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Some implications of predation to optimal management of Marine Resources

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Chapter 1

Introduction

"*When Italy declared war on Austria in 1915, both sides feared an invasion of their ports. Mines were set in many of the seaports in the Adriatic Sea, preventing fishing for the duration of the war. When the war ended three years later and the mines were removed, it was expected that the fisherman would have a better than usual catch since the fish stocks had had three years to replenish. Surprisingly, the opposite was true.*" The story is taken from Illner [18].

Traditionally, fishery research has been studying each specie in isolation, although ecological system have been studied for many years through mathematical models, e.g. Volterra(1928) and Lotka (1925, he gave an explanation of the phenomenon described above in a study of predator-prey systems). Often the models have plenty of allowance for exogenous influence. However, awareness is growing that some of those influences might be interactions with other species.

The income from the export of Norway's fish resources is very important. Both from an environmental and economic point of view, the government has an enormous responsibility to make the right harvesting decisions. The main intention with marine management is to ensure conservation of the fishery resource into the future. We will give an introduction on management of renewable resources where we include interactions between two species.

1.1 Marine Predators

Predation is defined as: *Consumption of an organism(prey) by another organism(predator) and prey is still alive when attacked by predator.*

People are natural predators of fish, much as fish of different species prey on themselves and on fish of other species. People are on the top of an amazingly complex web of predator-prey chain. If we want to maximize the stock of cod, it would be advantageous to consider the predators of cod (i.e. seal). It could pay off to heavily deplete sea mammals to increase the surplus production of fish resources for man, Flaaten [10].

Pomarenko [25] studied the predation effects on capelin in the Barents sea from cod and haddock. They found that the annual consumption amounted to between 6.6 and 9.8 million m.t. in the years 1974-1976. In comparison, in the same period, the annual catches were 1.4 million m.t., or only 15-20 present of the consumption by cod and haddock. Sergeant [30] studied interactions between seals and fish stocks in the Atlantic ocean and found that Harp seal in the Northwest Atlantic daily consumes 5 present of its body weight. Flaaten and Stollery [12], stated that the estimated average cost per North Eastern Atlantic Minke whale in 1991-1992 was between $US 1780 and $US 2370.

For a single fish stock the biological theory says that MSY(maximum sustainable yield) harvesting is an optimal choice (see figure 1.1). In equilibrium we harvest the growth, and this level yields the largest harvest. However, they do not consider the economic benefit and costs of fisheries. From an economic point of view, harvesting below, at or above the MSY can all be optimal. For a more detailed discussion, please turn to Clark [5].

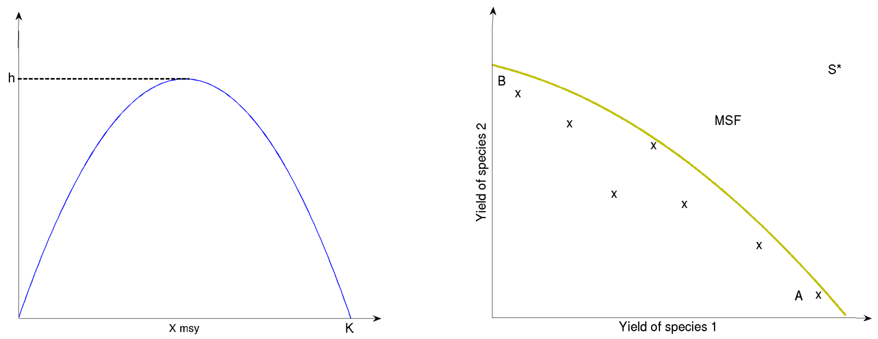

Figure 1.1: Left figure: MSY harvesting tells us that the harvest should equal it's maximum growth. Right figure: The maximum sustainable yield frontier (MSF) gives the maximum possible yield of one species for a given yield of the other.

When harvested species have strong interactions we cannot have optimal exploitation to each species individually as a guiding principle, as harvesting of one level in the chain influences the next level. The theory is that the yield for predator is maximized when yield of prey is zero, that is without harvesting prey. The maximized yield of prey is obtained when the predator stock is depleted. The existence of predator will for instance shift the stock of the prey to a level below the optimal, which will obviously not be sustainable. Moving from single-species to two-species models the biological constraint changes to, for example, the MSF (figure 1.1).

The MSF curve gives the absolute sustainable yield of either population for a specified yield of the other. Suppose the yield for the predator is given and our main goal is to maximize the yield of the prey. Then it is obviously better to overe xploit than to under exploit the predator. Otherwise the predator will consume more of the prey, and thereby removing a potential yield prey. For similar reasons, it is more efficient to under exploit the prey to leave more food for the predator.

Every combination of the two species resulting at a point on or under this curve will be sustainable from a biologic point of view. The optimal choice depends also on prices, costs, etc. If we maximize the yield for each species independent of the other, the total yield would result in the point S. Clearly this is not sustainable. Yields to the north-east of the curve are possible for some period of time, but they are not sustainable. Which combination of yield that should be chosen depends on the management objective and the price of fish. Let species 2 be the predator and species 1 be the prey. If the *predator is valuable and the prey is a low net valued species*, in economic terms, it could be optimal to have a large yield of the predator and less yield of the prey, in vincity of B in the figure. In case of the opposite, *predator of low net value and prey of high net value* the optimal combined harvest could be a point close to A.

MSF(maximum sustainable yield frontier) harvesting thus implies that neither shall the predator be underexploited, nor shall the prey be overexploited. A condition for MSF is that the two species can be harvested selectively.

In this chapter we introduce and discuss our model. Next chapter contains a summary of Optimal Control Theory, the theory we employ on our model. Chapter 3 gives the results from applying the optimal control theory on our problem, after which we implement our method on one example. Chapter 4 provides a summary and discusses the results from our example. The appendix shows some derivations and the program listing.

1.2 The Model

We use an aggregated deterministic model, formulated in a continuous time setting. Our model considers two populations of fish, one of which is the predator of the other. The growth of prey and predator depends on the growth of the predator and prey stock, respectively. We consider fish stocks in a restricted area.

For the prey we use this growth function with depensation.

$$f_1(x) = r_1 x^2 \Big(1 - \frac{x}{K_1}\Big) \tag{1.1}$$

We adopt the logistic growth function for predator.

$$g_1(y) = r_2 y \Big(1 - \frac{y}{K_2}\Big) \tag{1.2}$$

x and y are the total stocks, r is the rate of natural increase and K is the carrying capacity[1] of the stock. We assume that the carrying capacities for our system are given constants. This function describes the growth of a stock without predation or human interaction. The growth function in the case of predation will be:

$$For \quad the \quad prey : F(x, y) = r_1 x^2 \Big(1 - \frac{x}{K_1}\Big) - axy \tag{1.3}$$

$$For \quad the \quad predator : G(x, y) = r_2 y \Big(1 - \frac{y}{K_2}\Big) + bxy \tag{1.4}$$

a and b are the coefficients with respect to the other species. The predator coefficient, a, tells which share of the prey stock one unit of the predator is consuming per unit of time. Then axy is the total rate of consumption. Similarly, the existence of prey causes an increase in the predator stock. The negative term $\frac{-x}{K_1}$ prevents the prey stock, x, from growing without bounds. Similarly for the predator. Note that in presence of predator, the growth of prey has critical depensation (it can have negative growth). Our equations embody the essential elements of an interactive predator-prey system.

When including fishing this results in the equations[2]

$$For \quad the \quad prey : \dot{x} = r_1 x^2 \Big(1 - \frac{x}{K_1}\Big) - axy - h_1 \tag{1.5}$$

$$For \quad the \quad predator : \dot{y} = r_2 y \Big(1 - \frac{y}{K_2}\Big) + bxy - h_2 \tag{1.6}$$

h_1 and h_2 denote the catch of prey and predator, respectively. We assume that the species can be harvested independently of each other, that is, the fishing effort targeted at one species catches just that one. The right hand side of these equations do not depend explicitly on time. Systems of this

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[1] Carrying capacity depends usually on availability of food, spawning and nursery areas

[2] Dot notation is used for the time derivative: $\dot{x} = \frac{dx}{dt}$

property are called autonomous.

Next we assume that price is a function which decreases with quantity, $P = p - Bh$, where p is the maximum market price. The cost of fishing is proportional with the effort E. The net income: Income-Costs $= Ph - cE = ph - Bh^2 - cE$. We have an economic production function $h = qEx \Rightarrow E = \frac{h}{qx}$. The costs are then $cE = \frac{c}{q} \cdot \frac{h}{x} = \frac{Ch}{x}$. Then, for both functions the utility or profit can be described by

$$\widehat{\Pi}(x_i, h_i) = \Big(p_i - \frac{C_i}{x_i}\Big)h_i - B_i h_i^2 \tag{1.7}$$

p,C,B are economic constants. This function is also referred to as the objective function. The total profit will be the sum of the utility functions for the predator and the prey.

It would be helpful to rewrite the population variables x,y in an appropriate dimensionless form, in order to highlight the combinations of parameters that are the key to the behavior of the system. Defining:

$$X = \frac{x}{K_1}, Y = \frac{y}{K_2}$$

$$\tau = r_2 t \quad \gamma = \frac{\delta}{r_2} \quad s = \frac{r_1 K_1}{r_2} \quad \alpha = \frac{aK_2}{r_2}, \quad \beta = \frac{bK_1}{r_2},$$

$$U = \frac{h_1}{K_1 r_2}, V = \frac{h_2}{K_2 r_2}, b_1 = \frac{B_1 r_2 K_1}{p_1}, b_2 = \frac{B_2 r_2 K_2}{p_2}, c_1 = \frac{C_1}{K_1 p_1}, c_2 = \frac{C_2}{K_2 p_2}$$

We can now rewrite the growth equations as

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} = sX^2(1 - X) - \alpha XY - U, \qquad \frac{\mathrm{d}Y}{\mathrm{d}\tau} = Y(1 - Y) + \beta XY - V$$

After the scaling, we get the profit functions:

$$\Pi_1(X, U) = U(1 - \frac{c_1}{X} - b_1 U) \qquad \Pi_2(Y, V) = V(1 - \frac{c_2}{Y} - b_2 V)$$

These profit functions are scaled separately. In order to combine them we need a parameter to define them in the same measure. We multiply Π_1 by σ.

$$\sigma = \frac{K_1 p_1}{K_2 p_2}$$

The total profit for the system will be a sum of the profit functions for each population. The value from harvesting can be found by integrating the utility function over the time period. We use infinity as the upper limit due to our focus on a sustainable development:

$$\int_0^\infty e^{-\gamma\tau}\Big(\sigma\Pi_1(X,U)+\Pi_2(Y,V)\Big)d\tau \tag{1.8}$$

The objective is to maximize discounted social net benefits over an infinite horizon, subject to resource growth constraints. Our aim is to maximize the utility, so our problem can be stated as:

$$\max_{U,V}\int_0^\infty e^{-\gamma\tau}\left[\sigma U\Big(1-\frac{c_1}{X}-b_1U\Big)+V\Big(1-\frac{c_2}{Y}-b_2V\Big)\right]d\tau \tag{1.9}$$

Under the conditions

$$\begin{aligned}\dot{X} &= sX^2(1-X)-\alpha XY-U=f(X,Y,U)\\ \dot{Y} &= Y(1-Y)+\beta XY-V=g(X,Y,V)\\ &\quad X,Y,U,V\geq 0\end{aligned} \tag{1.10}$$

The problem is well defined when we add proper initial conditions. Maximizing the present value of discounted future resource rent is the main economic management objective, when harvested fish is the only benefit to society.

We note that this is an optimal control problem with two state variables and two controls. In the next chapter we will introduce a method that can be used to solve this problem.

Figure 1.2 demonstrates the growth equations without harvesting(U=V=0). From the right figure we see that the growth of predator which is described by a logistic growth function increases at a high rate when the stock is at a low level. The growth curve with depensation on the left increases slowly when the stock is small.

1.3 Remarks

Note that if the predator stock becomes very large, this will result in a large reduction in the growth function for the prey stock. We may obtain that there will not be positive growth of prey at all (see figure 1.2).

The scaled fish stocks, X and Y, are expressed as densities.

We used the logistic growth function to describe the growth of the predator. This means that the population grows at a high rate for a

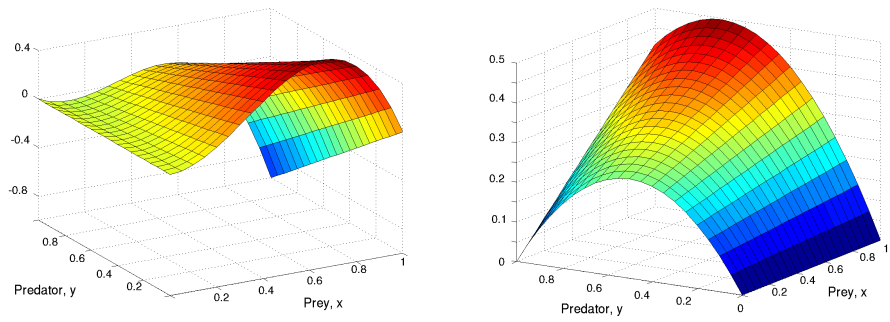


Figure 1.2: Demonstration of the growth functions. Left: The growth function for prey decline as the stock of predator increases. For a large stock of predator, the growth of prey can be negative. Right: Growth for predator increases as stock of prey is increasing.

population close to zero. We find that unrealistic. When the population is small it might have trouble replenish because it is unprotected, and process of spawning is slow. For a more detailed discussion, see Clark [5].

The scaled discount rate, $\gamma = \frac{\delta}{r_2}$, is the ratio between the intristic growth rate and the actual discount rate. When r_2 is not the intristic growth rate of an marine mammal, r_2 will usually dominate the discount rate δ. Optimal paths are therefore not very sensitive to changes in the discount rate.

Now we will give a presentation of the different equilibrium solution our model can have in a case *without harvesting*. The stability of a renewable resource exploitation problem may be of importance in policy making. A system with unstability may lead to destruction of the resource and must be managed more carefully than a stable one. We have the equations;

$$\dot{X} = sX^2(1-X) - \alpha XY \qquad \dot{Y} = Y(1-Y) + \beta XY$$

We seek an equilibrium $X > 0, Y > 0$

$$\begin{aligned}
\alpha Y &= s(X - X^2) \qquad and \qquad Y = 1 + \beta X &(1.11)\\
&or\\
0 &= sX^2 - (s - \alpha\beta)X + \alpha &(1.12)\\
\Leftrightarrow \quad 2sX &= s - \alpha\beta \pm \sqrt{[(s-\alpha\beta)^2 - 4\alpha s]} &(1.13)
\end{aligned}$$

We have an equilibrium if $(s - \alpha\beta)^2 \geq 4\alpha s$. In addition we need $s - \alpha\beta \geq 0$ for a solution $X \geq 0$

Now we need to study the Jacobian matrix of this system. Note that we have simplified it by using (1.11).

$$\mathbf{J} = \begin{pmatrix} 2\alpha Y - sX & -\alpha X \\ \beta Y & -Y \end{pmatrix}$$

$$\begin{aligned}
trJ &= \lambda_1 + \lambda_2 = (2\alpha - 1)Y - sX &(1.14)\\
det(J) &= \lambda_1 \cdot \lambda_2 = -2\alpha Y^2 + sXY + \alpha\beta XY &(1.15)\\
&= Y[(s+\alpha\beta)X - 2\alpha y] &(1.16)\\
&= Y[(s-\alpha\beta)X - 2\alpha] &(1.17)\\
&= Y[sX^2 - \alpha] &(1.18)
\end{aligned}$$

Let λ_1, λ_2 represent the eigenvalues of J. The value of these eigenvalues will give the character of the equilibrium point. The combinations of eigenvalues and their respective stability is given in table 1.3. If it exists a stable equilibrium point, it will be our solution. Note that this analysis is local, we study the stability of the equilibrium points.

Now we present the different cases of equilibrium. In a *unstable node* the trajectories go to infinity as $t \to +\infty$ and toward the equilibrium point as $t \to -\infty$, thus the trajectories move away from the equilibrium point. When the direction along the trajectories are reversed we call the equilibrium a *stable node*. A *saddle point* is an unstable equilibrium, but unlike an unstable node, two stable trajectories do converge to the equilibrium point. Except for the semiaxes, all trajectories "begin" at $t = -\infty$ and "end" at $t = +\infty$. A *center* is the case when ellipses are centered at the equilibrium point, it is neutrally stable. We could have a *stable focus*, spirals converging toward the equilibrium point.

From (1.13) we have

$$0 < 2sX \lesseqgtr s - \alpha\beta > 0 \quad or \quad 4s^2X^2 \lesseqgtr (s-\alpha\beta)^2 \geq 4\alpha s \qquad (1.19)$$

Values of λ_1, λ_2	Character of Equilibrium Point
$\lambda_1, \lambda_2 > 0$	Unstable node
$\lambda_1, \lambda_2 < 0$	Stable node
$\lambda_1 < 0 < \lambda_2$	Saddle point
$\lambda_2 < 0 < \lambda_1$	Saddle point
λ_1, λ_2 complex, $\text{Re}\lambda_i > 0$	Unstable focus
λ_1, λ_2 complex, $\text{Re}\lambda_i < 0$	Stable focus
λ_1, λ_2 complex, $\text{Re}\lambda_i = 0$	Center

Table 1.1: The eigenvalues give the stability of the equilibrium point.

The largest root gives: $4s^2X^2 \geq 4\alpha s$ or $sX^2 \geq \alpha \Rightarrow det(J) \geq 0$. From (1.12) we have that the product of the roots are $X_1 \cdot X_2 = \frac{\alpha}{s}$ or $X_1^2 X_2^2 = \frac{\alpha^2}{s^2}$. Let X_2 be the largest root. We then know that $sX_2^2 > \alpha$ when we have two different roots. This gives us: $sX_1^2 \cdot sX_2^2 = \alpha^2 > sX_1^2 \cdot \alpha \Rightarrow sX_1^2 < \alpha \Rightarrow det(J) < 0$ for X_1. To summarize, we have two equilibriums. For the largest we have $\lambda_1 \cdot \lambda_2 > 0$ and for the smallest we have $\lambda_1 \cdot \lambda_2 < 0$.

We first analyze the smallest root, where $det(J) < 0$ for this root:

$$
\begin{aligned}
0 &= \lambda^2 + trJ\lambda + det(J) && (1.20)\\
\lambda &= -\frac{trJ}{2} \pm \sqrt{\frac{(trJ)^2}{4} - det(J)} && (1.21)\\
&= -\frac{trJ}{2} \pm \sqrt{\frac{(trJ)^2}{4} + |det(J)|} && (1.22)
\end{aligned}
$$

This gives the two eigenvalues $\lambda_1 > 0, \lambda_2 < 0 \Rightarrow$ a saddle point, which is unstable.

Next we analyze the largest root, where $\lambda_1 \cdot \lambda_2 > 0$

$$\lambda = -\frac{trJ}{2} \pm \sqrt{\frac{(trJ)^2}{4} - det(J)}$$

From trJ we see that if $\alpha < \frac{1}{2}$ both eigenvalues are positive. This would yield an unstable node. If $\alpha > \frac{1}{2}$ this equilibrium point can be a stable node or focus. Summarizing, for some set of parameter values our model describes a stable system of predator-prey and we can try to harvest the resource using reasonable management.

The main intention for this thesis is to study a predator prey relationship close to cod and capelin. Our growth functions are chosen to fit this relation. The cod function is well described by the logistic growth function, but the capelin stock is best described by growth function with depensation. Other stocks might be better described by different growth functions.

Chapter 2

Optimal Control Theory

In this chapter we will give an introduction to the theory of deterministic optimal control problems. Control theory is the study of how to adjust the parameters in the equations controlling a system in order to maximize its performance.

First we will give a presentation of the notation in optimal control problems. Then we will introduce the necessary conditions an optimal control must satisfy, given by Pontryagins Maximum Principle. We introduce feedback controls and give a short introduction to dynamic programming and the Hamilton-Jacobi-Bellmann equation both in continuous and discrete time. In the end of this chapter we will provide economic interpretations of the most important equations. This chapter is mainly inspired by Seierstad and Sydsæter [29] and Kamien and Schwartz [20].

2.1 Introduction

In optimal control problems we have two classes of variables. $\mathbf{x} = (x_1, \dots, x_n)$ defines the state of the system. The state could typically be a stock of capital. We now assume that the process in the economy (and hence the $x_i(t)$ variables) can be controlled to some extent, we have a control that influences the process. We define the control as $\mathbf{u} = (u_1, \dots, u_r)$. The control variables or decision variables could typically be different rates, quotas, etc. In our model the state describes the size of the stock and the control describes the harvest of the fish resource.

Now we need the laws governing the behavior through time, that is *the dynamics* of the system.

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{u}(t), t)$$

This equation is known as the state equation. The functions $\mathbf{f} = f_1, \ldots, f_n$ are given functions, typically growth functions. We assume that the rate of change of each state variable in general depends on all the state variables, all the control variables and on time explicitly. We need a starting point t_0 for the problem, but the end point t_1 is not necessarily fixed. In our model we have an infinite horizon.

The state of the system is known at time t_0, $\mathbf{x}(t_0) = \mathbf{x_0}$. In some problems the final state $\mathbf{x}(t_1)$ might be subject to certain bounds or conditions. We choose a certain admissible control function and substitute it into the state equation. This choice will result in a unique solution $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$, referred to as the response.

We define an objective functional

$$W = \int_{t_0}^{t_1} \Pi(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$\Pi(\mathbf{x}(t), \mathbf{u}(t), t)$ is a given, continuously differentiable function often referred to as the utility function. We start with the initial amount of capital, $\mathbf{x_0}$. We follow the policy $\mathbf{u}$ and the total result will be W.

The fundamental problem is now to determine a feasible control function $\mathbf{u}(t)$ to maximize the objective functional. A feasible control must satisfy the bounds on the control and the initial and terminal conditions for the corresponding response. This control, if it exists, is an optimal control and the associated path $\mathbf{x}(t)$ is an optimal path.

We now summarize the problem:

$$\max_{\mathbf{u}} \int_{t_0}^{t_1} \mathbf{\Pi}(\mathbf{x}(t), \mathbf{u}(t), t)dt \qquad \mathbf{x}(t_0) = \mathbf{x_0} \qquad \dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{u}(t), t) \quad (2.1)$$

Usually we have bounds on both the state and the control, i.e. they cannot vary freely. We need an admissible control. The class U of admissible controls is by definition the class of all piecewise-continuous real functions $\mathbf{u}(t) \in U$, where U is a given interval called the control set. These admissible controls will lead to meaningful states.

The maximizing u, the optimal control, is often denoted by $\mathbf{u}^*$. The corresponding optimal path is similarly denoted as $\mathbf{x}^*$. Below we will now see the necessary conditions that must be satisfied by an optimal control.

2.2 The Pontryagin Maximum Principle

L.S. Pontryagin[1] has a famous maximum principle that gives us the techniques of optimal control theory. We state the Pontryagin Maximum Principle for fixed time intervals. This maximum principle is a collection of necessary conditions for a control function to solve the problem and thus to be an optimal control. In short the principle says that if there exists a solution to our problem, then it must satisfy some conditions. In appendix A we have derived the necessary conditions for the simplest problem in optimal control theory.

Our problem is to find a piecewise continuous control function $\mathbf{u}(t) = (u_1(t), \cdots, u_k(t))$ and an associated continuous and piecewise differentiable state vector $\mathbf{x}(t) = (x_1(t), \cdots, x_n(t))$ defined on the fixed time interval $[t_0, t_1]$, that will

$$\max_{\mathbf{u}} \int_{t_0}^{t_1} \Pi(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.2}$$

subject to the differential equations

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = f_i(\mathbf{x}(t), \mathbf{u}(t), t), \quad i = 1, \dots, n \tag{2.3}$$

initial conditions

$$x_i(t_0) = x_i^0 \quad i = 1, \dots, n \tag{2.4}$$

terminal conditions

$$\left.\begin{array}{ll} x_i(t_1) = x_i^1 & \text{for } i = 1, \dots, p (x_i^1 \text{ fixed}) \\ x_i(t_1) \geq x_i^1 & \text{for } i = p+1, \dots, q (x_i^1 \text{ fixed}) \\ x_i(t_1) \text{ free} & \text{for } i = q+1, \dots, n \end{array}\right\} \tag{2.5}$$

and control variable restriction

[1] Lev Semonevich Pontryagin(1908-1988) graduated from the University of Moscow in 1929 despite that an explosion left him blind at the age of 14. He received many honors for his work. He was elected to the Academy of Sciences in 1939 and in 1970 elected Vice-president of the International Mathematical Union.

$$\mathbf{u}(t) = (u_1(t), \ldots, u_k(t)) \in U \subseteq \mathbb{R}^k \tag{2.6}$$

We introduce the Hamiltonian

$$H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = \lambda_0 \cdot \Pi(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}(t) \cdot f(\mathbf{x}(t), \mathbf{u}(t), t)$$

where λ_0 is a constant and $\boldsymbol{\lambda}(t) = (\lambda_1(t), \cdots, \lambda_n(t))$ are Lagrange multipliers, also known as *costates* or *shadowprices.*

The maximum principle transfers the problem of finding a $\mathbf{u}(t)$ that maximizes (2.2) subject to given constraints, to the problem of maximizing the Hamiltonian function w.r.t. $\mathbf{u} \in U$. In addition it tells us how to determine the $\lambda-$function.

The Pontryagin Maximum Principle : *Let* $\mathbf{u}^*(t)$ *be a piecewise continuous control defined on* $[t_0, t_1]$ *which solves (2.2-2.6) and let* $\mathbf{x}^*(t)$ *be the associated optimal path. Then there exists a constant* λ_0 *and a continuous function* $\boldsymbol{\lambda}(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ *where for all* $t_0 \leq t \leq t_1$ *we have*

$(\lambda_0, \boldsymbol{\lambda}(t)) \neq (0, \mathbf{0})$

$\mathbf{u}^*(\mathbf{t})$ *maximizes* $H(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t)$ for u ∈ U, *that is:*

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \geq H(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) \qquad \forall \mathbf{u}(t) \in U \tag{2.7}$$

Except at the points of discontinuities of $\mathbf{u}^*(t)$, for $i = 1, \ldots, n$

$$\dot{\lambda}_i(t) = -\frac{\partial H^*}{\partial x_i} = -\frac{\partial H}{\partial x_i}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \tag{2.8}$$

Furthermore

$$\lambda_0 = 0 \quad \mathit{or} \quad \lambda_0 = 1 \tag{2.9}$$

and finally, the following transversality conditions are satisfied.

$$\left.\begin{array}{ll} \lambda_i(t_1) \text{ no conditions} & \text{for } i = 1, \ldots, p \\ \lambda_i(t_1) \geq 0 (= 0 \text{ if } x_i^*(t_1) > x_i^1) & \text{for } i = p+1, \ldots, q \\ \lambda_i(t_1) = 0 & \text{for } i = q+1, \ldots, n \end{array}\right\} \tag{2.10}$$

For the proof of this theorem, please turn to Pontryagin et al. [24], Hestenes [15] or Lee and Markus [22]. Clark [5] gives an intuitive proof.

In the economic literature it is quite common to assume $\lambda_0 = 1$. However, it is possible that the Maximum Principle is satisfied with $\lambda_0 = 0$.

In Seierstad and Sydsæter [29] they refer to problems which have $\lambda_0 = 0$ as abnormal, as this makes it possible to replace the function Π by *any* other function without changing any of the conditions in the Maximum Principle. We will always assume that $\lambda_0 = 1$.

In terms of the Hamiltonian, our problem can now be stated as:

$$\left.\begin{array}{rcl} H_{\boldsymbol{\lambda}} & = & \dot{\mathbf{x}} \\ -H_{\mathbf{x}} & = & \dot{\boldsymbol{\lambda}} \\ \text{argmax}_{\mathbf{u}\in U} H & = & \mathbf{u} \end{array}\right\} \tag{2.11}$$

These are the first order conditions for an optimal solution. Note that in the last condition we need to maximize H with respect to $\mathbf{u}$, whereas $\mathbf{x}$ and t are fixed. If the optimal control $\mathbf{u}(t)$ is an inner solution (i.e, lie within the control interval U), then we can write: $H_{\mathbf{u}} = 0$. We have three unknown functions to determine: $\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)$. We use the three equations above to solve them.

Current value formulation:
Very often values are discounted back to time $t = 0$ by multiplying the profit function by $e^{-\delta t}$. Often it may be convenient to express the values in the current time, that is, the value at t rather than the value at the initial time.

With the discount term, the Hamiltonian will have the form:

$$H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = e^{-\delta t}\Pi(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}(t) \cdot f(\mathbf{x}(t), \mathbf{u}(t), t)$$

The current value Hamiltonian is the Hamiltonian multiplied with $e^{\delta t}$. This leads to the introduction of a current value multiplier function, $\mathbf{m}(t) = e^{\delta t}\boldsymbol{\lambda}(t)$. The new current value multiplier m(t) gives the marginal value of the state variable at time t in terms of values at t.

Then the current value Hamiltonian will be formulated as

$$\begin{array}{rcl} \mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{m}(t), t) & = & e^{\delta t} \cdot H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}, t) \\ & = & \Pi(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{m}(t) \cdot f(\mathbf{x}(t), \mathbf{u}(t), t) \end{array}$$

The conditions for this current value problem are: [2]

$$\left.\begin{array}{rcl} \mathscr{H}_{\mathbf{m}} & = & \dot{\mathbf{x}} \\ \delta\mathbf{m} - \mathscr{H}_{\mathbf{x}} & = & \dot{\mathbf{m}} \\ \text{argmax}_{\mathbf{u}\in U} \mathscr{H} & = & \mathbf{u} \end{array}\right\} \tag{2.12}$$

[2]Subscripts denote partial derivatives; $H_x = \frac{\partial H}{\partial x}$

If t is not an explicit argument of f or Π, the differential equation describing an optimal solution will be autonomous. In the problem we study in this assignment, we use the current value formulation.

2.3 Feedback rules

Usually in optimal control theory the control depends on time, costates and the states. We can use the maximum principle ($H_u = 0$) to eliminate the costate variable. When the optimal control u(t) is expressed directly as a function of time and the state variables, we have a feedback rule. For autonomous problems the control will only depend on the state variables.

A feedback rule has the quality that when the state is changing, the change in the control variable immediately follows. As Clark [5] describes them: *"Such control laws are simple to describe and to implement, and they are capable of responding to random fluctuations in the state variable and in the parameters of the problem."*

Models that are linear in the control gives rise to bang-bang policies. For one-dimensional problems, we define the switching function $s(t) = H_u$[3]. The most rapid approach will be to choose the control u that drives the population level $x = x(t)$ toward x^* as rapidly as possible. When u_{max} denotes the maximum feasible harvest rate, we have

$$u = \begin{cases} u_{max} & \text{whenever} \quad s(t) \geq 0 \\ u^* & \text{whenever} \quad s(t) = 0 \\ 0 & \text{whenever} \quad s(t) \leq 0 \end{cases}$$

When $s(t) = 0$ we have the optimal policy $u = u^*$. When $s(t) = 0$ cannot be sustained over an interval, we will have a bang-bang policy. Models resulting in bang-bang policy assume constant costs and prices and gives on/off policies. Bang-bang means that it is optimal to approach the steady state as quickly as possible. This is a special case of most rapid approach paths (MRAP) to reach the optimal solution.

In our model such a bang-bang solution is not realistic. Following a bang-bang policy we fish at a maximum when the fish stock is above it's optimal level. Over time the stock will be reduced to a level below the optimal. Now a prohibition against fishing is introduced. We will then wait for the stock to reach its optimal level before we again

[3] Subscripts denote partial derivatives; $H_u = \frac{\partial H}{\partial u}$

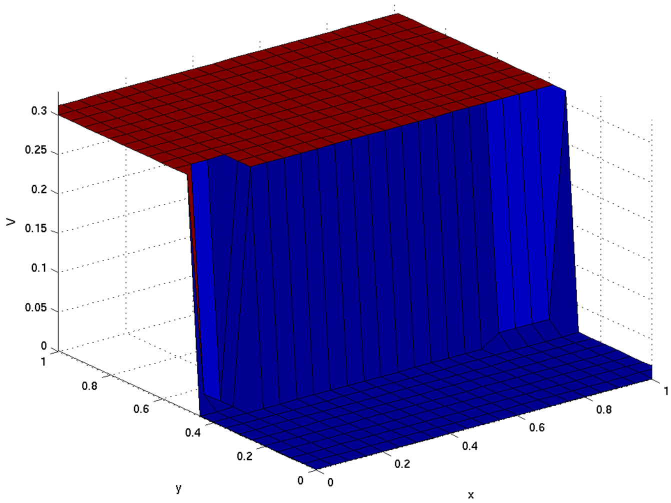


Figure 2.1: A bang-bang policy with the parameters taken from table 3.2 in chapter 3.3. This policy has been computed by ignoring the non-linear term in the Hamiltonian, i.e. b2=0. The results demonstrate the discontinuous bang-bang policy.

can start fishing. Such a bang-bang policy is very inconvenient for the fishery. Closing down a fishery completely seems to be an extreme action, particularly if the closure is expected to last for a longer period of time. However, a closure of the fishery would probably be accepted for a low stock level in order to let the stock replenish. But this bang-bang policy introduces a moratorium for all stock level below it's optimal (including relatively large levels close to it's optimal). As Sandal and Steinshamn [26] puts it: *"Although MRAP's have been shown to perform well for special cases (Clark[1976]), such paths are highly unmanageable and unrealistic in practice and are usually a result of oversimplification of the problem"*. However, linear models are very useful as advice on determining quotas (in relatively rare cases) when the stock is close to its optimal size .

When the Hamiltonian is nonlinear in the control we will have a nontrivial feedback control, characterized by an asymptotic approach to the equilibrium. We demonstrate nontrivial feedback controls in our example in the next chapter.

2.4 Dynamic programming

Dynamic programming can be applied on both continuous and discrete time problems and was developed by Richard Bellmann[4]. First we will arrive at the Hamilton-Jacobi-Bellmann-equation in continuous time setting, later we present it in discrete time setting. The HJB-equation is the fundamental partial differential equation for all problems in dynamic programming.

Dynamic programming is based on the principle of optimality:
If we have an optimal path, then the problem has to be optimal from every point on this path. That is; if we stop at some point on this curve, then the control over the remaining period must be optimal for the remaining problem. The initial conditions for this remaining problem is the state resulting from the early decisions.

2.4.1 Hamilton-Jacobi-Bellmann equation

We will now use this principle of optimality to arrive at the Hamilton-Jacobi-Bellmann equation.

We define $V(x_0, t_0)$ as the best value that can be obtained from the starting time t_0 in the state x_0.

$$\left.\begin{aligned} V(x_0,t_0) = \quad & \max_{\mathbf{u}} \int_{t_0}^{T} \Pi\big(\mathbf{x}(t),\mathbf{u}(t),t\big)dt + \varphi(\mathbf{x}(T),T) \\ & \dot{\mathbf{x}} = f(\mathbf{x}(t),\mathbf{u}(t),t) \quad ,\mathbf{x}(t_0) = x_0 \end{aligned}\right\} \tag{2.13}$$

This function is defined for all $0 \leq t_0 \leq T$ and for any possible x that may arise. It follows that

$$V(x(T),T) = \varphi(x(T),T)$$

Breaking up the integral

$$V(x_0,t_0) = \max_{\mathbf{u}} \Bigg(\int_{t_0}^{t_0+\Delta t} \Pi dt + \int_{t_0+\Delta t}^{T} \Pi dt + \varphi \Bigg) \tag{2.14}$$

where Δt is very small and positive. From the optimality principle we can argue that the control $\mathbf{u}$ should be optimal for the problem beginning at $t_0 + \Delta t$ in state $\mathbf{x}(t_0 + \Delta t) = x_0 + \Delta x$. Hence,

[4]Richard Bellmann(1920-1984) was an mathematician focusing on applied mathematics. He invented dynamic programming in 1953 and he is also known for important contributions in other fields of mathematics. In 1946 he received his PhD. at Princeton

$$V(x_0, t_0) = \max_{\mathbf{u}, t_0 \le t \le t_0+\Delta t} \left(\int_{t_0}^{t_0+\Delta t} \Pi dt + V(x_0 + \Delta x, t_0 + \Delta t) \right) \quad (2.15)$$

subject to $\dot{x} = f, \quad x(t_0 + \Delta t) = x_0 + \Delta x$

We now approximate the integral in (2.15) by $\Pi(x_0, \mathbf{u}, t_0) \cdot \Delta t$. Since Δt is very small we consider the control to be constant on the interval $(t_0, t_0+\Delta t)$. Further we assume V is sufficiently smooth and we expand the second term on the right by Taylor's theorem. We only consider terms of first order. By subtracting $V(x_0, t_0)$ from each side, then dividing through by Δt and letting $\Delta t \to 0$ we get[5]

$$\max_{\mathbf{u}} \left(\Pi(\mathbf{x}, \mathbf{u}, t) + V_t(\mathbf{x}, t) + V_{\mathbf{x}}(\mathbf{x}, t) \cdot \dot{\mathbf{x}} \right) = 0 \quad (2.16)$$

We then use the condition $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$ and get

$$\max_{\mathbf{u}} \left(\Pi(\mathbf{x}, \mathbf{u}, t) + V_x(\mathbf{x}, t) \cdot f(\mathbf{x}, \mathbf{u}, t) \right) + V_t(\mathbf{x}, t) = 0 \quad (2.17)$$

That is

$$-V_t = \max_{\mathbf{u}} \left(\Pi + V_x \cdot f \right) \quad (2.18)$$

This equation is known as the **Hamilton − Jacobi − Bellmann** equation for the continuous time setting.

It can be shown that this equation is consistent with the necessary conditions for an optimal solution. The first term on the right hand side is today's profit. V_x is the change in the value of the resource as the state changes, f is the growth function. Thus the equation says that the changes in the value through time must equal the maximized todays profit(Π) and change in the value according to a change in the state.

The Hamilton-Jacobi-Bellmann equation is a modified version of the Hamilton-Jacobi equation, known from classical physics.

From this equation we see that the optimal control is given by the state. This is the feedback solution: We consider a given time, observe the state and we seek the optimal policy based on these observations. This implies that the policy is a function of the state and the time, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$.

[5] Subscript denote partial derivatives: $V_t = \frac{\partial V}{\partial t}, V_x = \frac{\partial V}{\partial x}$

2.4.2 Discretization

Our model is formulated in continuous time, but is solved in discrete time setting. The continuous time model involves the assumption that the response of the population to external forces is instantaneous. Thus delay effects is not included in these models. We therefore introduce discrete-time models.

In this section we present a summary of the discretization technique given in the note by Grûne and Semmler [13]. We will not apply an adaptive grid since we obtain satisfactory solutions without it. The discretization procedure goes back to Capuzzo Dolcetta [2] and Falcone [8]. Further information can be found from Capuzzo Dolcetta and Falcone [3] or Bardi and Capuzzo Dolcetta [1]. The basic discretization technique is done in two steps. First we shift the time, then space. Note that the problem is almost autonomous, time is only present in the discount term. We have the following problem:

$$V(\mathbf{x}) = \max_{\mathbf{u}\in U} \int_0^\infty e^{-\delta t}\Pi(\mathbf{x}(t), \mathbf{u}(t), t)dt \qquad (2.19)$$

subject to $\frac{\partial}{\partial t}\mathbf{x}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$.

Step 1 :[6]

$$V_h(\mathbf{x}) = \max_{\mathbf{u}\in U} J_h(\mathbf{x}, \mathbf{u}) \qquad J_h(\mathbf{x}, \mathbf{u}) = h\sum_{i=0}^{\infty} \beta^i \Pi(\mathbf{x}_h(i), u_i), \qquad (2.20)$$

where $\beta = 1 - \delta h$ and $\mathbf{x}_h$ is defined by

$\mathbf{x}_h(0) = \mathbf{x}, \quad \mathbf{x}_h(i+1) = \varphi_h(\mathbf{x}_h(i), \mathbf{u}_i) := \mathbf{x}_h(i) + hf(\mathbf{x}_h(i), \mathbf{u}_i)$
h>0 is the discretization time step.

The optimal value function V_h is the unique solution of the discrete Hamilton-Jacobi-Bellman equation

$$V_h(\mathbf{x}) = \max_{\mathbf{u}\in U} \left\{ h\Pi(\mathbf{x}, \mathbf{u}) + \beta V_h(\varphi(\mathbf{x}, \mathbf{u})) \right\} \qquad (2.21)$$

We define the dynamic programming operator T_h by

$$T_h(V_h)(\mathbf{x}) = \max_{\mathbf{u}\in U} \left\{ h\Pi(\mathbf{x}, \mathbf{u}) + \beta V_h(\varphi(\mathbf{x}, \mathbf{u})) \right\} \qquad (2.22)$$

[6] In this case V_h does not denote partial derivatives.

then we can express V_h as the unique solution of the fixed point equation

$$V_h(\mathbf{x}) = T_h(V_h)(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathbb{R} \tag{2.23}$$

Step 2: we now approximate the solution on a grid Γ covering a compact subset Ω of the state space. We assume that for any point $x \in \Omega$ there exists at least one control value u such that $x + hf(x,u) \in \Omega$ is valid. We search an approximation $V_h^\Gamma(x^i)$ satisfying

$$V_h^\Gamma(x^i) = T_h(V_h^\Gamma)(x^i) \qquad \forall x^i \in \Gamma \tag{2.24}$$

When the points of evaluation, x, are not grid points, we determine them by interpolation. Note that we can obtain a feedback rule based on this approximation. (The control is given as a function of the state). We choose the value, u^* which maximize, equation (2.21). It is shown that this procedure will converge to the correct solution. For a rigorous convergence analysis of this discretization scheme, please turn to Bardi and Capuzzo Dolcetta [1] and Falcone and Giorgi [7].

2.5 Economic Interpretation

In this section we will give some economic interpretations of the multiplier function, the Hamiltonian and the necessary conditions for optimal solutions.

$\boldsymbol{\lambda}$: In Kamien and Schwartz [20] it is shown that along the optimal path, $\lambda(t)$ is the marginal value of the capital stock, x, at time t. That is, if the stock is reduced one unit, its value at time t will be reduced by $\lambda(t)$. $\lambda(t)$ is also referred to as the "shadow price", it is not the direct sale price, but loss of value for future productivity.

The Hamiltonian:

$$H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = \Pi(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}(t) \cdot f(\mathbf{x}(t), \mathbf{u}(t), t)$$

is the rate of increase of total assets. Π is the cash flow in the value function. f expresses the investment in capital, $\boldsymbol{\lambda}(t)$·f express the value of investment.

We have the first order condition: $H_u = 0$ for inner solutions. This maximum principle assures that the optimal control maximizes the rate of increase of total assets.

$$\frac{\partial \Pi}{\partial u} + \lambda \frac{\partial f}{\partial u} = 0$$

Clearly, from the state equation, the choice of u determines x. A decision taken at any time has two effects. It influences the profit earned at that time, and the change in the capital stock. Dorfman [6]interprets this condition as: *"It says that along the optimal path of the decision variable at any time the marginal short-run effect of a change in the decision must just counter-balance the effect of that decision on the total value of the capital stock an instant later."* Further, he states that the control should at every moment be chosen *"[]... so that the marginal immediate gain just equals the marginal long-run cost..."*

The second condition is: $\dot{\lambda} = -H_x$:

$$-\dot{\lambda}(t) = \frac{\partial \Pi}{\partial x} + \lambda \frac{\partial f}{\partial x}$$

$-\dot{\lambda}(t)$: Expresses the rate of depreciation of the capital(Dorfman [6]). The rate of depreciation along an optimal path should be equal to the marginal net increase of the value of the capital.

Now it is easier to understand that if T is free, then we must have $\lambda(T) = 0$. We exploit the resources as long as the marginal value is positive and terminate the project when it becomes zero. If the adjoint variable had a positive value in the terminal time, this implies that profit would be increased by further exploiting the stock. This is also the case with an infinite time horizon, the discount factor ensures that the present value of stock declines asymptotically to zero, i.e. $\lambda(T) \to 0$ as T$\to \infty$.

The use of a discount rate in the calculations of the optimal harvest have been criticized for lack of importance of future value. We discount all values back to time zero, which implies that we value the present generation's utility of the resource higher than a later generation's utility.

Chapter 3

Results

In this chapter we will use the theory from chapter two on our problem. First we apply the Pontryagin Maximum Principle on our problem with two control variables and two state variables. In chapter two we stated the principle for a fixed time interval, the case with an infinite horizon follows from it, please turn to Seierstad and Sydsæter [29]. We then analyze the equilibrium and obtain analytical results. Thereafter we employ our model on an example and obtain feedback solution by the Hamilton-Jacobi-Bellmann equation. We now turn to lowercase letters for the scaled variables. If nothing else is stated, it is the scaled variables that are displayed.

3.1 The Problem in equilibrium

The problem was defined as

$$\max_{u,v} \int_0^\infty e^{-\delta t} \Pi(x, y, u, v) dt \tag{3.1}$$

$$\Pi(x, y, u, v) = \sigma\Big(\Big(1 - \frac{c_1}{x}\Big)u - b_1 u^2\Big) + \Big(1 - \frac{c_2}{y}\Big)v - b_2 v^2$$

Subject to:

$$\begin{aligned} \dot{x} &= sx^2(1-x) - \alpha xy - u && (3.2)\\ \dot{y} &= y(1-y) + \beta xy - v && (3.3)\\ x, y, u, v &\geq 0, \quad x(0) = x_0, \quad y(0) = y_0 && (3.4) \end{aligned}$$

We first formulate the current value Hamiltonian:

$$\mathscr{H} = \Pi(x,y,u,v) + m[sx^2(1-x) - \alpha xy - u] + n[y(1-y) + \beta xy - v]$$

The first-order conditions for optimum yields the following equations:

$$\begin{aligned}
\dot{x} &= sx^2(1-x) - \alpha xy - u && (3.5)\\
\dot{y} &= y(1-y) + \beta xy - v && (3.6)\\
\dot{m} &= \gamma m - [\sigma \frac{c_1}{x^2} u + m(2sx - 3sx^2 - \alpha y) + \beta yn] && (3.7)\\
\dot{n} &= \gamma n - [\frac{c_2}{y^2} v - xm\alpha + n(1 - 2y + \beta x)] && (3.8)\\
u &= \text{argmax}\mathscr{H} && (3.9)\\
v &= \text{argmax}\mathscr{H} && (3.10)
\end{aligned}$$

We seek a steady-state equilibrium solution of the equations above, thus we put them equal to zero. Note that if the optimal control u(t) lies in the interior of the control interval, (3.9) implies that $\frac{\partial \mathscr{H}}{\partial u} = 0$. $u = v = 0$ has already been discussed(section 1.3). We seek an equilibrium where $u > 0$, $v > 0$. Thus, the remaining alternatives are $u > 0, v = 0$ or $v > 0, u = 0$. We assume that we have inner solutions and the last two equations can be written as $\mathscr{H}_u = 0$ and $\mathscr{H}_v = 0$.

$$\begin{aligned}
0 &= sx^2(1-x) - \alpha xy - u && (3.11)\\
0 &= y(1-y) + \beta xy - v && (3.12)\\
0 &= \gamma m - [\sigma \frac{c_1}{x^2} u + m(2sx - 3sx^2 - \alpha y) + \beta yn] && (3.13)\\
0 &= \gamma n - [\frac{c_2}{y^2} v - xm\alpha + n(1 - 2y + \beta x)] && (3.14)\\
0 &= \sigma\Big(1 - \frac{c_1}{x} - 2b_1 u\Big) - m && (3.15)\\
0 &= 1 - \frac{c_2}{y} - 2b_2 v - n && (3.16)
\end{aligned}$$

- From (3.11) and (3.12) it is obvious that the catch equals the growth for both the predator and the prey in equilibrium.

$$\begin{aligned}
u &= sx^2(1-x) - \alpha xy && (3.17)\\
v &= y(1-y) + \beta xy && (3.18)
\end{aligned}$$

- In the case without harvesting equation (3.12) possesses two equilibrium solutions, $y = 0$ and $y = \beta x + 1$. In the single species model the predators carrying capacity is 1, positive growth from predation increases it in the two species model. Equation (3.11) gives that $x = \frac{1 \pm \sqrt{1 - \frac{4\alpha y}{s}}}{2}$ or $x = 0$ in the case of no harvesting. This gives three equilibrium solutions, $x = 0$, $x = K_0$ (obtained by the minus sign above) and $x = \widehat{K_1}$ (by the positive sign above). In the single species model carrying capacity fro prey is 1, the presence of predator decreases its upper limit to a level below. The value $x = K_0$ is called the minimum viable population level. If the prey stock is below K_0, it will be depleted (critical depensation). For a more detailed discussion, please turn to Clark [5].

- (3.11) gives that $y \leq \frac{sx(1-x)}{\alpha}$, (3.12) gives that $y \leq 1 + \beta x$

  If the predator stock exceeds this limit for small prey stocks it could drive the prey to extinction. It is expected to have an upper limit for the predator stock derived from the stock level of prey in order to avoid extinction of prey.

- From (3.15) and (3.16) we notice that the shadow price is equal to the marginal profit and that we have upper bounds on the shadow prices in equilibrium. Remember the interpretation of m and n as the marginal value of the state variable at time t.

  (3.15) $\Rightarrow$

$$0 = \sigma\left(1 - \frac{c_1}{x} - 2b_1 u\right) - m \tag{3.19}$$

$$m = \sigma\left(1 - \frac{c_1}{x} - 2b_1 u\right) \tag{3.20}$$

$$m = \sigma\frac{\partial \Pi_1}{\partial u} \quad \text{or} \quad m \leq \sigma\left(1 - \frac{c_1}{x}\right) \tag{3.21}$$

  In the right expression we have used the fact that $0 \leq 2b_1 u$

  (3.16) $\Rightarrow$

$$0 = 1 - \frac{c_2}{y} - 2b_2 v - n \tag{3.22}$$

$$n = 1 - \frac{c_2}{y} - 2b_2 v \tag{3.23}$$

$$n = \frac{\partial \Pi_2}{\partial v} \quad \text{or} \quad n \leq 1 - \frac{c_2}{y} \tag{3.24}$$

In the right expression we have used the fact that $0 \leq 2b_2v$ These bounds in equilibrium sound reasonable because if investment pays off, we should stop harvesting and invest in natural assets at the maximum rate.

- (3.14) $\Rightarrow \quad 0 = \gamma n - \frac{c_2}{y^2}v + xm\alpha - n(1-2y+\beta x)$

$$\begin{aligned} 0 &\leq \gamma n + xm\alpha - n(1-2y+\beta x) \\ \beta xn - xm\alpha &\leq n(\gamma - 1 + 2y) \\ x &\leq \frac{1-\gamma-2y}{\frac{m\alpha}{n} - \beta} \end{aligned}$$

  We will have an upper bound on the prey stock. We have used the fact that $\frac{c_2}{y^2}v \geq 0$

- It is obvious that we have to assume that the fishery will have an equilibrium in the case of no harvesting. We have the equations

$$0 = sx^2(1-x) - \alpha xy \tag{3.25}$$
$$0 = y(1-y) + \beta xy \tag{3.26}$$

  From the discussion in section 1.3 we found that we had an equilibrium if $(\beta\alpha - s)^2 \geq 4\alpha$. α and β are unsure parameters and we assume that they can be chosen to fit this equation.

The equations (3.11)-(3.16) consists of six equations and six unknowns that determines the equilibrium. By substitution the system can be reduced to two equations in x and y. The roots of these polynomials are the mathematically possible equilibriums. We choose the values that are meaningful for our problem. These polynomials can be used to study how the equilibrium depend on different parameters in the problem. We will use the solve command of Maple. A Maple code is given in the appendix.

We now set the parameters to study how equilibrium changes with the scaled discount rate $\gamma = \frac{\delta}{r_2}$. An increase in γ results from an increase in the discount rate δ or a decrease in r_2(intristic growth rate for predator). In this section the discussion of the discount rate implies discussion of the scaled discount rate. The parameters we use are given in table 3.2.

The interval of the scaled discount rate implies an unrealistic span of discount rates, however the interesting values lie inside this interval, see figure 3.1. Traditionally one expects that an increase in the discount rate will result in an increase in harvest and a following decrease in the optimal stock level. This is true for one-dimensional linear models. In nonlinear multidimensional models, Sandal and Steinshamn [28] show that this is not necessarily true. A higher discount rate leads to a higher out-take in economic terms, but this does not necessarily lead to a lower stock. If the demand is inelastic, a lower harvest would lead to higher prices and lower harvest would result in a higher stock level if we are to the right of the maximum sustainable yield. We will now study the effect of different discount rates on the equilibrium in our model.

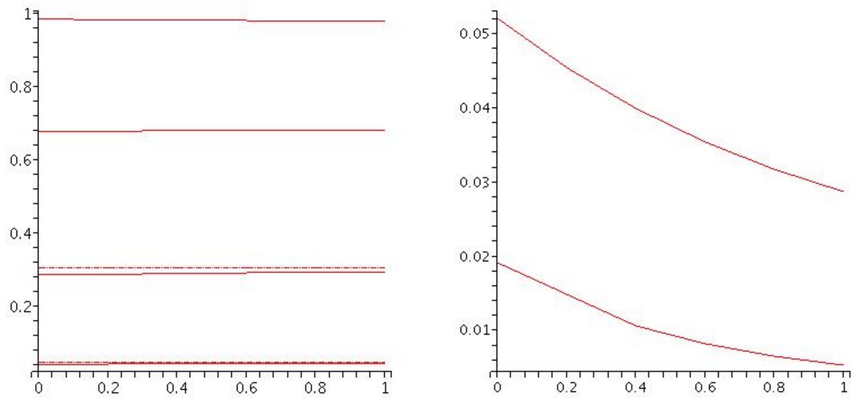


Figure 3.1: *Equilibrium as a function of the scaled discount rate, γ.* The left figure shows equilibrium for x,y,u,v, state and harvest for prey and predator, respectively. x is the upper curve, y is the second upper curve, v_{max} is the middle curve, v is the second lower curve and u_{max} and u is the lower curve. In the left figure the costates n(upper curve) and m are displayed, associated with y and x, respectively.

Our first conclusion from figure 3.1 is that the changes in the steady state stocks with the discount rate are relatively small.

The harvest of predator(v) increases as the discount rate increases, which results in a lower stock level of predator. This result is clearly a parallel to the single species case. The increased discount rate makes it more costly to keep a large stock, and we transmute a part of it into capital, yielding rent as expressed by the discount rate. For the prey we also notice that the harvest(u) increases with the discount rate. In a single species model we would expect a subsequent decrease in the stock level. However, we have a decreasing predator stock which affects the growth positively.

When we maximize the net revenue w.r.t. the control, this is clearly the best we can hope for. We plotted this control (static optimum, we denote it by u_{max}, v_{max}) as dotted lines in the figure. As the discount rate increases, future is less important and we expect the harvest to be closer to this level. It is shown that the harvest of the predator approaches v_{max} asymptotically, as the discount rate increases. This also goes for the prey where the harvest tracks u_{max} quite closely with increased discount rate.

The right figure in 3.1 shows the changes in the shadow prices as a function of the discount rate. As the discount rate increases, it makes today's income more important than tomorrow's. This implies that the marginal value of investment in the resource decreases.

3.2 How we obtain feedback solutions

In this section we will describe the method we use to solve our problem. The problem is given in a continuous time setting, such that the first step is to formulate the problem in discrete time, as described in section 2.4.2 on page 20. We then calculate the solution numerically by using the Hamilton-Jacobi-Bellmann equation, introduced in the same section. The Hamilton-Jacobi-Bellmann equation is given as:

$$V_h(\mathbf{x}) = \max_{\mathbf{u}} \left(h\Pi(\mathbf{x}, \mathbf{u}) + \beta V_h(\varphi(\mathbf{x}, \mathbf{u})) \right) \tag{3.27}$$

where $\beta = 1 - \delta h$, h>0 is the time step, δ is the discount rate and

$$\varphi(\mathbf{x}, \mathbf{u}) = \mathbf{x} + hf(\mathbf{x}, \mathbf{u})$$

In this discrete time problem, the time is divided into two periods; The first period which is the instant time, (it could be a year, a season, a day) and the next period which is all future. h is the time step, that is, the length for the first period. The state in the current period is x, the state in the next period is $\varphi(\mathbf{x}, \mathbf{u})$. The left hand side is the value of our problem. The first term on the right hand side is h times current profit, β is the discount term multiplied by the future value. This implies that the value of our system is the current profit plus the discounted future value of the resource, evaluated in the maximizing policy. Our original problem is given in continuous time, so we will choose this h small to get the approximation as close as possible.

Experience shows that it is most efficient to combine two iteration methods to solve the HJB-equation. In the policy iteration we maximize the right hand side of (3.27) with respect to the control variable, and store the maximizing control as the optimal policy in a matrix. Value iterations assume a fixed policy and apply the Hamilton-Jacobi-Bellmann equation;

$$V_h(\mathbf{x}) = h\widehat{\Pi}(\mathbf{x}) + \beta V_h(\widehat{\varphi}(\mathbf{x})) \tag{3.28}$$

Obviously, policy iteration evaluates much slower than value iteration. Usually policy iteration will settle with a lot fewer iterations than the value iteration, but without the maximization the value iteration evaluates much faster. We therefore perform many value iterations until the value has settled. We next perform some policy iterations until the policy has settled. As long as the estimated error(given as the difference in the value between the two last approximations) is larger than the tolerance error given in input, the program will keep on doing many value iterations and some policy iterations.

It has been shown that this scheme will converge for all initial conditions, (see section 2.4.2). We could only perform policy iterations and still reach the solution, but we know that it is much more efficient to combine policy and value iterations.

We summarize our algorithm:

1. Perform one policy iteration
2. Perform several value iterations
3. Perform a few policy iterations
4. Repeat step 2 and 3 until the estimated error is small enough

The value of V_h for points x which are not grid points are determined by interpolation. If the vector field given by $\varphi(\mathbf{x}, \mathbf{u})$ points out of the state space, we set it equal to the boundary. In the discretization process h is the time step. Small h gives high accuracy, we have used h=0.05 in the example. From our scaling in chapter one we have the new time, $\tau = r_2 t$. Our time step equal to 0.05 is very small, and in reality difficult to handle. It would be a challenge to measure the stock and regulate the harvest based on this short interval.

3.3 Examples

In this section we were planning to present an example related to Norwegian Fishery Policy. However, in lack of literature on correct parameters this example needs to be considered as results from our model used on a general predator-prey relationship. We apply our method on a fishery similar to North-East Arctic Cod(NEAC, Gadus morhua), the single most important fishery in Norway and its prey, Barents Sea Capelin(Mallotus villosus).

3.3.1 Predator(Cod) and one of it's Prey

The NEAC is the most important cod stock in Norwegian fisheries and its main habitat is in the Barents Sea. (ICES fishing area Ia, IIb1 and IIa2, figure B.1, on page 61 cover most of this area). The Barents Sea Capelin is the most important prey item for the stock of NEAC in the Barents Sea, and cod is also the most important predator on capelin in this area. The Barents Sea is one of the richest ocean areas. It is capable of maintaining large fish populations including cod, capelin and approximately 150 other fish species.

In this section our main aim is to find the optimal levels of exploitation of the fish resources, studied in a predator-prey relationship. The code for programming in Matlab is given in the Appendix. We use estimated parameters given in table 3.1. Note that these are probably not correct parameter for NEAC and Barents Sea Capelin, but they are chosen to be in the direction of this fishery. The profit function for predator(NEAC) is taken from the article by Kugarajh, Sandal and Berge [21]. The discount rate is set at 5 per cent. We scale our problem and give the calculated values of the scaled parameters in table 3.2.

We know that the cod is a very important and valuable fishery. In fact, the Barents Sea cod stock is potentially the largest cod stock in the world (Jakobsson [19]). For the capelin we know that during the 1970s and 1980s between 90 and 99% of the landings were used as reduction to fish meal and oil. The rest has mostly been used for fresh and frozen products and for roe production. This leads to the conclusion that we consider a valuable predator and a "cheep" prey. From our introduction on Marine Predators in 1.1, we can then conclude that the combination of yields should be closer to the area B in figure 1.1. That is, we expect a greater harvest of cod than capelin. Note that we also need to consider the value of the prey eaten by the predator, the biological cost. It might not be a good investment to let a large part of the prey be eaten

Parameter	Value	Description
r_1	0.0002	Intristic growth rate for prey
r_2	0.48	Intristic growth rate for predator
K_1	11500	Carrying capacity for prey
K_2	5000	Carrying capacity for predator
a	0.000096	loss because of predator
b	0.000017	gain from prey
p_1	1	Price parameter for prey
p_2	10.527	Price parameter for predator
C_1	0	Parameter for prey
C_2	8864	Parameter for predator
B_1	0.002	Cost parameter for prey
B_2	0.005973	Cost parameter for predator
σ	0.2185	Parameter to measure the profits equal

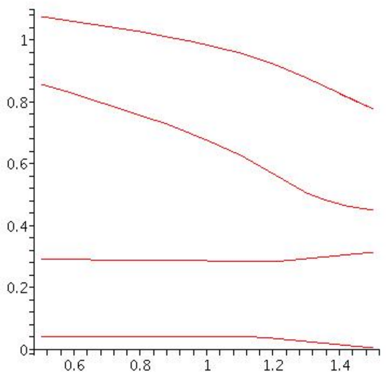
Table 3.1: Parameter values

Parameter	Definition	Value
γ	$\frac{\delta}{r_2}$	0.10
s	$\frac{r_1 * K_1}{r_2}$	4.79
α	$\frac{aK_2}{r_2}$	1
β	$\frac{bK_1}{r_2}$	0.41
b_1	$\frac{B_1 r_2 K_1}{p_1}$	11.04
b_2	$\frac{B_2 r_2 K_2}{p_2}$	1.36
c_1	$\frac{C_1}{K_1 p_1}$	0
c_2	$\frac{C_2}{K_2 p_2}$	0.17

Table 3.2: Scaled parameter values

by the predator, it could be more profitable to harvest the prey directly.

On the left side in figure 3.2 we show equilibrium values as a function of the coefficient, α, which is the scaled predator coefficient. A larger α would yield lower growth for the prey and both equilibrium stock level and harvest are decreasing. It follows that a decrease in the prey stock in time would yield higher catch and a lower stock of predator. In the right figure we show the equilibrium levels for stock level and harvest of predator as a function of the coefficient, β. From the figure it is clear that the stock level and harvest of predator are increasing as β is increasing. This is also expected, a larger β will increase the growth curve for predator. For the prey, an increased stock level of



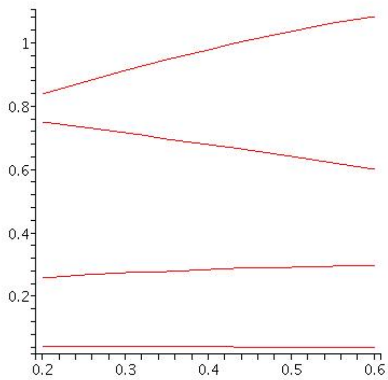


Figure 3.2: In the left figure: Equilibrium values as a function of α. In the right figure: Equilibrium as a function of β. Stock level of cod is the upper curve, level of capelin the second upper curve, harvest of cod is the second lower curve and the lower curve is harvest of capelin.

predator would yield a higher predation pressure, and both stock level and harvest decreases.

Prey appears in shoals, and we assume that harvesting cost does not depend on the size of its stock. This is why we set $C_1 = 0$. The figure 3.3 demonstrates the optimal harvest policy(u) for prey as a feedback policy. At the first sight the figure looks strange, we have a “gap” in the middle of it. For small prey stocks the figure tells us to harvest at a high rate. Depending on the size of the predator stock we do not harvest for larger prey stocks. Further increase in the prey stock yield positive harvest. It is obvious that we have a critical level for prey given by the size of the predator stock (critical depensation). If the prey stock is below this level it will anyhow be depleted, and we harvest at a high rate. A prey stock close to its critical level from above should not be harvested and as we see from the figure, a moratorium is introduced. Positive harvest is introduced for higher stock levels of prey when the stock is not threatened by extinction. From equation (3.11) we found that the prey had a minimum viable population. If the population for some reason is below this level, it will lead to extinction of the stock. A larger predator stock increases the minimum viable population for prey. When the stock of predator is close to zero(along the x-axis), we notice from the figure that we can harvest at a high rate for all stock levels of prey.

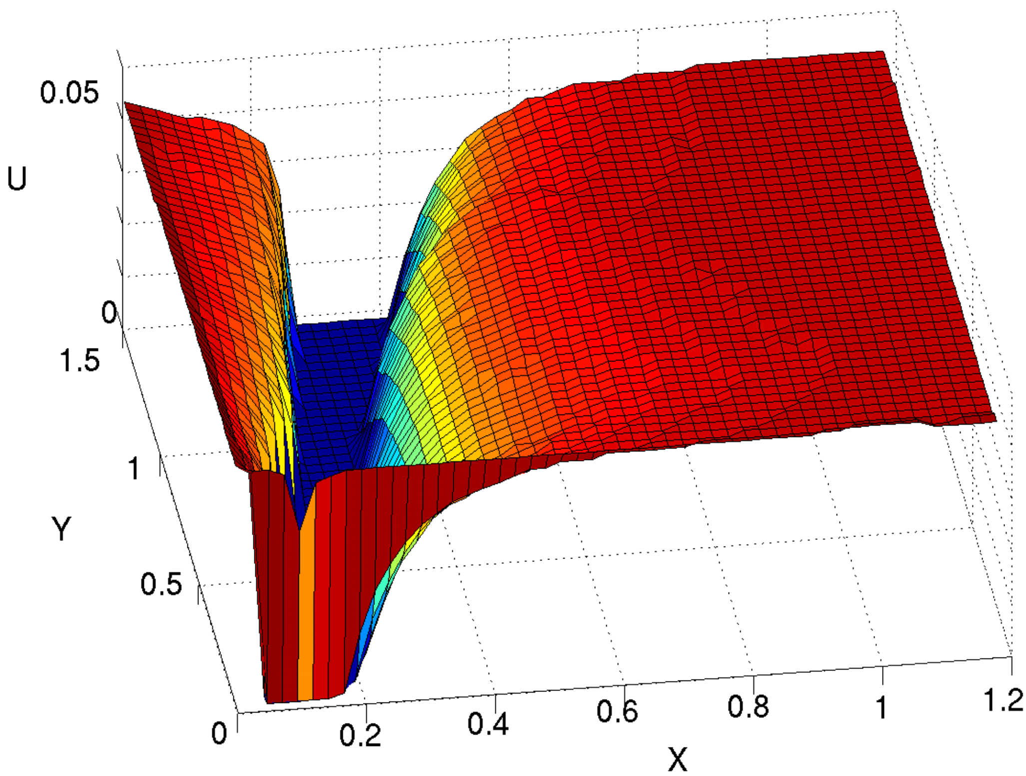


Figure 3.3: Optimal harvest for prey, x is the prey stock and y is the predator stock. Prey is modeled by depensation, which explains harvest for a small prey stock.

From figure 3.4 we see that harvest of predator is increasing as its standing stock increases. The harvesting cost of predator increases as its stock decreases, and harvest for a low stock level of predator is not profitable. We notice that when prey has reached a level of approximately 0.15 and the predator stock is at a level of 0.3, the harvest(v) increases further. But as the prey stock grows to a level above, it is somehow reduced. This appears for larger stock values as well. The increase and the following decrease looks like a wave in the policy. This needs to be investigated more closely.

When we study the stock levels where the harvest(v) is “on the wave”, we notice that these values correspond to the “gap” in the optimal harvest of prey, figure 3.3. The explanation is clear: In the gap prey is

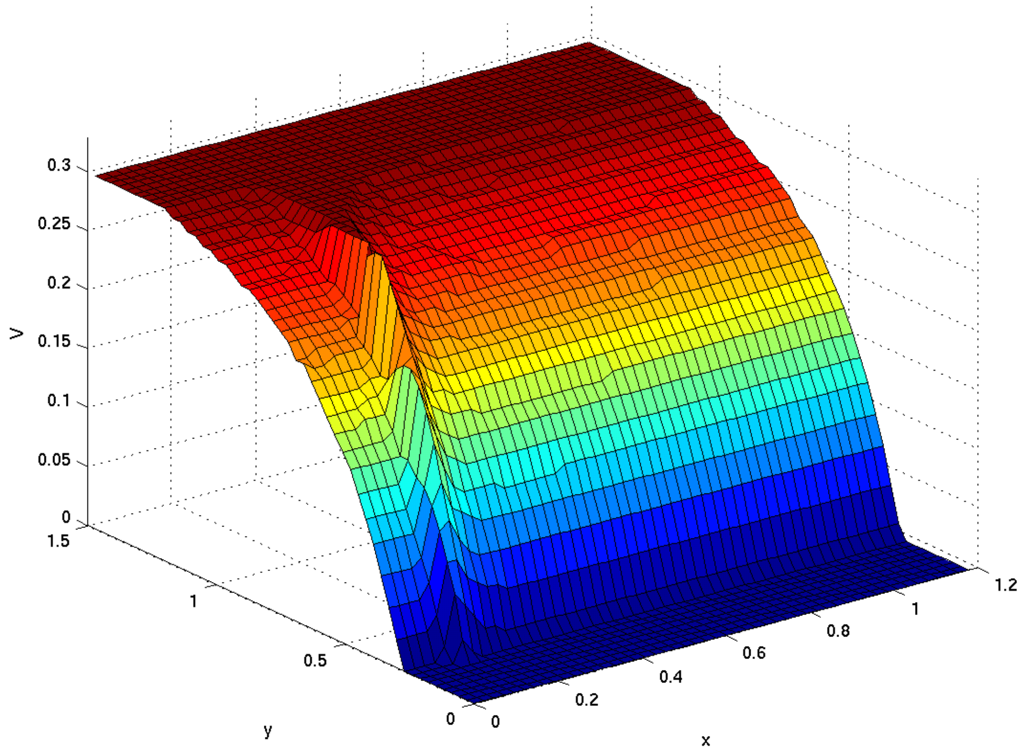


Figure 3.4: Optimal harvest for predator (cod), x is the prey stock and y is the predator stock. We observe a "wave" for corresponding values where we had a "gap" in the optimal harvest of prey, figure 3.3.

close to being extinct and we have to be careful harvesting it. In addition, the predator stock reaches a level where it could threaten prey and we find it optimal to increase the harvest of the predator.

We introduced the static optimum in a previous section. It is found by maximizing the net revenue (w.r.t. the policy) and is clearly the best we can hope for. The static optimum for predator is

$$\frac{\partial \Pi_2}{\partial v} = 0 \Rightarrow v_{max} = \frac{1 - \frac{c_2}{y}}{2 \cdot b_2} = \frac{1 - \frac{0.1684}{y}}{2.7236}$$

In figure 3.5 we plot the optimal harvest for predator in the same figure as v_{max}. From the figure we notice that our optimal policy for predator exceeds its static optimum for some stock levels. The level where the

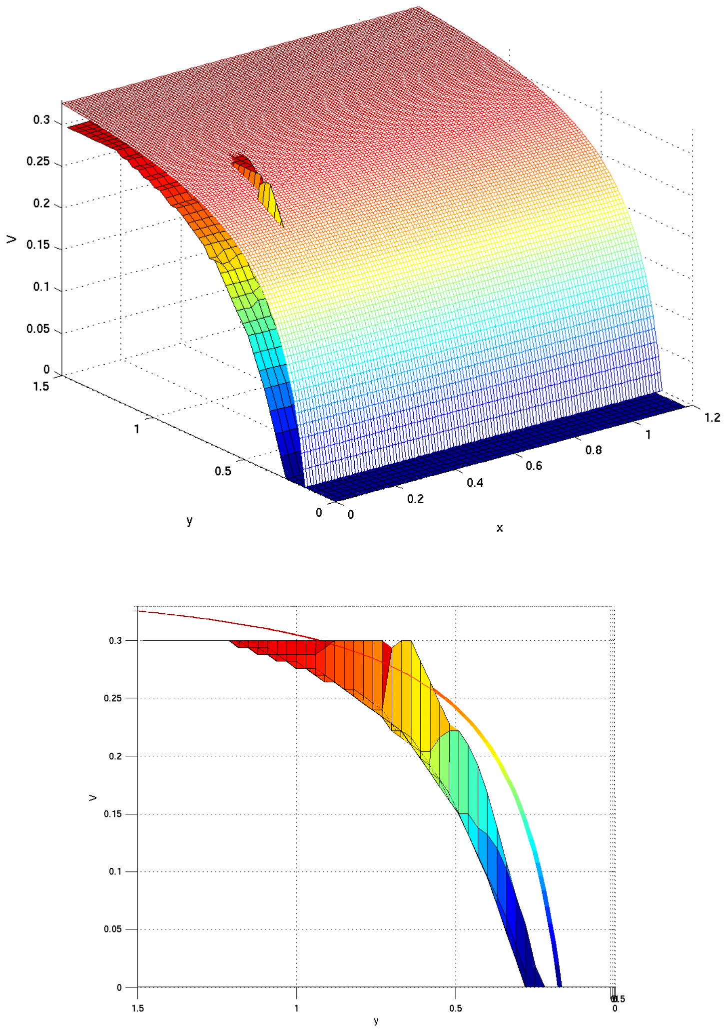


Figure 3.5: Static optimum harvest for predator, v_{max}, compared to its optimal harvest. In an aerea where prey is close to be driven to extinction we observe that our optimal harvest of predator exceeds its static optimum.

optimal policy(v) exceeds v_{max} corresponds to levels in the left bottom of the gap in figure 3.3. This implies that prey is close to its critical level, and it is then optimal to harvest more of the predator in order to reduce the predator pressure. This action might save the prey from extinction.

The optimal harvest for prey is below or equals $u_{max} = 0.0453$ for all stock levels. Along the x and y-axis and for large values of prey its optimal harvest is practically equal to the static optimum of prey.

The profit function for predator is depending on the stock size, y, which gives predator an economic protection. When the stock decreases it will become less profitable to harvest. For some stock level the profit will become negative and harvesting will not be optimal, i.e, v=0. The profit from harvesting prey does not depend on the stock size, x ($C_1 = 0$). For a small prey stock, our solution tells us to harvest at a high rate, a moratorium is introduced when we have a stock of prey large enough to survive.

We know that the prey will be exterminated for small stock levels. The reason for this could be consumption by the predator or our harvest. If the extinction is caused by the predator, we could increase the harvest of predator for small stock levels of prey. In addition a dependency of the prey stock, x, in the utility function ($C_1 > 0$) would lead to a decreased harvest for small stock levels of prey and probably introduce a moratorium on the prey fishery for a small stock. This could might rescue the prey from depletion. However, with our utility functions, a moratorium for small stock levels of prey is not optimal.

From the value plot, figure 3.6, we see that the value does not have the same form in the x-direction and in the y-direction. This is caused by the different growth functions. The prey, x, is modeled by using a growth function with depensation, that is, when the sea is short of prey the growth is small and in presence of predator it can be negative. A stock of prey large enough to survive gives an increase in the profit, which is displayed as a wave in the x-direction. We notice that for a larger predator stock the wave "starts" at a higher prey level, the critical prey level increases with the predator stock. To the left of the wave the prey stock has a small or no value caused by critical depensation. For the predator we used the logistic growth function. The growth is large even when the stock(y) is at a low level, the predator stock can recover from any positive level. This growth function yields a high increase in the value, displayed as the steep wall along the x-axis. The value increases from a low level when the predator stock(y) is zero, to a much higher level in a very short time as the predator stock increases. The value does not reach the expected value zero, an empty sea should not be profitable. This is due to our harvest of prey for a small standing stock.

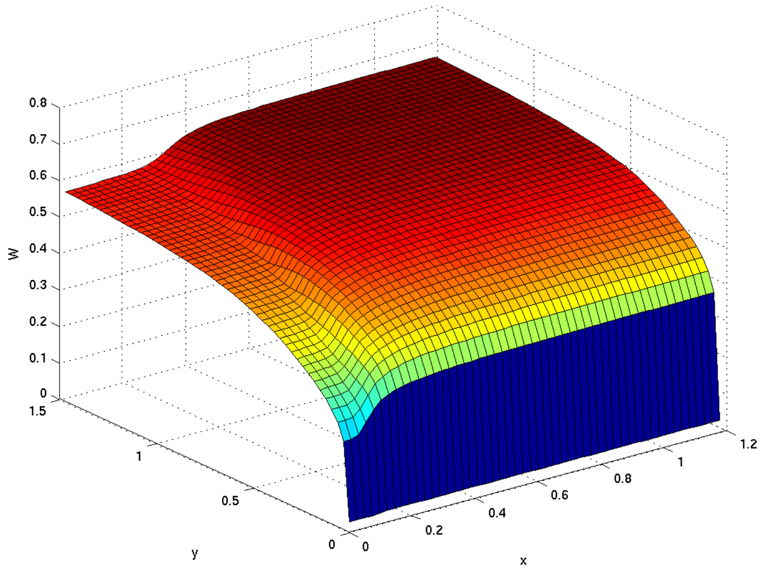


Figure 3.6: Value of the fishery

The vector field, figure 3.7, shows how the state will move in time when our optimal policy is applied. We observe two equilibrium points in the vector field. One is unstable, and occurs for a low prey level. The other one is stable and occurs at (x,y)=(0.68,0.985). It exists a stable equilibrium and this point will be our solution. More data on the steady state is found in table 3.3. When we first arrive at the stable equilibrium point, these stock levels can be sustained for all future. Note that we will probably never reach the exact equilibrium point, but the stock is close to the theoretic steady state, and in practice equal to it. From the vector field it is clear that for small prey levels, prey could be driven to extinction by following our optimal policy.

We next compare some starting points in the vector field. We know that the Barents Sea Capelin have had two collapses, in the first one in 1987 with stock levels ($y = 0.224$, $x = 0.0093$). When we start at this collapse level, we end up along the y-axis, i.e. the prey will be depleted. We thereafter compare starting points close to the gap in the policy for prey, figure 3.3. We notice that if we start in a point to the left of the gap, prey will be depleted by following our optimal policy. A starting point in the gap or to the right of the gap will give a solution path which ends in the stable equilibrium point.

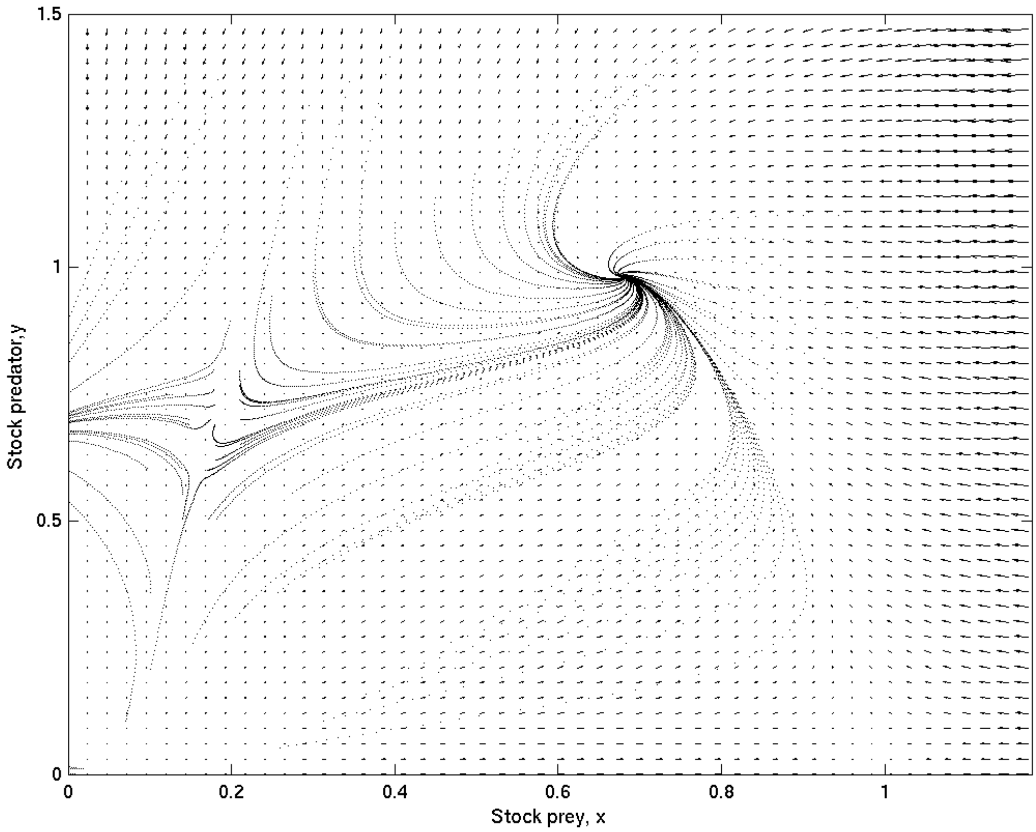


Figure 3.7: Vector field

In our model a collapse as we have had in the history of Barents Sea Capelin would drive the prey stock to extinction. However, the government introduced a moratorium on the prey fishery. This gave the stock of prey a chance to replenish. In our model we harvest capelin at a low stock level, and we introduce a moratorium as optimal when the prey stock is large enough to survive. For the predator stock fixed at $y = 0.22$ (which was the level of predator during the collapse) we have a moratorium for approximately $0.1 \leq x \leq 0.22$. During the collapse we had $x = 0.0093$.

The vector field shows the direction of the solution, but the time it will take to reach the equilibrium is not displayed. Figure 3.8 shows the solution curve in the vector field and the developments of stocks and harvests over time. ICES 2005 [16] gives the actual stocks for NEAC and Barents Sea Capelin. We evaluate the average from the years 1990-2000 and find $x = 0.22$, $y = 0.32$. From the vector field we observe that the system will reach the equilibrium solution after about 15 years.

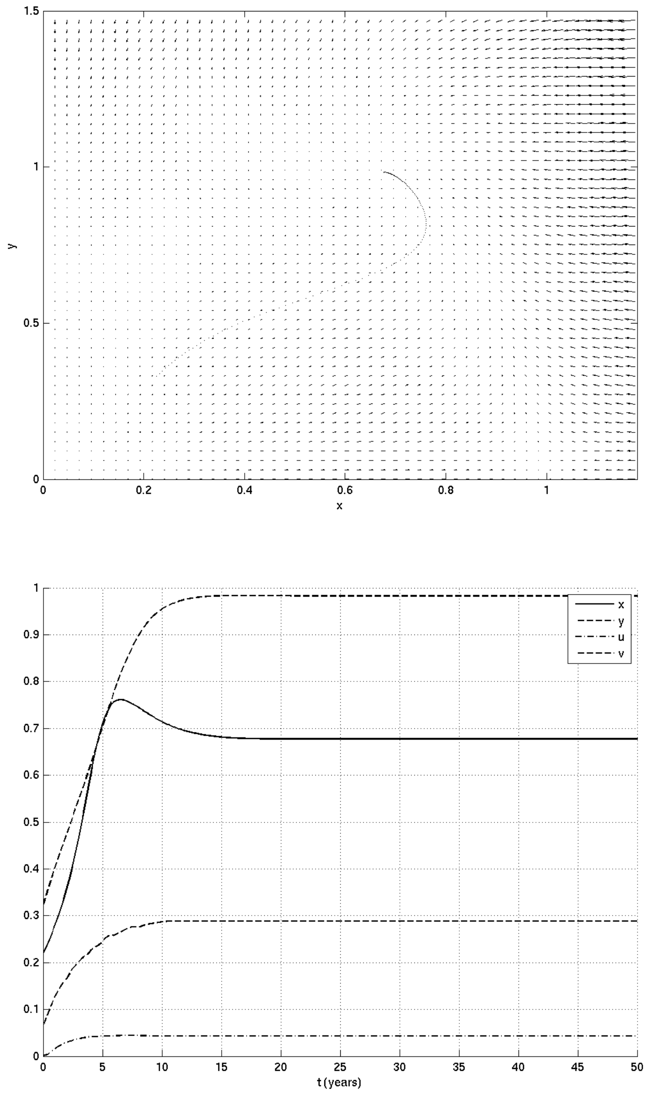


Figure 3.8: Gives the solution path in the vector field and the corresponding development of state and harvests as a function of time. The initial stocks are (x,y)=(0.22,0.32). x is the stock of prey, y is the stock of predator.

	Scaled values	*Real values*
Standing stock Prey	0.6775	7 792 000 tonnes
Standing stock Predator	0.9850	4 925 000 tonnes
Annual catch Prey	0.0419	231 000 tonnes
Annual catch Predator	0.2866	688 000 tonnes

Table 3.3: Steady state equilibrium for predator and prey. These values are given by the Maple program (Appendix B).

In the case without harvesting the phase plane consists of four isoclines. The intersection of the isoclines are the equilibrium points of the system. From $\dot{x} = 0$ we have two isoclines, the y-axis($x = 0$) or the parabola from equation (3.11) with $u = 0$. From $\dot{y} = 0$ we have the x-axis($y = 0$) and the straight line from equation (3.12) with $v = 0$. This results in five equilibrium points; (0,0),(0,1),(1,0) and the intersections of the parabola and the straight line. (0,0),(0,1) are unstable, (1,0) have trajectories towards the point when x is weak. The remaining points have been discussed in section 1.3. In the case of harvesting it is clearly seen from our vector field (figure 3.7), that the equilibrium points are modified from the case without harvesting. We now have the equilibriumpoints; (0,0),(0,1),(1,0) and the intersection of equations (3.11) and (3.12) following our optimal policy. The lower point of intersection between the line and the parabola is a saddle point, the entry to the equilibrium point is the tangent of the parabola and the vertical line through the equilibrium point. The upper point of intersection is a stable focus, the curves are spirals converging toward the point.

The next figure provides plots that shows the equilibrium as a function of the parameter b_1 and b_2. A larger b implies that the market is more sensitively on quantity and result in a price reduction. The left figure in 3.9 the equilibrium as a function of b_1 is displayed. An increased b_1 makes the prey less profitable, and the optimal harvest is at a lower level. This gives a higher stock level for prey and a following increased stock level for predator. From equation (3.20) the shadow price for prey decreases as b_1 increases, a lower price yields a lower marginal profit. A lower b_1 would probably fit the corresponding parameter for Barents Sea Capelin better. The harvest of prey would then be at a higher level, which is closer to the expected optimal harvest of Barents Sea Capelin.

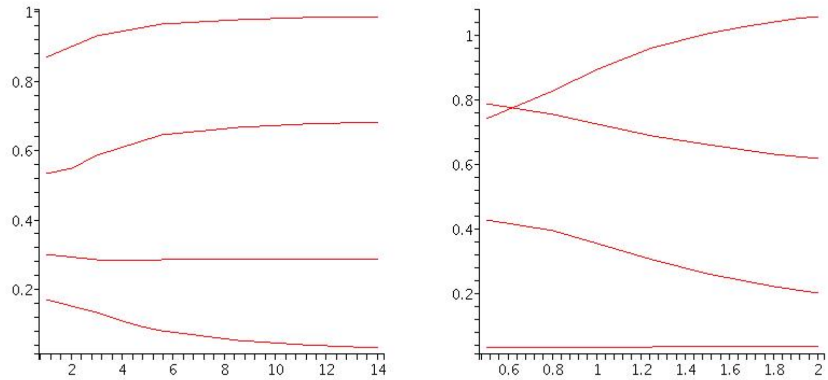


Figure 3.9: On the left: Equilibrium as a function of b_1. On the right: Equilibrium as a function of b_2. In both figures the upper curve represents stock level of cod(y), the second upper curve is stock level of capelin(x) the second lower curve shows harvest of cod(v) and the lower curve is harvest of capelin(u).

The right figure in 3.9 displays the changes in the steady state equilibrium as the b_2 parameter varies. A higher b_2 results in a lower marginal profit from predator harvest. The harvest of predator decreases and we obtain an increase in the predator stock. This will increase the predator pressure and the steady state for prey stock declines. A lower marginal profit for predator indicate a decrease of its shadow price.

Comparison

Now we compare our optimal harvest with the actual harvest for the predator. We have chosen our parameters for the predator close to the actual parameters for NEAC. ICES 2005 [16] gives the history of stock sizes and harvest for NEAC, we used data from the years 1972-2004. Now we compare it with our estimated optimal harvest for predator. We plotted the actual harvest as diamonds in the grid, see figure 3.10.

In the figure it is clearly seen that the harvest of NEAC has always exceeded our optimal harvest. Our results is in accordance with scientific results, the actual harvest of NEAC has exceeded its optimal harvest (Kugarajh, Sandal and Berge [21]).

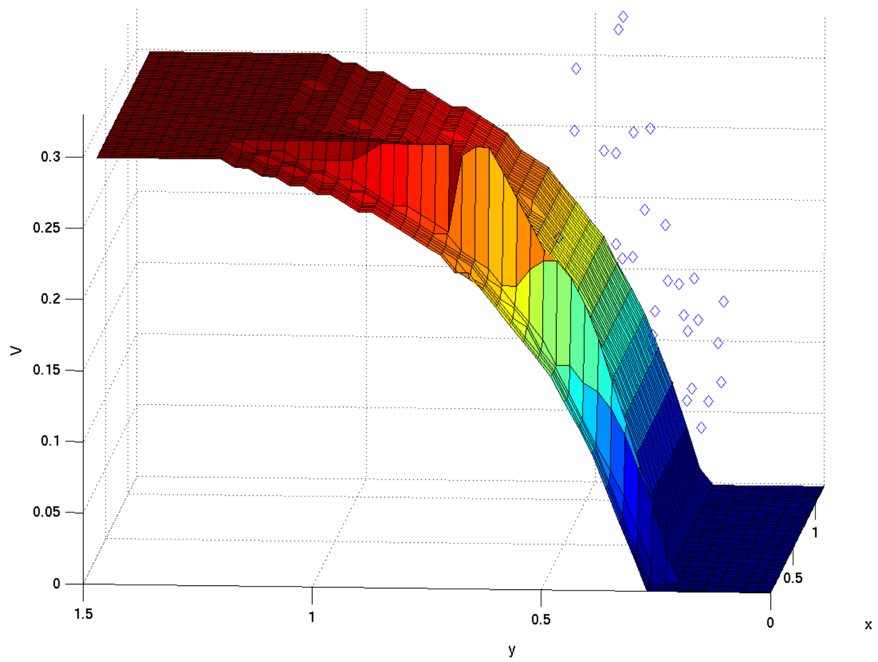


Figure 3.10: Comparison of our optimal harvest for the predator and actual harvest for NEAC. It is clear that the actual harvest exceeds our optimal harvest for all years.

Chapter 4

Conclusions

In the first section of this chapter we summarize and discuss the results from the previous chapter in the setting of the existing literature on this field. Secondly, we present some possible extensions of our model.

4.1 Summary and discussion

In chapter 1 we introduced our model. We introduced the possibilities for equilibrium points. Chapter two is mainly theory on how to solve a optimal control problem. In Chapter 3 we presented equilibrium solutions and feedback solutions.

We opened for extinction of prey. Fisheries, which shape exploitation paths, are facts. However, depletion is usually a result of several factors. Richard Cashin [4] gives six reasons for the collapse of the Canadian Northern Cod stock. They are (1) overly high Total Allowable Catch (TAC); (2) underreporting of catches and misleading data for management; (3) destructive fishing practices such as dumping; (4) foreign overfishing; (5) failure to control expansion of fishing effort; and (6) unforeseen ecological changes, including cooling water temperature, changes in water salinity, and shifting predator-prey relationships, particularly among seals, capelin and cod.

When the discount rate increases the traditional theory from single species models would yield increased equilibrium harvest and a following decrease in the steady state stock level. As mentioned earlier this is not necessarily true for the nonlinear multidimensional case. In our two species model we observe an increase in both harvests, the steady state predator stock decreases but equilibrium prey stock level increases. This is in accordance with the results from Flaaten [10]:

"The joint harvesting of a predator-prey ecological system can give the traditional result of a rise in the discount rate, decreased optimal stocks, or the untraditional result of an increase in one of the stocks".

We observe a gap in the policy for the prey, and a corresponding wave in the feedback policy for the predator. Our model will drive the prey to extinction if the prey stock for some reason decreases under its critical level(critical depensation). This explains the increasing harvest for small x values. We could leave the remaining prey stock in the sea as food for predator, but we increase our profit by harvesting prey and sell it on the market. The feedback policy for the predator changes with x. When x(prey) is close to its critical level from above, we (do not harvest prey and) increase harvest of predator in order to reduce the predator pressure on prey.

Flaaten [9] formulates: *"In the case of predator-prey interactions it is well known from the ecological literature that the reduction of the predator stock level may increase the surplus production of the prey."* This is the result we see from our feedback policy.

Our optimal harvest of prey reflects a possible weakness in the model. If the predator stock is large compared to prey, it will threaten the prey stock. When the stock is close to zero our optimal policy tells us to harvest at a high rate, caused by critical depensation. In our model this is optimal, our profit is maximized by harvesting prey. A small prey stock does not affect the harvesting costs. We have a problem of deciding when to introduce a moratorium. It might be too late when our model introduces it. For a smaller prey stock a moratorium on the prey fishery combined with increased harvest of predator might rescue the prey. However, this is not optimal with our utility functions. In the Norwegian history, a moratorium on Barents Sea Capelin has been a fact for several years with a low stock level. And the history of stock levels shows that after some years the stock has recovered. This implies that a growth with depensation may be a relatively rough approximation for capelin. Lately, the capelin stock has been at a low level, and last year and the current year it has not been allowed to harvest the Barents Sea Capelin.

Note that if one of our stocks is depleted (X=0 or Y=0) our problem reduces to known single species model, see equation 1.11.

Summarizing, we have studied the equilibrium of our model and computed feedback policies for both the predator and the prey. We have demonstrated a method that yields the optimal policy and for right starting points leads to the optimal steady state. Furthermore, we have given the dynamical behavior of the system as a function of time.

4.2 Further Work

In reality prices vary and costs are influenced by factors as technological advancements, union negotiations, government policies and taxation rates. Our model is deterministic, we could improve it by introducing a stochastic growth function. Stochastic models covers uncertainties s.a. shocks in the dynamics, stocks, catches or price.

Second, our thesis only involve one predator-prey relationship. Obviously it would be a better approximation to involve several species. The MULTISPEC model from the Institute of Marine Research(IMR), Bergen(see Tjelmeland and Bogstad, 1998) is a biological model for the Barents Sea fish/sea mammal system. It is used to study the species interactions quantitatively with the aim of improving the management of species. It includes cod, capelin, herring, minke whale, harp seal and species of zoo plankton.

Many, if not most renewable natural resources are harvested on a seasonal basis, there is a period when the resource can grow undisturbed and a harvesting season. Our model in discrete time could be extended to include this property. The population next season can be expressed in terms of this year's. We can use this estimate to predict maximal sustainable catch, optimal fleet size, etc. The total stock is the stock in the period without harvesting where it grows undisturbed minus the harvest. We assume that the fishing season is relatively short with respect to the closure period, the difference between them will be the catch. r_1 is the intristic growth rate r, multiplied by the time interval for unharvested stock, $r_1 = r\delta t$

$$\widehat{x}_n = x_{n-1} + r_1 x_{n-1}\left(1 - \frac{x_{n-1}}{K}\right) \tag{4.1}$$

$$x_n = \widehat{x}_n - h \tag{4.2}$$

Though NEAC spend most of its life in the Barents Sea, it migrates both as juvenile and as a mature spawning cod. Including migration would make the thesis more realistic. Another difficulty is where one of the stocks is “transboundary”, i.e. the fish stock migrates across the boundary of two countries. If one country has the main jurisdiction of the prey and the other country the main jurisdiction of the predator, it will complicate the model.

Appendix A

1.order Conditions

A.1 Necessary Conditions

In this section we will derive the necessary conditions for optimum in the simplest optimal control problem. The derivations follow Kamien and Schwartz [20].

We defined the system in equation (2.1). For simplicity, we derive the conditions for one state variable and one control variable. The calculations for several state and control variables is straight forward. Let the multiplier function, $\lambda(t)$, be any continuously differentiable function. From the equation we get the equality

$$\int_{t_0}^{t_1} \Pi(x(t), u(t), t)dt = \int_{t_0}^{t_1} \Big[\Pi(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t) - \lambda(t)\dot{x}(t)\Big] dt$$

The coefficient of $\lambda(t)$ must be zero if equation (2.1) is satisfied. We integrate the last term by parts:

$$-\int_{t_0}^{t_1} \lambda(t)\dot{x}(t)dt = -\lambda(t_1)x(t_1) + \lambda(t_0)x(t_0) + \int_{t_0}^{t_1} x(t)\dot{\lambda}(t)dt$$

Then we substitute this into the first expression and get

$$\begin{aligned}\int_{t_0}^{t_1} \Pi(x(t), u(t), t)dt &= \int_{t_0}^{t_1} \Big[\Pi(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t) + x(t)\dot{\lambda}(t)\Big] dt \\ &\quad -\lambda(t_1)x(t_1) + \lambda(t_0)x(t_0)\end{aligned}$$

It is easily seen that when u(t) is decided, we find x(t) from the differential equation (2.1) and the initial condition.

Assume that $u^*(t)$ is the optimal control and $x^*(t)$ is the corresponding optimal state. We now construct a family of comparison curves $u^*(t)+ah(t)$, a is a parameter and h(t) is arbitrary, but fixed. Now we define $y(t,a)$ as the state variable with the control: $u^*(t)+ah(t)$. Obviously

$$y(t,0)=x^*(t), \quad y(t_0,a)=x_0 \quad t_0 \le t \le t_1$$

for all a. Now we hold u^*, x^*, h fixed, and study the value depending on the parameter a. Thus, we write:

$$\begin{aligned} J(a) &= \int_{t_0}^{t_1} \Pi(y(t,a),u^*(t)+ah(t),t)dt \\ &= \int_{t_0}^{t_1} \Big[\Pi(y(t,a),u^*(t)+ah(t),t)+\lambda(t)f(y(t,a),u^*(t)+ah(t),t)+y(t,a)\dot{\lambda}(t)\Big]dt \\ &\quad -\lambda(t_1)y(t_1,a)+\lambda(t_0)y(t_0,a) \end{aligned}$$

Since u^* is a maximizing control, the function J(a) assumes its maximum at $a=0$, hence $J_a(0)=0$.[1] We differentiate w.r.t. a and evaluate at $a=0$:

$$J_a(0)=\int_{t_0}^{t_1} \Big[(\Pi_x+\lambda f_x+\lambda')y_a+(\Pi_u+\lambda f_u)h\Big]dt-\lambda(t_1)y_a(t_1,0)=0$$

$\Pi_x, f_x, \Pi_u, f_u, y_a$ denotes the partial derivatives w.r.t. the index.
$a=0$ so we have evaluated along $(x^*(t),u^*(t),t)$. $\lambda(t)$ was only required to be differentiable. To simplify the calculations we choose $\lambda(t)$ s.t. we don't need to determine difficult terms.

$\dot{\lambda}(t)=-(\Pi_x(x^*,u^*,t)+\lambda f_x(x^*,u^*,t))$
$\lambda(t_1)=0$

Substituting for λ:

$$\int_{t_0}^{t_1} \Big[(\Pi_u(x^*,u^*,t)+\lambda f_u(x^*,u^*,t)\Big]hdt=0$$

[1]Subscript denote the derivative: $J_a(a)=\frac{dJ(a)}{da}$

h(t) is an arbitrary function. We choose h(t) equal to the inside of the parenthesis above, $h(t) = \Pi_u(x^*, u^*, t) + \lambda f_u(x^*, u^*, t)$ and find a necessary condition for optimum:

$$\Pi_u(x^*(t), u^*(t), t) + \lambda f_u(x^*(t), u^*(t), t) = 0$$

We formulate the Hamiltonian:

$$H(x(t), u(t), \lambda(t), t) = \Pi(x(t), u(t), t) + \lambda(t) \cdot f(x(t), u(t), t)$$

To summarize, we have now shown that if the functions $x^*(t), u^*(t)$ maximize (2.1), then there exists a continuously differentiable function $\lambda(t)$ s.t. $x^*(t), u^*(t), \lambda(t)$ satisfy

$$\left.\begin{array}{l} H_\lambda = \dot{x} \\ \dot{\lambda} = -H_x \\ H_u = 0 \end{array}\right\} \tag{A.1}$$

These are called the first order conditions (FOC) for optimality.

Appendix B

Programming

In this chapter we give the code for the data program that have been used to calculate the solutions. To solve the equilibrium, Maple is used. To find the feedback solution Matlab m-files are used.

B.1 Equilibrium solution

Pontryagin gives us some necessary conditions for an optimal solution. We have six equations to solve, which in equilibrium equal zero, 3.11-3.16. It gets complicated and is difficult to solve by hand. Maple is a useful tool to calculate this equilibrium. Note that this code solves our scaled problem and the parameters a(alpha),b(beta),s,c1,c2,gamma,b1,b2 are the scaled variables.

```
>restart;

We first define our scaled equilibrium equations:
>lkv:=(x,y,u,v,m,n,a,b,s,c1,c2,gamma,b1,b2)->
      {s*x*x*(1-x)-a*x*y-u=0,y*(1-y)+b*x*y-v=0,
      r*m-(0.2185*(c1*u/(x*x))+m*(2*s*x-3*s*x*x-a*y)+n*b*y)=0,
      r*n-(c2*v/(y*y)-m*a*x+n*(1-2*y+b*x))=0,
      0.2185*(1-c1/x-2*b1*u)-m=0,
      1-c2/y-2*b2*v-n=0,
      x>=0, y>=0, u>=0, v>=0};



We then create a procedure that inserts the parametervalues
and computes our solution
```

```
>lsn:=proc(a,b,s,c1,c2,gamma,b1,b2)
      local test,x,y,u,v,m,n;
      if not type ([a,b,s,c1,c2,gamma,b1,b2], list(numeric))
       then return('lsn(a,b,s,c1,c2,gamma,b1,b2)')
      else test:=solve(lkv(x,y,u,v,m,n,a,b,s,c1,c2,gamma,b1,b2),
         {x,y,u,v,m,n});
      end if;
      assign(test);
      [x,y,u,v,m,n];
      end proc;

A call that starts the computation. The parameters inserted must be numeric.
>lsn(a,b,s,c1,c2,gamma,b1,b2);
```

B.2 Feedback solution

HJB is the main program to find our feedback solutions. First we need to define our grid size. We give the vectors x1,y1 for the state grid and u1,v1 for the policy grid as input. HJB maximizes the Hamilton-Jacobi-Bellmann equation with respect to the policy. To make the procedure most effective, it combines iterations with respect to the HJB equation (Piteration) and a simplified version of it(Viteration). We would obtain the same result by performing only iterations w.r.t the policy(Piteration), but it would take much longer time. When both the value and the policy have settled, HJB draws the optimal policy for both the predator and the prey stock, the value of the fishery, the vector field and the change in time for the system.

```
function [W,u,v]=HJB(x1,y1,u1,v1,h,beta,wtol,utol,vtol,d)
%
% function [W,u,v]=HJB(x1,y1,u1,v1,h,beta,wtol,utol,vtol,d)
%  is the main program
%
% Input:
%  - x1 and y1 are row vectors defining the size of the state-grid
%  - u1 and v1 are row vectors defining the size of the policy-grid.
%  - h is the time step in the discrete time
%  - beta is the discount rate, given as 1-delta*h (delta is the
%    real rate of return)
%  - wtol gives the accuracy for the valuematrix
%  - utol gives the accuracy for the policymatrix of prey
```

```
%  - vtol gives the accuracy for the policymatrix of predator
%  - d is used as distance to plot the vector field. We used 1
%
% Output:
%  - W is the new valuematrix
%  - u is the optimal policy for the prey
%  - v is the optimal policy for the predator
%

  tic; % measures the time our program uses

  % This function defines the growth functions and the profit
  % functions for both the predator and the prey.
  [f,g,PI1,PI2]=deffunksj();

  % This Opprett function makes the grids for the system and
  % stores the computed growth and profit functions in all grid
  % point in the matrix M.
  [X,Y,U,V,XX,YY,UU,VV,M]=Opprett(x1,y1,u1,v1,h,f,g,PI1,PI2);

  [m,n]=size(X);

  W=zeros(m,n);
  u=zeros(m,n);
  v=zeros(m,n);
  uplass=ones(size(XX));

 % We first call Piteration once to get away from the zero policy.
 [W,u,v,uplass]=Piteration(M,W,u,v,X,Y,XX,YY,UU,VV,beta,x1,y1,uplass);

 wfeil=10*wtol; ufeil=10*utol; vfeil=10*vtol;
 Q=W; q1=u; q2=v;
 it=0;


 % If the error is too large, we perform more policy and value iterations
while ( wfeil(end)>wtol || ufeil(end)>utol || vfeil(end)>vtol || (it<5) )

  it=it+1;
  wfeil(end+1)=10*wtol;

  while (wfeil(end)>wtol)
```

```
      for i=1:45
        Q=Viteration(Q,XX,YY,X,Y,beta,x1,y1,M,uplass);
      end
      wfeil(end+1)=max(max(abs(Q-W)));
      W=Q;
   end

      ufeil(end+1)=10*utol;
      vfeil(end+1)=10*vtol;

   while (ufeil(end)>utol) || (vfeil(end)>vtol)

      for i=1:4
  [Q,q1,q2,uplass]=Piteration(M,Q,q1,q2,X,Y,XX,YY,UU,VV,beta,x1,y1,uplass);
      end

      ufeil(end+1) = max(max(abs(q1-u)));
      vfeil(end+1) = max(max(abs(q2-v)));
      wfeil(end+1)=max(max(abs(Q-W)));

      u=q1;  v=q2; W=Q;
   end


   while (wfeil(end)>wtol)
       for i=1:20
         Q=Viteration(Q,XX,YY,X,Y,beta,x1,y1,M,uplass);
       end

       wfeil(end+1)=max(max(abs(Q-W)));
       W=Q;

   end

end

% Plots the optimal policy for prey
figure
surf(X,Y,u),xlabel('x'),ylabel('y'),zlabel('U')

% Plots the optimal policy for predator
figure
```

```
surf(X,Y,v),xlabel('x'),ylabel('y'),zlabel('V')%meshz

% Plots the value of the fishery
figure
surf(X,Y,W),xlabel('x'),ylabel('y'),zlabel('W')

% Plots the vector field for our solution
vektorfelt2(X,Y,h,d,f,g,uplass,[0.22:0.01:1.2],[0.32:0.01:1.5],u,v)
vektorfelt4(X,Y,h,d,f,g,u,v)

toc;
```

Opprett defines the grid of the system. We define the size of the grid by vectors x1,y1,u1,v1. It uses meshgrid to make one state grid and one policy grid. For every grid point in the state grid it calculates the growth and profit functions for every grid point in the policy grid. It stores these values in M.

```
function [X,Y,U,V,XX,YY,UU,VV,M]=Opprett(x1,y1,u1,v1,h,f,g,PI1,PI2)
%
%  [X,Y,U,V,XX,YY,UU,VV,M]=Opprett(x1,y1,u1,v1,h,f,g,PI1,PI2) make
%  two grids and compute the growth and profit functions in every
%  grid point.
%
%
% Output:
%  -X and Y is the original state-grid.
%  -U and V is the original policy-grid
%  -XX,YY,UU,VV are the grids given in a compact form
%  -M gives h times the computed values for the growth- and profit functions
%   for every state and policy grid points.
%
% Input:
%  -x1,x2 are row vectors defining the size of the state-grid
%  -u1,v1 are row vectors defining the size of the policy-grid
%  -h is the time step, we used 0.05
%  -f,g are growthfunctions for prey and predator, respetively
%  -PI1 and PI2 are the profit functions for prey and predator, respetively


  % The grids are made by meshgrid
  [X,Y]=meshgrid(x1,y1);
  [U,V]=meshgrid(u1,v1);

  % Here we introduce the compact form to ease the calculations later
  YY=Y(:);
  XX=X(:);
  UU=U(:);
  VV=V(:);


  m=length(XX);
  o=length(UU);
```

```
  %Initiation
  M=zeros(m,o,4);

  % For every point in the state grid, it computes the growth and
  % profit functions in every point of the policy grid
 for i=1:m
     for k=1:o

M(i,k,1)=h*f(XX(i),YY(i),UU(k),VV(k));
M(i,k,2)=h*g(XX(i),YY(i),UU(k),VV(k));
M(i,k,3)=h*PI1(XX(i),YY(i),UU(k),VV(k));
M(i,k,4)=h*PI2(XX(i),YY(i),UU(k),VV(k));


     end

  end
```

Piteration perform a policy iteration. For every point in the state grid it runs through every point in the policy grid. It saves the policy that maximizes the Hamilton-Jacobi-Bellmann equation. The best value is stored in W and the optimal policy in u and v for prey and predator, respectively.

```
function [W,u,v,uplass]=Piteration(M,W,u,v,X,Y,XX,YY,UU,VV,beta,x1,y1,uplass)
%
% [W,u,v,uplass]=Piteration(M,W,u,v,X,Y,XX,YY,UU,VV,beta,x1,y1,uplass)
% iterates to find the optimal policy.
%
% Output:
% -W is the new valuematrix.
% -u and v is the new policy matrix.
% -uplass gives the corresponding optimal gridpoint for every
%  gridpoint in the state grid.
%
% Input:
% -M( , ,1) is h*f(X,Y,U,V), the time step multiplied with the
%  growth function for prey.
% -M( , ,2) is h*g(X,Y,U,V), the time step multiplied with the
%  growth function for the predator
% -M( , ,3) is the profit function for the prey
% -M( , ,4) is the profit function for the predator
% -W is the "old" value matrix. u and v are the "old" optimal
%  policy.
% -X, Y is the original state grid
% -XX,YY is the "compact" state grid and UU,VV is the "compact" policy grid
% -beta is the discount term
% -x1 and y1 are row vectors defining the size of the state grid
% -uplass contains the optimal policy gridpoint corresponding to XX
%  and YY

  [m,n]=size(X);
  Q=length(UU);

  minx=x1(1); maxx=x1(end);
  miny=y1(1); maxy=y1(end);

  % Stores the optimal policy grid-point in uplass

 for i=1:m
  for j=1:n
```

```
    % r is the index change from an m*n matrix to an mn*1 matrix
    r=i+(j-1)*m;
    x=XX(r);
    y=YY(r);

    for q = 1:Q

      nyu=UU(q);
      nyv=VV(q);

       newx=x+M(r,q,1);
       newx=max(minx,newx);
       newx=min(maxx,newx);

       newy=y+M(r,q,2);
       newy=max(miny,newy);
       newy=min(maxy,newy);

       % Because newx and newy may not be grid point, we
       % interpolate to find their corresponding value.
       neww=interp2(X,Y,W,newx,newy,'linear');

       % The HJB-equation gives the value of our
       % problem. We need to maximize it w.r.t the policy, u
      test= M(r,q,3)+ M(r,q,4) +  beta*neww;

  % If this new value is larger then the value stored in the
  % value matrix, we save this one.

      if(test>W(i,j))
W(i,j)=test;
u(i,j)=nyu;
v(i,j)=nyv;
uplass(r)=q;
      end

   end

  end
 end
```

Viteration computes a value iteration. For every point in the state grid, it uses the optimal policy given from Piteration to evaluate the value of the Hamilton-Jacobi-Bellmann equation.

```
function W = Viteration(W,XX,YY,X,Y,beta,x1,y1,M,uplass)
%
% W = Viteration(W,XX,YY,X,Y,beta,x1,y1,M,uplass) computes a value iteration
%
% Output: W is the new value matrix
%
% Input:
% -W is the value matrix before the iteration
% -XX,YY,UU,VV gives the grid for the problem
% -M(...,1) is h*f, the time step multiplied with the growth
%  function for prey
% -M(...,2) is h*g, the time step multiplied with the growth
%  function for predator
% -M(...,3) and M(...,4) are the profittfunctions for prey and
%  predator, respectively
% -beta is the discount term
% -x1,y1 are row vectors defining the state grid
% -uplass contains the optimal policy gridpoint corresponding to XX
%  and YY


  [m,n]=size(X);

 % We use these later to make sure that we are on the grid
  minx=x1(1);
  maxx=x1(end);

  miny=y1(1);
  maxy=y1(end);


  % Computes the new value by running through the
  % Hamilton-Jacobi-Bellmann equation in every position

for i=1:m
  for j=1:n

  % r is the index change from an m*n matrix to an mn*1 matrix
  r=i+(j-1)*m;
```

```
  x=XX(r);
  y=YY(r);

  % This is the corresponding optimal policy
  q=uplass(r);

  newx=x+M(r,q,1);
  newx=max(minx,newx);
  newx=min(maxx,newx);

  newy=y+M(r,q,2);
  newy=max(miny,newy);
  newy=min(maxy,newy);

  % Because newx and newy may not be grid point, we
  % interpolate to find their corresponding value.
  neww=interp2(X,Y,W,newx,newy);

  % This is the Hamilton-Jacobi-Bellmann equation, which computes
  % the value of our system
  W(i,j)=M(r,q,3) + M(r,q,4) + beta*neww;

  end

end
```

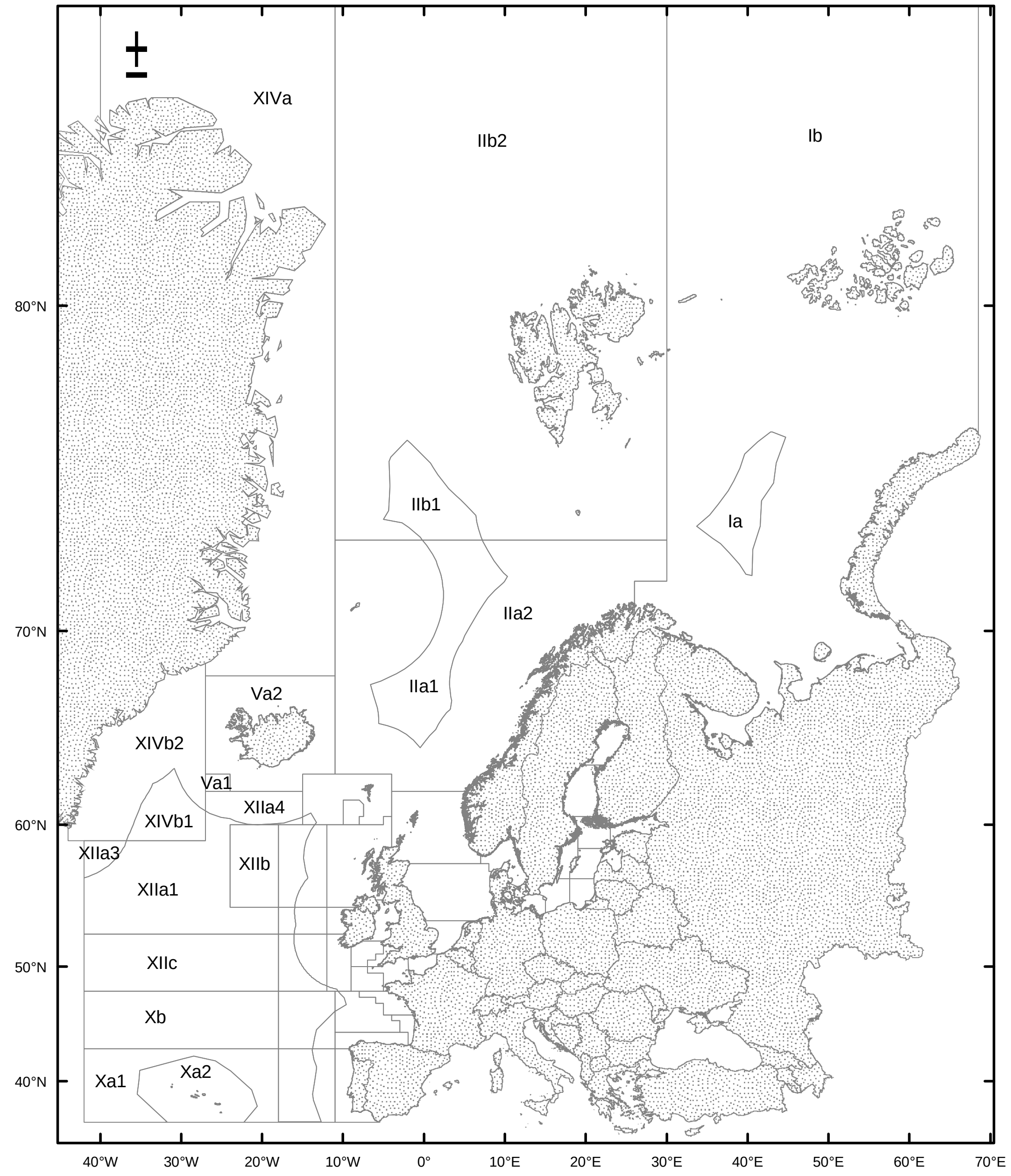


Figure B.1: ICES fishing areas [17]

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