committed to the contract, and after he is informed about the investment cost, k. The reason we present $\tilde{V}^P(s,k)$ instead of $V^P(s)$ is that by doing so we are able to study efficiency losses for different values of the investment cost k, as was done in chapter 2.

Figure 3.2 shows that the principal's value function under asymmetric information is convexly increasing in s, and convexly decreasing in k, in the area where postponing the option to invest is the optimal decision. When immediate investment is the optimal decision, the investor's value increases convexly in the output value s, and is independent of changes in k. The reason for the independence is explained by the optimal compensation function as shown in equation (3.28):

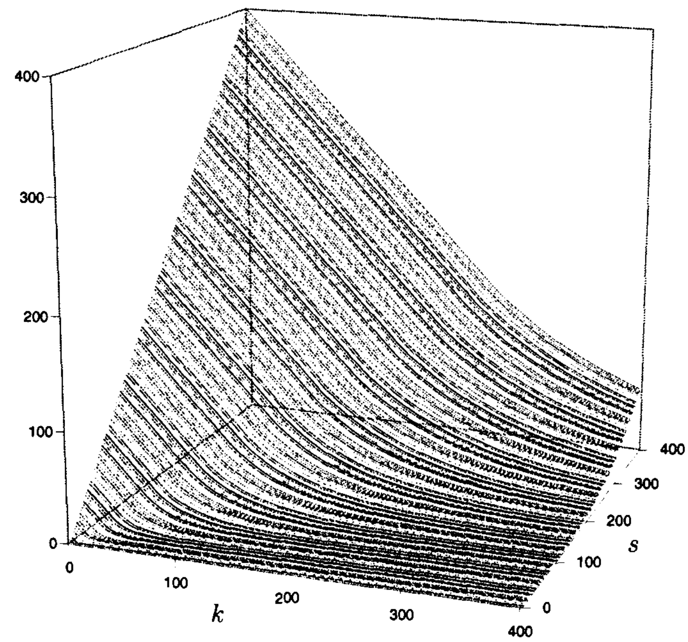

Figure 3.1: The investor's value under full information, corresponding to the value of an exchange option of American type, $V^{P}_{sym}(s,k)$.

The compensation function is independent of k, and as the investor's payoff from immediate investment equals $s - X^*(s)$, the investor's value does not change with respect to changes k. These effects are consistent with the results for the principal's value in chapter 2, Figures 2.2, 2.4 and 2.8.

As the compensation is not dependent upon the investment cost, the agent alone profits on decreases in the investment cost (and analogously bears the whole cost if the investment cost increases) in the region where immediate investment is the optimal decision. This effect is seen from Figure 3.3, where the agent's value of the contract is drawn. We find the same effect in the agent's value function in chapter 2, confer Figure 2.8. Furthermore, Figure 3.3 shows that in the area where

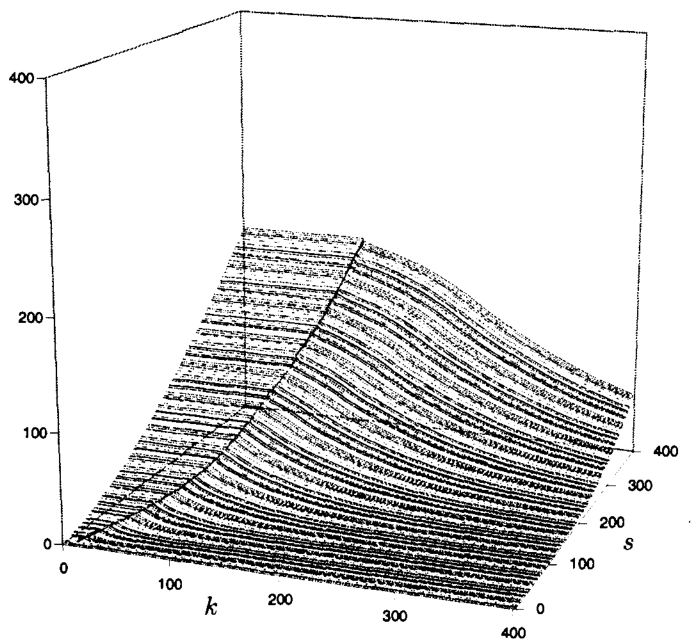

Figure 3.2: The investor's value under asymmetric information, $\tilde{V}^P(s,k)$.

it is optimal to postpone the investment decision, the value increases convexly in s, and decreases convexly in k, as is the case in the numerical example of the model in chapter 2, see Figures 2.2 and 2.8.

As long as the state variables lead to the same investment strategy whether the information is complete or asymmetric, the investor's value under full information in Figure 3.1 equals the sum of both parties' values under asymmetric information, shown in Figures 3.2 and 3.3. When the state variables result in second-best investment strategies, we have a positive dead-weight loss, equal to $V^P_{sym}(s,k) + V^A_{sym}(s,k) - (V^P(s,k) + V^A(s,k))$.[8]

[8]This definition of the dead-weight loss corresponds to the evaluated dead-weight loss in equation (2.38).

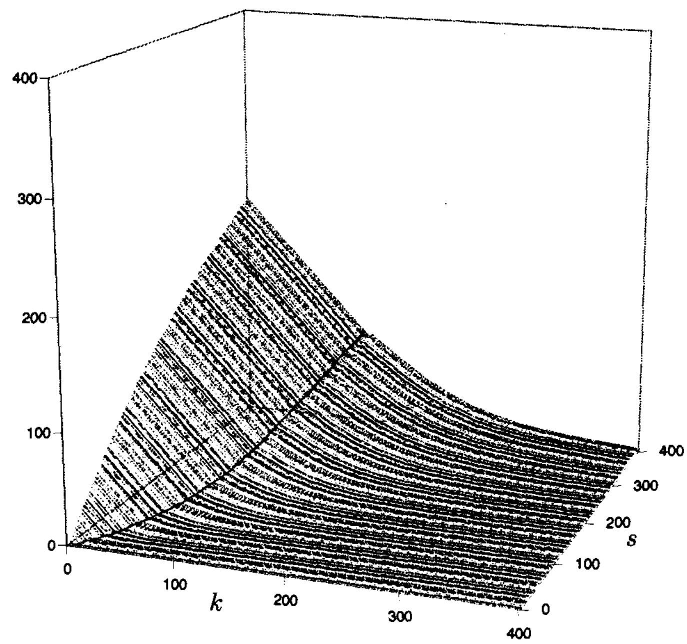

Figure 3.3: The agent's value of private information, $V^A(s,k)$.

The optimal investment trigger is in Figures 3.2 and 3.3 shown as the line that forms the regime switches for the different combinations of s and k. In Figure 3.4 the same investment trigger function $\vartheta^*(s)$ is plotted, having a concave and increasing form. The linear curve in Figure 3.4 represents the optimal investment trigger in the full information case, corresponding to the investor's value function in Figure 3.1. The two curves in Figure 3.4 are to be interpreted as follows. If s is equal to 200 in the full information case, it is optimal to invest when k is lower than 126. Analogously, when $s = 200$ and we have asymmetric information, it is optimal to invest when k is lower than 96.

When the asset value s is higher the 100, the curve representing the investment strategy under asymmetric information is below the investment trigger function

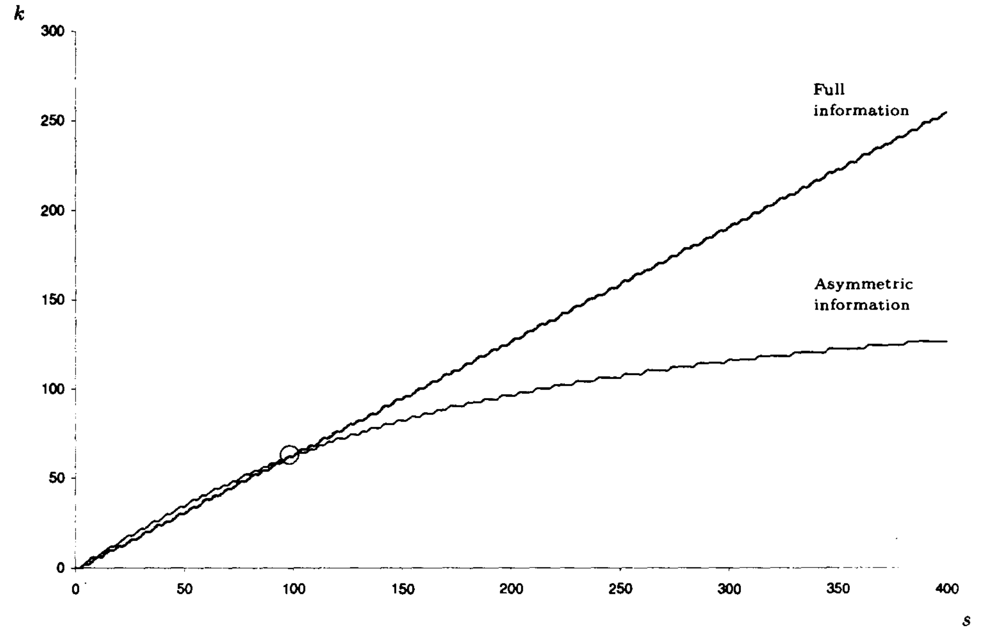

Figure 3.4: Optimal investment strategies $\vartheta^*(s)$ under full information and asymmetric information, respectively. Base case.

under full information. Thus, in this interval asymmetric information leads to under-investment. In the interval where s is lower than 100, over-investment is the result, whereas the optimal investment strategy under asymmetric information is neutral when $s = 100$.

The concave form of the "inverse" investment strategy $\vartheta^*(s)$, and therefore the convexly increasing $\psi(k)$, is consistent with our earlier guess. The "technical" reason for the concavely increasing investment strategy, is that the total cost $k + \frac{F(k)}{f(k)}$ in the optimization function (equation (3.27)) increases in k, thus leading to an investment strategy $\psi^*(k)$ that increases convexly in k, and thereby a concavely increasing inverse entry threshold, $\vartheta^*(s)$, as shown in Figure 3.4. The economic interpretation for the convex form of $\psi^*(k)$ is explained in the following[9]. In order to find a contract that induces the agent to follow the investment strategy preferred by the principal, an increase in the agent's investment cost k must be followed by an increase in the payment to the agent. Thus, if the principal increases the payment for a certain level of k, to ensure incentive compatibility, the principal needs to increase the payment for all lower levels of k, as well. The result is that an increase in the investment cost of one unit, implicitly increases the payment from the principal to the agent by more than one unit. Hence, as the principal's payment increases convexly in the cost k, the trigger function $\psi^*(k)$ increases convexly in the investment cost k as well.

Compared to the investment strategy under full information, represented by the linear curve in Figure 3.4, the concavely increasing $\vartheta^*(s)$ tends to under-investment as the asset value s increases. Figure 3.4 shows that when the asset value s is higher than 100, the incentive problem leads to under-investment.

A perhaps more surprising result is that private information for some parameter values leads to over-investment, as is the case in Figure 3.4 when s has a lower value than 100. The explanation is that the incentive problem, tending to under-investment, decreases the value of waiting. When the effect that reduces the value of waiting (the option value) is higher than the effect caused by the incentive problem, over-investment is the consequence. This result is in contrast to the second-best investment strategy in chapter 2: when the private information about the investment cost is constant, the optimal investment strategy may lead to under-investment only. The reason for the different results is that the principal's expenses at the investment time is a constant, equal to $K + F(K)/f(K)$, for all states in chapter 2, whereas in the model in chapter 3 the expenses $k + F(k)/f(k)$ may increase very much for increases in the investment cost k. Thus, the prospects of a high expenses lead to early investments.

In Figure 3.5 the optimal investment strategies are drawn for different values of the volatility due to technical uncertainty, σ_θ. The linear curves are the optimal investment strategies given full information, and the concave curves represent the asymmetric information case. In the case of an American exchange option (corresponding to the full information case), the trigger function decreases in the volatility parameter σ_θ. The same effect applies for the case of asymmetric

[9]Confer Laffont and Tirole (1993), page 65, for a similar explanation.

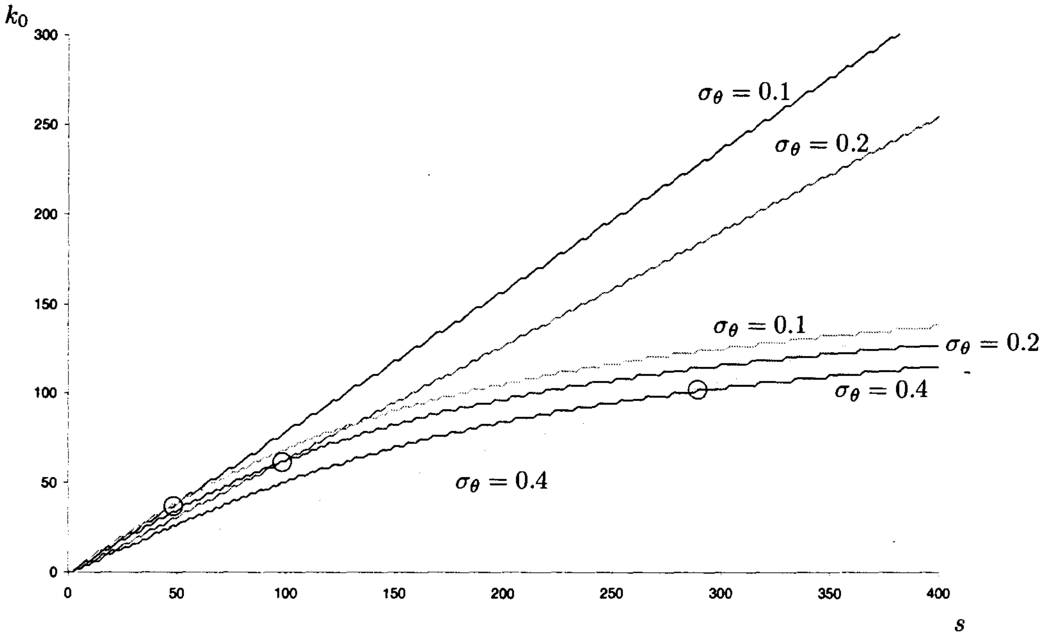

Figure 3.5: Optimal investment strategies $\vartheta^*(s)$ for different parameter values of volatility σ_θ, and varying asset value s.

information. However, the increase is larger for the full information case than the asymmetric case. This implies that as σ_θ increases, the over-investment effect gets larger. The neutral asset values s, at which neither over-investment nor under-investment take place, increase in the volatility parameter.

An explanation for the relatively small effects on the investment strategies when we change the volatility parameter value, is that the fraction $\frac{F(\cdot)}{f(\cdot)}$ increases rapidly in the investment cost k for all values of the volatility parameter. This tends to reduce the exercise regions.

In Figure 3.6 the optimal compensation function X is drawn as a function of the asset value s, and for different values of the volatility parameter σ_θ. The concave

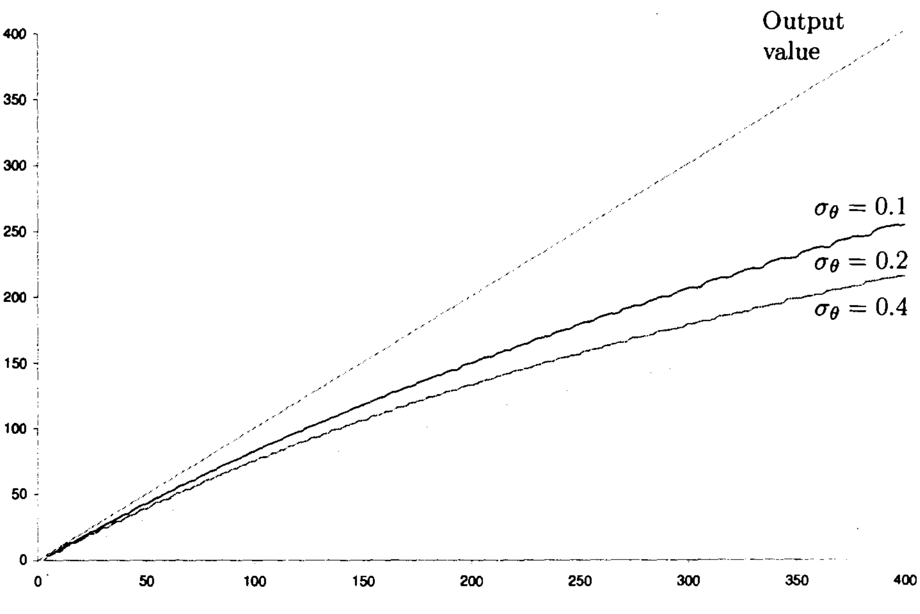

Figure 3.6: The optimal compensation function for different parameter values of volatility σ_θ, and for varying asset value s.

form on the compensation function is a result of the concavely increasing trigger function $\vartheta^*(s)$, confer equation (3.28). A corresponding interpretation is that as s increases, the probability that s is higher than k increases as well. Thus, the compensation function has a decreasing slope as the asset value increases: for high values of s the principal can pay the agent a lower share of the total value, as the agent will be induced to invest even though the share left to him is smaller for a high value of s than for a low value of s.

Figure 3.6 also shows that the compensation is decreasing in the volatility σ_θ. An explanation is the following. As σ_θ increases, the fraction $F(k)/f(k)$ increases in the relevant interval, thus increasing the principal's expenses at the investment time. To induce the agent not to invest at "too" high levels of $k + F(k)/f(k)$,

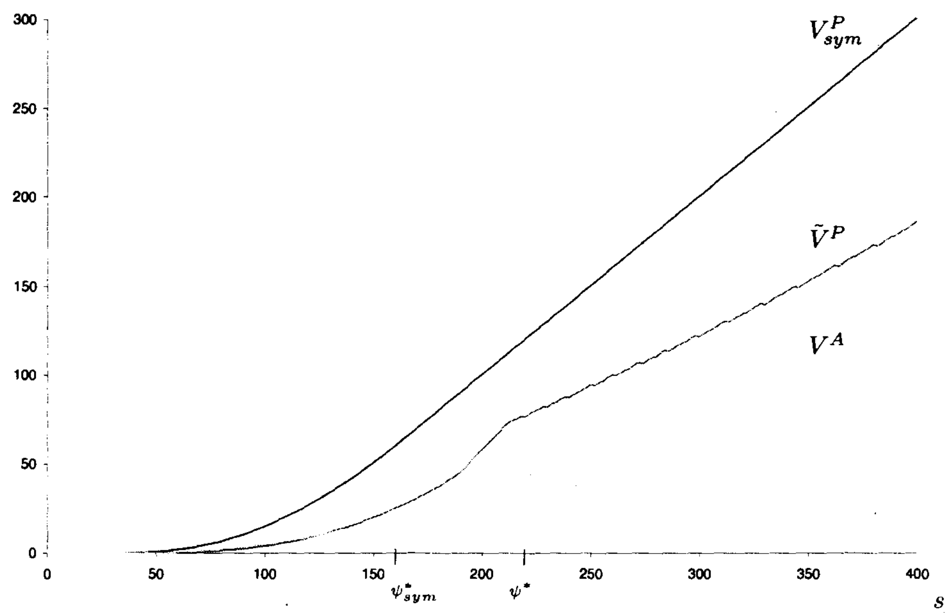

Figure 3.7: The principal's and the agent's respective value functions, V^P_{sym}, $\tilde{V}^P$ and V^A for varying asset value s. Base case.

the compensation is lower the higher the value of σ_θ.

The concave form on the compensation function in Figure 3.6 is similar to the compensation function when the private information is constant, as shown in Figure 2.1, although for different reasons. Recall that in the numerical example of chapter 2 the compensation is concave because of the agent's decreasing value of private information as the uniformly distributed investment cost approaches the upper level cost.

If we compare the compensation function to the linear curve in Figure 3.6, representing the asset value s, we see that the compensation consists of a large part of the total value.

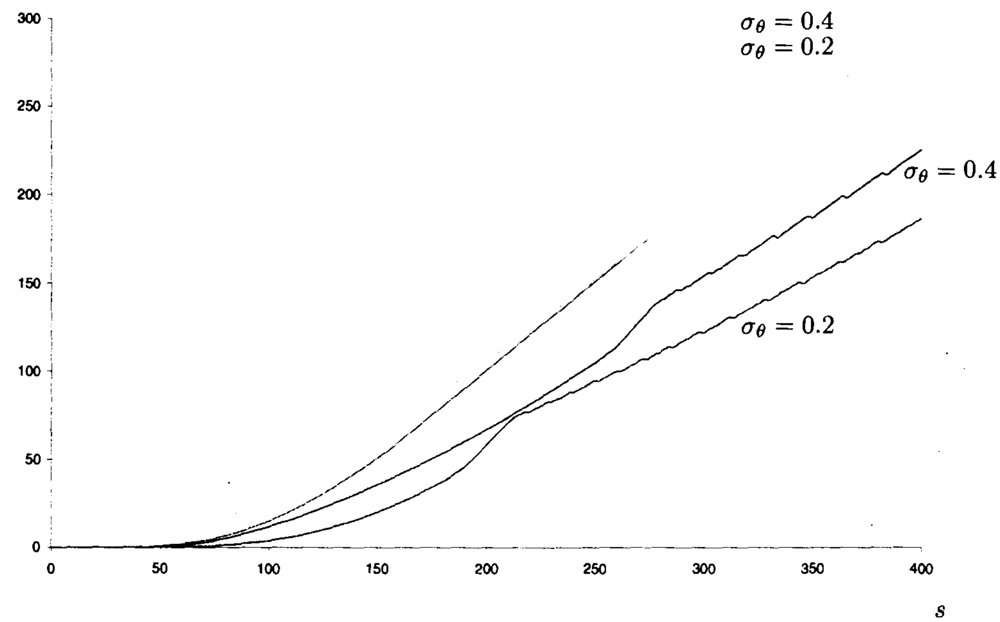

Figure 3.8: The principal's value functions V^P_{sym} and $\tilde{V}^P$ for different values of volatility σ_θ and varying asset value s.

Figure 3.7 plots the parties' values as functions of the asset value s. The values corresponds to the parties' values shown in Figures 3.1-3.3 for the case where the investment cost $k = 100$. The upper curve is the principal's value when the agent has no private information, whereas the middle curve represents the principal's value when private information exists. The lower curve represents the agent's value of the contract. Figure 3.7 corresponds to the principal's value functions in Figure 2.2 in chapter 2. By comparison of the two figures, we find that the value functions have similar forms.

Figure 3.7 shows that in the base case example, where $k = 100$, the incentive problems lead to under-investment: The optimal investment trigger under full information ψ^*_{sym} equals 158, whereas the optimal entry threshold under asym-

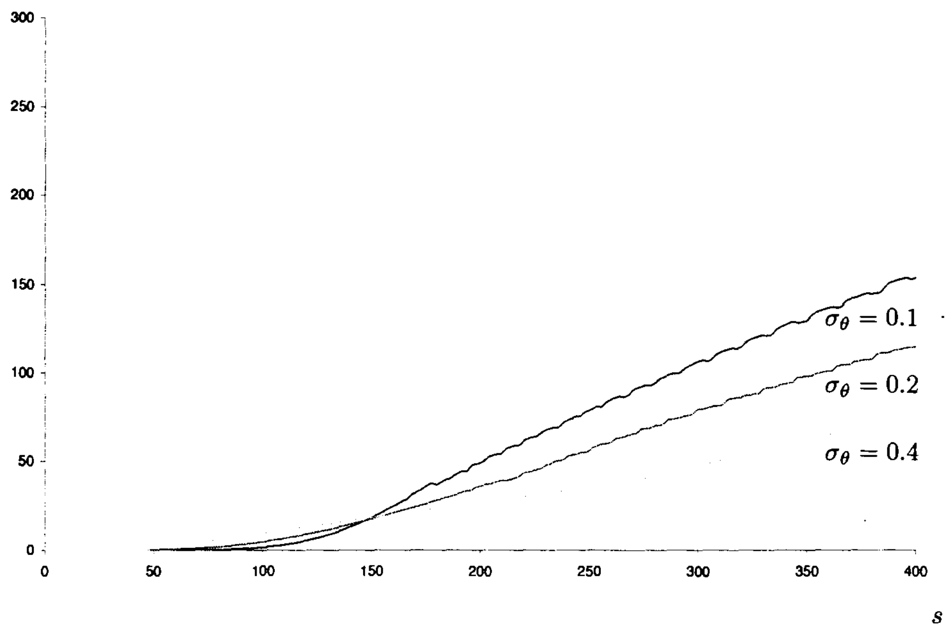

Figure 3.9: The agent's value function V^A for different values of volatility σ_θ and varying asset value s.

metric information ψ^* is equal to 218. The investor's loss because of asymmetric information amounts to the difference between the two upper curves. The figure shows that the investor's value is substantially reduced due to the incentive problems.

Under asymmetric information both parties' values increases convexly in s when $s \leq \psi^* = 218$. In the interval where $s > \psi^* = 218$ the principal's value increases convexly, whereas the agent's value increases concavely, as is shown in Figure 3.9. The concave increase in the agent's value is due to a fact earlier mentioned: because of the incentive compatibility constraint the principal's payments are convexly increasing in the investment cost k, which means that the compensation increases concavely with respect to the output value s. The reason for the convex increase in the principal's value function when $s > \psi^*$ is as follows. If $s > \psi^*_{sym} = 158$ the investment strategy is efficient under asymmetric information as the investment decision is the same whether there exists private information or not. Thus, the principal's value under full information equals the sum of the parties' values under asymmetric information. As the principal's value under full information is linear in s, the difference between this full information value and the agent's value yields a convexly increasing value function $\tilde{V}^P$ when $s > \psi^*$.

Figure 3.8 illustrates the investor's value of the investment option under full information and asymmetric information, respectively, for different values of volatility σ_θ. The two upper curves plot the investor's values in the case of full information. The value functions meet at $s = 282$. The two lower curves represent the investor value under asymmetric information. Both under asymmetric and full information the principal's value functions increase in the volatility parameter σ_θ. When the volatility parameter σ_θ equals 0.2, it is optimal to invest when $s > 218$, whereas the optimal entry threshold is 278 when $\sigma_\theta = 0.4$. Note that, in contrast to the full information case, the investor's value under asymmetric information is dependent on the volatility in the interval where immediate investment is optimal. The reason is that the optimal compensation function X is dependent on the volatility parameter.

In Figure 3.9 the agent's value of the contract is drawn for different values of the volatility parameter σ_θ. As opposed to the effect for the principal, the incentive problems tend to a decrease in the agent's value function as the volatility increases, as shown in the figure for s higher than approximately 150. The intuition for the decrease in the agent's value for an increase in the volatility σ_θ corresponds to the similar effect of the compensation function, shown in Figure 3.6: to induce the agent to follow the optimal investment strategies given in Figure 3.5, the compensation must be reduced for increasing parameter values of the volatility σ_θ. This means that increases in the volatility decreases the agent's value of private information for $s > 150$.

In Figures 3.5-3.6 and 3.8-3.9 we have presented how the value and the corresponding optimal investment strategies are affected by changes in the volatility parameter of the privately observed cost, σ_θ. It can, by numerical examples, be shown that the effects of changing the volatility parameter σ_S are qualitatively similar to the results of changing the parameter values of σ_θ.

3.8 Conclusion

In this chapter we study effects of stochastically changing private information on an optimal stopping problem. We formulate a principal-agent model where an agent has private information about a stochastic investment cost.

Under the assumption of a binding contract between a principal and an agent, we find an optimal compensation function that induces the agent to follow the dynamic investment strategy preferred by the principal. The compensation is dependent on the observable variables included in the model, and is independent of the agent's private information. Note that, because of the agent's private information, the principal cannot do better than to enter into a contractual relationship as described by the optimal compensation function we have found.

From the numerical illustrations in section 3.7, we derive several results, most of which are consistent with the results from the model in chapter 2, where the private information is constant.

One result that contrasts the model in chapter 2 is that the stochastically changing private information yields two opposite effects on the optimal investment strategy: The incentive problem increases the investor's cost of exercising the option to invest, thereby leading to under-investment. On the other side, the incentive problem implies that the value of postponing the investment is reduced, which tends to over-investment. As the volatility increases, the over-investment effect gets larger in our numerical example. In the model of chapter 2, where the private information is a constant, we find that asymmetric information leads to under-investment only.

As in chapter 2 we find that the compensation function increases concavely in the observable asset value. An effect is that the agent's value is decreasing in the volatility for some parameter values. However, the principal's value under asymmetric information increases in volatility, similarly to the effect of an investor's option value as a function of volatility when we have full information. In the numerical example we find that the investor's value of the investment is substantially reduced because of private information.

The private information discussed in this chapter concerns only the investment cost of the project. An interesting extension is to examine how the contract is changed when we assume that the output value is privately observed by the agent. For a special case, this extension is addressed in chapter 5.

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Chapter 4

Asymmetric Information about an Investment Cost: Competing Agents

For the investment projects modelled in the previous chapters, we assumed that only one agent has private information. However, for many types of real options, there will typically be more than one agent having private information. Thus, in this chapter we study the value of the investment project, and the corresponding optimal investment strategies, when two or more agents compete about the contract.

The chapter is split into two parts. In the first part we assume that the agents have private information about constant investment cost levels, extending the model in chapter 2. The second part studies the contract under the assumption that the investment costs are given by stochastic variables, as is the assumption in chapter 3. In both cases we find that the optimal investment strategies are identical to the respective optimal entry thresholds under the assumption of no competition. However, we find that the agent's value of private information is lower under competition than in the principal-agent models.

4.1 Introduction

In chapters 2 and 3 we assume that only one agent has private information concerning an investment project. However, in most cases there will be more than one agent having private information. For example, in the case of an investor owning petroleum resources, typically there exists more than one supplier having private information about technical solutions for producing the resources. Below we analyze how the investor's value of the contract is changed when two or more agents compete about a contract that gives the winner the right to manage an investment project.

We extend the principal-agent models of chapters 2 and 3 by assuming that there are n agents competing about the management of the investment project. Furthermore, we assume that the investment cost of each agent may be different, reflecting that the agents' qualifications may not be identical. In section 4.2 we extend the principal-agent model in chapter 2 to the case where n agents have private information about their respective, different constant investment costs. In section 4.3 we assume that n agents' private cost information is stochastic, i.e., we extend the model in chapter 3 to the case where n agents compete about the task of managing the investment project. We will analyze how the optimal contracts in the principal-agent models are changed when competition is introduced, and also find each agent's value of private information.

The incorporation of competition follows an approach similar to Laffont and Tirole (1987). Laffont and Tirole (1987) assume that the respective agents' private information is constant and formulate their model as a *second-price sealed-bid private-values auction*, also called a *Vickrey auction*. In such an auction, each bidder simultaneously submits a bid, without seeing others' bids, and the contract is given to the bidder who makes the best bid. However, the contract is priced according to the second-best bidder. Although we apply a Vickrey auction in the presentation below, it can be shown by the *revenue equivalence theorem* that under the assumptions we use the results does not depend on the organization of the auction[1].

[1]For a discussion of the auction model and the revenue equivalence theorem, see chapter 1, page 17, or the survey article in Klemperer (1999).

In the first model the private information is constant, and each agent reports only once. Thus, in this case the approach by Laffont and Tirole (1987) can be used almost directly: The winner of the contract is the agent with the lowest cost, and the parties know who is the winner immediately after the reports are given.

In the second model, the agents' private information changes continuously, and their private information is likewise continuously reported. Hence, the auction must be adapted to the changes in private information. We organize the model as an auction of Vickrey type, where new reports are continuously given, until one or more agents report a cost low enough to trigger investment. At this point in time the agent with the lowest cost report wins the contract.

Note that when the private information changes stochastically, it is not optimal to assign the contract to any of the agents before the time of investment. The reason is that the investment cost is given by a stochastic process with independent increments over time. This means that the agent reporting the lowest cost at one point in time, does not necessarily have the lowest cost at a later point in time. Therefore, we assume that all the agents participate in the auction, until a cost is reported that is low enough to immediately trigger investment. As the winner of the contract is not chosen prior to the time when the investment decision is made, in the case of changing private information, competition implies that the investment decision cannot be delegated to the winning agent.

4.2 Competition when each agent's private information is constant

Assumptions. We assume that n agents compete about a contract that gives the winner the right to manage the investment strategy (or more specifically, gives the winner the right to decide on an optimal stopping strategy), and to receive a predetermined compensation. Note that most of the assumptions below are similar to the ones in chapter 2. However, many terms associated with a particular agent carry an i-superscript, compared with chapter 2.

Each agent i has private information about his own cost of the investment, K^i, but has no private information about the competitors' costs. We define the competitors' costs by the vector $K^{-i} = (K^1, ..., K^{i-1}, K^{i+1}, ..., K^n)$. The investor is now called the auctioneer (and he is identical to the principal in the principal-agent models). The auctioneer does not observe any of the n agents' investment cost parameter values, but it is common knowledge that the values are drawn independently from the same distribution, having a cumulative distribution function $F(\cdot)$ on the interval $[\underline{K}, \overline{K}]$.[2] We assume that $F(\cdot)$ is absolutely continuous. As $F(\cdot)$ is common knowledge, agent i's knowledge about the competitors' true investment cost is identical to the auctioneer's knowledge. As in chapter 2 we assume that the fraction $F(\cdot)/f(\cdot)$ is non-decreasing.

The assumptions with respect to the investment option and the value of the asset in place are identical to the model in chapter 2. For convenience we repeat the main assumptions below.

The option to invest in the project is perpetual. As in chapter 2, the output value (the value of the "asset in place") from the investment project is denoted S_t, and is known by all the participants in the auction, including the auctioneer. The output value S_t is a stochastic process, defined by a complete probability space $(\Omega, \mathcal{F}, P)$ and state space $(0, \infty)$. Under the equivalent martingale measure Q the stochastic process is given by

$$dS_t = (rS_t - \delta(S_t))dt + \sigma(S_t)dB_t^S, \quad s \equiv S_0. \tag{4.1}$$

The parameter r denotes the risk free rate, $\delta(\cdot)$ denotes the convenience yield function, $\sigma(\cdot)$ is the volatility function, and B_t^S is a standard Brownian motion

[2]The assumptions that the cost parameters are different for the agents, and that the parameter values are independently drawn from the same distribution, are important for the results. An alternative assumption we could make about the agents' information, is that the true value is the same for everyone, but that the agents' have different information about the true value. In this case one agent learns about the true value if he observes another agent's signal. If these assumptions are made, the game is analyzed in a *pure common-value* model, whereas our assumptions about the agents' information above yield a *private-value* model, see an overview of auction theory by Klemperer (1999). We can also assume models where both kinds of information is present, i.e., where the value of an object differs from agent to agent (for example because of subjective valuation), and where at the same time each agent learns more about the value from others' signals. Klemperer (1999) refers to any model in which the value depends on some extent on others' bids, as *common-value* models.

The revenue equivalence theorem applies only in the case of a private-value model, or if the bidders' signals are independent.

with respect to the equivalent martingale measure. We make the same assumptions about the output value S_t as in chapter 2: The functions $\delta(\cdot) > 0$ and $\sigma(\cdot) > 0$ are Lipschitz continuous. Moreover, the stochastic process in (4.1) is a linear diffusion.

It is assumed that the investor's information at time t is given by $\mathcal{F}_t^S$, generated by $\{S_\xi, \xi \leq t\}$. Each agent i's information at time t is given by $\mathcal{F}_t^{S,K^i}$ generated by $\{S_\xi, K^i, \xi \leq t\}$.

Define the vector of reports by $\hat{K} = (\hat{K}^1, ..., \hat{K}^n)$. Each agent i's expected compensation $X^i(S_t, \hat{K})$ is received at the time the investment is exercised. Observe that the compensation function may be dependent on the vector of all reports $\hat{K} = (\hat{K}^1, ..., \hat{K}^n)$, in addition to all the observable quantities.

The investment strategy, if agent i wins the contract, is given by the optimal stopping time $\tau^i_{\hat{K}}$, and based on the reports given by the agents, as well as the value of S_t. Moreover, the investment strategy is time independent, as the option to invest is perpetual and S_t is driven by a time-homogeneous stochastic process. We denote the critical price by $S^i(\hat{K})$. When $S_t > S^i(\hat{K})$ the strategy prescribes immediate investment, whereas the investment is postponed if $S_t \leq S^i(\hat{K})$. Note that as the investment strategy $S^i(\hat{K})$ may be dependent on all the cost reports, the investment strategy is stochastic to each agent i.

The auction is organized such that the agents simultaneously report their investment cost $\hat{K} \equiv (\hat{K}^1, ..., \hat{K}^n)$ to the auctioneer. The agents do not know the other agents' reports.

We introduce a control variable $y^i(\cdot)$ that depends on the vector of the agents' reports $\hat{K}$. By $y^i(\cdot)$ the auctioneer decides on the winner of the contract. Thus, the variable can be interpreted as a probability, where $y^i(\hat{K})$ is the probability that agent i wins the contract. We make the following restrictions:

$$\textstyle\sum_{i=1}^n y^i(\hat{K}) \leq 1 \quad \text{for any } \hat{K}, \tag{4.2}$$

i.e., the sum of each agent's probability of winning the contract cannot exceed one. In addition, as probabilities are always negative, we assume that

$$y^i(\hat{K}) \geq 0 \quad \text{for any } \hat{K}. \tag{4.3}$$

Incentive mechanisms. We shall see that the results of the auction lead to the same outcome whether it is the auctioneer or the winning agent who decides on the investment strategy. However, in order to solve the problem, we now assume that the auctioneer decides on the investment strategy (i.e., on the optimal stopping time) based on the winning agent's cost report. Thus, the incentive scheme is given by $(X^i(\hat{K}), \tau^i_{\hat{K}}, y^i(\hat{K}))$. As we look for truth telling equilibria, we approach the problem in the same way as for an analogous principal-agent problem. More specifically, we look for mechanisms $(X^i(\hat{K}), \tau^i_{\hat{K}}, y^i(\hat{K}))$ that induce truth telling Bayesian Nash equilibria[3].

As in chapter 2, section 2.2, we assume that the investment cost is not correlated to capital markets.

Agent i's value function $v^i(\cdot)$ is given by the value of the compensation function reduced by the expected investment cost, where the expected investment cost is adjusted for the probability of winning the contract, i.e.,

$$v^i(s, K^i; \hat{K}^i) = E\left[e^{-r\tau^i_{\hat{K}}} \left(X^i(S_{\tau^i_{\hat{K}}}, \hat{K}) - y^i(\hat{K})K^i \right)^+ \middle| \mathcal{F}_0^{S,K^i} \right], \qquad (4.4)$$

where $\tau^i_{\hat{K}}$ is a stopping time for agent i. By comparing equation (4.4) to the agent's value function when there is no competition, equation (2.3), we see that the two functions are similar. However, there are some differences. First, the investment cost is now corrected for the probability that the agent obtains the contract. This implies that the expected cost is lower than under no competition. Next, we have included in (4.4) that the compensation may depend on the agents' reports. Note that the compensation $X^i(\cdot)$ and the investment strategy $\tau^i_{\hat{K}}$ may now be dependent on the competitors' reports as well as the report of each agent i. Hence, the investment strategy and the value of the compensation may be stochastic to the winning agent.

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[3]In a Bayesian Nash equilibrium each agent's reporting strategy is a function of his own information, and each agent maximizes his value function given the other agents' strategies, and given his beliefs about the other agents' information. In our model the agents' beliefs about the others' private information is given by the probability density $f(\cdot)$ together with the limits $\underline{\theta}$ and $\overline{\theta}$. A Bayesian Nash equilibrium is the appropriate equilibrium concept in auctions because of the presence of asymmetric information.

The auctioneer's value function is given by

$$v^P(s;\hat{K}) = E\left[\sum_{i=1}^{n} e^{-r\tau^i_{\hat{K}}}\left(y^i(\hat{K})S_{\tau^i_{\hat{K}}} - X^i(S_{\tau^i_{\hat{K}}};\hat{K})\right)^+ \middle| \mathcal{F}^S_0\right]. \tag{4.5}$$

The auctioneer's value of the investment depends on the net present value of future cash flows, reduced by the sum of the transfer functions $X^i(\cdot)$. The value corresponds to the principal's value of the contract in equation (2.2). The term $y(\hat{K})S_{\tau^i_{\hat{K}}}$ is the output value the auctioneer obtains at the investment time, adjusted for the probability that agent i wins the contract. To find the auctioneer's expected value of the output from the project, we need to sum up over all the agents participating in the contract, as done in (4.5). The compensation X^i is the amount paid to each agent i.

The optimization problem. We are now ready to state the auctioneer's optimization problem:

$$V^P(s;\hat{K}) = \sup_{X^i(\cdot),\tau^i,y^i(\cdot)} v^P(s;\hat{K}), \tag{4.6}$$

subject to each agent i's optimization problem

$$V^i(s,K^i;\hat{K}^i) = \sup_{\hat{K}^i} v^i(s,K^i;\hat{K}^i). \tag{4.7}$$

The optimization problem corresponds to the principal's optimization problem in (2.2)-(2.3) in the principal-agent model in chapter 2, with the exception that in (2.2)-(2.3) we have not incorporated the direct mechanism $\hat{K}$. Apart from this exception, the two optimization problems are identical in the case where $n = 1$.

Valuation of the expected, future cash flows. In a similar way to the uncertainty evaluation in section 2.3 of chapter 2, we now evaluate the stochastic, future cash flows. Define $\mathcal{F}^{S,K}_t$ as the information set at time t under full information, generated by $\{S_\xi, K, \xi \leq t\}$. Similarly to equation (2.4), the value of the "discounting factor" of agent i is expressed as[4]

$$E\left[e^{-r\tau^i_{\hat{K}}}|\mathcal{F}^{S,K}_0\right] = \begin{cases} \frac{\phi(s)}{\phi(S^i(\hat{K}))} & \text{if } s \leq S^i(\hat{K}) \\ 1 & \text{if } s > S^i(\hat{K}) \end{cases} \tag{4.8}$$

[4]See footnote 6 in chapter 2.

where $\phi(\cdot)$ is a strictly positive and increasing function. Defining $u(s) = E\left[e^{-r\tau_{\hat{K}}^i}|\mathcal{F}_0^{S,K}\right]$, the value of the discounting factor satisfies the ordinary differential equation

$$\frac{1}{2}(\sigma(s))^2\frac{\partial^2 u}{\partial s^2} + (rs - \delta(s))\frac{\partial u}{\partial s} - ru(s) = 0,$$

similarly to equation (2.5), and with boundaries $\lim_{s\downarrow S^i(\hat{K})} u(s) = 0$ and $\lim_{s\uparrow S^i(\hat{K})} u(s) = 1$. We interpret equation (4.8) as the value of the discounting factor given that the vector of investment cost reports is known.

Using the result in equation (4.8), agent i's value function may be formulated as (computed in appendix C.1),

$$\begin{aligned} v^i(s, K^i; \hat{K}^i) &= E\left[\tfrac{\phi(s)}{\phi(S^i(\hat{K}))}\left(X^i(S^i(\hat{K}), \hat{K}) - y^i(\hat{K})K^i\right)\mathrm{I}_{\{s\leq S^i(\hat{K})\}}\right. \\ &\qquad \left. + \left(X^i(s, \hat{K}) - y^i(\hat{K})K^i\right)\mathrm{I}_{\{s>S^i(\hat{K})\}}\middle| \mathcal{F}_0^{S,K^i}\right]. \end{aligned} \tag{4.9}$$

Each agent i's value as formulated in equation (4.9) corresponds to the agent's value functions (2.7) and (2.22). A difference between the agent's value in the principal-agent model and agent i's value in the auction model, is that the direct mechanism now may be stochastic as agent i only observes his own report, and not the others. This means that agent i's value function in the auction model does not consist only of "deterministic" functions, as the agent's value function in the principal-agent model in (2.22) does.

If the auctioneer does not observe the agents' cost parameter, his value function is given by,

$$\begin{aligned} v^P(s; \hat{K}) &= E\left[\sum_{i=1}^n \left\{\tfrac{\phi(s)}{\phi(S^i(\hat{K}))}\left(y^i(\hat{K})S^i(\hat{K}) - X^i(S^i(\hat{K}), \hat{K})\right)\mathrm{I}_{\{s\leq S^i(\hat{K})\}}\right.\right. \\ &\qquad \left.\left. + \left(y^i(\hat{K})s - X^i(s, \hat{K})\right)\mathrm{I}_{\{s>S^i(\hat{K})\}}\right\}\middle| \mathcal{F}_0^S\right], \end{aligned} \tag{4.10}$$

derived in appendix C.2. The auctioneer's value in (4.10) corresponds to the principal's value function (2.6) in the principal-agent model of chapter 2.

The reformulations of the auctioneer's and the agents' respective value functions simplify the optimization problem given by (4.6) to (4.7), as the value functions no longer are stochastic with respect to the value of the variable S_t. However, the value functions are still uncertain with respect to the auctioneer's and the agents' respective vectors of unobservable investment cost parameters.

4.2.1 The agents' reporting behavior

Similarly to the approach in the principal-agent models we find a truth telling equilibrium, implying that the first-order condition for the report $\hat{K}$ must be satisfied for each agent i at the point where $\hat{K}^i = K^i$, i.e.,

$$\left.\frac{\partial v^i(s, K^i; \hat{K}^i)}{\partial \hat{K}^i}\right|_{\hat{K}^i = K^i} = 0. \tag{4.11}$$

Hence, for the truth telling condition to hold, reporting the true cost is optimal for each agent i when the condition in (4.11) is satisfied.

Let now $v^i(s, K^i)$ be each agent i's value function given truth telling. The value function of agent i under truth telling is written as

$$\begin{aligned} v^i(s, K^i) &= E\left[\tfrac{\phi(s)}{\phi(S^i(K))}\left(X^i(S^i(K), K) - y^i(K)K^i\right) \mathrm{I}_{\{s \le S^i(K)\}}\right. \\ &\quad \left. + \left(X^i(s, K) - y^i(K)K^i\right) \mathrm{I}_{\{s > S^i(K)\}} \right| \mathcal{F}_0^{S,K^i}\Big], \end{aligned} \tag{4.12}$$

which is equal to equation (4.9) with the exception that the vector $\hat{K}$ is replaced by the vector K.

By the envelope theorem, the first-order condition in (4.11) is found, similarly to (2.24) and (3.11), as

$$\frac{dv^i(s, K^i)}{dK^i} = E\left[-\frac{\phi(s)}{\phi(S^i(K))} y^i(K) \mathrm{I}_{\{s \le S^i(K)\}} - y^i(K) \mathrm{I}_{\{s > S^i(K)\}} \right| \mathcal{F}_0^{S,K^i}\Bigg]. \tag{4.13}$$

The second-order condition mimics to the second-order condition for truth telling in the model of chapter 2, cf. appendix A.5.

Integration of both sides of the first-order condition in (4.13) leads to an expression of agent i's value of private information,

$$\begin{aligned} &v^i(s, K^i) \\ &= E\left[\int_{K^i}^{\overline{K}} \tfrac{\phi(s)}{\phi(S^i(K^{-i}, u))} y^i(K^{-i}, u) du \mathrm{I}_{\{s \le S^i(K)\}} + \left(\int_{K^i}^{\vartheta(s, K^{-i})} y^i(K^{-i}, u) du \right.\right. \\ &\quad \left.\left. + \int_{\vartheta(s, K^{-i})}^{\overline{K}} \tfrac{\phi^i(s)}{\phi^i(S^i(K^{-i}, u))} y^i(K^{-i}, u) du \right) \mathrm{I}_{\{s > S^i(K)\}} \right| \mathcal{F}_0^{S,K^i}\Big]. \end{aligned} \tag{4.14}$$

In equation (4.14) we have formulated agent i's value of private information without including the unknown compensation function $X^i(\cdot)$. Agent i's value of private information differs from the agent's value in the principal-agent models, formulated in (2.25) and (3.13), because the auction model adjusts each agent's value of private information for the probability of winning the contract. Also, the value of private information is stochastic as each agent does not observe the other agents' private information.

4.2.2 The auctioneer's optimization problem

In this section we solve the auctioneer's optimization problem, i.e., we choose the winner of the auction and find the optimal investment strategy. In order to do so, we approach the problem in the same way as earlier: we substitute the compensation function, $X^i(\cdot)$, by agent i's value function in equation (4.12). Then the auctioneer's optimization problem in (4.10) is reformulated as

$$\begin{aligned} & V^P(s,K) \\ = \; & \sup_{S^i(\cdot),y^i(\cdot)} E\left[\sum_{i=1}^n \left\{\frac{\phi(s)}{\phi(S^i(K))} y^i(K)\left(S^i(K)-K^i\right)\mathrm{I}_{\{s\leq S^i(K)\}}\right.\right. \qquad (4.15) \\ & \left.\left. + y^i(K)\left(s-K^i\right)\mathrm{I}_{\{s>S^i(K)\}} - v^i(s,K^i)\right\} \Big| \mathcal{F}_0^S\right], \end{aligned}$$

where $v^i(s, K^i)$ is given by (4.14).

Observe that the optimization problem could be simplified if the trigger price S^i were dependent only on agent i's cost level K^i, instead the vector of all costs, K. The reason is that if $S^i(K^i)$ equals $S^i(K)$ we can optimize the auctioneer's value with respect to each agent i separately. In appendix C.3, it is shown that this is the optimal solution indeed, i.e., $S^{i*}(K^i) = S^{i*}(K)$, where $S^{i*}(\cdot)$ is defined as the optimal entry threshold of agent i. The idea of this simplification is based on Laffont and Tirole (1987), where a similar argument is used to show that a random incentive scheme is not optimal in the solution of their problem.

The auctioneer's value function is linearly dependent upon the probability that agent i is the winner of the contract, $y^i(K)$. Thus, we can substitute $y^i(K)$ by defining $Y^i(K^i) = E\left[y^i(K)|\mathcal{F}_0^{S,K^i}\right]$ in the optimization problem (4.15), where the function $Y^i(K^i)$ is interpreted as agent i's probability of winning the contract.

Define $\hat{V}^P(s;K^i) = \sup_{S^i(\cdot),y^i(\cdot)} \hat{v}^P(s;K^i)$ as the auctioneer's optimization problem when $S^i(K)$ is replaced by $S^i(K^i)$. For given $y^i(\cdot)$, and hence for given $Y^i(\cdot)$, the auctioneer's optimization problem (derived in appendix C.4), is given by

$$
\begin{aligned}
&\hat{V}^P(s;K^i) \\
&= \sup_{S^i(\cdot)} \sum_{i=1}^{n} \left\{ \int_{\underline{K}}^{\overline{K}} \left[\frac{\phi(s)}{\phi(S^i(K^i))} Y^i(K^i) \left(S^i(K^i) - K^i - \frac{F(K^i)}{f(K^i)} \right) \mathrm{I}_{\{s \le S^i(K^i)\}} \right. \right. \\
&\quad \left. \left. + \left(Y^i(K^i) \left(s - K^i - \frac{F(K^i)}{f(K^i)} \right) \right) \mathrm{I}_{\{s > S^i(K^i)\}} \right] f(K^i) dK^i \right\}.
\end{aligned}
\tag{4.16}
$$

Note that each of the n optimization problems given by (4.16) are very similar to the principal's problem in chapter 2, equation (2.28). Observe that we now can separate the problem of finding the optimal critical price $S^{i*}(K^i)$, and the problem of choosing a winner of the contract. Thus, the optimal investment strategy is identical to the optimal investment strategy in equation (2.30), as will be seen by optimization of the auctioneer's simplified optimization problem in (4.16), with respect to $S^i(K^i)$, i.e.,

$$
S^{i*}(K^i) - K^i - \frac{F(K^i)}{f(K^i)} = \frac{\phi(S^{i*}(K^i))}{\phi'(S^{i*}(K^i))}. \tag{4.17}
$$

The function $\phi'(S^{i*}(K^i))$ denotes the derivative of $\phi(\cdot)$ with respect to the optimal investment strategy S^{i*}. The left-hand side of equality (4.17) represents the net value of the auctioneer's payoff at the time when the investment is exercised. The right-hand side is interpreted as the opportunity cost of exercising the option with payoff value equal to $S^{i*}(K^i) - K^i - \frac{F(K^i)}{f(K^i)}$.

The control variable $y^i(K)$ is linear in the auctioneer's problem of finding the investment strategy of agent i. Therefore, we choose an optimal $y^{i*}(K)$ such that

$$
y^{i*}(K) = \begin{cases} 1 & \text{if } K^i < \min_{j \neq i} K^j \\ 0 & \text{if } K^i > \min_{j \neq i} K^j. \end{cases} \tag{4.18}
$$

Thus, the agent with the lowest cost wins the contract, provided it is sufficiently low. If $K^i = \min_{j \neq i} K^j$ the auctioneer is indifferent between which agent to choose as a winner of the contract.

As the optimal investment strategy given by (4.17) equals the optimal investment strategy in the one-agent case, the efficiency is not improved when competition is introduced. However, the winner of the contract in the competition probably has a lower investment cost than the agent in a principal-agent model, and thereby the investment will probably take place at a lower cost. Moreover, if the number of competing agents gets large, the winner's cost level gets close to the lowest possible cost, $\underline{K}$. When the winner's cost level converges to $\underline{K}$, the cumulative distribution $F(\cdot)$ converges to zero, which leads to no inefficiency in the investment strategy.

4.2.3 Implementation of the contract

Using (4.14), (4.17) and (4.18), agent i's value of private information is found as

$$V^i(s;K^i) = \begin{cases} \int_{K^i}^{\overline{K}} \frac{\phi(s)}{\phi(S^{i*}(u))} Y^{i*}(u)du & \text{if } s \leq S^{i*}(K^i) \\ \\ \int_{K^i}^{\vartheta^*(s)} Y^{i*}(u)du + \int_{\vartheta^*(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^{i*}(u))} Y^{i*}(u)du & \text{if } s > S^{i*}(K^i) \end{cases} \tag{4.19}$$

Hence, we find that agent i's optimal value of the compensation $X^{i*}(s, K^i)$ is given by

$$X^{i*}(s,K^i) = K^i Y^{i*}(K^i) + \int_{K^i}^{\vartheta^{i*}(s)} Y^{i*}(u)du + \int_{\vartheta^{i*}(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^{i*}(u))} Y^{i*}(u)du, \tag{4.20}$$

when $s > S^{i*}(K^i)$. Otherwise, $X^{i*}(s, K^i) = 0$. Equation (4.20) represents the expected compensation of each agent participating in the auction. The main difference between each agent's value of the compensation in the auction and the agent's compensation in the principal-agent model, given by equation (2.34), is that the compensation function in the auction model is adjusted for the probability of winning the contract, $Y^{i*}(K^i)$. As the probability is lower than one, each agent's expected compensation in the auction model is lower than in the principal-agent model.

In the above expression of the optimal compensation function, each agent's strategy is optimal based on "average quantities", i.e., the strategy depends on $Y^i(K^i) = E\left[y^i(K)|\mathcal{F}_0^{S,K^i}\right]$ and $S^i(K^i) = E\left[S^i(K)\mathrm{I}_{\{y^i(K)=1\}}|\mathcal{F}_0^{S,K^i}\right]$.

Now, construct a dominant strategy auction[5] where each agent has a reporting strategy that is optimal for any reports by the other agents. We formulate a second-price sealed-bid private values auction (or a Vickrey auction)[6] that implements the optimal investment strategy, and selects the agent with the lowest cost. We denote the compensation function $\tilde{X}^i$, and its value is given by

$$\tilde{X}^i(s,K) = \begin{cases} \vartheta^{i*}(s) + \int_{\vartheta^{i*}(s)}^{K^j} \frac{\phi(s)}{\phi(S^{i*}(u))} du & \text{if } S^{i*}(K^i) < s \leq S^{i*}(K^j) \\ K^j & \text{if } s > S^{i*}(K^j), \end{cases} \tag{4.21}$$

if $K^i = \min_h K^h$ and $K^j = \min_{h \neq i} K^h$. If $s \leq S^{i*}(K^i)$, $\tilde{X}^i(s,K) = 0$. Thus, $\tilde{X}^i$ is the optimal and implementable compensation to agent i, given that agent i is the winner of the contract. Note that $\tilde{X}^i$ is the optimal compensation to the winner of the contract, whereas X^{i*} is each agent i's expected value of participating in the auction. In appendix C.5 it is shown that each agent's expected value of the compensation function in (4.21), $E\left[\tilde{X}^i(s,K)|\mathcal{F}_0^{S,K^i}\right]$, equals the value in (4.20), i.e., $X^{i*}(s,K^i) = E\left[\tilde{X}^i(s,K)|\mathcal{F}_0^{S,K^i}\right]$.

The implementable compensation $\tilde{X}^i$ ensures that the agent having the lowest investment cost obtains the contract. When agent i wins the contract, the agent's compensation equals the value of his private information when the distribution is truncated at K^j. Thus, competition for the best agent amounts to a truncation of the interval $(\underline{K}, \overline{K})$ to $(\underline{K}, K^j)$, where K^j is the second-lowest report.

To sum up, we see that the optimal compensation in (4.21) is formally identical to the optimal compensation when there is only one agent, given by equation (2.34), with the exception that the truncation is changed from $\overline{K}$ to the second-lowest report K^j. Truth telling is an optimal strategy, whether there are competing agents or not. The only difference between the principal-agent model and the auction is that the upper level of possible reports is changed, leading to a lower value of the agent's private information.

[5]A dominant strategy auction is an auction in which each agent has a strategy that is optimal for any strategies of its competitors.

[6]For definitions, confer for instance Klemperer (1999) or chapter 1, page 17 of this dissertation.

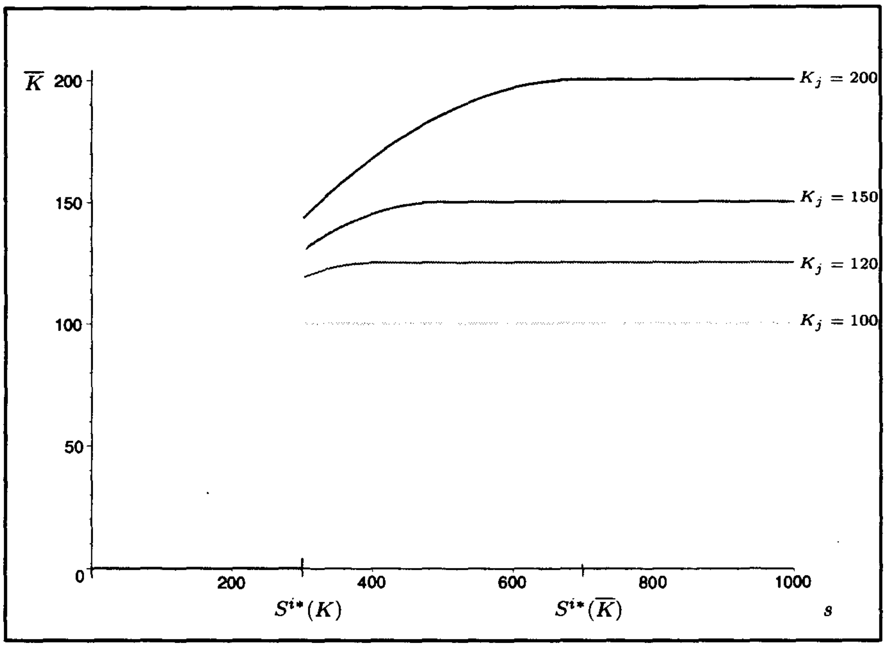

Figure 4.1: The compensation $\tilde{X}^i$ as a function of the asset value s for different values of second-lowest cost report K_j.

4.2.4 Numerical illustration of the effect of competition

Under the assumptions that the value of the asset in place is driven by a geometric Brownian motion, and the unobservable investment cost parameters K^i are uniformly distributed, we illustrate some effects of competition. The parameter values are identical to the base case parameter values in the numerical examples of the similar model in chapter 2, page 43.

In Figure 4.1 the winner's compensation function $\tilde{X}^i$ is drawn for different levels of the second-lowest cost report K_j. We assume that agent i is the winner, and that agent j gives the second-lowest cost report. In the case where the cost of the agent with the second-lowest report equals 200, i.e., $K_j = 200$, the winner's compensation is equal to the compensation in the principal-agent model, shown

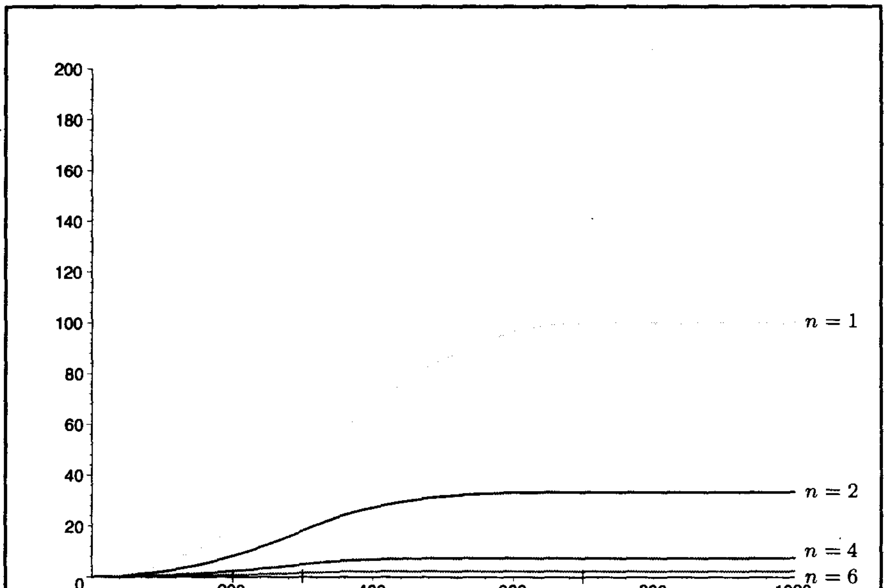

Figure 4.2: The winner's value V^i as a function of the asset value s. The number of competitors is denoted by n.

in Figure 2.1. The compensation functions are equal in the two models in this case because agent j's cost level coincide with the upper level cost $\overline{K}$. As agent j's cost level gets closer to the winner's investment cost $K^i = 100$, the value of the agent's private information decreases. Moreover, as agent j's cost level decreases, the interval where the compensation is independent of the asset value s gets larger. This is the effect from reducing the possible cost reports from $[\underline{K}, \overline{K}]$ to $[\underline{K}, K_j]$. In the limiting case, where $K_j = 100$, the winner's value of the contract is zero, as the winner only obtains a compensation equal to his cost level for all asset values s. This situation is illustrated in the lower curve in Figure 4.1. Observe that although the agent's value is zero, the situation does not necessarily coincide with the full information case (except when $K_j = \underline{K}$) as the optimal investment strategy is not the same as under full information.

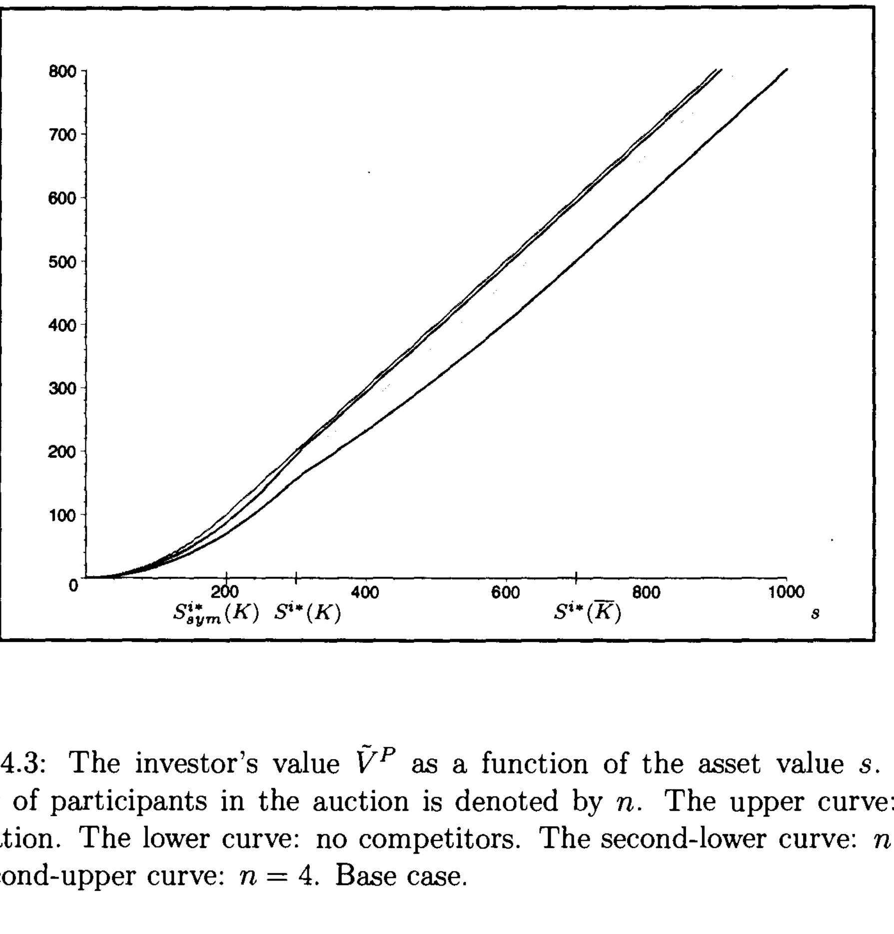

Figure 4.3: The investor's value $\tilde{V}^P$ as a function of the asset value s. The number of participants in the auction is denoted by n. The upper curve: full information. The lower curve: no competitors. The second-lower curve: $n = 2$. The second-upper curve: $n = 4$. Base case.

Figure 4.2 illustrates the effect from competition on the agent's value of private information. In the figure we draw four curves representing the winner's contract value when there is no competition ($n = 1$), and when there are 2,4 and 6 competitors, respectively. The value function represented by the upper curve, showing the case of no competition, is identical to the agent's value in the principal-agent model, found in Figure 2.2. As the number of competitors increases, the winner's value of the contract falls rapidly. In our example, the winner's value falls by about two thirds when we go from no competition to two competitors. When there are six competitors the value of each auction participant is close to zero.

Figure 4.3 illustrates the investor's value of the contract under competition. The upper curve is the full information case, whereas the lower curve is the value when there is only one agent having private information. Thus, the lower curve is identical to the investor's value under asymmetric information in the principal-agent model, illustrated in Figure 2.2. The second-lower curve and the second-upper curve are the investor's values in the case of asymmetric information and when there are two and four competitors, respectively. From Figure 4.3 we see that as the number of competitors increases the agent's value gets close to zero, implying that the auctioneer's value gets closer to the full information value. However, even when the winner's value is close to zero because of competition, the optimal investment strategy is not efficient as long as the winner's cost is above the lower limit $\underline{K}$. The effect is illustrated in Figure 4.3. When there are four competitors (corresponding to the second-upper curve) the investor's value almost coincides with the value under full information in the interval where it is optimal to invest immediately, i.e., when $s > S^{i*}(K^i) = 300$. However, in the interval where $s \leq S^{i*}(K^i)$, the difference between the full information case and the auctioneer's value when $n = 4$ is larger.

4.3 Competition when the agents' private information changes stochastically

4.3.1 Problem formulation

Now we assume that each agent's private information changes stochastically. The agents' respective private information is given by independent stochastic processes. Once again, we organize the model as an auction of the Vickrey type. However, the auction model is slightly changed compared to the case where the agents' private information is constant: as the private information now changes continuously, we assume that each agent participating in the auction gives new reports simultaneously and continuously, until one or more agents report a cost low enough to trigger investment. At this point in time, the agent with the lowest cost report wins the contract, receives the compensation, and invests immediately.

It is not optimal to assign the contract to any of the agents prior to the time of the investment. As mentioned introductorily, the reason is that the investment cost is given by a stochastic process that has independent increments over time. Thus, if we are restricted to choose a winner of the contract prior to the time of the investment, the contract value is lower than in the situation where the auctioneer choose the winner at the investment time. In the model below we assume that all the agents continuously participate in the auction, until a cost is reported that is low enough to trigger immediate investment. At this time the agent reporting the lowest cost wins the contract. As the winner is not chosen prior to the investment time, the investment decision cannot be delegated to the winner of the contract.

The model is an extension of the principal-agent model in chapter 3, where we assume that the single agent's private information is stochastic. In chapter 3 the stochastic processes in the principal-agent model follows geometric Brownian motions. Now we assume that the stochastic processes in the auction are time-homogeneous Ito diffusions. However, the main results are very similar whether we make use of geometric Brownian motions, or more general time-homogeneous Ito diffusions.

The investment cost of each agent i is given by the function $K_t^i \equiv h(C_t, \theta_t^i)$, where the investment cost is a function of a commonly observed variable C_t, and a variable θ_t^i, observable to agent i only[7]. We assume that $h(C_t, \theta_t^i)$ is increasing in both variables.

The one-dimensional and nonnegative cost variable C_t is specified by,

$$dC_t = rC_t dt + \bar{\sigma}_C(C_t) dB_t^C, \tag{4.22}$$

where B_t^C is a standard Brownian motion under the Q measure, $\bar{\sigma}_C(\cdot)$ represents a given Lipschitz continuous volatility function, and r is the risk-free rate. A special case of the market component of the cost is the geometric Brownian motion in chapter 3, equation (3.2).

The variables θ_t^i are independently distributed between the agents. Agent i's one-dimensional and nonnegative variable θ_t^i is given by

$$d\theta_t^i = \bar{\alpha}(\theta_t^i)dt + \sigma_\theta\left(\theta_t^i\right) dB_t^i. \tag{4.23}$$

[7]Recall that in chapter 3 the cost function has the simpler multiplicative form $K_t = C_t\theta_t$.

The functions $\bar{\alpha}(\cdot)$ and $\bar{\sigma}_\theta(\cdot)$ are given Lipschitz continuous functions. All the agents face the same volatility function $\bar{\sigma}_\theta(\cdot)$. The stochastic process of the technical cost component θ_t in chapter 3 is a special case of the process θ_t^i in (4.23).

As earlier, each agent i obtains the compensation $X^i(\cdot)$, which is paid at the time the investment is made.

Each agent reports a value $\hat{\theta}_t^i$ related to his observed value θ_t^i at time t. We assume that each agent cannot observe the others' reports. The vector of reports at time t is denoted $\hat{\theta}_t = (\hat{\theta}_t^1, ..., \hat{\theta}_t^n)$, with lower and upper limits equal to $\underline{\theta}$ and $\overline{\theta}$, respectively, and $\underline{\theta}, \overline{\theta} \in [0, \infty)$. Agent i's investment strategy is given by a stopping time $\tau_{\hat{\theta}}^i \in [0, \bar{\tau}]$ where $\bar{\tau} \in (0, T]$ represents the time when the option to invest expires, $T < \infty$, and T is the time horizon.

At the time the investment is made, agent i has a probability $y^i(\hat{\theta}_t)$ of winning the auction, where $\hat{\theta}_t = (\hat{\theta}_t^1, ..., \hat{\theta}_t^n)$ is the vector of reports at time t. We make the assumption

$$\textstyle\sum_{i=1}^n y^i(\hat{\theta}_t) \leq 1 \quad \text{for any } \hat{\theta}_t,\ 0 \leq t \leq \tau_{\hat{\theta}}^i. \tag{4.24}$$

Furthermore, we need

$$y^i(\hat{\theta}_t) \geq 0 \quad \text{for any } \hat{\theta}_t,\ 0 \leq t \leq \tau_{\hat{\theta}}^i. \tag{4.25}$$

The function $y^i(\cdot)$ has the same interpretation in this section as in the auction model where the investment cost is constant, section 4.2.

The set of incentive mechanisms at each point in time is formulated by $(X^i(\cdot), \tau_{\hat{\theta}}^i, y^i(\cdot))$.

Assume that $(\Omega, \mathcal{F}, Q)$ is a given probability space for the $(2+n)$-dimensional Brownian motion $(B_t^S, B_t^C, B_t^\theta)$, where B_t^θ is the n-dimensional Brownian process given by $B_t^\theta = (B_t^1, ..., B_t^n)$, and B_t^S and B_t^C are both one-dimensional Brownian processes. Agent i observes the variables S_t, C_t and θ_t^i. The stochastic process of the variable S_t is given by equation (4.1). Agent i does not observe the $\theta_t^{-i} = (\theta_t^1, ..., \theta_t^{i-1}, \theta_t^{i+1}, ..., \theta_t^n)$, but the expectation and the variance of the competitors' private information is common knowledge, i.e., the stochastic processes given by (4.23) for i from 1 to n are known. Agent i's information set on time t is denoted by $\mathcal{F}_t^{S,C,\theta^i}$ and generated by $\{S_\xi, C_\xi, \theta_\xi^i, \xi \leq t\}$.

As in chapter 3 we now denote (s, c, θ^i) equal to (S_t, C_t, θ^i_t). Given an initial time t, agent i's value function is represented by

$$v^i(s, c, \theta^i, t; \hat{\theta}^i) = E\left[g^i(S_{\tau_{\hat{\theta}}}, C_{\tau_{\hat{\theta}}}, \theta^i_{\tau_{\hat{\theta}}}, \tau_{\hat{\theta}})^+ | \mathcal{F}^{S,C,\theta^i}_t\right], \tag{4.26}$$

where

$$g^i(S_t, C_t, \hat{\theta}_t, t; \hat{\theta}_t) = E\left[e^{-rt}\left(X^i(S_t, C_t, t; \hat{\theta}^i_t) - y^i(\hat{\theta}_t)h(C_t, \theta^i_t)\right) | \mathcal{F}^{S,C,\theta^i}_t\right]. \tag{4.27}$$

The function $g^i(\cdot)$ is understood as the agent's expected payoff at the time the investment is made. The stopping times of agent i, $\tau^i_{\hat{\theta}}$, may be dependent on the reports of all the agents participating in the auction. Compared to agent i's value when the investment cost is constant in equation (4.4), the investment cost is now a function of two stochastic variables, and the compensation function depends on the observable cost component C_t, in addition to the output value S_t and the vector of privately observed cost components. Also, now the reports change stochastically over time.

The auctioneer's information at time t is given by $\mathcal{F}^{S,C}_t$. This means that the auctioneer does not observe the vector $\theta_t = (\theta^1_t, ..., \theta^n_t)$. However, he knows the expectation and the variance of each agent's private information θ^i_t. The auctioneer's value function at time t is given by

$$v^P(s, c, t) = E\left[\sum_{i=1}^{n} g^P_i(S_{\tau^i_{\hat{\theta}}}, C_{\tau^i_{\hat{\theta}}}, \tau^i_{\hat{\theta}})^+ | \mathcal{F}^{S,C}_t\right], \tag{4.28}$$

where

$$g^P_i(S_t, C_t, t; \hat{\theta}_t) = E\left[e^{-rt}\left(y^i(\hat{\theta}_t)S_t - X^i(S_t, C_t, t; \hat{\theta}_t)\right) | \mathcal{F}^{S,C}_t\right]. \tag{4.29}$$

The function $g^P_i(\cdot)$ is interpreted as agent i's contribution to the auctioneer's value at the time the investment is made.

The auctioneer's optimization problem is now given by

$$V^P(s, c, t) = \sup_{X^i(\cdot), y^i(\cdot), \tau^i} v^P(s, c, t), \tag{4.30}$$

subject to all the n agents' optimization problems, each agent having the following optimization problem,

$$V^i(s, c, \theta^i, t; \hat{\theta}^i) = \sup_{\hat{\theta}^i} v^i(s, c, \theta^i, t; \hat{\theta}^i), \tag{4.31}$$

for any $0 \leq t \leq \bar{\tau}$, $\hat{\theta}^i \in [\underline{\theta}, \overline{\theta}]$.

Recall that in this auction, the auctioneer makes the investment decision, as the decision cannot be delegated. This restriction is in contrast to the principal-agent model where the private information changes stochastically, in chapter 3, as well as to the auction model in section 4.2, where the private information is constant.

In sections 4.3.2-4.3.4 we derive the value and optimal strategy for the contract. The procedure for the derivations resemble the derivations in section 4.2 and in chapter 3.

4.3.2 The agents' optimal reporting strategies

We define a trigger function $\psi^i(C_t, t; \hat{\theta}_t)$ with similar properties to the analogous function in the one-agent case of chapter 3: The investor chooses to invest when $S_t > \psi^i(C_t, t; \hat{\theta}_t)$, and wait otherwise. However, the investment strategy $\psi^i(C_t, t; \hat{\theta}_t)$ is stochastic to each agent, as it depends on the vector of reports given by all the agents.

Below we find each agent's truth telling condition. For this purpose we formulate agent i's the value of the contract as follows,

$$\begin{aligned} & v^i(S_t, C_t, \theta_t^i, t; \hat{\theta}_t^i) \\ = \; & E\Big[w^i(S_t, C_t, \theta_t^i, t; \hat{\theta}_t^i) \mathrm{I}_{\{S_t \leq \psi^i(C_t, t; \hat{\theta}_t)\}} \\ & + \Big(X^i(S_t, C_t, t; \hat{\theta}_t) - y^i(\hat{\theta}_t) h(C_t, \theta_t^i)\Big) \mathrm{I}_{\{S_t > \psi^i(C_t, t; \hat{\theta}_t)\}} | \mathcal{F}_t^{S,C,\theta^i}\Big], \end{aligned} \tag{4.32}$$

where $w^i(\cdot)$ denotes agent i's value in the region where the investment option is not exercised.

Analogously to the assumption in chapter 3 we assume that only reports at the investment time affect the compensation function. Thus, incentive compatibility requires that all the n agents' payoff values satisfy the following first order condition,

$$\left. \frac{\partial v^i(S_t, C_t, \theta_t^i, t; \hat{\theta}_t^i)}{\partial \hat{\theta}_t^i} \right|_{\hat{\theta}_t^i = \theta_t^i} = 0, \tag{4.33}$$

similarly to the corresponding condition in (4.11).

Let $v^i(S_t, C_t, \theta_t^i, t)$ be agent i's value when truth telling is optimal. The vector of the agents' true investment cost at time t is denoted by $\theta_t = (\theta_t^1, ..., \theta_t^n)$. Then we state agent i's payoff when truth telling is optimal by

$$\begin{aligned} & v^i(S_t, C_t, \theta_t^i, t) \\ = \; & E\left[w^i(S_t, C_t, \theta_t^i, t)\mathrm{I}_{\{S_t \leq \psi^i(C_t,t,\theta_t)\}}\right. \\ & \left. + \left(X^i(S_t, C_t, t, \theta_t) - y^i(\theta_t)h(C_t, \theta_t^i)\right)\mathrm{I}_{\{S_t > \psi^i(C_t,t,\theta_t)\}} | \mathcal{F}_t^{S,C,\theta^i}\right]. \end{aligned} \tag{4.34}$$

By the envelope theorem, agent i's first-order condition is written as

$$\begin{aligned} \frac{dv^i(S_t,C_t,\theta_t^i,t)}{d\theta_t^i} = \; & E\left[\frac{\partial w^i(S_t,C_t,\theta_t^i,t)}{\partial\theta_t^i}\mathrm{I}_{\{S_t \leq \psi^i(C_t,t,\theta_t)\}}\right. \\ & \left. - y^i(\theta_t)h_{\theta_t^i}(C_t, \theta_t^i)\mathrm{I}_{\{S_t > \psi^i(C_t,t,\theta_t)\}} | \mathcal{F}_t^{S,C,\theta^i}\right], \end{aligned} \tag{4.35}$$

where $h_{\theta_t^i}(\cdot)$ denotes the differentiation of $h(\cdot)$ with respect to θ_t^i. The truth telling condition in (4.35) is similar to the truth telling condition in the principal-agent model of chapter 3, equation (3.11). The main difference is that the condition in (4.35) includes the probability $y^i(\theta_t)$, which implies that the optimal compensation is not independent of the agents' reports. In the case where the investment cost $h(C_t, \theta_t^i)$ is a constant, the truth telling condition in (4.35) equals the condition in (4.13) when investment is made immediately.

4.3.3 The auctioneer's optimization problem

To solve the auctioneer's optimization problem we proceed as in section 4.2. and incorporate the n agents' truth telling restrictions as given by (4.35) into the auctioneer's optimization problem in (4.30)-(4.31).

We replace the compensation function in (4.29) by agent i's value of private information given investment at time t. Define $Y^i(\theta_t^i) = E\left[y^i(\theta_t)|\mathcal{F}^{S,C,\theta^i}\right]$. We reformulate equation (4.29) to (derivation shown in appendix C.6),

$$g_i^P(S_t, C_t, t) = E\left[e^{-rt}Y^i(\theta_t^i)\left(S_t - h(C_t, \theta_t^i)\right) - g^i(S_t, C_t, \theta_t^i, t)|\mathcal{F}_t^{S,C}\right]. \tag{4.36}$$

The next step in finding an unconstrained optimization problem for the auctioneer, is to incorporate an expression of agent i's value of private information into the auctioneer's optimization problem. In appendix C.7 it is shown that each agent i's value of private information when the investment is made at time t can be expressed as

$$g^i(S_t, C_t, \theta_t^i, t) = -\int_{\theta_t^i}^{\overline{\theta}} w_u^i(S_t, C_t, u, t)du. \tag{4.37}$$

We insert the expression in (4.37) into (4.36), and by simple derivations (see appendix C.8) we find that

$$g_i^P(S_t, C_t, t) = E\left[e^{-rt}Y^i(\theta_t^i)\left(S_t - h(C_t, \theta_t^i) - h_{\theta_t^i}(C_t, \theta_t^i)\frac{F(\theta_t^i|t)}{f(\theta_t^i|t)}\right)\middle| \mathcal{F}_t^{S,C}\right], \tag{4.38}$$

where $f(\theta_t^i|t)$ is the probability density of θ_t^i, and $F(\theta_t^i|t)$ is the cumulative distribution function of θ_t^i.

By equation (4.29) we know that the control variable $y^i(\cdot)$ is linearly dependent on agent i's contribution to the auctioneer's value. This means that the optimal value of $y^i(\cdot)$ is given by 0 or 1. Furthermore, equation (4.38) depends only on agent i's report, and not at the other agents' reports, which means that the optimal stopping time depends only on the winner of the contract. Thus, the optimization problems with respect to y^i and a stopping time τ^i can be separated, which means that the optimal stopping time in (4.28) can be split into n programs, where each program i only depends on the cost reports of agent i.

Suppose that $y^i(\theta_t) = 1$, i.e., agent i is the winner of the contract. Combining (4.28) and (4.38) the auctioneer's optimization problem if $y^i = 1$ is given by

$$\sup_{\tau^i} E\left[E\left[e^{-r\tau^i}\left(S_{\tau^i} - h\left(C_{\tau^i}, \theta_{\tau^i}^i\right) - h_{\theta^i}\left(C_{\tau^i}, \theta_{\tau^i}\right)\frac{F(\theta_{\tau^i}^i|\tau^i)}{f(\theta_{\tau^i}^i|\tau^i)}\right)\middle| \mathcal{F}_t^{S,C,\theta^i}\right]\middle| \mathcal{F}_t^{S,C}\right]. \tag{4.39}$$

For simplicity we have suppressed the subscript θ^i in τ^i in equation (4.39). As in section 4.2 we find that the optimal investment trigger depends only on the winner's cost level. This means that the critical price for investment is given by $\psi(c, \theta^i, t)$ instead of $\psi(c, \theta, t)$.

The principal's optimization problem given by equation (3.20) in chapter 3, where the stochastic processes of S_t, C_t and θ_t^i are driven by geometric Brownian motions, is a special case of the auctioneer's optimization problem with respect to agent i. In the case where the auction model and the principal-agent model both are driven by geometric Brownian processes, the auctioneer's and the principal's respective optimization problems lead to the same optimal investment strategy. This means that, similarly to the case where the private information is constant, the optimal investment strategy is the same whether there is competition or not.

Let $\tilde{v}_i^P(S_t, C_t, \theta_t^i, t)$ be the auctioneer's value function when agent i wins the contract, the auctioneer is committed to a truth telling contract, and the auctioneer's information at time t is given by the information set $\mathcal{F}_t^{S,C,\theta^i}$, i.e.,

$$\begin{aligned} &\tilde{v}_i^P(s, c, \theta^i, t) \\ &= E\left[e^{-r\tau^i}\left(S_{\tau^i} - h\left(C_{\tau^i}, \theta^i_{\tau^i}\right) - h_{\theta^i}\left(C_{\tau^i}, \theta^i_{\tau^i}\right)\frac{F(\theta^i_{\tau^i}|\tau^i)}{f(\theta^i_{\tau}|\tau^i)}\right)\Big|\, \mathcal{F}_t^{S,C,\theta^i}\right]. \end{aligned} \tag{4.40}$$

Also define

$$\tilde{g}^P(s, c, \theta^i, t) = e^{-rt}\left(s - h(c, \theta^i) - h_{\theta^i}(c, \theta^i)\frac{F(\theta^i|t)}{f(\theta^i|t)}\right).$$

Thus, we solve the optimal stopping problem *as if* we know the private information θ^i, similarly to the approach used in chapter 3, equations (3.23)-(3.25).

We find the optimal investment strategy by optimizing equation (4.40) with respect to the optimal stopping time. Using the approach in chapter 3, equations (3.23)-(3.25). The optimal solution must satisfy the variational inequalities[8]:

$$\tilde{v}_i^P(s, c, \theta^i, t) \geq \tilde{g}_i^P(s, c, \theta^i, t) \tag{4.41}$$

$$L\tilde{v}_i^P(s, c, \theta^i, t) - r\tilde{v}_i^P(s, c, \theta^i, t) \leq 0 \tag{4.42}$$

$$\max\left\{L\tilde{v}_i^P(s, c, \theta^i, t) - r\tilde{v}_i^P(s, c, \theta^i, t), \tilde{g}_i^P(s, c, \theta^i, t) - \tilde{v}_i^P(s, c, \theta^i, t)\right\} = 0. \tag{4.43}$$

The differential operator L is given by

$$\begin{aligned} L\tilde{v}_i^P(s, c, \theta^i, t) &= \frac{\partial \tilde{v}_i^P}{\partial t} + (rs - \delta(s))\frac{\partial \tilde{v}_i^P}{\partial s} + \frac{1}{2}\sigma(s)^2\frac{\partial^2 \tilde{v}_i^P}{\partial s^2} + \bar{\alpha}(\theta)\frac{\partial \tilde{v}_i^P}{\partial \theta} \\ &+ \frac{1}{2}\sigma_\theta(\theta^i)^2\frac{\partial^2 \tilde{v}_i^P}{\partial \theta^{i2}} + rc\frac{\partial \tilde{v}^P}{\partial c} + \frac{1}{2}\bar{\sigma}_C(c)^2\frac{\partial^2 \tilde{v}^P}{\partial c^2} + \rho(s, c)\sigma(s)\bar{\sigma}_C(c)\frac{\partial \tilde{v}_i^P}{\partial s \partial c}, \end{aligned} \tag{4.44}$$

[8]For explanation of the solution procedure, confer the discussion following (3.23)-(3.26) in chapter 3.

where $\rho(s,c)$ is the correlation between the standard Brownian motions B_t^S and B_t^C. The difference between the operator here and in (3.26) is that now we assume time-homogeneous Ito diffusions rather than processes driven by geometric Brownian motions as is the assumption in (3.26).

Denote ψ^{i*} as the optimal critical price. As the control variable $y^i(\cdot)$ is linear in the auctioneer's payoff values, we find that at any time $t \in [0,T]$, the optimal solution of $y^i(\theta_t)$ is given by

$$y^{i*}(\theta_t) = \begin{cases} 1 & \text{if } \theta_t^i < \min_{j\neq i}\theta_t^j \text{ and } s_t > \psi^{i*}(C_t,\theta_t^i,t),\ t \le \tau^i \\ 0 & \text{if } \theta_t^i > \min_{j\neq i}\theta_t^j. \end{cases} \tag{4.45}$$

The optimal solution says that the winner of the contract is the agent with the lowest cost at the time it is optimal to invest.

4.3.4 Implementation

Let the optimal investment strategy found from optimization of (4.39) be given by $\psi^{i*}(C_t,\theta_t^i,t)$. Denote $\vartheta^{i*}(S_t,C_t,t)$ as the optimal inverse entry threshold of agent i, i.e., we invest immediately if $\theta_t^i < \vartheta^{i*}(S_t,C_t,t)$ and wait if $\theta_t^i \ge \vartheta^{i*}(S_t,C_t,t)$. Note that ϑ^* now is the critical price for θ^i, whereas it was the critical price for $K_t = C_t\theta_t$ in the principal-agent model in chapter 3. Furthermore, $Y^{i*}(\theta_t^i)$ is defined as the optimal $Y^i(\theta_t^i)$.

Agent i's optimal compensation function X^{i*} when $S_t > \psi^{i*}(C_t,\theta_t^i,t)$, is found to be equal to (see the appendix, section C.9 for derivation of the result)

$$\begin{aligned} X^{i*}(S_t,C_t,\theta_t^i,t) &= h(C_t,\theta_t^i)Y^{i*}(\theta_t^i) + \int_{\theta_t^i}^{\upsilon^{i*}(S_t,C_t,t)} h_u(C_t,u)Y^{i*}(u)du \\ &\quad - \int_{\upsilon^{i*}(S_t,C_t,t)}^{\overline{\theta}} w_u^i(S_t,C_t,u,t)du. \end{aligned} \tag{4.46}$$

if $S_t > \psi^{i*}(S_t,\theta_t^i,t)$. Otherwise, $X^{i*}(S_t,C_t,\theta_t^i,t) = 0$. Note that the expected compensation function $X^i(\cdot)$ is dependent on θ_t^i in the auction model, i.e., the expected compensation is implemented by a direct mechanism.

If we compare the optimal expected compensation of agent i in (4.46) with the corresponding compensation function when the investment cost is constant, in equation (4.20), we find that the compensation functions are similar even though the contracts are different in the two cases: In the case where the investment cost is constant, the winner of the contract may be chosen prior to the time the investment is made, whereas the winner is chosen at the investment time when the investment cost changes stochastically. However, if we let $h(C_t, \theta_t^i)$ be equal to K, the compensation function in (4.46) converges to the function given by (4.20).

So far we have only found each agent's optimal reporting strategy on average, i.e., given the other agents' strategies through the expectation $Y^i(\cdot)$. Now, we construct a dominant strategy auction that implements the same investment strategy as the one found from optimizing equation (4.39). In addition, the dominant strategy auction selects the firm with the lowest investment cost at the time of investment. This approach is similar to the one in the case of constant private information in section 4.2. Let

$$\tilde{X}^i(S_t, C_t, \theta_t, t)$$

$$= \begin{cases} h\left(C_t, v^{i*}(S_t, C_t, t)\right) \\ + \int_{v^{i*}(S_t,C_t,t)}^{\theta_t^j} w_u^i(S_t, C_t, u, t)du & \text{if } \psi^*(C_t, \theta_t^i, t) < S_t \leq \psi^{i*}(C_t, \theta_t^j, t) \\ h\left(C_t, \theta_t^j\right) & \text{if } S_t > \psi^{i*}(C_t, \theta_t^j, t). \end{cases} \tag{4.47}$$

if $\theta_t^i = \min_l \theta_t^l$, $\theta_t^j = \min_{l \neq i} \theta_t^l$ and $\tilde{X}^i(S_t, C_t, \theta_t^i, t) = 0$ otherwise.

Note that the function $\tilde{X}^i$ in (4.47) is the compensation to agent i, given that he wins the contract, whereas the function X^{i*} in (4.46) represents each agent's expected compensation of participating in the auction.

If we mimic the approach in appendix C.5 we find that

$$X^{i*}(S_t, C_t, \theta_t^i, t) = E\left[\tilde{X}^i(S_t, C_t, \theta_t, t)|\mathcal{F}_t^{S,C,\theta^i}\right].$$

Thus, we conclude that the contract given by equation (4.47) is the optimal contract under competition.

The winner's compensation is dependent on the cost of the agent having the second-lowest investment cost. Thus, competition implies that the interval of possible reports, $[\underline{\theta}, \overline{\theta}]$, is truncated to $[\underline{\theta}, \theta_t^j]$, where θ_t^j is the second-lowest report at time t, in the same way as for the case of a constant information cost, given by equation (4.21).

In the subsection below we illustrate the model by assuming that the stochastic processes are geometric Brownian motions. We compare the result of the auction model to the principal-agent model in chapter 3.

4.3.5 Illustration of the results: Geometric Brownian motions

We illustrate the results by assuming that the stochastic processes follow the same processes as in chapter 3. This implies that the dynamics of $S(t)$ is given by

$$dS_t = (r - \delta_s)S_t dt + \sigma_s S_t dB_t^S,$$

rather than by (4.1), the observable part of the investment cost, C_t, follows the process,

$$dC_t = rC_t dt + \sigma_C C_t dB_t^c,$$

rather than (4.22), and the unobservable part of the investment cost, θ_t^i, follows,

$$d\theta_t^i = \alpha\theta_t^i dt + \sigma_\theta \theta_t^i dB_t^i,$$

rather than (4.23). The investment cost function of agent i is given by $K_t^i \equiv C_t\theta_t^i$. Furthermore, assume that the limits of the admissible cost reports are given by $\underline{\theta}$ equals to 0 and $\overline{\theta}$ approaching ∞.

Using equation (4.47) we find that when the stochastic processes follow the geometric Brownian motions above, the contract is given by

$$\tilde{X}^i(S_t, C_t, \theta_t, t)$$

$$= \begin{cases} cv^{i*}(S_t, C_t, t) \\ + \int_{v^{i*}(S_t,C_t,t)}^{\theta_t^j} w_u^i(S_t, C_t, u, t)du & \text{if } \psi^{i*}(C_t, \theta_t^i, t) < s \leq \psi^{i*}(C_t, \theta_t^j, t) \\ \\ C_t\theta_t^j & \text{if } S_t > \psi^*(C_t, \theta_t^j, t), \end{cases} \tag{4.48}$$

if $\theta_t^i = \min_l \theta_t^l$, $\theta_t^j = \min_{l \neq i} \theta_t^l$ and $\tilde{X}^{i*}(S_t, C_t, \theta_t^i, t) = 0$ otherwise.

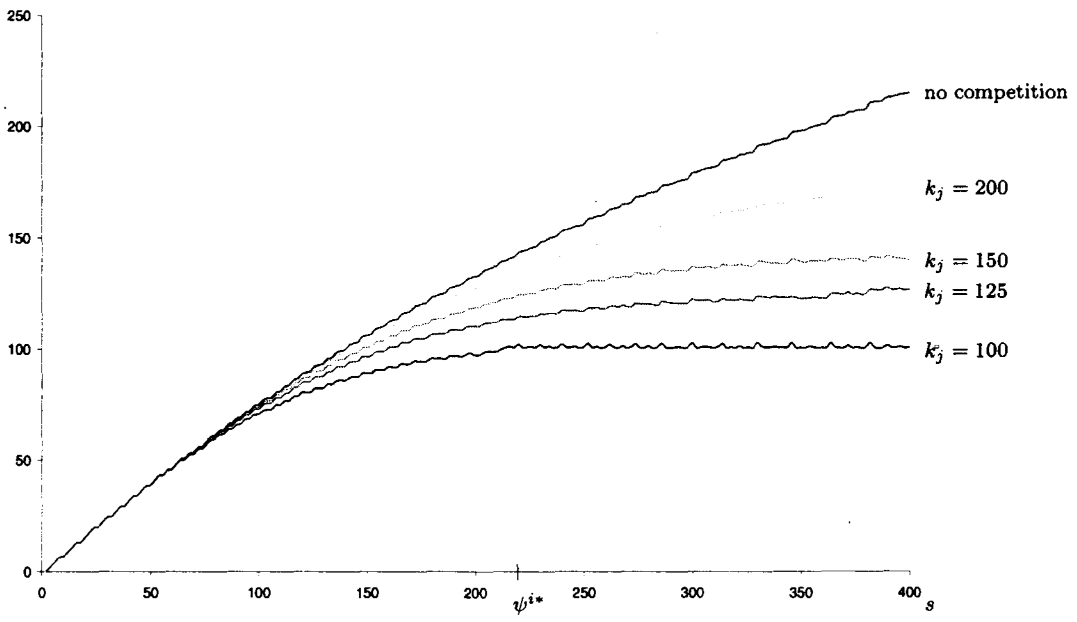

Figure 4.4: The winner's optimal compensation for different levels of the second-lowest cost report k_j at the investment time, and for varying asset value s.

Compare the compensation functions under competition, (4.48), to the optimal compensation function in the principal-agent model in (3.28). The main difference is that the compensation under competition cannot be higher than $K_t^j = C_t\theta_t^j$. Therefore, the transferred amount is lower under competition. The closer the investment cost reported by the second-best agent is to the winner's report, the lower is the winner's information rent. This implies a higher value to the auctioneer.

We illustrate the effect from competition on the winner's compensation in Figure 4.4. The parameter values used are identical to the parameter values for the numerical examples in chapter 3, see the base case table on page 72. We assume that the winner's cost level k^i is identical to k in the table. The figure shows the compensation given different cost levels k^j of the agent with the second-lowest cost level at the time the investment is made. The upper curve corresponds to the compensation in the principal-agent model, compare Figure 3.6, the base case where $\sigma_\theta = 0.2$. As the cost levels of agent j (i.e., the agent with the second-lowest cost report) gets lower, the compensation decreases as well. The effect is similar to the corresponding effect on the compensation in the case where the private information is constant in Figure 4.1. In the case where agent j reports a cost level equal to the winner's cost level, i.e., when $k_i = k_j = 100$, the winner's compensation is independent of the asset value s in the interval where immediate investment is optimal. This corresponds to the interval where $s > 218$ in Figure 4.4. In this interval the winner's contract value equals zero as his compensation just covers his costs of the investment.

In the special case of geometric Brownian motions we assume that the admissible set of investment cost variables of the agents have a lower level of zero and no upper level. Thus, if the number of competing agents gets very large, the agent with the lowest cost will approach the lower cost level of zero. This is in most cases not a realistic assumption. Thus, in this case we need to assume that the number of agents participating in the auction is not "too" large.

4.3.6 Auction outcome when the competing agents have different volatility parameters

Let us assume that the stochastic processes are given by geometric Brownian motions as in section 4.3.5, but that the agents have different volatility parameters related to the unobservable cost component. The volatility parameter of each agent is known.

When the volatility parameters are different for each agent, agent i's stochastic process with respect to the true investment cost K_t^i is given by

$$dK_t^i = (r - \alpha)K_t^i dt + \sigma_C K_t^i dB^C(t) + \sigma_\theta^i K_t^i dB_t^i, \tag{4.49}$$

with the volatility parameter value σ_θ^i being different for each agent participating in the auction.

Each agent's truth telling, first-order condition is identical to the one given by

(4.35), with the exception that now the volatility parameter of each agent may be different.

Furthermore, the formulation of the auctioneer's optimization problem is the same as the one in equation (4.39). However the optimal stopping time τ^i is now different for each i, i.e.,

$$\tau_{\hat{\theta}}^{i*} = \inf\left\{t \in [0,T] | S_t > \psi^{i*}(C_t, t; \sigma_\theta^i, \hat{\theta})\right\},$$

where different σ_θ^i's yield different optimal investment triggers $\psi^{i*}(C_t, t; \sigma_\theta^i, \hat{\theta})$ for each agent i. Thus, the auctioneer will follow the investment strategy of the agent that has the volatility parameter value that results in the highest value to the auctioneer.

In the numerical examples of the investor's value of the contract, Figure 3.8, the investor's value increases in the value of σ_θ^i. Hence, in this case the auctioneer's optimal investment strategy is to follow the investment strategy $\psi^i(c, t; \sigma_\theta, \hat{\theta})$, where agent i is assumed to have the highest parameter value of the volatility σ_θ. As in the case of identical volatility parameters, the winner of the auction is the agent who reports the lowest cost in the interval where immediate investment is the optimal strategy. The winner is paid according to the investment strategy followed by the auctioneer. Suppose agent h is the winner of the contract. Then he is paid according to (confer equation (4.48)) the optimal compensation function

$$\tilde{X}^h(S_t, C_t, \theta_t, t)$$

$$= \begin{cases} cv^{i*}(S_t, C_t, t; \sigma_\theta^i) \\ + \int_{v^{i*}(S_t,C_t,t;\sigma_\theta^i)}^{\theta_t^j} w_u^i(S_t, C_t, u, t)du & \text{if } \psi^{i*}(C_t, \theta_t^h, t; \sigma_\theta^i) < s \le \psi^{i*}(C_t, \theta_t^j, t; \sigma_\theta^i) \\ C_t\theta_t^j & \text{if } S_t > \psi^*(C_t, \theta_t^j, t; \sigma_\theta^i), \end{cases}$$

if $\theta_t^h = \min_l \theta_t^l$, $\theta_t^j = \min_{l \neq h} \theta_t^l$ and $\tilde{X}^{h*}(S_t, C_t, \theta_t^h, t) = 0$ otherwise.

Summing up, the outcome of the auction is the qualitatively the same whether the competing agents have different volatilities or not: all agents participate in the auction until a cost is reported that is low enough to trigger investment. At this time the agent with the lowest cost report wins the contract. However, the winning agent h implicitly follows the optimal investment strategy for agent i, who has the highest volatility parameter value. Moreover, if one of the agent's volatility function is higher than the previously assumed common volatility σ_θ, then the auctioneer's option value is now higher, whereas the agent's value is reduced, compared to the case where all the agents have identical volatilities.

4.4 Concluding remarks

In this chapter we have extended the principal-agent models in chapters 2 and 3 to the case of n agents having private information. Similarly to the solutions of the principal-agent models, we find optimal contracts via direct, truthful mechanisms. As is to be expected, competition leads to a higher contract value to the investor, although the optimal investment strategy is identical whether we have competition or not. The compensation, however, is lower in the case of competition.

A difference from the principal-agent models is that the contract in the case where the private information is stochastic, the investment decision cannot be delegated to an agent. The reason is that in this case, the winner of the contract is not chosen prior to the investment time.

A remaining question is how robust the model is of changes in some of the assumptions. In section 4.3.6 we examined the special case of different volatility parameters. We find that the auction outcome is the same (i.e., we find that the agent with the lowest cost at the time the investment is exercised, is the winner of the contract) compared to the case where the competing agents have identical volatilities. However, the auctioneer's value from the option is higher when the agents have different volatility parameters, because the auctioneer chooses the investment strategy of the agent who has the volatility parameter value that gives the highest value to the auctioneer.

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Chapter 5

Sequential investments: Private learning

We formulate a sequential investment problem where new investment phases can be temporarily stopped if the value of the project is low, and later restarted when the profitability prospects improve. As new investment phases are realized the manager (the agent) of the investments learns more about the profitability of the project than the investor (the principal) does. The costs and the value of the project are shared between the principal and the agent, depending on the agent's information rent. We assume that the agent has private information about the output value of the project, instead of about the investment cost as in earlier chapters.

5.1 Introduction

In this chapter we return to an investment project within the framework of a principal-agent model. However, we change the assumptions with respect to the private information, and with respect to the number of investment decisions made. Now we assume that an agent has private information about the *output value*[1] of an investment project, and that we have a *sequence* of investment

[1]As in previous chapters, the terms output value and value of the "asset in place" are used interchangeably.

decisions to make.

The manager (the agent) of a sequential investment project privately learns more about the value of the project as new investment phases are realized: for each investment phase that is finished, the agent observes an updated signal of the output value. The output value is a stochastic variable that is completely revealed when the last investment phase is completed. The investor (the principal) does not obtain these signals about the output value of the project, and does not observe the realized value of the completed project. Thus, in this model it is the agent who obtains the output value from the project, and therefore the contracted amounts are now paid from the agent to the principal.

In previous chapters we have assumed that the principal's payment to the agent may depend only on the report made at the time when the compensation is payable. Now we will investigate whether it may be optimal to let the payments depend on previous reports, as well. As earlier, we assume that a principal enters into a binding agreement with an agent, and that the agent has private information. We find that it is not optimal to let the contract be dependent on previous reports. The reason is that we assume that the privately observed signals are driven by a Markov process.

As we formulate an investment project where investment decisions are made sequentially, there is a possibility to temporarily stop the project if the value of the completed project falls[2]. The model is similar to the one given in Majd and Pindyck (1987), where a firm may invest continuously until the project is completed, and where the investments at no cost can be stopped and later restarted. In both models there is a maximum rate at which investments can proceed, which means that it takes "time to build".

In our problem the sequential investments are not made continuously as in Majd and Pindyck (1987). Instead the investments are made in "bulks", and each investment phase takes a specified time to complete. However, if the amount of time that each investment phase takes gets very small, our model converges to the model in Majd and Pindyck (1987).

The sequential investment model partly draws upon MacKie-Mason (1985), and

[2]I.e., the investment project is analogous to a compound option.

MacKie-Mason's model has a similar structure to ours. A main difference is that MacKie-Mason (1985) does not formulate an optimal stopping problem, as in his model it is never optimal to postpone further investments: one either continues with a new investment phase when the previous one is finished, or abandons the project forever.

The model is formulated in the next section. Section 5.3 examines the agent's optimization problem. The principal's optimization problem is studied in section 5.4, and the optimal contract is characterized in section 5.5.

5.2 Model formulation

Assume that an investor (a principal) delegates the management of an investment project to an agent. The project consists of N investment phases. Each phase j takes time $\Delta_j > 0$ to finish, and has an investment cost $K_j > 0$. The project's value is realized when all the investment phases are completed. If all the investment phases are finished at time t, the value $S(t) = \pi(t)q(t)$ is realized. We interpret $S(t)$ as the value of the investment project if it were finished at time t.

The value $\pi(t)$ represents market price of output, whereas the variable $q(t)$ may be interpreted as, for example, "noise", volume, technological uncertainty or demand. The values $\pi(t)$ and $q(t)$ are both driven by geometric Brownian motions. As in previous chapters we assume that the capital market is dynamically complete, and that both the principal and the agent are well diversified. We assume that the market price $\pi(t)$ is spanned in capital markets, whereas the variable $q(t)$ is not correlated with capital markets. These assumptions simplify the problem, as the investment project can be evaluated by risk-neutral pricing.

Thus, the stochastic process of $\pi(t)$ is measured under the equivalent martingale measure Q, i.e.,

$$d\pi(t) = (r - \delta)\pi(t)dt + \sigma\pi(t)dB^{\pi}(t), \quad \pi \equiv \pi(0). \tag{5.1}$$

The parameters r, δ and σ are the risk-free rate, the proportional convenience yield and the volatility parameter, respectively, whereas $B^{\pi}(t)$ is a standard

Brownian motion under the Q measure. All the parameter values are positive. Thus, the market price of output $\pi(t)$ is observable to both parties.

The variable $q(t)$ is only observable by the agent at each point in time when the agent has finished an investment phase. The stochastic process $q(t)$ is given by the geometric Brownian motion

$$dq(t) = \kappa q(t)dt + \nu q(t)dB^q(t). \tag{5.2}$$

The standard Brownian motion $B^q(t)$ is independent of $B^\pi(t)$. Hence, the process of $q(t)$ has identical processes under the true and the risk-adjusted measures. The drift parameter is denoted κ, and ν is the volatility parameter of $q(t)$.

The contract is entered into at time t_0. The agent's observation of $q(t)$ is specified by the variable $\zeta(t)$, where

$$\zeta(t) = \begin{cases} q_0 & \text{if } t_0 \leq t < \tau_1 + \Delta_1 \\ \\ q_j & \text{if } \tau_j + \Delta_j \leq t < \tau_{j+1} + \Delta_{j+1}. \end{cases} \tag{5.3}$$

The stopping time τ_j is the time when investment phase j is started. This means that $\tau_j + \Delta_j$ is the time when investment phase j is completed, and $q_j = q(\tau_j + \Delta_j)$ is observed, $j = 1, 2, ..., N$. The agent's observations of q_j is interpreted as updated signals of the value when the project is completed. The privately observed value of the realized project is given by $q_N = q(\tau_N + \Delta_N)$. Thus, $q(t)$ is a hidden Markov process that is only observable to the agent at the points in time when a new investment phase is completed[3].

The principal knows the expectation and the variance of q_0 at time t_0, as well as the stochastic process in equation (5.2). The time elapsed since the last observation of $q(t)$ is given by the process,

$$\eta(t) = \begin{cases} t - t_0 & \text{if } t_0 \leq t < \tau_1 + \Delta_1 \\ \\ t - (\tau_j + \Delta_j) & \text{if } \tau_j + \Delta_j \leq t < \tau_{j+1} + \Delta_{j+1}. \end{cases} \tag{5.4}$$

[3] An article that treats the problem of valuation of contingent claims when the observation of a stochastic process is incomplete, is Childs, Ott, and Riddiough (2001). The valuation problems in the article are solved using optimal filtering results.

To sum up, we describe the time-homogeneous system of states by

$$\Gamma(t) = \begin{bmatrix} \pi(t) \\ \varsigma(t) \\ \eta(t) \end{bmatrix} \quad \text{when } \tau_j + \Delta_j \leq t < \tau_{j+1} + \Delta_{j+1}, \tag{5.5}$$

where the state variables are, respectively, the market price of the output, the signals observed by the agent, and the time passed since the last signal observation. Define $\Gamma(t^-)$ as the left limit of $\Gamma(t)$, i.e., $\Gamma(t^-)$ is interpreted as the value of the stochastic variable immediately before the value at time t. When any investment phase j is completed, the system changes from

$$\Gamma((\tau_j + \Delta_j)^-) = \begin{bmatrix} \pi((\tau_j + \Delta_j)^-) \\ q_{j-1} \\ \eta((\tau_j + \Delta_j)^-) \end{bmatrix}, \tag{5.6}$$

to

$$\Gamma(\tau_j + \Delta_j) = \begin{bmatrix} \pi(\tau_j + \Delta_j) \\ q_j \\ 0 \end{bmatrix}. \tag{5.7}$$

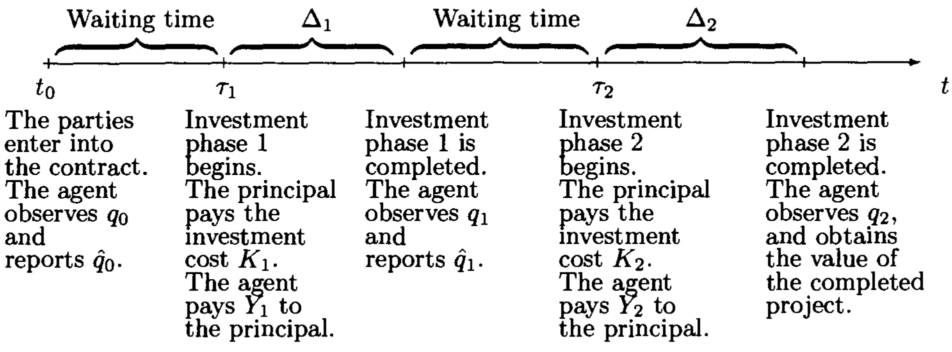
The investment cost of each investment phase, K_j, is paid by the principal at the point in time when a new investment phase is started. We assume that the agent cannot start a new investment phase before the previous one is finished.

The sequence of optimal stopping times is given by $\hat{\tau} = \{\tau_1, ..., \tau_N\}$. The agent's payments are denoted $Y = \{Y_1, ..., Y_N\}$.

Analogously to the previous analyzes in this dissertation, we apply a direct truth telling mechanism. Application of this mechanism implies that we, in order to solve the problem, let the agent report the privately observed signals to the principal, and we let the principal make the investment decision based on the reports. When we have found an optimal contract, we find a delegation-based contract that replicates the direct mechanism. The reports are defined by $\hat{q} = \{\hat{q}_0, ..., \hat{q}_{N-1}\}$.

In figure 5.1 a scheme of the model is drawn for the case of two investment phases, i.e., $N = 2$. When the parties enter into a contract at time t_0, the agent privately observes q_0, reports that the signal is $\hat{q}_0$, and makes the first investment decision based on this signal. At a stopping time τ_1 investment phase 1 is exercised, and

Figure 5.1: An illustration of the investment project in the case of two investment phases



the agent pays the pre-specified amount $Y_1(\cdot)$ to the principal. Furthermore, when investment phase 1 is completed, the agent observes q_1, reports $\hat{q}_1$, and makes his next investment decision based on this updated information. Investment phase 2 is started at a stopping time τ_2, and the contracted amount $Y_2(\cdot)$ is simultaneously paid to the principal. When the last investment phase is finished, the agent obtains the realized value from the project, i.e., he receives the "asset in place".

Except from the signals $\zeta(t)$, privately observed by the agent, the principal has the same information as the agent. The principal also knows when the agent obtains new signals. Similarly to the approach in the previous chapters we define the principal's available information at time t by $\mathcal{F}_t^{\pi,\eta}$, generated by $(\pi(\xi), \eta(\xi) : \xi \leq t)$, whereas the agent's information at time t is specified by $\mathcal{F}_t^{\Gamma}$, generated by $(\Gamma(\xi); \xi \leq t)$.

The principal's optimization problem at the time when the contract is entered into is given by

$$\begin{aligned} & V^P(\pi, \eta; \hat{q}) \\ = \; & \sup_{Y(\cdot),\hat{\tau}} E\left[\sum_{j=1}^{N} e^{-r\tau_j}\left[Y_j\left(\pi(\tau_j), \eta(\tau_j); \hat{q}\right) - K_j\right] | \mathcal{F}_0^{\pi,\eta}\right], \end{aligned} \tag{5.8}$$

subject to the agent's optimization problem,

$$\begin{aligned} V^A(\pi,\zeta,\eta;\hat{q}) &= \sup_{\hat{q}} E\left[e^{-r(\tau_N+\Delta_N)}\pi(\tau_N+\Delta_N)q(\tau_N+\Delta_N)\right. \\ &\qquad \left. -\sum_{j=1}^{N} e^{-r\tau_j}Y_j\left(\pi(\tau_j),\eta(\tau_j);\hat{q}\right)|\mathcal{F}_0^{\Gamma}\right], \end{aligned} \tag{5.9}$$

and the participation constraint,

$$V^A(\pi,\zeta,\eta) \geq 0. \tag{5.10}$$

The principal's value function in equation (5.8) depends on the investment cost K_j and the contracted payments Y_j, paid and received, respectively, each time an investment phase j is exercised. The amount Y_N can be interpreted as the principal's "output value" of the project, whereas the contracted amounts $Y_0,...,Y_{N-1}$ (partly) finance the investment costs K_j. The amounts Y_j mirror the agent's value of participating in the contract: Because of the private information, the agent obtains the realized value of the investment project when it is finished, and the values of the privately observed signals reflect how much the agent is willing to pay for participating in the contract payments, and at the same time be induced to report truthfully. Recall that the principal never observes the realized value q_N. This is the reason why the agent obtains the value of the finished project, as formulated in the agent's value function (5.9). The contracted amounts Y_j are the agent's costs. The participation constraint is included in order to ensure that the agent is willing to enter into a contractual agreement with the principal.

Below we characterize the model in more details. The optimization problem in equations (5.8)-(5.10) consists of N compound optimal stopping problems, and N optimization problems with respect to the reported signals. Thus, the model is formulated similarly to a stochastic impulse control problem. The agent's optimal reports can be formulated as *impulses*. At any time we can intervene our system of states by starting a new investment phase, and the value of this intervention is dependent on the report (i.e., the impulse) given.

Define $v_i^P(\cdot)$ and $v_i^A(\cdot)$ as the principal's and the agent's respective (arbitrary) value functions, $i = 1,...,N$, when we have completed $N-i$ investment phases, i.e., when we have i investment phases left. Furthermore, define *switching* functions $g_{i-1}^P(\cdot)$ and $g_{i-1}^A(\cdot)$ as the principal's and the agent's respective values from exercising investment phase $j = N-i+1$. The switching functions are interpreted as the values of switching from having i options left, to having $i-1$ options[4]. Observe that we denote i as the number of options left, starting with N options when the contract is entered into, and numbered decreasingly to 1 option left. The notation j represents the number of investment phases exercised, going from 1 to N. When we are about to invest in investment phase j the agent's updated signal is represented by q_{j-1}. Thus, we have the relationships

$$v_i^P(\pi,\eta;\hat{q}_0,..,\hat{q}_{j-1}) = E\left[e^{-r\tau}g_{i-1}^P\left(\pi(\tau),\eta(\tau);\hat{q}_0,..,\hat{q}_{j-1}\right)|\mathcal{F}_0^{\pi,\eta}\right],$$

and

$$v_i^A(\pi,q_{j-1},\eta;\hat{q}_0,..,\hat{q}_{j-1}) = E\left[e^{-r\tau}g_{i-1}^A\left(\pi(\tau),q(\tau),\eta(\tau);\hat{q}_0,..,\hat{q}_{j-1}\right)|\mathcal{F}_0^{\Gamma}\right],$$

where τ is a stopping time.

In the formulation of the value functions above, we have not included a time variable. This means that we implicitly have guessed that the problem is time-homogeneous. Given time-homogeneity the agent's value function can be represented by

$$\bar{v}_i^A(\xi,\pi,q_{j-1},\eta) = e^{-r\xi}v_i^A(\pi,q_{j-1},\eta),$$

where ξ represents a time variable. Let $L^{q_{j-1}}\bar{v}_i^A$ denote the partial differential operator, defined by

$$\begin{aligned} & L^{q_{j-1}}\bar{v}_i^A(\xi,\pi,\eta) \\ = \; & \tfrac{\partial\bar{v}_i^A}{\partial\xi}(\xi,\pi,\eta) + (r-\delta)\pi\tfrac{\partial\bar{v}_i^A}{\partial\pi}(\xi,\pi,\eta) + \tfrac{1}{2}\sigma^2\pi^2\tfrac{\partial^2\bar{v}_i^A}{\partial\pi^2}(\xi,\pi,\eta) + \tfrac{\partial\bar{v}_i^A}{\partial\eta}(\xi,\pi,\eta). \end{aligned}$$

By the definition $\bar{v}_i^A(\xi,\pi,q_{j-1},\eta) = e^{-r\xi}v_i^A(\pi,q_{j-1},\eta)$ we find that

$$L^{q_{j-1}}\bar{v}_i^A(\xi,\pi,\eta) = e^{-r\xi}L^{q_{j-1}}v_i^A(\pi,\eta),$$

where

$$L^{q_{j-1}}v_i^A(\pi,\eta) = (r-\delta)\pi\frac{\partial v_i^A}{\partial\pi}(\pi,\eta) + \frac{1}{2}\sigma^2\pi^2\frac{\partial^2 v_i^A}{\partial\pi^2}(\pi,\eta) + \frac{\partial v_i^A}{\partial\eta}(\pi,\eta) - rv_i^A(\pi,\eta). \tag{5.11}$$

By this transformation we get rid of the time dependence reflected in the $\partial\bar{v}_i^A/\partial\xi$ term. Thus, the "discounting costs" are incorporated through the term $-rv_i^A$

[4]Valuation problems when there is a decreasing number of options is also analyzed in Ekern (1993) and Sødal (2001).

in (5.11), even though we do not include the time variable ξ in the definition of $L^{q_{j-1}}v_i^A$. The partial differential operator in (5.11) is similar to the partial differential operator for arbitrage-free contingent claims, as presented in, for example, Duffie (1996), section 5.G.

In order to find an optimal contract, the following variational inequalities must be satisfied. For the agent's optimization problem we need, for $i, j = 1, .., N$, where $j = N - i + 1$,

$$v_i^A \geq g_{i-1}^A \tag{5.12}$$

$$L^{q_{j-1}}v_i^A \leq 0 \tag{5.13}$$

$$\max\{g_{i-1}^A - v_i^A, L^{q_{j-1}}v_i^A\} = 0, \tag{5.14}$$

where the inequalities must hold for all states. The inequality in (5.12) says that the value function v_i^A is always higher than or equal to the switching functions g_i^A as it consists of the decision flexibility (i.e., options to wait), in addition to the expected value from exercising the option. When we have equality in (5.12), it is optimal to invest. As long as it is not optimal to intervene by exercising investment phase j, we need $L^{q_{j-1}}v_i^A = 0$. This requirement is stated in equation (5.14).

The principal's optimization problem consists both of N optimal stopping problems, and of incentive problems because of the agent's privately observed signals. The incentive problems are solved by the revelation principle. When this truthful mechanism is satisfied, we can solve the principal's optimal stopping problems *as if* he observes the agent's private information. Thus, we define $\tilde{v}_i^P(\cdot)$ as the principal's value function when there are i investment phases left to complete, for a *given* value of the signal q_{j-1}. The reason we use the value $\tilde{v}_i^P(\pi, q_{j-1}, \eta)$, instead of $v_i^P(\pi, \eta)$, is that we need to find the optimal investment strategy given the private information q_{j-1}. As long as we have incorporated the agent's truth telling restrictions into the optimization problem given by $\tilde{v}_i^P(\cdot)$, we are allowed to optimize the value function $\tilde{v}_i^P(\cdot)$, instead of $v_i^P(\cdot)$. We have the relationship

$$v_i^P(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1}) = \int_0^\infty \tilde{v}_i^P(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1}) f(q_{j-1}|\hat{q}_0, .., \hat{q}_{j-2}, \eta) dq_{j-1},$$

where $f(\cdot|\hat{q}_0, ..., \hat{q}_{j-2}, \eta)$ is the probability density of a signal q_{j-1}, possibly dependent on the agent's previous reports $\hat{q}_0, ..., \hat{q}_{j-2}$. Note that, if the report q_{j-2} is incentive compatible (i.e., if it is truth telling), only this report in $f(\cdot|\hat{q}_0, ..., \hat{q}_{j-2})$ is of value, as the variable $q(t)$ follows a Markov process.

Similarly to the agent's optimization problem, the principal's problem needs to satisfy the variational inequalities, for $i, j = 1, .., N$, where $j = N - i + 1$,

$$\tilde{v}_i^P \geq \tilde{g}_{i-1}^P \tag{5.15}$$

$$L^{q_{j-1}} \tilde{v}_i^P \leq 0 \tag{5.16}$$

$$\max\{\tilde{g}_{i-1}^P - \tilde{v}_i^P, L^{q_{j-1}} \tilde{v}_i^P\} = 0, \tag{5.17}$$

for all states. The partial differential operator $L^{q_{j-1}}$ is identical to the operator in equation (5.11). For the truth telling condition to hold, we require that the variational inequalities in (5.12)-(5.14) and (5.15)-(5.17) result in the same optimal investment strategy: As the investment strategy is based on the agent's reports, we need to ensure that the agent is induced to follow the investment strategy preferred by the principal. Otherwise, truth telling will not be optimal for the agent.

The value of the investment project is shared between the principal and the agent, and the respective shares are dependent upon the agent's value of private information. As the number of unfinished investment phases decreases, we expect to find that the value of the project increases, i.e., we guess that $v_0^P > v_1^P > ... > v_N^P$, and $v_0^A > v_1^A > ... > v_N^A$. We guess that the continuation region D_j has the form

$$D_j = \left\{(\xi, \pi, q_{j-1}, \eta); q_{j-1} < q_{j-1}^*(\pi, \eta; \hat{q})\right\},$$

i.e., the continuation region is time-homogeneous. The function $q_{j-1}^*(\cdot)$ is understood as the entry threshold of starting investment phase $j = N - i + 1$. The increasing value functions as the investment project approaches completion imply that the investment triggers are decreasing, such that $q_0^* > q_1^* > ... > q_{N-1}^*$.

The optimal contract is derived from the agent's value of private information. As already mentioned the problem is solved with help of the revelation principle. Hence, in the next section we study the agent's truth telling constraints, which will be used as constraints in the principal's optimization problem in section 5.4.

5.3 The agent's truth telling constraints

In our model we have N truth telling constraints that need to hold, each constraint corresponding to the agent's private observation of an updated signal.

First, note that we do not assume that any contracted payments are payable when all the investment phases are finished, i.e., when the value $q_N = q(\tau_N + \Delta_N)$ is realized. As all the costs are sunk, there is no way for the principal to detect a lie.

Below we examine the incentive compatibility constraints for all the privately observed signals.

The agent's switching function g_{i-1}^A denotes the value obtained by exercising investment phase $j = N - i + 1$, and is defined by

$$\begin{aligned} & g_{i-1}^A(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1}) \\ = \; & \sup_{\hat{q}_{j-1}} E\left[e^{-r\Delta_j} v_{i-1}^A(\pi(\Delta_j), q(\Delta_j), \eta(\Delta_j); \hat{q}_0, .., \hat{q}_{j-1}) \right. \\ & \left. -Y_j(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1}) | \mathcal{F}_0^\Gamma\right], \end{aligned} \tag{5.18}$$

for i, j from 1 to N. The value when investment phase j starts, $g_{i-1}^A(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1})$, consists of the value obtained after the investment phase is started, $v_{i-1}^A(\cdot)$, reduced by the amount paid to the principal, Y_j. The function $g_{i-1}^A(\cdot)$ may depend on the reported current signal $\hat{q}_{j-1}$, as well as all previously given reports, $\hat{q}_0, ..., \hat{q}_{j-2}$. The agent optimizes the switching functions with respect to the reported signals, as shown in equation (5.18).

For use below, we also define the agent's value function when we have i investment phases left,

$$\begin{aligned} & v_i^A(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1}) \\ = & \begin{cases} w_i^A(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1}) & \text{if } q_{j-1} \leq q_{j-1}^*(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1}) \\ g_{i-1}^A(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1}) & \text{if } q_{j-1} > q_{j-1}^*(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1}), \end{cases} \end{aligned} \tag{5.19}$$

where $w_i^A(\cdot)$ denotes the value function when we have i compound options left, given that the investment decision is postponed. Recall that $q_{j-1}^*(\cdot)$ is defined as a critical value such that it is optimal to invest when $q_{j-1} > q_{j-1}^*(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1})$.

Now we study the truth telling conditions. At the stage where the agent has i options left, the most recent signal is given by q_{j-1}. Thus, the agent optimizes the function v_i^A with respect to the current signal $\hat{q}_{j-1}$. For notational simplicity we denote the agent's value function $v_i^A(\pi, q_{j-1}, \eta; \hat{q}_{j-1}) \equiv v_i^A(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1})$, where all the reports, except for the one given currently, are suppressed in the notation. In this model we take a slightly different, but equivalent, approach in order to find the incentive compatibility constraints. Using the approach presented below, it is easier to understand how the truth telling constraints affect the contract.

Truth telling implies that for any two values q_{j-1} and q'_{j-1} the following inequalities must be satisfied:

$$v_i^A(\pi, q_{j-1}, \eta; q_{j-1}) \geq v_i^A(\pi, q_{j-1}, \eta; q'_{j-1}),$$

and

$$v_i^A(\pi, q'_{j-1}, \eta; q'_{j-1}) \geq v_i^A(\pi, q'_{j-1}, \eta; q_{j-1}).$$

The inequalities tell us that the contract needs to be designed such that the value of reporting the truth is at least as high as the value of lying. Adding the inequalities we find

$$\begin{aligned} &v_i^A(\pi, q_{j-1}, \eta; q_{j-1}) - v_i^A(\pi, q'_{j-1}, \eta; q_{j-1}) \\ &\quad -[v_i^A(\pi, q_{j-1}, \eta; q'_{j-1}) - v_i^A(\pi, q'_{j-1}, \eta; q'_{j-1})] \geq 0. \end{aligned} \tag{5.20}$$

Suppose $q_{j-1} > q'_{j-1}$. MacKie-Mason (1985) interprets a condition similar to the one in (5.20) as follows: to ensure truth telling, the value of reporting the truth must be at least as great when the truth is good news as when the truth is bad news.

The incentive compatibility above can be evaluated further for the case where the signal q_{j-1} prescribes immediate investment, i.e., when $q_{j-1} > q^*_{j-1}$. Replace the function $v_i^A(\cdot)$ in (5.20) by g_{i-1}^A, and evaluate the inequality in (5.20) using equation (5.18). The calculations are presented in appendix D.1. This yields the following incentive compatibility constraint:

$$\frac{\partial v_{i-1}^A(q(\Delta_j); q_{j-1})}{\partial q(\Delta_j)} - \frac{\partial v_{i-1}^A(q(\Delta_j); q'_{j-1})}{\partial q(\Delta_j)} \geq 0. \tag{5.21}$$

The incentive compatibility constraint in (5.21) says that when there are $i-1$ investment phases left to exercise, and the agent's latest information is given by the signal q_j, the previous report $\hat{q}_{j-1}$ must have a positive impact on $\frac{\partial v_{i-1}^{A}(\pi(\Delta_j),q(\Delta_j,\eta(\Delta_j);\hat{q}_0,..,\hat{q}_j)}{\partial q(\Delta_j)}$ if the incentive compatibility constraint is to hold. What about the impact on v_{i-1}^{A} of the earlier reports $\hat{q}_0,..,\hat{q}_{j-2}$? In order to solve our problem we need to find the incentive compatible constraints for these reports as well.

Suppose investment phase j is exercised at time t. Then we know that the value of exercising this investment phase is given by

$$E\left[e^{-rt}g_{i-1}^{A}(\pi(t),q(t),\eta(t);\hat{q}_0,..,\hat{q}_{j-1})|\mathcal{F}_0^{\Gamma}\right]. \tag{5.22}$$

We have already found the incentive compatibility constraint for $\hat{q}_{j-1}$. As we want to study the incentive compatible constraint for each report $\hat{q}_k$, for $k=0,..,j-2$, we reformulate the above value to

$$E\left[e^{-rt}\int_0^{\infty} g_{i-1}^{A}(\pi(t),q(t),\eta(t);\hat{q}_k)f(q(t)|\hat{q}_k,t)dq(t)|\mathcal{F}_0^{\Gamma}\right]. \tag{5.23}$$

The expressions in (5.22) and (5.23) are equivalent[5]. The value in (5.22) is reformulated to (5.23) because we want to emphasize the dependence on the previously given report $\hat{q}_k$. By equation (5.23) we know that the incentive compatibility constraint for any earlier report has a similar impact as the report $\hat{q}_{j-1}$. This result is obtained by replacing the report $\hat{q}_{j-1}$ in equation (D.1) by $\hat{q}_k$, for $k=0,..,j-2$, as shown in the last part of appendix D.1. The truth telling constraint for each report $\hat{q}_k$, $k=0,..,j-2$, with respect to the value when there are j investment phases left, is given by

$$\frac{\partial v_{i-1}^{A}(q(t);q_k)}{\partial q(t)}-\frac{\partial v_{i-1}^{A}(q(t);q_k')}{\partial q(t)}\geq 0. \tag{5.24}$$

[5]The two expressions are equivalent as we can define

$$e^{-rt}\int_0^{\infty} g_{i-1}^{A}(\pi(t),q(t),\eta(t);\hat{q}_k)f(q(t)|\hat{q}_k,t)dq(t)\equiv E^{\zeta,\eta}\left[e^{-rt}g_{i-1}^{A}(\pi(t),q(t),\eta(t);\hat{q}_0,..,\hat{q}_{j-1})\right],$$

yielding

$$\begin{aligned}&E\left[e^{-rt}\textstyle\int_0^{\infty} g_{i-1}^{A}(\pi(t),q(t),\eta(t);\hat{q}_k)f(q(t)|\hat{q}_k,t)dq(t)|\mathcal{F}_0^{\Gamma}\right]\\&=E\left[E^{\zeta,\eta}\left[e^{-rt}g_{i-1}^{A}(\pi(t),q(t),\eta(t);\hat{q}_0,..,\hat{q}_{j-1})\right]|\mathcal{F}_0^{\Gamma}\right].\end{aligned}$$

The left-hand side of the equality is identical to (5.23), and the right-hand side equals (5.22) by conditional expectations.

Thus, the incentive compatibility constraints for all earlier reports, $\hat{q}_0, ..., \hat{q}_{N-i}$, require that each report has a positive impact on the agent's value $v^A_{i-1}(\cdot)$. In the next section the incentive compatibility constraints are incorporated in the principal's optimization problem.

5.4 The principal's optimization problem

The principal's switching function is given by

$$\begin{aligned} & g^P_{i-1}(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1}) \\ = \; & E\left[e^{-r\Delta_j}\tilde{v}^P_{i-1}(\Gamma(\Delta_j); \hat{q}_0, .., \hat{q}_{j-1}) + Y_j(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1}) - K_j | \mathcal{F}^{\pi,\eta}_0\right]. \end{aligned} \tag{5.25}$$

Thus, the principal's value of exercising investment phase j, in (5.25), consists of the value obtained upon investment, $E[e^{-r\Delta_j}\tilde{v}^P_{i-1}(\cdot)|\mathcal{F}^{\pi,\eta}_0]$, plus the amount received from the agent, Y_j, reduced by the investment cost K_j.

By substitution of Y_j in equation (5.25), using equation (5.18), we find that equation (5.25) can be formulated by

$$\begin{aligned} & g^P_{i-1}(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1}) \\ = \; & E\left[e^{-r\Delta_j}\left(\tilde{v}^P_{i-1}(\Gamma(\Delta_j); \hat{q}_0, .., \hat{q}_{j-1}) + v^A_{i-1}(\Gamma(\Delta_j); \hat{q}_0, .., \hat{q}_{j-1})\right)\right. \\ & \left. -g^A_{i-1}(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1}) - K_j | \mathcal{F}^{\pi,\eta}_0\right]. \end{aligned} \tag{5.26}$$

The optimization problem is solved iteratively. Thus, at the step where we are about to find an optimal investment strategy for exercising phase j, we have already solved for the values of $\tilde{v}^P_{i-1}$ and v^A_{i-1}. The function g^A_{i-1} represents the agent's value of exercising investment phase j. In a similar way to the reformulation from (5.22) to (5.23), equation (5.26) can be reformulated to

$$\begin{aligned} & g^P_{i-1}(\pi, \eta; \hat{q}_0, .., \hat{q}_{j-1}) \\ = \; & E\left[\int_0^\infty \left\{e^{-r\Delta_j}\left(\tilde{v}^P_{i-1}(\Gamma(\Delta_j); \hat{q}_0, .., \hat{q}_{j-1}) + v^A_{i-1}(\Gamma(\Delta_j); \hat{q}_0, .., \hat{q}_{j-1})\right)\right.\right. \\ & \left.\left. -g^A_{i-1}(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1}) - K_j\right\} f(q_{j-1}|\hat{q}_0, .., \hat{q}_{j-2}) dq_{j-1} | \mathcal{F}^{\pi,\eta}_0\right], \end{aligned} \tag{5.27}$$

where the expectation of the principal's value with respect to q_{j-1} is written on "integral form".

The value in (5.27) depends on the probability density $f(q_{j-1}|\hat{q}_0, .., \hat{q}_{j-2})$, which is the density of the signal q_{j-1}, given the previous reports $\hat{q}_0, .., \hat{q}_{j-2})$. As long as the agent has no incentives to lie, the probability density is given by $f(q_{j-1}|q_{j-2})$.[6] Below we will discuss whether or not it is optimal to let the contracted amounts $Y_j(\cdot)$ depend on the previous report $\hat{q}_{j-1}$. If not, we also need to check if it is optimal to let Y_j be dependent on any earlier reports.

We have already mentioned that the optimization problem is solved iteratively. Below we show that when $i = 1$, that is, when we have only one investment phase left, we are able to find a closed form solution.

Assume that $i = 1$, and denote $\hat{q}^{-(N-1)} = \{\hat{q}_0, ..., \hat{q}_{N-2}\}$, i.e., $\hat{q}^{-(N-1)}$ is the vector of all reports except the last one. The principal's value of starting the last investment phase is given by

$$\begin{aligned} &g_0^P(\pi, \eta) \\ &= \int_0^\infty \left\{ \pi q_{N-1} e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} - \int_0^{q_{N-1}} \frac{\partial w_1^A(\pi,u,\eta)}{\partial u} du - K_N \right\} f(q_{N-1}|\hat{q}^{-(N-1)}) dq_{N-1}. \end{aligned} \tag{5.28}$$

The result in (5.28) is a reformulated version of equation (5.27) when $i = 1$ and $j = N$. The derivation is presented in appendix D.2. In the case of full information the investor's value of exercising the last investment equals $q_{N-1}e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} - K_N$. Thus, in (5.28) the "full information" payoff is reduced by the agent's value of private information, $\int_0^{q_{N-1}} \frac{\partial w_1^A(\pi,u,\eta)}{\partial u} du$.

By some further calculations, shown in appendix D.3, the principal's value of exercising the last option is written as

$$\begin{aligned} &g_0^P(\pi, \eta) \\ &= \int_0^\infty \left[\pi e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} \left(q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})} \right) - K_N \right] f(q_{N-1}|\hat{q}^{-(N-1)}) dq_{N-1}. \end{aligned} \tag{5.29}$$

The fraction $\frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})}$ is the inverse hazard rate, and is interpreted as the costs of asymmetric information at the time the last investment is exercised. In equation (5.29) the principal's value of starting the last investment phase is reduced by this fraction.

[6] Because the last observation is a sufficient statistic when the variable $q(t)$ is driven by a Markov process.

The fraction $\frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})}$ corresponds to the inverse hazard rate $\frac{F(\cdot)}{f(\cdot)}$ in chapters 2-4. Recall that $\frac{F(\cdot)}{f(\cdot)}$ is interpreted as the increase in the principal's investment cost at the investment time, due to an agent's private information about an investment cost. Analogously, $\frac{1-F(\cdot)}{f(\cdot)}$ is the reduction in the principal's "output value" because of asymmetric information.

Define $M(t) = \pi(t)e^{\kappa(\eta(t)+\Delta_N)-\delta\Delta_N}$ for $\tau_{N-1} + \Delta_{N-1} \leq t < \tau_N + \Delta_N$. By Ito's lemma we find

$$dM(t) = (r - \delta + \kappa)M(t)dt + \sigma M(t)dB^\pi(t), \quad m = M(0).$$

Replacing $\pi e^{\eta+\Delta_N}$ by m, equation (5.29) is equivalently written as

$$g_0^P(m) = E\left[m\left(q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})}\right) - K_N|\mathcal{F}_0^{\pi,\eta}\right]. \tag{5.30}$$

The reason we substitute $\pi e^{\eta+\Delta_N}$ for m, is that it reduces the number of states from two (π and η) to one (m).

Denote V_1^P as the principal's optimal value when there is only one investment phase left before completion. The optimal investment strategy when there is one investment phase left, and when $q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})} > 0$, is found from

$$\begin{aligned} &V_1^P(m) \\ &= \sup_\tau E\left[E\left[e^{-r\tau}\left(M(\tau)\left(q_{N-1} - \tfrac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})}\right) - K_N\right)|\mathcal{F}_0^\Gamma\right]|\mathcal{F}_0^{\pi,\eta}\right]. \end{aligned} \tag{5.31}$$

Note that if $q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})} \leq 0$, the principal's value of the contract, $V_1^P(m)$, equals zero.

The optimization problem in (5.31) equals a perpetual American call option, where the stochastic variable is driven by a geometric Brownian motion. Hence, the closed form solution is a well known result. The solution is derived in appendix D.4. Define an investment trigger function M^*, taking only strictly positive values. It is optimal to invest when $m > M^*$, and to postpone the investment when $m \leq M^*$. The critical price M^* may depend on the current signal q_{N-1}, as well as previously reported signals $\hat{q}^{-(N-1)}$. We find that the optimal investment trigger M^* when $q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})} > 0$ is given by

$$M^*(q_{N-1};\hat{q}^{-(N-1)}) = \frac{\lambda}{\lambda-1}K_N\left(q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})}\right)^{-1}, \quad (5.32)$$

whereas $M^*(q_{N-1};\hat{q}^{-(N-1)})$ approaches infinity when $q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})} \leq 0$. The parameter λ is given by

$$\lambda = \frac{1}{\sigma_S^2}\left[\frac{1}{2}\sigma_S^2 - (r-\delta+\kappa) + \sqrt{\left((r-\delta+\kappa)-\frac{1}{2}\sigma_S^2\right)^2 + 2r\sigma_S^2}\right] > 1.$$

In order to ensure that $\lambda > 1$, we require that $\kappa < \delta$.

As the signal q_{N-1} is a constant in the optimal stopping problem in (5.31), it seems to be optimal for the principal to reject the contract when $q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})} \leq 0$. However, in the next section we shall see that this is not the case as long as it possible to find an inverse critical price $q_{N-1}^*(m;\hat{q}^{-(N-1)})$ to the critical price $M^*(q_{N-1};\hat{q}^{-(N-1)})$.

5.5 Characterization of the optimal contract

In this section we characterize the optimal contract by examination of the contracted payments Y_j, $j = 1,..,N$. Only in the case where $i = 1$ do we find a closed form solution of the payment, presented in section 5.5.1. However, for $i > 1$ we are able to characterize the contracted amounts, by studying how they depend on the agent's reports. This is done in section 5.5.2.

5.5.1 The optimal contracted payment when $i = 1$

In this section we find the optimal amount Y_N. First, define the inverse critical price for investment (using the result in (5.32)), $q_{N-1}^*(m;\hat{q}^{-(N-1)})$, such that the inverse trigger function satisfies the relationship

$$m = \frac{\lambda}{\lambda-1}K_N\left(q_{N-1}^* - \frac{1-F(q_{N-1}^*|\hat{q}^{-(N-i)})}{f(q_{N-1}^*|\hat{q}^{-(N-i)})}\right)^{-1},$$

when $q^*_{N-1}(m) - \frac{1-F(q^*_{N-1}(m)|\hat{q}^{-(N-1)})}{f(q^*_{N-1}(m)|\hat{q}^{-(N-1)})} > 0$. Appendix D.5 shows that the value of the contracted amount $Y_N(\cdot)$ is formulated as

$$Y_N(m, q_{N-1}; \hat{q}^{-(N-1)})$$

$$= \begin{cases} mq^*_{N-1}(m;\hat{q}^{-(N-1)}) - \int_{\underline{q}_{N-1}}^{q^*_{N-1}(m;\hat{q}^{-(N-1)})} \left(\frac{m}{M^*(u;\hat{q}^{-(N-1)})}\right)^{\lambda} M^*(u;\hat{q}^{-(N-1)})du \\ \qquad \text{if } q^*_{N-1} - \frac{1-F(q^*_{N-1}|\hat{q}^{-(N-1)})}{f(q^*_{N-1}|\hat{q}^{-(N-1)})} > 0 \\ C \quad \text{if } q^*_{N-1} - \frac{1-F(q^*_{N-1}|\hat{q}^{-(N-1)})}{f(q^*_{N-1}|\hat{q}^{-(N-1)})} \leq 0, \end{cases} \tag{5.33}$$

where $\underline{q}_{N-1}$ is the value of the signal q_{N-1} such that $\underline{q}^*_{N-1} - \frac{1-F(\underline{q}^*_{N-1}|\hat{q}^{-(N-1)})}{f(\underline{q}^*_{N-1}|\hat{q}^{-(N-1)})} = 0$, and C is a constant sufficiently high to prevent the agent from investing when $q^*_{N-1} - \frac{1-F(q^*_{N-1}|\hat{q}^{-(N-1)})}{f(q^*_{N-1}|\hat{q}^{-(N-1)})} \leq 0$.

In order to keep the principal's value of the contract positive, we need to prevent the agent from investing when $q^*_{N-1} - \frac{1-F(q^*_{N-1}|\hat{q}^{-(N-1)})}{f(q^*_{N-1}|\hat{q}^{-(N-1)})} \leq 0$, or when $q_{N-1} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})} \leq 0$. This is shown by studying the principal's optimal stopping problem in (5.31).

The contracted amount Y_N in equation (5.33) implies that the investment decision of the last investment phase can be delegated to the agent, as the amount is independent of the agent's current private information at the stage where there is only one investment phase left. However, so far we do not know whether or not it is optimal to let the contracted amount Y_N depend on any previous reports. This question is examined in the following. Recall that the incentive compatibility restrictions in equations (5.21) and (5.24) require that $\frac{\partial v_1^A(q(\Delta_{N-1});q_k)}{\partial q(\Delta_{N-1})} - \frac{\partial v_1^A(q(\Delta_{N-1});q'_k)}{\partial q(\Delta_{N-1})} \geq 0$ for $q_k > q'_k$, where $k = 0,..,N-2$. Evaluation of $v_1^A(q(\Delta_{N-1});q_{N-2}) \equiv v_1^A(M(\Delta_{N-1}), q(\Delta_{N-1}); \hat{q}_0, .., q_{N-2})$ leads to (using appendix D.5)

$$\frac{\partial v_1^A(M(\Delta_{N-1}), q(\Delta_{N-1}); \hat{q}^{-(N-1)})}{\partial q_{N-1}}$$

$$= \begin{cases} \left(\frac{M(\Delta_{N-1})}{M^*(q(\Delta_{N-1});\hat{q}^{-(N-1)})}\right)^{\lambda} M^*(q(\Delta_{N-1}); \hat{q}^{-(N-1)}) & \text{if } M(\Delta_{N-1}) \leq M^*(q(\Delta_{N-1}); \hat{q}^{-(N-1)}) \\ M(\Delta_{N-1}) & \text{if } M(\Delta_{N-1}) > M^*(q(\Delta_{N-1}); \hat{q}^{-(N-1)}). \end{cases} \quad (5.34)$$

When $M(\Delta_{N-1}) > M^*(q(\Delta_{N-1}); \hat{q}^{-(N-1)})$, in (5.34) we find that $\frac{\partial v_1^A(q(\Delta_{N-1});q_{N-2})}{\partial q(\Delta_{N-1})} = M(\Delta_{N-1})$, leading to $\frac{\partial v_1^A(q(\Delta_{N-1});q_{N-2})}{\partial q(\Delta_{N-1})} - \frac{\partial v_1^A(q(\Delta_{N-1});q'_{N-2})}{\partial q(\Delta_{N-1})} = 0$. If we substitute the right-hand side expression in (5.34) when $M(\Delta_{N-1}) \leq M^*$, into the incentive compatibility constraint in (5.21), we find that the restriction in (5.21) is given by

$$M(\Delta_{N-1})^{\lambda} \left((M^*(q(\Delta_{N-1}); q_k))^{1-\lambda} - (M^*(q(\Delta_{N-1}); q'_k))^{1-\lambda}\right) \geq 0. \quad (5.35)$$

The inequality shows that for the incentive compatibility constraint to be satisfied, we need $M^*(q_{N-1}); q_k) \leq M^*(q_{N-1}); q'_k)$.

The property of the investment trigger $M^*(q_{N-1}); q_k)$ is examined by differentiating $M^*(q_{N-1}; \hat{q}_k)$ with respect to the report $\hat{q}_k$, for k from 0 to $N-2$. This yields

$$\frac{\partial M^*(q_{N-1}; \hat{q}_k)}{\partial \hat{q}_k} > 0, \quad (5.36)$$

as

$$\frac{\partial \left(\frac{1-F(q_{N-1}|\hat{q}_k)}{f(q_{N-1}|\hat{q}_k)}\right)}{\partial \hat{q}_k} < 0.$$

The result in (5.36) contradicts the incentive compatibility constraint in (5.35) for all $\hat{q}_k$, $k = 0, ..., N-1$. Thus, we conclude that it is *not* optimal to let $Y_N(\cdot)$ depend on the reports $\hat{q}^{-(N-1)}$, and that this means that the optimal compensation is given by

$$Y_N(m, q_{N-1})$$

$$= \begin{cases} mq^*_{N-1}(m) - \int_{\underline{q}_{N-1}}^{q^*_{N-1}(m)} \left(\frac{m}{M^*(u)}\right)^{\lambda} M^*(u)du & \text{if } q^*_{N-1}(m) - \frac{1-F(q^*_{N-1}(m))}{f(q^*_{N-1}(m))} > 0 \\ C \text{ if } q^*_{N-1}(m) - \frac{1-F(q^*_{N-1}(m))}{f(q^*_{N-1}(m))} \leq 0, \end{cases} \quad (5.37)$$

where

$$M^*(q_{N-1}) = \frac{\lambda}{\lambda - 1} K_N \left(q_{N-1} - \frac{1 - F(q_{N-1})}{f(q_{N-1})} \right)^{-1} \tag{5.38}$$

when $q_{N-1} - \frac{1-F(q_{N-1}|\cdot)}{f(q_{N-1}|\cdot)} \leq 0$.

We have found that the contracted amount Y_N is given by a function of observable variables only, it is not path dependent, and it implements the optimal strategy when the informed agent makes the investment decision. Thus, the contracted amounts lead to the same type of contracts as in the situations in chapters 2 and 3, where the agent has private information about the costs of the investment project.

5.5.2 Characterization of the optimal contract when $i > 1$

A remaining question is if it is optimal to let $Y_j(\cdot)$, $j = 1, ..., N-1$, depend on previous reports. We do not find closed form solutions for the contracted amounts $Y_0, ..., Y_{N-1}$. However, we are able to find some properties of Y_j, by studying whether the amount depends on previously reports.

In the section above we examined the case where $i = 1$. Now we first assume that $i = 2$. The agent's value at the time the second last investment phase is exercised, is equal to

$$\begin{aligned} & g_1^A(\pi, q_{N-2}, \eta; \hat{q}_0, .., \hat{q}_{N-2}) \\ = \; & E\left[-r\Delta_{N-1} \int_0^\infty v_1^A(\pi(\Delta_{N-1}), q(\Delta_{N-1}), \eta(\Delta_{N-1})) f(q(\Delta_{N-1})|q^{-(N-1)}) dq(\Delta_{N-1}) \right. \\ & \left. -Y_{N-1}(\pi, \eta; \hat{q}^{-(N-1)}) | \mathcal{F}_0^\Gamma \right]. \end{aligned} \tag{5.39}$$

Equation (5.39) equals equation (5.27) for $i = 2$ and $j = N - 1$. Note that in equation (5.27) the function $v_{i-1}^A(\cdot)$ on the right-hand side of may depend on previous reports. However, in section 5.5.1 we have found that it is optimal *not* to let $v_{i-1}^A(\cdot)$ for $i = 2$, i.e., $v_1^A(\cdot)$, depend on previous reports.

Thus, in equation (5.39) only the contracted amount Y_{N-1} may depend on the previous reports. This actually means that it is not optimal to let Y_{N-1} depend on previous reports either: for any reports the incentive compatibility constraint in (5.24) equals zero. The conclusion is that the contracted amount Y_{N-1} is independent of reports previous to the current one, i.e., Y_{N-1} is path independent.

The conclusion implies that agent's value in (5.39) is independent of previous reports, which means that v_2^A cannot depend on earlier reports either. Moreover, by the same analyzes as above for $i = 3, ..., N$ this also means that none of the subsequent values, v_{i-1}^A, depend on previous reports. Thus, we conclude that it is not optimal to let any of the contracted amounts depend on reports previous to the most recent one.

Although the contracted amounts does not depend on previous reports, the agent's value of private information is still of value. The information rent can be expressed as

$$g_{i-1}^A(\pi, q_{j-1}, \eta; \hat{q}) = \int_0^{q_{N-i}} \frac{\partial w_i^A(\pi, u, \eta; \hat{q}^{-(j-1)})}{\partial u} du.$$

Using this expression, we find that the principal's switching function in equation (5.26) can be formulated as

$$\begin{aligned} & g_{i-1}^P(\pi, \eta) \\ = \; & E\left[\int_0^\infty \left\{\int_0^\infty e^{-r\Delta_j} \left(\tilde{v}_{i-1}^P(\Gamma(\Delta_j)) + v_{i-1}^A(\Gamma(\Delta_j))\right) f(q(\Delta_j|q_{j-1})dq(\Delta_j)\right.\right. \\ & \left.\left. - \tfrac{\partial w_i^A(\pi, q_{j-1}, \eta; \hat{q}_{-(j-1)})}{\partial q_{j-1}} \tfrac{1-F(q_{j-1}|\hat{q}^{-(j-1)})}{f(q_{j-1}|\hat{q}^{-(j-1)})}\right\} f(q_{j-1})dq_{j-1} - K_j | \mathcal{F}_0^{\pi,\eta}\right]. \end{aligned}$$

This value is the principal's value of exercising investment phase j. The optimal investment strategy when there is more than one investment phase left must be found numerically, and is left for future work. However, the problem is simplified compared to the problem we started with, as we have found that it is not path dependent.

5.6 Conclusion

In this chapter we have studied sequential investment decisions in presence of incentive problems. An agent has private information about the output value of the realized investment project. As in the principal-agent models of chapters 2 and 3 we find a delegation based contract, where the privately informed agent makes the investment decisions.

A difference from the optimal contracts found in chapters 2 and 3 is that now the agent pays a contracted amount to the principal each time a new investment phase is exercised. In addition, the assumption of private information about the *output* value has a different effect on the contract compared to the case where the agent has private information about the investment costs. In the case of a privately observed output value we need to let the contracted payment approach infinity for some levels of the observed signal, in order to ensure that the principal's value of the contract is positive.

Some of the calculations in this chapter have been rather tedious, but the results are simple: the contract does not depend on reports previous to the latest reports from the agent. When there is only one investment phase left, we find closed form solutions, as the contract then is formulated as a perpetual American call option.

Chapter 6

Summary and concluding remarks

This dissertation is a theoretical study of real options valuation and strategies in the presence of incentive problems. The investment projects are evaluated under the assumption that the asset values of the projects are uncertain, and we have formulated the investment decision problems as optimal stopping problems. We have assumed that some manager of some investment project is better informed than some investor about the profitability of the project. This creates incentive problems, which have been solved using a direct truth telling mechanism (the revelation principle).

Previous results on valuation and strategies of uncertain investments have been extended in the dissertation. We have shown that private information in our models may lead to second-best optimal investment strategies, resulting in lower values of the real options compared to the case where the agent has no private information.

In chapters 2, 3 and 5 the incentive problems have been examined within a principal-agent framework. In these chapters the models result in optimal contracts where the investment decisions are delegated to the agent, consistent with well known results in the principal-agent literature. Chapter 4 has discussed the incentive problems when more than one agent has private information about their respective investment cost levels. In chapter 4 it has been demonstrated that the investment decision can be delegated to the contract winner only in the case where the agents' private respective information is constant. When the private information changes stochastically the principal makes the investment decision as the contract winner is not chosen prior to the time the investment is made.

In chapter 2 we have studied how private information about a constant investment cost affects the strategies and value of an uncertain investment project. It has been demonstrated that the interaction of a privately observed constant investment cost and an asset value that is driven by an Ito diffusion, may result in under-investment compared to the case of full information.

The model in chapter 3 have been extended to the case where the private information changes stochastically: we let the privately observed investment cost and the commonly observed asset value be driven by geometric Brownian motions. In this case the conclusion with respect to the optimal investment decision was found to be ambiguous: depending on the parameter values private observation of the stochastically changing investment cost may result in over-investment as well as under-investment. The reason is two opposing effects: the incentive effect tends to higher costs of the project and thereby under-investment, whereas the increased cost decreases the value of waiting, thereby tending to over-investment.

The numerical examples in chapters 2 and 3 illustrate that the investor's loss can be substantially reduced because of asymmetric information.

We have found that the optimal strategies are not changed when we extend the principal-agent models in chapters 2 and 3 to the case where two or more privately informed agents compete about a contract that gives the winner the right to invest in the project, as is done in chapter 4. However, the winner's value of private information is lower in the case of competition than in the case of only one agent having private information. The numerical illustrations in chapter 4 show that the value of the agent's private information converges rapidly to zero as the number of competitors increase.

In the principal-agent models of chapters 2 and 3 we have studied how changes in volatility parameters affect the values of the principal and the agent. We have found that the agent's value as a function of the volatility parameter may be decreasing. This effect is in contrast to the well known result that the option value under full information increases as a function of volatility. In the case of a constant private information, modelled in chapter 2, we have illustrated that the principal's option value too may decrease as a function of the volatility parameter. The reason is that the effect from a second-best investment strategy for some parameter values dominates the effect of the prospects of higher future profits.

In chapter 5 we have formulated a sequential investment model where an agent, each time a new investment phase is finished, privately observes a signal about the output value of the realized investment project. We allow the contracted transfers between a principal and an agent to depend on reports about previously given reports. Thus, in this chapter the focus is on whether the optimal contract is path dependent, rather than examining the optimal strategies and corresponding values. We assume that the signals are driven by a geometric Brownian motion, and find that it is not optimal to let the contract depend on previously given reports.

Many simplifying assumptions have been made in the models. One limitation is that we only consider complete contracts, meaning that all variables that may have an impact on the contractual relationship are included. Secondly, we have assumed that all the contracts are binding for the contracted period. The relationship cannot be breached or renegotiated.

Another aspect we have not considered is moral hazard effects, i.e., effects of unobservable effort have not been included in the analyzes[1]. Furthermore, in the auction models of this dissertation we have not considered the situation where the agents have different degrees of private information. As long as one agent is better informed than the others, the better informed agent probably will have a positive value of private information, even under competition. It might be interesting to study how different degrees of observability affects option values and strategies.

For simplicity we have assumed that uncertain values either can be spanned in dynamically complete capital markets, or that the risk is independent of risk in

[1]However, in the case where both unobservable effort and private information is present, the problem can be reduced to a pure private information/adverse selection problem, see Laffont and Tirole (1993).

these markets. Moreover, the investment problems we have studied are limited to optimal stopping problems. Finally, in the principal-agent model of chapter 3, where it is assumed that the private information changes stochastically, we have only considered the case where the private observations are given by continuous processes. For many applications it is more realistic to allow jumps in the stochastically changing and privately observed variables.

Appendix A

Appendix for chapter 2

A.1 The second-order condition for a perpetual American call option

The second-order condition of the option in the case where the principal observes the investment cost parameter, is derived from the first-order condition in equation (2.12) (for notational simplicity, we write S^*_{sym} for $S^*_{sym}(K)$),

$$\begin{aligned}
&\frac{\partial^2 V^P_{sym}(s,K;S^*_{sym})}{\partial S^{*2}_{sym}} \\
&= -\frac{\phi(s)\phi'(S^*_{sym})}{\phi(S^*_{sym}(K))^2}\left[1-\frac{\phi'(S^*_{sym})}{\phi(S^*_{sym})}\left(S^*_{sym})-K\right)\right] \\
&\quad +\frac{\phi(s)}{\phi(S^*_{sym})}\left[-\frac{\phi''(S^*_{sym})\phi(S^*_{sym})-\phi'(S^*_{sym})^2}{\phi(S^*_{sym})}\left(S^*_{sym}-K\right)-\frac{\phi'(S^*_{sym})}{\phi(S^*_{sym})}\right].
\end{aligned}$$

Note that the first term on the right-hand side equals zero as $\left[1-\frac{\phi'(S^*_{sym})}{\phi(S^*_{sym})}\left(S^*_{sym})-K\right)\right]=0$ by the first-order condition in equation (2.12). Rearranging, we find

$$\frac{\partial^2 V^P_{sym}(s,K;S^*_{sym})}{\partial S^*_{sym}(K)^2}=-\frac{\phi(s)}{\phi(S^*_{sym})}\frac{\phi''(S^*_{sym})}{\phi(S^*_{sym})}\left(S^*_{sym}(K))-K\right),$$

which is non-positive when $\phi''(S^*_{sym})\geq 0$.

A.2 The optimality conditions of a perpetual American call option: The case of geometric Brownian motion

The first-order condition of the investor's value function in (2.20) is

$$\frac{\partial V^P_{sym}(s,K;S^*_{sym})}{\partial S^*_{sym}} = \begin{cases} \left(\frac{s}{S^*_{sym}}\right)^\beta \left(1-\beta+\beta\frac{K}{S^*_{sym}}\right) = 0 & \text{if } s \le S^*_{sym} \\ 0 & \text{if } s > S^*_{sym} \end{cases}$$

The second-order derivative is

$$\frac{\partial^2 V^P_{sym}(s,K;S^*_{sym})}{\partial (S^*_{sym})^2}$$

$$= \begin{cases} \left(\frac{s}{S^*_{sym}}\right)^\beta \left\{\left(-\frac{s}{(S^*_{sym})^2}\right)\left(1-\beta+\beta\frac{K}{S^*_{sym}}\right) - \beta\frac{K}{(S^*_{sym})^2}\right\} & \text{if } s \le S^*_{sym} \\ 0 & \text{if } s > S^*_{sym} \end{cases}$$

The first term on the right-hand side is zero because $\left(1-\beta+\beta\frac{K}{S^*_{sym}}\right)$ by the first-order condition. Thus, $\frac{\partial^2 V^P_{sym}(s,K;S^*_{sym})}{\partial (S^*_{sym})^2} \le 0$, and the second-order condition is satisfied.

A.3 The agent's second-order condition for optimization w.r.t. $\hat{S}(K)$

Differentiation of the first-order condition in equation (2.21) leads to

$$\begin{aligned} \frac{\partial^2 v^A(s,K;\hat{S}(K))}{\partial \hat{S}(K)} &= -\frac{\phi(s)\phi'(\hat{S}(K))}{\phi(\hat{S}(K))^2}\left[X'(\hat{S}(K)) - \frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\left(X(\hat{S}(K))-K\right)\right] \\ &+\frac{\phi(s)}{\phi(\hat{S}(K))}\left[X''(\hat{S}(K)) - \frac{\phi''(\hat{S}(K))\phi(\hat{S}(K))-\phi'(\hat{S}(K))^2}{\phi(\hat{S}(K))^2}\left(X(\hat{S}(K))-K\right)\right. \\ &\left.-\frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}X'(\hat{S}(K))\right]. \end{aligned}$$

The first term on the right-hand side equals zero because of the first-order condition in (2.21). Simplification yields

$$\frac{\partial^2 v^A(s,K;\hat{S}(K))}{\partial \hat{S}(K)} = \frac{\phi(s)}{\phi(\hat{S}(K))}\left\{\left[-\left(\frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\right)^2 - \frac{\phi''(\hat{S}(K))}{\phi(\hat{S}(K))}\right]\left(X(\hat{S}(K)) - K\right)\right.$$
$$\left. -\frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}X'(\hat{S}(K)) + X''(\hat{S}(K))\right\}.$$

A.4 The revelation principle applied to our model

Applied to the model in this chapter, the revelation principle can be shown by the following arguments (we follow the proof used by Salanié (1997), section 2.1.2).

The entry threshold $\hat{S}(\hat{K})$ and the compensation $X(\hat{S}(\hat{K}), \hat{K})$ are based on the report $\hat{K}$.

Let M be the space of admissible reports, and let $(X(\cdot), \hat{S}(\cdot), M)$ be a set of incentive mechanisms that implement the compensation function X^*, and the investment strategy S^*. Moreover, let $\hat{K}^*(K)$ be the optimal report, so that $X^* = X^*(\hat{K}^*)$ and $S^* = S^*(\hat{K}^*)$. Now consider the direct mechanism (X^*, S^*, K). If it did not implement a truthful report, then an agent would prefer to announce some K' other than K, and we would have

$$v^A(s,K;X^*(K'),S^*(K'),K') > v^A(s,K;X^*(K),S^*(K),K).$$

But by definition this would imply that

$$v^A(s,K;X^*(K^*(K')),S^*(K^*(K')),K') > v^A(s,K;X^*(K^*(K)),S^*(K^*(K)).K)$$

so that K^* would not be an optimal given that the true investment cost equals K. Hence the direct mechanism (X^*, S^*, K) must be truthful, and implement the compensation X^* and the investment strategy S^*.

A.5 The second-order condition for incentive compatibility

The second-order condition for $\hat{K}$ must be satisfied at $\hat{K} = K$, which implies the function $v^A(s, K)$ must be more convex than $v^A(s, K; \hat{K})$, i.e.,

$$\left.\frac{\partial^2 v^A(s, K; \hat{K})}{\partial \hat{K}^2}\right|_{\hat{K}=K} \leq \frac{\partial^2 v^A(s, K)}{\partial K^2}, \tag{A.1}$$

where $v^A(s, K; \hat{K})$ is given by equation (2.22), and $v^A(s, K)$ equals

$$v^A(s, K) = \begin{cases} \frac{\phi(s)}{\phi(\hat{S}(K))}\left(X(\hat{S}(K), K) - K\right) & \text{if } s \leq \hat{S}(K) \\ X(s, K) - K & \text{if } s > \hat{S}(K). \end{cases} \tag{A.2}$$

The first-order condition of $v^A(s, K; \hat{K})$ with respect to $\hat{K}$ when $\hat{K} = K$, and when $s \leq \hat{S}(K)$, is found to be equal to

$$\begin{aligned} &\left.\frac{\partial v^A(s, K; \hat{K})}{\partial \hat{K}}\right|_{\hat{K}=K} \\ = \; &\frac{\phi(s)}{\phi(\hat{S}(K))}\left[X_{\hat{S}}(\hat{S}(K), K)\hat{S}'(K) + X_K(\hat{S}(K), K) - \frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\left(X(\hat{S}(K), K) - K\right)\hat{S}'(K)\right]. \end{aligned} \tag{A.3}$$

Differentiating the first-order condition with respect to $\hat{K}$, when $\hat{K} = K$, yields,

$$\begin{aligned} &\left.\frac{\partial^2 v^A(s, K; \hat{K})}{\partial \hat{K}^2}\right|_{\hat{K}=K} \\ = \; &\frac{\phi(s)}{\phi(\hat{S}(K))}\Big[X_{\hat{S}\hat{S}}(\hat{S}(K), K)\hat{S}'(K)^2 + X_{\hat{S}}(\hat{S}(K), K)\hat{S}''(K) + X_{KK}(\hat{S}(K), K) \\ &- \frac{\phi''(\hat{S}(K))\hat{S}'(K)^2 + \phi'(\hat{S}(K))\hat{S}''(K) - \left(\phi'(\hat{S}(K))\hat{S}'(K)\right)^2}{\phi(\hat{S}(K))^2}\left(X(\hat{S}(K), K) - K\right) \\ &- \frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\hat{S}'(K)\left(X_{\hat{S}}(\hat{S}(K), K)\hat{S}'(K) + X_K\right)\Big]. \end{aligned} \tag{A.4}$$

The first-order derivative of equation (A.2) when $s \leq \hat{S}(K)$ is given by

$$\begin{aligned} &\frac{\partial v^A(s, K)}{\partial K} \\ = \; &\frac{\phi(s)}{\phi(\hat{S}(K))}\left[X_{\hat{S}}(\hat{S}(K))\hat{S}'(K) + X_K(\hat{S}(K), K) - 1 - \frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\left(X(\hat{S}(K), K) - K\right)\hat{S}'(K)\right], \end{aligned} \tag{A.5}$$

yielding the second-order derivative,

$$
\begin{aligned}
&\left.\frac{\partial^2 v^A(s,K;\hat{K})}{\partial K^2}\right|_{\hat{K}=K} \\
&= \tfrac{\phi(s)}{\phi(\hat{S}(K))}\Big[X_{\hat{S}\hat{S}}(\hat{S}(K),K)\hat{S}'(K)^2 + X_{\hat{S}}(\hat{S}(K),K)\hat{S}''(K) + X_{KK}(\hat{S}(K),K) \\
&\quad -\tfrac{\phi''(\hat{S}(K))\hat{S}'(K)^2+\phi'(\hat{S}(K))\hat{S}''(K)-\left(\phi'(\hat{S}(K))\hat{S}'(K)\right)^2}{\phi(\hat{S}(K))^2}\Big(X(\hat{S}(K),K)-K\Big) \\
&\quad -\tfrac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\hat{S}'(K)\Big(X'(\hat{S}(K),K)\hat{S}'(K)+X_K-1\Big)\Big].
\end{aligned}
\tag{A.6}
$$

Using the restriction in equation (A.1), this leads to the second-order condition,

$$
\frac{\partial^2 v^A(s,K)}{\partial K^2} - \left.\frac{\partial^2 v^A(s,K;\hat{K})}{\partial \hat{K}^2}\right|_{\hat{K}=K} = \frac{\phi(s)}{\phi(\hat{S}(K))}\frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\hat{S}'(K) \geq 0. \tag{A.7}
$$

When $s > \hat{S}(K)$ we find that $\left.\frac{\partial^2 v^A(s,K;\hat{K})}{\partial \hat{K}^2}\right|_{\hat{K}=K} = X_{KK}(s,K)$ and $\frac{\partial^2 v^A(s,K)}{\partial K^2} = X_{KK}(s,K)$, which satisfies the condition in equation (A.1).

A.6 The agent's value of private information, equation (2.25)

Integration of both sides of equation (2.24) when $s \leq \hat{S}(K)$,

$$
\int_K^{\overline{K}} \frac{dv^A(s,u)}{du} = -\int_K^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))}du,
$$

leads to

$$
v^A(s,\overline{K}) - v^A(s,K) = -\int_K^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))}du.
$$

When $s > \hat{S}(K)$, integration on both sides of (2.24),

$$
\int_K^{\vartheta_s} \frac{dv^A(s,u)}{du} = -\int_K^{\vartheta_s} 1du,
$$

results in

$$
v^A(s,\vartheta(s)) - v^A(s,K) = -(\vartheta(s) - K).
$$

Rearrangements leads to equation (2.25).

A.7 Partial integration

$$\int_{\underline{K}}^{\overline{K}} \left(\int_{K}^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))} du \right) f(K)dK$$

$$= \left[\int_{K}^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))} du F(K) \right]_{\underline{K}}^{\overline{K}} - (-) \int_{\underline{K}}^{\overline{K}} \frac{\phi(s)}{\phi(\hat{S}(u))} F(K)dK.$$

By inserting the bounds $\overline{K}$ and $\underline{K}$ in the first term on the right-hand side, we see that this term equals zero.

A.8 The optimal investment strategy

Differentiation of $v^P(s;\hat{S}(K))$ with respect to $\hat{S}(K)$ gives the first-order condition for the optimal exercise value $S^*(K)$,

$$\frac{\partial v^P(s;\hat{S}(K)}{\partial \hat{S}(K)} = \frac{\phi(s)}{\phi(\hat{S}(K))} \left[1 - \frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))} \left(\hat{S}(K) - K - \frac{F(K)}{f(K)} \right) \right] = 0. \quad \text{(A.8)}$$

As the fraction $\frac{\phi(s)}{\phi(\hat{S}(K))}$ is positive, the expression in square brackets must be zero. Equation (2.30) in the text is obtained by reorganizing and evaluating at the optimal trigger $S^*(K)$.

The optimality conditions for the trigger value are satisfied as long as the second-order condition

$$\frac{\partial^2 v^P(s;\hat{S}(K)}{\partial \hat{S}(K)^2}$$

$$= -\frac{\phi(s)}{\phi(\hat{S}(K))^2} \phi'(\hat{S}(K)) \left[1 - \frac{\phi_{\hat{S}}(\hat{S}(K))}{\phi(\hat{S}(K))} \left(\hat{S}(K) - K - \frac{F(K)}{f(K)} \right) \right]$$

$$+ \frac{\phi(s)}{\phi(\hat{S}(K))} \left[- \frac{\phi''(\hat{S}(K))\phi(\hat{S}(K)) - \phi'(\hat{S}(K))^2}{\phi(\hat{S}(K))^2} \left(\hat{S}(K) - K - \frac{F(K)}{f(K)} \right) - \frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))} \right] \leq 0.$$

The first term on the right-hand side equals zero because of the first-order condition. Rearrangement leads to

$$\begin{aligned}\frac{\partial^2 v^P(s;\hat{S}(K)}{\partial \hat{S}(K)^2} &= \frac{\phi(s)}{\phi(\hat{S}(K))}\left[-\frac{\phi''(\hat{S}(K))}{\phi(\hat{S}(K))}\left(\hat{S}(K) - K - \frac{F(K)}{f(K)}\right)\right.\\ &\left.-\frac{\phi'(\hat{S}(K))}{\phi(\hat{S}(K))}\left(1 - \frac{\phi_{\hat{S}}(\hat{S}(K))}{\phi(\hat{S}(K))}\left(\hat{S}(K) - K - \frac{F(K)}{f(K)}\right)\right)\right] \le 0.\end{aligned}$$

As the last term equals zero because of the first-order condition, we are left with the expression

$$\frac{\partial^2 v^P(s;\hat{S}(K))}{\partial \hat{S}(K)^2} = -\frac{\phi(s)}{\phi(\hat{S}(K))}\frac{\phi''(\hat{S}(K)}{\phi(\hat{S}(K))}\left(\hat{S}(K)) - K - \frac{F(K)}{f(K)}\right) \le 0,$$

which in turn means that $\phi''(\hat{S}(K)) \ge 0$.

A.9 Implicit differentiation of the optimal investment triggers

Define $A(S^*_{sym}, K) = S^*_{sym} - K - \frac{\phi(S^*_{sym})}{\phi'(S^*_{sym})} = 0$, where $S^*_{sym} \equiv S^*_{sym}(K)$, corresponding to the condition for the optimal entry threshold in the case of full information, given in equation (2.14). By implicit differentiation we know that

$$(S^*)'_{sym} = -\frac{A_K(S^*_{sym}, K)}{A_{S^*_{sym}}(S^*_{sym}, K)},$$

leading to

$$(S^*)'_{sym} = -\frac{-1}{1 - \frac{\phi'(S^*_{sym})^2 - \phi(S^*_{sym})\phi''(S^*_{sym})}{\phi'(S^*_{sym})^2}}.$$

Rearranging, we get

$$(S^*)'_{sym} = \frac{\phi'(S^*_{sym})^2}{\phi(S^*_{sym})\phi''(S^*_{sym})}.$$

Analogously, define $B(S^*, K) = S^* - K - \frac{F(K)}{f(K)} - \frac{\phi(S^*)}{\phi'(S^*)} = 0$, where $S^* \equiv S^*(K)$. corresponding to the condition for the optimal entry threshold in the case of asymmetric information, as written in equation (2.30). By implicit differentiation,

$$\begin{aligned}(S^*)' &= -\frac{B_K(S^*,K)}{B_{S^*}(S^*,K)}\\ &= -\frac{-1 - \frac{\partial(F(K)/f(K))}{\partial K}}{1 - \frac{\phi'(S^*)^2 - \phi(S^*)\phi''(S^*)}{\phi'(S^*)^2}}.\end{aligned}$$

Simplification of the above differentiation leads to

$$(S^*)' = \left[1 + \frac{\partial(F(K)/f(K))}{\partial K}\right] \frac{\phi'(S^*)^2}{\phi(S^*)\phi''(S^*)}.$$

A.10 Show that the communication-based compensation is equal to the delegation-based compensation

In order to show that $\frac{\phi(s)}{\phi(S^*(K))}X(S^*(K),K) = \frac{\phi(s)}{\phi(S^*(K))}X(S^*(K))$ we exploit the fact that we from $s = S^*(K)$ can find the inverse investment trigger $K = \vartheta(s)$ as long as $S^*(K)$ is a strictly continuous and increasing function in $K \in [\underline{K}, \overline{K}]$.

By partial integration we find, using equation (2.33) and given $s \leq S^*(K)$, that

$$\frac{\phi(s)}{\phi(S^*(K))}X(S^*(K),K)$$

$$= \frac{\phi(s)}{\phi(S^*(K))}K + \left[u\frac{\phi(s)}{\phi(S^*(u))}\right]_K^{\overline{K}} - \int_K^{\overline{K}} u\left(-\frac{\phi(s)\phi_{S^*}(S^*(u))}{(\phi(S^*(u)))^2}\right)(S^*)'(u)du$$

$$= \frac{\phi(s)}{\phi(S^*(\overline{K}))}\overline{K} - \int_K^{\overline{K}} u\left(-\frac{\phi(s)\phi_{S^*}(S^*(u))}{(\phi(S^*(u)))^2}\right)(S^*)'(u)du.$$

As K equals $\vartheta(S^*(K))$, we know that $u = \vartheta(S^*(u))$, leading to

$$\frac{\phi(s)}{\phi(S^*(K))}X(S^*(K)) = \frac{\phi(s)}{\phi(S^*(\overline{K}))}\overline{K} - \int_K^{\overline{K}} \vartheta(S^*(u))\left(-\frac{\phi(s)\phi_{S^*}(S^*(u))}{(\phi(S^*(u)))^2}\right)(S^*)'(u)du.$$

By substitution of integration variables we obtain

$$\frac{\phi(s)}{\phi(S^*(K))}X(S^*(K)) = \frac{\phi(s)}{\phi(S^*(\overline{K}))}\overline{K} - \int_{S^*(K)}^{S^*(\overline{K})} \vartheta(S^*(u))\left(-\frac{\phi(s)\phi_{S^*}(S^*(u))}{(\phi(S^*(u)))^2}\right)dS^*(u).$$

Partial integration leads to

$$\frac{\phi(s)}{\phi(S^*(K))}X(S^*(K)) = \vartheta(S^*(K))\frac{\phi(s)}{\phi(S^*(K))} + \int_{S^*(K)}^{S^*(\overline{K})} \vartheta'(S^*(u))\frac{\phi(s)}{\phi(S^*(u))}dS^*(u).$$

The value of the compensation is a function of $\hat{S}(K)$, only, and thus, we have shown that $\frac{\phi(s)}{\phi(S^*(K))}X(S^*(K),K) = \frac{\phi(s)}{\phi(S^*(K))}X(S^*(K))$. As $s \to S^*(K)$, implying

$\frac{\phi(s)}{\phi(S^*(K))} \to 1$, we see that

$$X(s) = s + \int_s^{S^*(\overline{K})} \vartheta'(S^*(u)) \frac{\phi(s)}{\phi(S^*(u))} dS^*(u),$$

which equals equation (2.34), because $\int_s^{S^*(\overline{K})} \vartheta'(S^*(u)) \frac{\phi(s)}{\phi(S^*(u))} dS^*(u) = \int_{\vartheta(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^*(u))} du.$

A.11 The dead-weight loss $\tilde{L}(s, K)$

When $s \leq S^*_{sym}(K)$ the dead-weight loss equals

$$\begin{aligned}
&\tilde{L}(s,K) \\
&= V^P_{sym}(s,K) + V^A_{sym}(s,K) - \tilde{V}^P(s,K) - V^A(s,K) \\
&= \tfrac{\phi(s)}{\phi(S^*_{sym}(K))} \left(S^*_{sym}(K) - K\right) + 0 \\
&\quad - \left[\tfrac{\phi(s)}{\phi(S^*_K)} \left(S^*(K) - K\right) - \int_{\underline{K}}^{\overline{K}} \tfrac{\phi(s)}{\phi(S^*(u))} du\right] - \int_{\underline{K}}^{\overline{K}} \tfrac{\phi(s)}{\phi(S^*(u))} du \\
&= \tfrac{\phi(s)}{\phi(S^*_{sym}(K))} \left(S^*_{sym}(K) - K\right) - \tfrac{\phi(s)}{\phi(S^*_K)} \left(S^*(K) - K\right)
\end{aligned}$$

In the interval $S^*_{sym}(K) < s \leq S^*(K)$, we obtain

$$\begin{aligned}
&\tilde{L}(s,K) \\
&= V^P_{sym}(s,K) + V^A_{sym}(s,K) - \tilde{V}^P(s,K) - V^A(s,K) \\
&= s - K - \left[\tfrac{\phi(s)}{\phi(S^*_K)} \left(S^*(K) - K\right) - \int_{\underline{K}}^{\overline{K}} \tfrac{\phi(s)}{\phi(S^*(u))} du\right] - \int_{\underline{K}}^{\overline{K}} \tfrac{\phi(s)}{\phi(S^*(u))} du \\
&= s - K - \tfrac{\phi(s)}{\phi(S^*_K)} \left(S^*(K) - K\right).
\end{aligned}$$

If $S^*(K) < s \leq S^*(\overline{K})$ the dead-weight loss is

$$\begin{aligned}
&\tilde{L}(s,K) \\
&= V^P_{sym}(s,K) + V^A_{sym}(s,K) - \tilde{V}^P(s,K) - V^A(s,K) \\
&= s - K + 0 - \left[s - \vartheta(s) - - \int_{\vartheta s}^{\overline{K}} \tfrac{\phi(s)}{\phi(S^*(u))} du\right] - \left[\vartheta(s) + \int_{\vartheta s}^{\overline{K}} \tfrac{\phi(s)}{\phi(S^*(u))} du - K\right] \\
&= 0.
\end{aligned}$$

And finally, when $s > S^*(K)$ we find that

$$\begin{aligned}\tilde{L}(s,K) &= V^P_{sym}(s,K) + V^A_{sym}(s,K) - \tilde{V}^P(s,K) - V^A(s,K)\\ &= s - K + 0 - (s - \overline{K}) - (\overline{K} - K)\\ &= 0.\end{aligned}$$

Collecting these values lead to the dead-weight loss in equation (2.38).

A.12 The value of the contract compared to the value of selling the real option ex ante

If the principal sells the investment project to the agent, instead of entering into a contractual relationship, the agent's gross value of the project after sale equals $V^P_{sym}(s,K)$, as the agent has full information. This means that the agent accepts to buy the investment project if the principal's price is lower than, or equal to, $V^P_{sym}(s,K)$, given that entering into a contractual relationship is not an alternative. The principal's price is denoted γ.

The principal's outcome from a sale is given by

$$\gamma \mathrm{I}_{\{V^P_{sym}(s,K)\geq\gamma\}} + 0\mathrm{I}_{\{V^P_{sym}(s,K)<\gamma\}},$$

yielding the principal's value of sale, i.e., the principal obtains the value γ if the agent accepts the price, whereas the outcome equals zero if the agent does not accept the principal's price. Thus, the principal's value of selling the real option ex ante, is given by

$$\begin{aligned}V^P_{sale}(s) &= \gamma \mathrm{Prob}(V^P_{sym}(s,K)\mathrm{I}_{\{s\leq S^*_{sym}(K)\}} \geq \gamma)\mathrm{Prob}(s \leq S^*_{sym}(K))\\ &\quad+\gamma \mathrm{Prob}(V^P_{sym}(s,K)\mathrm{I}_{\{s> S^*_{sym}(K)\}} \geq \gamma)\mathrm{Prob}(s > S^*_{sym}(K)).\end{aligned} \tag{A.9}$$

If sale is to be profitable to the principal, we need $V^P_{sale}(s) \geq V^P(s)$.

In the case where $s > S^*(\overline{K})$, we know that $\mathrm{Prob}(s > S^*_{sym}(K)) = 1$, and $\mathrm{Prob}(s \leq S^*_{sym}(K)) = 0$. Thus, the principal's value of sale in the interval $s > S^*(\overline{K})$ is given by

$$V^P_{sale}(s)\mathrm{I}_{\{s>S^*(\overline{K})\}} = \gamma \mathrm{Prob}(s - K \geq \gamma).$$

The principal's value of contracting when $s > S^*(\overline{K})$ is (using equation (2.43)) given by

$$V^P(s)\mathrm{I}_{\{s>S^*(\overline{K})\}} = s - \overline{K}.$$

This means that if sale is to be profitable, we need $V^P_{sale}(s)\mathrm{I}_{\{s>S^*(\overline{K})\}} = \gamma\mathrm{Prob}(s - K \geq \gamma) \geq V^P(s)\mathrm{I}_{\{s>S^*(\overline{K})\}}$, which leads to the inequality

$$\gamma\mathrm{Prob}(s - K \geq \gamma) \geq s - \overline{K}.$$

Note that if we let γ be equals to $s-\overline{K}$, we obtain $V^P_{sale}(s)\mathrm{I}_{\{s>S^*(\overline{K})\}} = V^P(s)\mathrm{I}_{\{s>S^*(\overline{K})\}}$.

The principal's value of sale when $s > S^*(\overline{K})$ is expressed as

$$\begin{aligned} V^P_{sale}(s)\mathrm{I}_{\{s>S^*(\overline{K})\}} &= \gamma\mathrm{Prob}(s - K \geq \gamma) \\ &= \gamma\mathrm{Prob}(K \leq s - \gamma) \\ &= \gamma F(s - \gamma) \\ &= \gamma\tfrac{s-\gamma-\underline{K}}{\overline{K}-\underline{K}}, \end{aligned}$$

where the last equality applies under the assumption that the investment cost is uniformly distributed.

To optimize γ, we differentiate $V^P_{sale}(s, \gamma)$, leading to

$$\frac{\partial V^P_{sale}(s;\gamma)}{\partial\gamma}\mathrm{I}_{\{s>S^*(\overline{K})\}} = \frac{s - \underline{K} - 2\gamma}{\overline{K} - \underline{K}},$$

for $\gamma \in [s - \overline{K}, s - \underline{K}]$. For all possible values of γ, the first-order condition is negative. Thus we conclude that the optimal sales price γ equals $s - \overline{K}$ when $s > S^*(\overline{K})$ and K is uniformly distributed, which means that $V^P_{sale}(s) = V^P(s)$.

In the case where $s \leq S^*(\overline{K})$, we find by evaluation of equation (A.9), and comparison of (A.9) and the principal's value function under asymmetric information shows that selling the option ex ante is never optimal.

A.13 Properties of the value functions

The first- and second-order conditions of $X(s)$ in equation (2.41):

$$\frac{\partial X(s)}{\partial s} = \begin{cases} \frac{1}{2}\left[1 - \frac{\beta}{s}\left(\frac{s}{S^*(\overline{K})}\right)^{\beta} \frac{2\overline{K}-K}{\beta-1}\right] > 0 & \text{if } s \leq S^*(\overline{K}) \\ 0 & \text{if } s > S^*(\overline{K}) \end{cases}$$

$$\frac{\partial^2 X(s)}{\partial s^2} = \begin{cases} -\frac{1}{2}\frac{\beta(\beta-1)}{s^2}\left(\frac{s}{S^*(\overline{K})}\right)^{\beta} \frac{2\overline{K}-K}{\beta-1} < 0 & \text{if } s \leq S^*(\overline{K}) \\ 0 & \text{if } s > S^*(\overline{K}) \end{cases}$$

The first- and second-order conditions of $V^P_{sym}(s,K)$ in equation (2.20) with respect to s are given by

$$\frac{\partial V^P_{sym}(s,K)}{\partial s} = \begin{cases} \beta s^{\beta-1}(S^*_{sym}(K))^{-\beta} K \frac{1}{\beta-1} > 0 & \text{if } s \leq S^*_{sym}(K) \\ 1 & \text{if } s > S^*_{sym}(K), \end{cases}$$

and

$$\frac{\partial^2 V^P_{sym}(s,K)}{\partial s^2} = \begin{cases} \beta(\beta-1) s^{\beta-2}(S^*_{sym}(K))^{-\beta} K \frac{1}{\beta-1} > 0 & \text{if } s \leq S^*_{sym}(K) \\ 0 & \text{if } s > S^*_{sym}(K). \end{cases}$$

For $V^A(s,K)$ in equation (2.42) the first- and second-order conditions are formulated as

$$\frac{\partial V^A(s,K)}{\partial s} = \begin{cases} \beta s^{\beta-1} S^*(K)^{-\beta} \frac{1}{2}\left[\frac{2K-K}{\beta-1} - \left(\frac{S^*(K)}{S^*(\overline{K})}\right)^{\beta} \frac{2\overline{K}-K}{\beta-1}\right] > 0 & \text{if } s \leq S^*(K) \\ \frac{1}{2}\left[1 - \left(\frac{s}{S^*(\overline{K})}\right)^{\beta-1}\right] > 0 & \text{if } S^*(K) < s \leq S^*(\overline{K}) \\ 0 & \text{if } s > S^*(\overline{K}), \end{cases}$$

and

$$\frac{\partial^2 V^A(s,K)}{\partial s^2} = \begin{cases} \beta(\beta-1) s^{\beta-2} S^*(K)^{-\beta} \frac{1}{2}\left[\frac{2K-K}{\beta-1} - \left(\frac{S^*(K)}{S^*(\overline{K})}\right)^{\beta} \frac{2\overline{K}-K}{\beta-1}\right] > 0 & \text{if } s \leq S^*(K) \\ -\frac{1}{2}(\beta-1) s^{\beta-2} S^*(\overline{K})^{1-\beta} < 0 & \text{if } S^*(K) < s \leq S^*(\overline{K}) \\ 0 & \text{if } s > S^*(\overline{K}). \end{cases}$$

The first- and second-order conditions for $\tilde{V}^P(s,K)$ in equation (2.43) are equal to

$$\frac{\partial \tilde{V}^P(s,K)}{\partial s}$$

$$= \begin{cases} \beta s^{\beta-1} S^*(K)^{-\beta}\frac{1}{2}\left[S^*(K) - \underline{K} + \left(\frac{S^*(K)}{S^*(\overline{K})}\right)^{\beta}\frac{2\overline{K}-\underline{K}}{\beta-1}\right] > 0 & \text{if } s \leq S^*(K) \\ \frac{1}{2}\left[1 + \beta s^{\beta-1} S^*(\overline{K})^{-\beta}\frac{2\overline{K}-\underline{K}}{\beta-1}\right] > 0 & \text{if } S^*(K) < s \leq S^*(\overline{K}) \\ 1 & \text{if } s > S^*(\overline{K}), \end{cases}$$

and

$$\frac{\partial^2 \tilde{V}^P(s,K)}{\partial s^2}$$

$$= \begin{cases} \beta(\beta-1) s^{\beta-2} S^*(K)^{-\beta}\frac{1}{2}\left[S^*(K) - \underline{K} + \left(\frac{S^*(K)}{S^*(\overline{K})}\right)^{\beta}\frac{2\overline{K}-\underline{K}}{\beta-1}\right] > 0 & \text{if } s \leq S^*(K) \\ \frac{1}{2}\beta(\beta-1) s^{\beta-2} S^*(\overline{K})^{-\beta}\frac{2\overline{K}-\underline{K}}{\beta-1} > 0 & \text{if } S^*(K) < s \leq S^*(\overline{K}) \\ 0 & \text{if } s > S^*(\overline{K}). \end{cases}$$

Appendix B

Appendix for chapter 3

B.1 The agent's value of private information, equation (3.13)

When $\hat{K} = k$ and $s \leq \psi(c,k,t)$ the agent's first-order condition in equation (3.11) is equivalently written in the form

$$\int_k^\infty \frac{dv^A(s,c,u,t)}{du} du = \int_k^\infty w_u^A(s,c,u,t) du,$$

where we have integrated the first-order condition on both sides of the equality. When $k \to \infty$, we obtain $v^A(\cdot) \to 0$. Thus, evaluating the integration on the left-hand side of the equality above, and changing the sign at both sides of the equality, leads to

$$v^A(s,c,k,t) = -\int_k^\infty w_u^A(s,c,u,t) du$$

for $s \leq \psi(c,k,t)$, as stated in equation (3.13).

When $s > \psi(c,k,t)$, integration of the first-order condition in (3.11) gives

$$\int_k^{\vartheta(s,c,t)} \frac{dv^A(s,c,u,t)}{du} du = \int_k^{\vartheta(s,c,t)} -e^{-rt} du.$$

Evaluation of the integrals and change of signs yield

$$v^A(s,c,k,t) - v^A(s,c,\vartheta(s,c,t),t) = e^{-rt}\left(\vartheta(s,c,t) - k\right).$$

Observe that $v^A(s,c,\vartheta(s,c,t),t) = -\int_{\vartheta(s,c,t)}^{\infty} w_u^A(s,c,u,t)du$. If we replace $v^A(s,c,\vartheta(s,c,t),t)$ by $-\int_{\vartheta(s,c,t)}^{\infty} w_u^A(s,c,u,t)du$ and rearrange the above equality, we find that

$$v^A(s,c,k,t) = e^{-rt}\left(\vartheta(s,c,t) - k\right) - \int_{\vartheta(s,c,t)}^{\infty} w_u^A(s,c,u,t)du,$$

equal to the expression in equation (3.13).

B.2 The principal's payoff value, equation (3.18)

If we replace $X(s,c,t)$ in (3.17) by the right-hand side of equation (3.14), we find that

$$g^P(s,c,t) = \int_0^{\infty} \left\{ e^{-rt}\left(s - \vartheta(s,c,t)\right) + \int_{\vartheta(s,c,t)}^{\infty} w_u^A(s,c,u,t)du \right\} f(k|c,t)dk.$$

When $k = \vartheta(s,c,t)$ we obtain

$$g^P(s,c,t) = \int_0^{\infty} \left\{ e^{-rt}(s-k) + \int_k^{\infty} w_u^A(s,c,u,t)du \right\} f(k|c,t)dk. \qquad \text{(B.1)}$$

Partial integration of $\int_0^{\infty}\int_k^{\infty} w_u^A(s,c,u,t)duf(k|c,t)dk$ leads to

$$\int_0^{\infty}\int_k^{\infty} w_u^A(s,c,u,t)duf(k|c,t)dk$$
$$= \left[\int_k^{\infty} w_u^A(s,c,u,t)\,duF(k|c,t)\right]_0^{\infty} - (-)\int_0^{\infty} w_k^A(s,c,k,t)F(k|c,t)dk.$$

The first term equals zero, which implies that the equality above is given by

$$\int_0^{\infty}\int_k^{\infty} w_u^A(s,c,u,t)duf(k|c,t)dk = \int_0^{\infty} w_k^A(s,c,k,t)F(k|c,t)dk.$$

Hence, we find that the right-hand side of equation (B.1) can be formulated as

$$g^P(s,c,t) = \int_0^{\infty} \left\{ e^{-rt}(s-k) + w_k^A(s,c,k,t)\frac{F(k|c,t)}{f(k|c,t)} \right\} f(k|c,t)dk,$$

as given in equation (3.18).

B.3 Reformulation of the principal's value, equation (3.27)

We want to show that $s - c\left(\theta + \frac{F(\theta,t)}{f(\theta,t)}\right) = s - k - \frac{F(k|c,t)}{f(k|c,t)}$, i.e., that

$$c\frac{F(\theta,t)}{f(\theta,t)} = \frac{F(k|c,t)}{f(k|c,t)}. \tag{B.2}$$

As the variables c, θ, and k are log-normally distributed, we know that $\ln(c) \sim \mathrm{N}(m_c, \gamma_c)$, $\ln(\theta) \sim \mathrm{N}(m_\theta, \gamma_\theta)$ and $\ln(k) \sim \mathrm{N}(m_k, \gamma_k)$, where m_i and γ_i, for $i = c, \theta, k$ denotes the expectation and the variance, respectively, of a normally distributed variable $\ln(i)$.

Recall that the investment cost k is defined as $k = c\theta$, and define $\ln(k|c) \sim \mathrm{N}(m_{k|c}, \gamma_{k|c})$. Thus, $\ln(k|c) = \ln(\theta) + \ln(c)$, where, from the principal's point of view, $\ln(\theta)$ is an uncertain variable, whereas $\ln(c)$ is observed. Then we obtain $m_{k|c} = m_\theta + \ln(c)$ and $\gamma_{k|c} = \gamma_\theta$.

The conditional density of k given c is defined by

$$f(k|c) = \frac{1}{k}\frac{1}{\sqrt{2\pi\gamma_{k|c}}}e^{-\frac{1}{2}\left(\ln(k|c)-m_{k|c}\right)^2\frac{1}{\gamma_{k|c}}}.$$

Observe that

$$\ln(k|c) - m_{k|c} = [\ln(\theta) + \ln(c)] - [m_\theta + \ln(c)] = \ln(\theta) - m_\theta,$$

and $\gamma_{k|c} = \gamma_\theta$. Substitution in the conditional density above leads to

$$\begin{aligned} f(k|c) &= \tfrac{1}{c\theta}\tfrac{1}{\sqrt{2\pi\gamma_\theta}}e^{-\frac{1}{2}(\ln(\theta)-m_\theta)^2\frac{1}{\gamma_\theta}} \\ &= \tfrac{1}{c}f(\theta). \end{aligned}$$

Furthermore,

$$F(k|c) = F\left(\frac{c\theta}{c}\right) = F(\theta).$$

Hence, $\frac{F(k|c,t)}{f(k|c,t)} = c\frac{F(\theta,t)}{f(\theta,t)}$.

B.4 Numerical implementation

The results in chapter 3 are implemented using an implicit finite difference method for the two state variables S_t and K_t. Recall that in the numerical examples we assume that $C_t = 1$ for all t, i.e., C_t is not a stochastic variable.

In order to find numerical solutions we solve the partial differential equation $L\tilde{v}^P(s,k,t) - r\tilde{v}^P(s,k,t) = 0$ by an implicit finite difference method. However, before we approximate the partial differential equation by discrete steps, we transform the partial differential equation $L\tilde{v}^P(s,k,t) - r\tilde{v}^P(s,k,t) = 0$, as well as the "payoff" function $\tilde{g}(s,k,t)$ in (3.22).

When $C_t = 1$, the investment cost K_t in (3.3) is driven by the stochastic process $dK_t = \alpha K_t dt + \sigma_\theta K_t dB_t^\theta$. The output value S_t is driven by the process in (3.4). Define the discounted processes $\bar{S}_t = e^{-rt}S_t$ and $\bar{K}_t = e^{-rt}K_t$. By Ito's lemma we obtain the dynamics

$$d\bar{S}_t = -\delta_S \bar{S}_t dt + \sigma_S \bar{S}_t dB^S$$

and

$$d\bar{K}_t = (\alpha - r)\bar{K}_t dt + \sigma_\theta \bar{K}_t dB_t^\theta.$$

Next, we express the "payoff" function as a function dependent on the discounted stochastic variables only, i.e., we define $h(\bar{s},\bar{k},t) \equiv \tilde{g}^P(s,k,t)$, equal to

$$h(\bar{s},\bar{k},t) = \bar{s} - \bar{k} - \frac{H(\bar{k},t)}{f(\bar{k},t)},$$

where

$$H(\bar{k},t) = \int_0^{\bar{k}e^{rt}} \frac{1}{u}\frac{1}{\sqrt{2\pi\gamma_k(t)}} e^{-\frac{1}{2}(\ln(u)-m_{\bar{k}}(t))^2 \frac{1}{\gamma_k(t)}},$$

$\gamma_k(t) = \mathrm{Var}\left[\ln(\bar{k})\right]$, and $m_{\bar{k}}(t) = E[\ln(\bar{k})]$. The function $H(\bar{k},t) = F(k,t)$ represents a log-normal distribution of k. The upper integral variable $e^{rt}\bar{k}$ equals k, and the function $F(k,t)$ is presented in appendix B.3. The distribution $H(\bar{k},t)$ is numerically found by an approximation of the standard normal distribution, presented in Abramowitz and Stegun (1965).

Define $X_1(t) = \ln(\bar{S}_t)$ and $X_2(t) = \ln(\bar{K}_t)$. The stochastic processes of $X_1(t)$ and $X_2(t)$ are, by Ito's lemma,

$$dX_1(t) = (-\delta_S - \frac{1}{2}\sigma_S^2)dt + \sigma_S dB_t^S$$

and

$$dX_2(t) = (\alpha - r - \frac{1}{2}\sigma_\theta^2)dt + \sigma_\theta dB_t^\theta.$$

Furthermore, define $(x_1, x_2) = (X_1(t), X_2(t))$ and $v(x_1, x_2, t) = e^{-rt}\tilde{v}^P(s, k, t)$. Thus, the partial differential operator $L\tilde{v}^P$ introduced in (3.26) is transformed to

$$\begin{aligned} Lv(x_1, x_2, t) \;\; = \;\; & \tfrac{\partial v}{\partial t}(x_1, x_2, t) + (-\delta_s - \tfrac{1}{2}\sigma_S^2)\tfrac{\partial v}{\partial x_1}(x_1, x_2, t) + \tfrac{1}{2}\sigma_S^2\tfrac{\partial^2 v}{\partial x_1^2}(x_1, x_2, t) \\ & +(\alpha - r - \tfrac{1}{2}\sigma_\theta^2)\tfrac{\partial v}{\partial x_2}(x_1, x_2, t) + \tfrac{1}{2}\sigma_\theta^2\tfrac{\partial^2 v}{\partial x_2^2}(x_1, x_2, t). \end{aligned}$$

We approximate the partial differential equation above based on an extension of the implicit finite difference method presented in Clewlow and Strickland (1998).

Denote $i = (0, ..., N)$ as a time step, and Δt as the discrete time interval, i.e., $t = i\Delta t$. Furthermore, $j = (-N_j, ..., N_j)$ and $k = (-N_k, ..., N_k)$ represent the respective levels of the values of x_1 and of x_2 relative to their initial values. The differences between two subsequent values of x_1 and x_2 are denoted Δx_1 and Δx_2, respectively. The approximations are given by

$$\frac{\partial v}{\partial t}(x_1, x_2, t) \approx \frac{v_{i+1,j,k} - v_{i,j,k}}{\Delta t},$$

$$\frac{\partial v}{\partial x_1}(x_1, x_2, t) \approx \frac{v_{i,j+1,k} - v_{i,j-1,k}}{2\Delta x_1},$$

$$\frac{\partial^2 v}{\partial x_1^2}(x_1, x_2, t) \approx \frac{v_{i,j+1,k} - 2v_{i,j,k} + v_{i,j-1,k}}{\Delta x_1^2},$$

$$\frac{\partial v}{\partial x_2}(x_1, x_2, t) \approx \frac{v_{i,j,k+1} - v_{i,j,k-1}}{2\Delta x_2},$$

and

$$\frac{\partial^2 v}{\partial x_2^2}(x_1, x_2, t) \approx \frac{v_{i,j,k+1} - 2v_{i,j,k} + v_{i,j,k-1}}{\Delta x_2^2}.$$

Replacing the derivatives of $v(x_1, x_2, t)$ by the finite differences above, the discrete approximation of the partial differential equation $Lv(x_1, x_2, t) - rv(x_1, x_2, t) = 0$ equals

$$v_{i+1,j,k} = p_{um}v_{i,j+1,k} + p_{mu}v_{i,j,k+1} + p_{mm}v_{i,j,k} + p_{dm}v_{i,j-1,k} + p_{md}v_{i,j,k-1}, \quad \text{(B.3)}$$

where

$$p_{um} = (\delta_S + \frac{1}{2}\sigma_S^2)\frac{1}{2}\Delta t\frac{1}{\Delta x_1} - \frac{1}{2}\sigma_S^2\Delta t\frac{1}{\Delta x_1^2},$$

$$p_{mu} = (-(\alpha - r) + \frac{1}{2}\sigma_\theta^2)\frac{1}{2}\Delta t\frac{1}{\Delta x_2} - \frac{1}{2}\sigma_\theta^2\Delta t\frac{1}{\Delta x_2^2},$$

$$p_{mm} = 1 + \sigma_S^2\Delta t\frac{1}{\Delta x_1} + \sigma_\theta^2\Delta t\frac{1}{\Delta x_2^2},$$

$$p_{dm} = (-\delta_S - \frac{1}{2}\sigma_S^2)\frac{1}{2}\Delta t\frac{1}{\Delta x_1} - \frac{1}{2}\sigma_S^2\Delta t\frac{1}{\Delta x_1^2},$$

and

$$p_{md} = (\alpha - r - \frac{1}{2}\sigma_\theta^2)\frac{1}{2}\Delta t\frac{1}{\Delta x_2} - \frac{1}{2}\sigma_\theta^2\Delta t\frac{1}{\Delta x_2^2}.$$

Define $K_k = k\exp(k\Delta x_2)$, where k is assumed to be the initial cost level, and $S_j = s\exp(j\Delta x_1)$, where s is the initial output value. At each step i, we solve the equations in (B.3) for j from $-N_j + 1$ to $N_j - 1$, and k from $-N_k + 1$ to $N_k - 1$, together with the boundary conditions,

$$v_{i,N_j,k} - v_{i,N_{j-1},k} = S_{N_j} - S_{N_{j-1}}, \quad \text{(B.4)}$$

$$v_{i,-N_{j+1},k} - v_{i,-N_j,k} = 0, \quad \text{(B.5)}$$

$$v_{i,j,N_k} - v_{i,j,N_{k-1}} = 0, \quad \text{(B.6)}$$

$$v_{i,j,-N_{k+1}} - v_{i,j,-N_k} = 0. \quad \text{(B.7)}$$

The boundary condition in equation (B.4) is explained as follows. When the output value S_j is at its maximum value S_{N_j}, a change in the value of one unit equals the difference $S_{N_j} - S_{N_j-1}$ as there is a high probability that the option is exercised. However, this boundary overvalues the change in the option value for high levels of cost K_j. For high levels of K_j the payoff $S_{N_j} - K_j - \frac{H(K_j)}{f(K_j)}$ becomes negative.

When the investment cost equals the maximum level K_{N_k} as in (B.5), or the output level is at its minimum level S_{-Nj} as in (B.6), a difference in the states of one unit, has an effect on the option value close to zero, as the option value is very low for these states.

The boundary condition in equation (B.7) represents the effect on the option value of a small change in the investment cost when the cost level equals K_{-N_k}. A more correct right-hand side of this boundary condition equals $K_{-N_k} + \frac{H(K_{-N_k},i)}{f(K_{-N_k},i)} - K_{-N_{k+1}} - \frac{H(K_{-N_{k+1}},i)}{f(K_{-N_{k+1}},i)}$, instead of zero. However, in order to simplify the calculations, we approximate the right-hand side by setting it equal to zero. This simplification is justified for a small difference in K_{-N_k} and K_{-N_k+1}, as will be the case when $-N_k$ is large and Δx_2 is small.

The solution procedure is as follows. The problem is solved backwards, which means that first we solve for the optimal investment strategy at the time horizon, i.e.,

$$v_{N,j,k} = \max \left[0, S_j - K_k - \frac{H(K_k, N)}{f(K_k, N)} \right].$$

For each $i < N$, we solve the set of equations, as described by equation (B.3) for j from $-N_j$ to N_j, and k from $-N_k$ to N_k, and the boundary conditions in equations (B.4)-(B.7). This set of equations has a band diagonal structure[1], and is solved using routines described in Press, Teukolsky, Vetterling, and Flannery (1992), chapter 2.4. Furthermore, at each time step i, and for each j and k we apply the early exercise condition,

$$v^*_{i,j,k} = \max \left[v_{i,j,k}, S_j - K_k - \frac{H(K_k, i)}{f(K_k, i)} \right].$$

The parameter values used in the numerical examples of chapter 3, in addition to those specified in section 3.7, are $N = N_j = N_k = 30$ and $\Delta x_1 = \Delta x_2 = 0.04$. With these numbers the max increase in the output value over the five year horizon is 232 per cent, and the max decrease is 70 per cent. The probabilities that the values of $S_{\bar{T}} = S_5$ and $K_{\bar{T}} = K_5$ exceed these limits are close to zero, given our parameter values.

[1] A band diagonal system of linear equations has non-zero elements only along a few diagonal lines, adjacent (above and below) to the main diagonal.

Appendix C

Appendix for chapter 4

C.1 Derivation of agent i's value function in equation (4.9)

The value function in (4.9) is found as follows. Equation (4.4) equals

$$v^i(s, K^i; \hat{K}^i) = E\left[e^{-r\tau^i_{\hat{K}}}\left(X^i(S_{\tau^i_{\hat{K}}}, \hat{K}) - y^i(\hat{K})K^i\right)^+ \middle| \mathcal{F}_0^{S,K^i}\right].$$

By conditional expectations, the value function is formulated as,

$$v^i(s, K^i) = E\left[E\left[e^{-r\tau^i_{\hat{K}}}\left(X^i(S_{\tau^i_{\hat{K}}}, \hat{K}) - y^i(\hat{K})K^i\right)^+ \middle| \mathcal{F}_0^{S,K}\right] \middle| \mathcal{F}_0^{S,K^i}\right].$$

Time-homogeneity implies that the value of the discounting factor can be written independently of the value of the options' payoff, i.e.,

$$v^i(s, K^i) = E\left[E\left[e^{-r\tau^i_{\hat{K}}} \middle| \mathcal{F}_0^{S,K}\right] E\left[\left(X^i(S_{\tau^i_{\hat{K}}}, \hat{K}) - y^i(\hat{K})K^i\right)^+ \middle| \mathcal{F}_0^{S,K}\right] \middle| \mathcal{F}_0^{S,K^i}\right].$$

From equation (4.8) we know that

$$E\left[e^{-r\tau^i_{\hat{K}}} | \mathcal{F}_0^{S,K}\right] = \frac{\phi(s)}{\phi(S^i(\hat{K}))} \mathrm{I}_{\{s \leq S^i(\hat{K})\}} + 1\mathrm{I}_{\{s > S^i(\hat{K})\}}.$$

We exploit the relationship above, and replace $S_{\tau^i_{\hat{K}}}$ by the critical price $S^i(\hat{K})$, leading to

$$\begin{aligned} v^i(s,K^i) &= E\Big[E\Big[\tfrac{\phi(s)}{\phi(S^i(\hat{K}))}\Big(X^i(S^i(\hat{K}),\hat{K}) - y^i(\hat{K})K^i\Big)\mathrm{I}_{\{s\le S^i(\hat{K})\}} \\ &\quad + \Big(X^i(s,\hat{K}) - y^i(\hat{K})K^i\Big)\mathrm{I}_{\{s>S^i(\hat{K})\}}\Big|\mathcal{F}_0^{S,K}\Big]\Big|\mathcal{F}_0^{S,K^i}\Big]. \end{aligned}$$

By conditional expectations we obtain,

$$\begin{aligned} v^i(s,K^i) &= E\Big[\tfrac{\phi(s)}{\phi(S^i(\hat{K}))}\Big(X^i(S^i(\hat{K}),\hat{K}) - y^i(\hat{K})K^i\Big)\mathrm{I}_{\{s\le S^i(\hat{K})\}} \\ &\quad + \Big(X^i(s,\hat{K}) - y^i(\hat{K})K^i\Big)\mathrm{I}_{\{s>S^i(\hat{K})\}}\Big|\mathcal{F}_0^{S,K^i}\Big], \end{aligned}$$

identical to equation (4.9) in the text.

C.2 Deriving the auctioneer's value function in equation (4.10)

The auctioneer's value function is in (4.5) given by

$$v^P(s;\hat{K}) = E\left[\sum_{i=1}^n e^{-r\tau^i_{\hat{K}}}\Big(y^i(\hat{K})S_{\tau^i_{\hat{K}}} - X^i(S_{\tau^i_{\hat{K}}},\hat{K})\Big)^+\Bigg|\mathcal{F}_0^S\right].$$

Conditional expectations lead to

$$v^P(s;\hat{K}) = E\left[E\left[\sum_{i=1}^n e^{-r\tau^i_{\hat{K}}}\Big(y^i(\hat{K})S(\tau^i_{\hat{K}}) - X^i(S(\tau^i_{\hat{K}}),\hat{K})\Big)^+\Bigg|\mathcal{F}_0^{S,K}\right]\Bigg|\mathcal{F}_0^S\right].$$

Because of time-homogeneity we are allowed to separate the expression of the discounting term from the option's payoff as follows,

$$\begin{aligned} &v^P(s;\hat{K}) \\ &= E\Big[E\Big[\textstyle\sum_{i=1}^n e^{-r\tau^i_{\hat{K}}}\Big|\mathcal{F}_0^{S,K}\Big]E\Big[y^i(\hat{K})S_{\tau^i_{\hat{K}}} - X^i(S_{\tau^i_{\hat{K}}},\hat{K})\Big|\mathcal{F}_0^{S,K}\Big]\Big|\mathcal{F}_0^S\Big]. \end{aligned}$$

Next, we insert the value of the discounting factor as expressed in equation (4.8), given an (arbitrary) value of the investment trigger, $S^i(\hat{K})$,

$$\begin{aligned} v^P(s;\hat{K}) &= E\Big[E\Big[\textstyle\sum_{i=1}^n\Big\{\tfrac{\phi(s)}{\phi(S^i(\hat{K}))}\Big(y^i(\hat{K})S^i(\hat{K}) - X^i(S^i(\hat{K}),\hat{K})\Big)\mathrm{I}_{\{s\le S^i(\hat{K})\}} \\ &\quad + \Big(y^i(\hat{K})s - X^i(s,\hat{K})\Big)\mathrm{I}_{\{s>S^i(\hat{K})\}}\Big\}\Big|\mathcal{F}_0^{S,K}\Big]\Big|\mathcal{F}_0^S\Big]. \end{aligned}$$

This expression is equivalent to

$$
\begin{aligned}
v^P(s;\hat{K}) &= E\Big[\textstyle\sum_{i=1}^{n}\Big\{\frac{\phi(s)}{\phi(S^i(\hat{K}))}\Big(y^i(\hat{K})S^i(\hat{K})-X^i(S^i(\hat{K}),\hat{K})\Big)\,\mathrm{I}_{\{s\le S^i(\hat{K})\}} \\
&\qquad + \Big(y^i(\hat{K})s-X^i(s,\hat{K})\Big)\,\mathrm{I}_{\{s>S^i(\hat{K})\}}\Big\}\Big|\,\mathcal{F}_0^S\Big],
\end{aligned}
$$

which is identical to equation (4.10).

C.3 Properties of the optimal investment strategy

We now prove that in optimum we have $S^{i*}(K) = S^{i*}(K^i)$.

Suppose that agent i is the winner of the contract, i.e., $y^i(K) = 1$. Define agent i's expected critical price as $S^i(K^i) = E\left[S^i(K)\mathrm{I}_{\{y^i(K)=1\}}|\mathcal{F}_0^{S,K^i}\right]$. For $s \le S^i(K)$, the principal's value if agent i wins the contract can be written as (from (4.15))

$$
E\left[\frac{\phi(s)}{\phi(S^i(K))}\left(S^i(K)-K^i\right)\mathrm{I}_{\{s\le S^i(K)\}} + (s-K^i)\mathrm{I}_{\{s>S^i(K)\}} - v^i(s,K^i)|\mathcal{F}_0^S\right].
$$

Observe that, by Jensen's inequality,

$$
E\left[\phi(S^i(K))\mathrm{I}_{\{y^i(K)=1\}}|\mathcal{F}_0^S\right] \ge \phi(S^i(K^i)),
$$

under the assumption that $\phi(\cdot)$ is a convex function and

$$
\phi(S^i(K^i)) = \phi(E[S^i(K)\mathrm{I}_{\{y^i(K)=1\}}|\mathcal{F}_0^S]).
$$

This implies that

$$
\begin{aligned}
&\frac{\phi(s)}{\phi(S^i(K^i))}\left(S^i(K^i)-K^i\right) - v^i(s,K^i) \\
&\ge E\left[\left(\tfrac{\phi(s)}{\phi(S^i(K))}\left(S^i(K)-K^i\right) - v^i(s,K^i)\right)\mathrm{I}_{\{y^i(K)=1\}}|\mathcal{F}_0^S\right].
\end{aligned}
$$

Thus, the auctioneer's value function can be replaced by a larger quantity, substituting $S^i(K)$ by $S^i(K^i)$. From this result we see that a stochastic mechanism as given by $S^i(K)$ is not optimal.

C.4 The auctioneer's simplified optimization problem

Define $\hat{v}^P$ as the auctioneer's arbitrary value function when $S^i(K^i) = S^i(K)$. Replace the investment triggers $S^i(K)$ by $S^i(K^i)$, in the principal's value function specified by equation (4.15), leading to

$$\begin{aligned}\hat{v}^P(s,K) &= E\Big[\textstyle\sum_{i=1}^n \Big\{\frac{\phi(s)}{\phi(S^i(K^i))} y^i(K)\,(S^i(K^i)-K^i)\,\mathrm{I}_{\{s\leq S^i(K^i)\}} \\ &\quad + y^i(K)\,(s-K^i)\,\mathrm{I}_{\{s>S^i(K^i)\}} - v^i(s_0,K^i)\Big\}\Big|\,\mathcal{F}_0^S\Big].\end{aligned}$$

Furthermore, conditional expectations yield

$$\begin{aligned}\hat{v}^P(s,K) &= E\Big[\textstyle\sum_{i=1}^n E\Big[\Big\{\frac{\phi(s)}{\phi(S^i(K^i))} y^i(K)\,(S^i(K^i)-K^i)\,\mathrm{I}_{\{s\leq S^i(K^i)\}} \\ &\quad + y^i(K)\,(s-K^i)\,\mathrm{I}_{\{s>S^i(K^i)\}} - v^i(s_0,K^i)\Big\}\Big|\,\mathcal{F}_0^{S,K^i}\Big]\Big|\,\mathcal{F}_0^S\Big],\end{aligned}$$

which, by exploiting the definition $Y^i(K^i) = E\left[y^i(K)|\mathcal{F}_0^{S,K^i}\right]$, is written as

$$\begin{aligned}\hat{v}^P(s,K) &= E\Big[\textstyle\sum_{i=1}^n E\Big[\Big\{\frac{\phi(s)}{\phi(S^i(K^i))} Y^i(K^i)\,(S^i(K^i)-K^i)\,\mathrm{I}_{\{s\leq S^i(K^i)\}} \\ &\quad + Y^i(K^i)\,(s-K^i)\,\mathrm{I}_{\{s>S^i(K^i)\}} - v^i(s_0,K^i)\Big\}\Big|\,\mathcal{F}_0^{S,K^i}\Big]\Big|\,\mathcal{F}_0^S\Big].\end{aligned}$$

Each agent's "contribution" to the auctioneer's value is an expression that depends only on each agent's report K^i (i.e., the direct mechanism is not stochastic), which means that the outer expectation operator is superfluous, resulting in

$$\begin{aligned}\hat{v}^P(s,K^i) &= \textstyle\sum_{i=1}^n E\Big[\frac{\phi(s)}{\phi(S^i(K^i))} Y^i(K^i)\,(S^i(K^i)-K^i)\,\mathrm{I}_{\{s\leq S^i(K^i)\}} \\ &\quad + Y^i(K^i)\,(s-K^i)\,\mathrm{I}_{\{s>S^i(K^i)\}} - v^i(s,K^i)|\mathcal{F}_0^{S,K^i}\Big].\end{aligned}$$

The above expression can equivalently be written

$$\begin{aligned}\hat{v}^P(s,K^i) &= \textstyle\sum_{i=1}^n \Big\{\int_{\underline{K}}^{\overline{K}} \Big[\frac{\phi(s)}{\phi(S^i(K^i))} Y^i(K^i)\,(S^i(K^i)-K^i)\,\mathrm{I}_{\{s\leq S^i(K^i)\}} \\ &\quad + Y^i(K^i)\,(s-K^i)\,\mathrm{I}_{\{s>S^i(K^i)\}} - v^i(s,K^i)\Big]\,f(K^i)dK^i\Big\}.\end{aligned}$$

Substituting the expression in (4.14) into the value above, and replacing $S^i(K)$ by $S^i(K^i)$, leads to

$$\begin{aligned}
&\hat{v}^P(s,K) \\
&= \sum_{i=1}^n \left\{ \int_{\underline{K}}^{\overline{K}} \left[\left(\frac{\phi(s)}{\phi(S^i(K^i))} Y^i(K^i)\left(S^i(K^i)-K^i\right) - \int_{K^i}^{\overline{K}} \frac{\phi(s)}{\phi(S^i(u))} Y^i(u)du \right) \mathrm{I}_{\{s \leq S^i(K^i)\}} \right.\right. \\
&\quad \left.\left. + \left(Y^i(K^i)(s-K^i) - \int_{K^i}^{\vartheta^i(s)} Y^i(u)du + \int_{\vartheta(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^i(u))} Y^i(u)du \right) \mathrm{I}_{\{s > S^i(K^i)\}} \right] f(K^i)dK^i \right\} \\
&= \sum_{i=1}^n \left\{ \int_{\underline{K}}^{\overline{K}} \left[\frac{\phi(s)}{\phi(S^i(K^i))} Y^i(K^i) \left(S^i(K^i) - K^i - \frac{F(K^i)}{f(K^i)} \right) \mathrm{I}_{\{s \leq S^i(K^i)\}} \right.\right. \\
&\quad \left.\left. + \left(Y^i(K^i) \left(s - K^i - \frac{F(K^i)}{f(K^i)} \right) \right) \mathrm{I}_{\{s > S^i(K^i)\}} \right] f(K^i)dK^i \right\},
\end{aligned}$$

where the last equality follows from partial integration of

$$\int_{\underline{K}}^{\overline{K}} \int_{K^i}^{\overline{K}} \frac{\phi(s)}{\phi(S^i(u))} Y^i(u) \mathrm{I}_{\{s \leq S^i(K^i)\}} du f(K^i) dK^i$$

and

$$\int_{\underline{K}}^{\overline{K}} \left[\int_{K^i}^{\overline{K}} Y^i(u) + \int_{\vartheta^i(s)}^{\overline{K}} \frac{\phi^i(s)}{\phi^i(S^i(u))} Y^i(u)du \right] \mathrm{I}_{\{s > S^i(K^i)\}} du f(K^i) dK^i,$$

respectively.

C.5 Equality between the two approaches of finding the optimal compensation function

The probability $Y^{i*}(K^i)$ given the principal's optimal choice of the winner of the contract, equals $[1-F(K^i)]^{n-1}$, which is understood as the probability of having the lowest cost in a sample of n. Substitution of $Y^{i*}(K^i) = [1-F(K^i)]^{n-1}$ in (4.20), leads to

$$\begin{aligned}
&X^{i*}(s,K^i) \\
&= K^i\left[1-F(K^i)\right]^{n-1} + \int_{K^i}^{\vartheta^{i*}(s)} \left[1-F(u)\right]^{n-1} du + \int_{\vartheta^{i*}(s)}^{\overline{K}} \frac{\phi^i(s)}{\phi^i(S^{i*}(u))} \left[1-F(u)\right]^{n-1} du
\end{aligned} \tag{C.1}$$

if $s > S^i(K^{i*})$.

We will now show that $X^{i*}(s, K^i) = E\left[\tilde{X}^i(s,K)|\mathcal{F}_0^{S,K^i}\right]$. We treat K^j as the first-order statistic in a sample of size $n-1$, which means that we assume that K_j is the lowest cost parameter in a sample of $n-1$ parameters. We find $E\left[\tilde{X}^i(s,K)|\mathcal{F}_0^{S,K^i}\right]$ as follows

$$\begin{aligned} E\left[\tilde{X}^i(s,K)|\mathcal{F}_0^{S,K^i}\right] &= \int_{K^i}^{\vartheta^{i*}(s)} K^j d\left(-[1-F(K^j)]^{n-1}\right) \\ &\quad + \int_{\vartheta^{i*}(s)}^{\overline{K}} \left(\vartheta^{i*}(s) + \int_{\vartheta^*(s)}^{K^j} \frac{\phi(s)}{\phi(S^{i*}(u))} du\right) d\left(-[1-F(K^j)]^{n-1}\right) \end{aligned} \tag{C.2}$$

when $s > S^{i*}(K^i)$. Partial integration of equation (C.2) leads to

$$\begin{aligned} &E\left[\tilde{X}^i(s,K)|\mathcal{F}_0^{S,K^i}\right] \\ &= K^i[1-F(K^i)]^{n-1} + \int_{K^i}^{\vartheta^{i*}(s)}[1-F(u)]^{n-1}du + \int_{\vartheta^{i*}(s)}^{\overline{K}} \frac{\phi(s)}{\phi(S^{i*}(u))}[1-F(u)]^{n-1}du \end{aligned} \tag{C.3}$$

if $s > S^{i*}(K^i)$. We see that equation (C.3) equals equation (C.1), and thus equals equation (4.20).

C.6 Agent *i*'s contribution to the auctioneer's value, stochastic private information

By equation (4.27), and given truth telling, we find that

$$E\left[e^{-rt}X^i(S_t,C_t,\theta_t,t)|\mathcal{F}_t^{S,C,\theta^i}\right] = E\left[e^{-rt}y^i(\theta_t)h(C_t,\theta_t^i)|\mathcal{F}_t^{S,C,\theta^i}\right] + g^i(S_t,C_t,\theta^i,t).$$

Substitute $X^i(S_t,C_t,\theta_t,t)$ in equation (4.29) by the right-hand side expression above. This leads to

$$\begin{aligned} g_i^P(S_t,C_t,t) &= E\left[E\left[e^{-rt}\left(y^i(\theta_t)S_t - X^i(S_t,C_t,\theta_t,t)\right)|\mathcal{F}_t^{S,C,\theta^i}\right]|\mathcal{F}_t^{S,C}\right] \\ &= E\left[E\left[y^i(\theta_t)\left(S_t - h(C_t,\theta_t^i)\right) - g^i(S_t,C_t,\theta_t^i,t)|\mathcal{F}_t^{S,C,\theta^i}\right]|\mathcal{F}_t^{S,C}\right] \\ &= E\left[e^{-rt}y^i(\theta_t)\left(S_t - h(C_t,\theta_t^i)\right) - g^i(S_t,C_t,\theta_t^i,t)|\mathcal{F}_t^{S,C}\right], \end{aligned}$$

when truth telling is ensured. Since y^i is linearly dependent on g_i^P, we simplify the equation to being dependent on the uncertainty with respect to θ_t^i, only, i.e.,

$$g_i^P(S_t, C_t, t) = E\left[e^{-rt}Y^i(\theta_t^i)\left(S_t - h(C_t, \theta_t^i)\right) - g^i(S_t, C_t, \theta_t^i, t)|\mathcal{F}_t^{S,C}\right],$$

as given in equation (4.36).

C.7 Agent i's incentive compatible value function

Suppose the investment is made at time t. Then the truth telling condition in (4.35) can be written

$$\frac{dv^i(S_t, C_t, \theta_t^i, t)}{d\theta_t^i}\mathrm{I}_{\{S_t=\psi^i(C_t,t,\theta_t)\}} = \frac{\partial w^i(S_t, C_t, \theta_t^i, t)}{\partial \theta_t^i}\mathrm{I}_{\{S_t=\psi^i(C_t,t,\theta_t)\}}.$$

Integration on both sides of the equality yields

$$\int_{\theta_t^i}^{\overline{\theta}} \frac{dv^i(S_t, C_t, u, t)}{du} du\mathrm{I}_{\{S_t=\psi^i(C_t,t,\theta_t)\}} = \int_{\theta_t^i}^{\overline{\theta}} \frac{\partial w^i(S_t, C_t, u, t)}{\partial u} du\mathrm{I}_{\{S_t=\psi^i(C_t,t,\theta_t)\}}.$$

Evaluation and reformulation leads to

$$v^i(S_t, C_t, \theta_t^i, t)\mathrm{I}_{\{S_t=\psi^i(C_t,t,\theta_t)\}} = -\int_{\theta_t^i}^{\overline{\theta}} \frac{\partial w^i(S_t, C_t, u, t)}{\partial u} du\mathrm{I}_{\{S_t=\psi^i(C_t,t,\theta_t)\}}.$$

As $g^i(S_t, C_t, \theta_t^i, t)\mathrm{I}_{\{S_t=\psi^i(C_t,t,\theta_t)\}} = v^i(S_t, C_t, \theta_t^i, t)\mathrm{I}_{\{S_t=\psi^i(C_t,t,\theta_t)\}}$, we obtain the result in (4.37).

C.8 Derivation of equation (4.38).

By inserting (4.37) into (4.36), we find that

$$\begin{aligned} g_i^P(S_t, C_t, t) &= E\left[Y^i(\theta_t^i)\left(S_t - h(C_t, \theta_t^i)\right) - \textstyle\int_{\theta_t^i}^{\overline{\theta}} w_u^i(S_t, C_t, u, t)du\right| \mathcal{F}_t^{S,C}\Big] \\ &= \textstyle\int_{\underline{\theta}}^{\overline{\theta}} \left\{Y^i(\theta_t^i)\left(S_t - h(C_t, \theta_t^i)\right) - \int_{\theta_t^i}^{\overline{\theta}} w_u^i(S_t, C_t, u, t)du\right\} f(\theta_t|t)d\theta_t^i. \end{aligned}$$

Partial integration of the term $\int_{\underline{\theta}}^{\overline{\theta}} \int_{\theta_t^i}^{\overline{\theta}} w_u^i(S_t, C_t, u, t) du f(\theta_t^i|t) d\theta_t^i$ leads to

$$g_i^P(S_t, C_t, t) = \int_{\underline{\theta}}^{\overline{\theta}} Y^i(\theta_t^i) \left(S_t - h(C_t, \theta_t^i) - w_{\theta_t^i}^i(S_t, C_t, \theta_t^i, t) \frac{F(\theta_t^i|t)}{f(\theta_t^i|t)} \right) f(\theta_t|t) d\theta_t^i. \tag{C.4}$$

If agent i's compensation function is to be truth telling, among other requirements, smooth pasting must be satisfied. This means that at the trigger $S_t = \psi(C_t, \theta_t, t)$, we need to have

$$w_{\theta_t^i}^i(\phi(C_t, \theta_t, t), C_t, \theta_t^i, t) = -h_{\theta_t^i}(C_t, \theta_t^i) Y^i(\theta_t^i),$$

where $w_{\theta_t^i}^i$ denotes the first-order derivative of $w^i(\cdot)$ with respect to θ_t^i. Replace $w_{\theta_t^i}^i(S_t, C_t, \theta_t^i, t)$ by $-h_{\theta_t^i}(C_t, \theta_t^i) Y^i(\theta_t^i)$ in equation (C.4), which yields the result in (4.38).

C.9 The compensation function under competition and stochastic private information

From equation (4.34) we find that agent i's payoff value, given that the optimal investment strategy is implemented, equals

$$g^i(S_t, C_t, \theta_t^i, t) = E\left[e^{-rt} \left(X^i(S_t, C_t, \theta_t, t) - Y^i(\theta_t^i) h(C_t, \theta_t^i) \right) |\mathcal{F}_t^{S,C,\theta^i} \right],$$

which can be written as

$$e^{-rt} X^i(S_t, C_t, \theta_t^i, t) = e^{-rt} Y^i(\theta_t^i) h(C_t, \theta_t^i) - g^i(S_t, C_t, \theta_t^i, t). \tag{C.5}$$

The last relationship is found by observing that $X^i(S_t, C_t, \theta_t^i, t) = E\left[X^i(S_t, C_t, \theta_t, t) |\mathcal{F}_t^{S,C,\theta^i} \right]$, as the right-hand side of equation (C.5) only depends on θ_t^i, and not on the vector θ_t.

The value of $g^i(S_t, C_t, \theta_t^i, t)$ is found by integration on both sides of the equality in (4.35) when $S_t > \psi^i(C_t, \theta_t^i, t)$, i.e.,

$$\int_{\theta^i}^{\vartheta^i(S_t, C_t, t)} \frac{dg^i(S_t, C_t, u, t)}{du} du = - \int_{\theta^i}^{\vartheta^i(S_t, C_t, t)} h_u(C_t, u) Y^i(u) du.$$

Evaluation of the equality above leads to

$$g^i(S_t, C_t, \theta_t^i, t) = \int_{\theta_t^i}^{\vartheta^i(S_t,C_t,t)} h_u(C_t, u)Y^i(u)du + g^i\left(S_t, C_t, \vartheta^i(S_t, C_t, t), t\right).$$

Next, replace $g^i\left(S_t, C_t, \vartheta^i(S_t, C_t, t), t\right)$ using the expression in equation (4.37), yielding

$$g^i(S_t, C_t, \theta_t^i, t) = \int_{\theta^i}^{\vartheta^i(S_t,C_t,t)} h_u(C_t, u)Y^i(u)du - \int_{\vartheta^i(S_t,C_t,t)}^{\bar{\theta}} w_u^i(S_t, C_t, u, t)du.$$

Hence, we find that

$$X^i(S_t, C_t, \theta_t^i, t)$$

$$= Y^i(\theta_t^i)h(C_t, \theta_t^i) + \int_{\theta_t^i}^{\vartheta^i(S_t,C_t,t)} h_u(C_t, u)Y^i(u)du - \int_{\vartheta^i(S_t,C_t,t)}^{\bar{\theta}} w_u^i(S_t, C_t, u, t)du.$$

Evaluation at the optimal Y^{i*} and ϑ^{i*} yield (4.46) in the text.

Appendix D

Appendix for chapter 5

D.1 The incentive compatibility constraints

For simplicity, denote $g^A_{i-1}(\pi, q_{j-1}, \eta; \hat{q}_{j-1}) = g^A_{i-1}(\pi, q_{j-1}, \eta; \hat{q}_0, .., \hat{q}_{j-1})$. Equation (5.18) can be reformulated as

$$\begin{aligned} & g^A_{i-1}(\pi, q_{j-1}, \eta; \hat{q}_{j-1}) \\ = \; & E\left[e^{-r\Delta_j} \int_0^\infty v^A_{i-1}(\Gamma(\Delta_j); \hat{q}_{j-1}) f(q(\Delta_j)|q_{j-1}, \eta) dq_j \right. \\ & \left. -Y_j(\pi, \eta; \hat{q}_{j-1}) | \mathcal{F}^\Gamma_0 \right]. \end{aligned} \tag{D.1}$$

For $q_{j-1} > q^*_{j-1}$, we know that $v^A_i = g^A_{i-1}$. Replace v^A_i in (5.21) by the right-hand side of equation (D.1). Then the condition in (5.20) requires that for any q_{j-1} and q'_{j-1}, we need

$$\begin{aligned} & E\left[e^{-r\Delta_j} \int_0^\infty v^A_{i-1}(\Gamma(\Delta_j); q_{j-1}) f(q(\Delta_j)|q_{j-1}, \eta) dq_j - Y_j(\pi, \eta; q_{j-1}) | \mathcal{F}^\Gamma_0 \right] \\ & -E\left[e^{-r\Delta_j} \int_0^\infty v^A_{i-1}(\Gamma(\Delta_j); q_{j-1}) f(q(\Delta_j)|q'_{j-1}, \eta) dq_j - Y_j(\pi, \eta; q_{j-1}) | \mathcal{F}^\Gamma_0 \right] \\ & -E\left[e^{-r\Delta_j} \int_0^\infty v^A_{i-1}(\Gamma(\Delta_j); q'_{j-1}) f(q(\Delta_j)|q_{j-1}, \eta) dq_j - Y_j(\pi, \eta; q'_{j-1}) | \mathcal{F}^\Gamma_0 \right] \\ & +E\left[e^{-r\Delta_j} \int_0^\infty v^A_{i-1}(\Gamma(\Delta_j); q'_{j-1}) f(q(\Delta_j)|q'_{j-1}, \eta) dq_j - Y_j(\pi, \eta; q'_{j-1}) | \mathcal{F}^\Gamma_0 \right] \\ & \geq 0. \end{aligned}$$

Rearrangement of the above inequality leads to

$$E\left[e^{-r\Delta_j}\int_0^\infty \left\{v_{i-1}^A(\Gamma(\Delta_j);q_{j-1}) - v_{i-1}^A(\Gamma(\Delta_j);q'_{j-1})\right\}\right.$$
$$\left.\cdot\left(f(q(\Delta_j)|q_{j-1},\eta) - f(q(\Delta_j)|q'_{j-1},\eta)\right)dq_j|\mathcal{F}_0^\Gamma\right] \geq 0.$$

Evaluation of the integral leads to

$$E\left[e^{-r\Delta_j}\left[\left\{v_{i-1}^A(\Gamma(\Delta_j);q_{j-1}) - v_{i-1}^A(\Gamma(\Delta_j);q'_{j-1})\right\}\left(F(q(\Delta_j)|q_{j-1},\eta) - F(q(\Delta_j)|q'_{j-1},\eta)\right)\right]_0^\infty\right.$$
$$\left.-\int_0^\infty\left\{\frac{\partial v_{i-1}^A(\Gamma(\Delta_j;q_{j-1})}{\partial q(\Delta_j)} - \frac{\partial v_{i-1}^A(\Gamma(\Delta_j;q'_{j-1})}{\partial q(\Delta_j)}\right\}\left(F(q(\Delta_j)|q_{j-1},\eta) - F(q(\Delta_j)|q'_{j-1},\eta)\right)dq_j|\mathcal{F}_0^\Gamma\right]$$
$$\geq 0.$$

In the above inequality the first term equals zero as $F(\infty|q_{j-1},\eta) - F(\infty|q'_{j-1},\eta) = 0$ and $F(0|q_{j-1},\eta) - F(0|q'_{j-1},\eta) = 0$. Thus, the above inequality is simplified to.

$$-E\left[\int_0^\infty\left(\frac{\partial v_{i-1}^A(\Gamma(\Delta_j);q_{j-1})}{\partial q(\Delta_j)} - \frac{\partial v_{i-1}^A(\Gamma(\Delta_j);q'_{j-1})}{\partial q(\Delta_j)}\right)\right.$$
$$\left.\cdot\left[F(q(\Delta_j)|q_{j-1}) - F(q(\Delta_j)|q'_{j-1})\right]dq(\Delta_j)|\mathcal{F}_0^\Gamma\right] \geq 0. \qquad \text{(D.2)}$$

When $q_{j-1} > q'_{j-1}$, we obtain $F(q(\Delta_j)|q_{j-1}) - F(q(\Delta_j)|q'_{j-1}) < 0$. Thus, for the incentive compatibility constraint in (D.2) to be satisfied, we need

$$\frac{\partial v_{i-1}^A(q(\Delta_j);q_{j-1})}{\partial q(\Delta_j)} - \frac{\partial v_{i-1}^A(q(\Delta_j);q'_{j-1})}{\partial q(\Delta_j)} \geq 0,$$

as is stated in (5.21).

The truth telling constraints for the earlier reports, $\hat{q}_0, .., \hat{q}_k$, where $k = 0, ..., j-2$, are found below, by mimicking the procedure above for $k = j-1$.

Suppose $q_k > q'_k$. Then the truth telling constraints for each report $\hat{q}_k$ with respect to the value of $g_{i-1}^A(\cdot)$ is given by

$$E\left[e^{-rt}\int_0^\infty \left(e^{-r\Delta_j}v_{i-1}^A(\Gamma(t);q_k) - Y_j(\pi(t),\eta(t);q_k)\right) f(q(t)|q_k,\eta)dq_k|\mathcal{F}_0^\Gamma\right]$$

$$-E\left[e^{-rt}\int_0^\infty \left(e^{-r\Delta_j}v_{i-1}^A(\Gamma(t);q_k) - Y_j(\pi(t),\eta(t);q_k)\right) f(q(t)|q_k',\eta)dq(t)|\mathcal{F}_0^\Gamma\right]$$

$$-E\left[e^{-rt}\int_0^\infty \left(e^{-r\Delta_j}v_{i-1}^A(\Gamma(t);q_k') - Y_j(\pi(t),\eta(t);q_k')\right) f(q(t)|q_k,\eta)dq(t)|\mathcal{F}_0^\Gamma\right]$$

$$+E\left[e^{-rt}\left(e^{-r\Delta_j}\int_0^\infty v_{i-1}^A(\Gamma(t);q_k') - Y_j(\pi(t),\eta(t);q_k')\right) f(q(t)|q_k',\eta)dq(t)|\mathcal{F}_0^\Gamma\right]$$

$$\geq 0.$$

Rearrangement of this inequality leads to the constraint in (5.24), as

$$-E\left[\int_0^\infty \left(\frac{\partial v_{i-1}^A(\Gamma(t);q_k)}{\partial q(t)} - \frac{\partial v_{i-1}^A(\Gamma(t);q_k')}{\partial q(t)}\right)\left[F(q(t)|q_k) - F(q(t)|q_k')\right]dq(t)|\mathcal{F}_0^\Gamma\right] \geq 0.$$

D.2 The principal's switching function when $i = 1$

When $i = 1$ and $j = N$ equation (5.25) is given by

$$g_0^P(\pi,\eta) = E\left[e^{-r\Delta_N}\left(v_0^P(\pi(\Delta_N),\eta(\Delta_N);\hat{q}) + v_0^A(\pi(\Delta_N),q(\Delta_N),\eta(\Delta_N))\right)\right.$$
$$\left. -g_0^A(\pi,q_{N-1},\eta;\hat{q}) - K_N|\mathcal{F}_0^{\pi,\eta}\right].$$

When all the investment phases are completed, the agent obtains the realized value, whereas the principal receives nothing. Hence, $v_0^P = 0$ and $v_0^A(\pi(\Delta_N),q(\Delta_N),\eta(\Delta_N) = \pi(\Delta_N)q(\Delta_N)$. It follows that

$$g_0^P(\pi,\eta) = E\left[e^{-r\Delta_N}\pi(\Delta_N)q(\Delta_N) - g_0^A(\pi,q_{N-1},\eta;\hat{q}) - K_N|\mathcal{F}_0^{\pi,\eta}\right].$$

Evaluation of the expression above leads to

$$g_0^P(\pi,\eta) = E\left[\pi q_{N-1}e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} - \int_0^{q_{N-1}}\frac{\partial w_1^A(\pi,u,\eta)}{\partial u}du - K_N|\mathcal{F}_0^{\pi,\eta}\right], \quad \text{(D.3)}$$

where

$$E\left[e^{-r\Delta_N}\pi(\Delta_N)|\mathcal{F}_0^\Gamma\right] = E\left[e^{-r\Delta_N}\pi e^{(r-\delta-\frac{1}{2}\sigma^2)\Delta_N+\sigma(B^\pi(\Delta_N)-B\pi(0))}|\mathcal{F}_0^\Gamma\right]$$
$$= \pi e^{-\delta\Delta_N},$$

and

$$\begin{aligned} E\left[q(\Delta_N)|\mathcal{F}_0^\Gamma\right] &= q_{N-1}e^{\kappa\eta}E\left[e^{(\kappa-\frac{1}{2}\nu^2)\Delta_N+\nu(B^q(\Delta_N)-B^q(0))}|\mathcal{F}_0^\Gamma\right] \\ &= q_{N-1}e^{\kappa(\eta+\Delta_N)} \end{aligned}$$

Furthermore, the agent's value of private information at the time the last investment is exercised can be expressed as

$$g_0^A(\pi, q_{N-1}, \eta) = \int_0^{q_{N-1}} \frac{\partial w_1^A(\pi, u, \eta)}{\partial u} du, \tag{D.4}$$

since the first-order condition of incentive compatibility when $i = 1$ requires that

$$\frac{dv_1^A(\pi, q_{N-1}, \eta)}{dq_{N-1}} = \begin{cases} \frac{\partial w_1^A(\pi, q_{N-1}, \eta)}{\partial q_{N-1}} & \text{when } q_{N-1} \leq q_{N-1}^* \\ \pi e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} & \text{when } q_{N-1} > q_{N-1}^*. \end{cases}$$

The value in equation (D.3) can be reformulated to

$$\begin{aligned} &g_0^P(\pi, \eta) \\ &= \int_0^\infty \left\{\pi q_{N-1}e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} - \int_0^{q_{N-1}} \frac{\partial w_1^A(\pi,u,\eta)}{\partial u} du - K_N\right\} f(q_{N-1}|\hat{q}^{-(N-1)}dq_{N-1}. \end{aligned} \tag{D.5}$$

D.3 The principal's value of exercising the last option

Partial integration of $\int_0^\infty \int_0^{q_{N-1}} \frac{\partial w_1^A(\pi,u,\eta)}{\partial u} du f(q_{N-1}|\hat{q}^{-(N-1)}$ in equation (D.5) leads to

$$\begin{aligned} &g_0^P(\pi, \eta) \\ &= \int_0^\infty \left\{\pi q_{N-1}e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})} \frac{\partial w_1^A(\pi, q_{N-1}, \eta)}{\partial q_{n-1}} - K_N\right\} f(q_{N-1}|\hat{q}^{-(N-1)}dq_{N-1}. \end{aligned}$$

At the time when the investment is exercised, the smooth pasting condition requires that $\pi e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} = \frac{\partial w_1^A(\pi,q_{N-1},\eta)}{\partial q_{n-1}}$. Hence,

$$\begin{aligned} &g_0^P(\pi, \eta) \\ &= \int_0^\infty \left\{\pi \left(q_{N-1}e^{\kappa(\eta+\Delta_N)-\delta\Delta_N} - \frac{1-F(q_{N-1}|\hat{q}^{-(N-1)})}{f(q_{N-1}|\hat{q}^{-(N-1)})}\right) - K_N\right\} f(q_{N-1}|\hat{q}^{-(N-1)}dq_{N-1}, \end{aligned}$$

which is identical to the result in (5.29).

D.4 The optimal investment trigger M^*

We guess on the solution $\tilde{v}_1^P(m, q_{N-1}) = Am^\lambda$ when $m \le M^*$, where A and λ are constants. The parameter λ is the positive root of

$$(r - \delta + \kappa)\lambda + \frac{1}{2}\sigma^2\lambda(\lambda - 1) - r = 0,$$

derived from the constraint $L^{q_{N-1}}\tilde{v}_0^P(m) = 0$, where

$$L^{q_{N-1}}\tilde{v}_0^P(m) = (r - \delta + \kappa)m\frac{\partial \tilde{v}_0^P}{\partial m}(m) + \frac{1}{2}\sigma^2 m^2 \frac{\partial^2 \tilde{v}_0^P}{\partial m^2}(m) - r\tilde{v}_0^P(m).$$

At the entry threshold M^*, we know by the variational constraint in (5.17) that $\tilde{v}_1^P = \tilde{g}_0^P$ when $m = M^*$, leading to

$$A(M^*)^\lambda = M^*\left(q_{N-1} - \frac{1 - F(q_{N-1}|\hat{q}_0, ..\hat{q}_{N-2})}{f(q_{N-1}|\hat{q}_0, ..\hat{q}_{N-2})}\right) - K_N.$$

Furthermore, at the trigger where $m = M^*$ the first-order derivative must be continuous, i.e.

$$\lambda A(M^*)^{\lambda-1} = q_{N-1} - \frac{1 - F(q_{N-1}|\hat{q}_0, ..\hat{q}_{N-2})}{f(q_{N-1}|\hat{q}_0, ..\hat{q}_{N-2})}.$$

Solving these two equalities with respect to the two constants A and M_N^*, yields $A = (M^*)^{1-\lambda}$, and M^* given by equation (5.32).

D.5 The agent's value of private information and the contracted amount Y_N when $i = 1$

Using equations (5.18) and (5.19), we find that when $i = 1$ and $j = N$, the agent's value function equals

$$v_1^A(m, q_{N-1}; \hat{q}^{-(N-1)}) = E\left[e^{-r\tau}\left(M(\tau)q_{N-1} - Y_N(M(\tau); \hat{q}^{-(N-1)})\right)|\mathcal{F}_0\Gamma\right], \quad \text{(D.6)}$$

whereas the agent's switching function is represented by

$$g_0^A = mq_{N-1} - Y_N(m; \hat{q}^{-(N-1)}). \quad \text{(D.7)}$$

By the same approach as in section D.4, we guess on the solution $v_1^A = Bm^\gamma$, where B is a constant. At the critical price M^* the value matching condition needs to hold, i.e.,

$$B(M^*)^\gamma = M^* q_{N-1} - Y_N(M^*; \hat{q}^{-(N-1)}),$$

leading to

$$v_1^A(m, q_{N-1}) = \begin{cases} \left(\frac{m}{M^*}\right)^\lambda \left(M^* q_{N-1} - Y_N(M^*, \hat{q}^{-(N-1)})\right) & \text{if } m \leq M^* \\ m q_{N-1} - Y_N(m, \hat{q}) & \text{if } m > M^*. \end{cases} \tag{D.8}$$

By equation (D.8), we find a necessary condition for truth telling,

$$\frac{dv_1^A(m, q_{N-1})}{dq_{N-1}} = \begin{cases} \left(\frac{m}{M^*(q_{N-1}; \hat{q}^{-(N-1)})}\right)^\lambda M^*(q_{N-1}; \hat{q}^{-(N-1)}) & \text{if } m \leq M^*(q_{N-1}; \hat{q}^{-(N-1)}) \\ m & \text{if } m > M^*(q_{N-1}; \hat{q}^{-(N-1)}). \end{cases} \tag{D.9}$$

Recall that we define $M^*(q_{N-1}; \hat{q}^{-(N-1)})$ only for $q^*_{N-1} - \frac{1-F(q^*_{N-1}|\hat{q}^{-(N-1)})}{f(q^*_{N-1}|\hat{q}^{-(N-1)})} > 0$. For q_{N-1} such that $q^*_{N-1} - \frac{1-F(q^*_{N-1}|\hat{q}^{-(N-1)})}{f(q^*_{N-1}|\hat{q}^{-(N-1)})} \downarrow 0$, the critical price M^* approaches infinity, which implies that $\left(\frac{m}{M^*}\right)^\lambda \to 0$. Thus, in order to find an expression for the agent's value of private information, we find the lower value of the signal q_{N-1}, denoted by $\underline{q}_{N-1}$, such that $\bar{q}_{N-1} - \frac{1-F(\bar{q}_{N-1}|\hat{q}^{-(N-1)})}{f(\bar{q}_{N-1}|\hat{q}^{-(N-1)})} = 0$.

Then, integration on both sides of the equality in equation (D.9) leads to the agent's value of private information when $i = 1$,

$$v_1^A(m, q_{N-1}) = \begin{cases} \int_{\underline{q}_{N-1}}^{q_{N-1}} \left(\frac{m}{M^*(u,\hat{q})}\right)^\lambda M^*(u, \hat{q}) du & \text{if } m \leq M^* \\ m(q_{N-1} - q^*_{N-1}) + \int_{\underline{q}_{N-1}}^{q^*_{N-1}} \left(\frac{m}{M^*(u,\hat{q})}\right)^\lambda M^*(u, \hat{q}) du & \text{if } m > M^*. \end{cases} \tag{D.10}$$

By the value of private information as formulated above, and equation (D.7), the contracted amount Y_N is found to be equal to

$$\begin{aligned} & Y_N(m, q_{N-1}; \hat{q}^{-(N-1)}) \\ = \; & m q^*_{N-1}(m; \hat{q}^{-(N-1)}) - \int_{\underline{q}_{N-1}}^{q^*_{N-1}(m; \hat{q}^{-(N-1)})} \left(\frac{m}{M^*(u; \hat{q}^{-(N-1)})}\right)^\lambda M^*(u; \hat{q}^{-(N-1)}) du. \end{aligned}$$

when $q^*_{N-1} - \frac{1-F(q^*_{N-1}|\hat{q}^{-(N-1)})}{f(q^*_{N-1}|\hat{q}^{-(N-1)})} > 0$, identical to the contracted amount in equation (5.33).

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