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Master Thesis

Geometric Analysis of the Capital Asset Pricing Model

Thomas Kure

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Thesis Advisor: Xunhua Su

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Abstract

The derivation of the capital asset pricing model is in most literature limited to a graphical analysis. Since this method avoids a complicated mathematical framework the derivation is more intuitive to people who are unfamiliar to this topic. This approach, however, can result in misleading or even wrong results if the analysis is imprecise. Some of the main mistakes seem to be already established in financial textbooks.
This thesis gives a deeper analysis of the so often used graphical framework used to derive the Capital Asset Pricing Model and indicates some pitfalls. First we present the derivation of a small market containing only few securities before we expand it to one with arbitrary amount of securities.

Contents

1. Introduction

In economics the financial theory covers a manifold of different issues related to the investment decision problem of an investor. If we assume the existence of a market containing different kinds of assets the decision would be based on available information about certain attributes the investor could extract out of the market and on its own preferences and abilities. The gathering of information is thus the first step. The second is the evaluation with respect to personal preferences. Each one of both steps contains problems which are subject to the financial theory. But if we simply assume that we have a general model of the decision process then we could determine the demand of all investors on the market and compare it with the supply of assets. This general model could therefore give the evidence, if the market is in an equilibrium state or not. Additionally, also the movement can be predicted. Conversely, if we assume a market in an equilibrium state the same model could be used as a tool for investors during their decision process.

The Capital Asset Pricing Model (CAPM) is based on the idea of portfolio selection published by Harry Markowitz (1952). He applied assumptions about the steps mentioned above in the decision process and reasoned that only a minor asset combination is efficient within these assumptions. The CAPM extends these considerations under the same assumptions. One main element of the CAPM is a separation theorem defined by James Tobin (1958). It shows that within the efficient set of asset combinations exists an asset combination of particular importance the so-called market portfolio. By knowledge of this market portfolio and another asset, the so-called risk-free security, this theory can determine if the market is in an equilibrium state or as a tool for investors during their decision process.

Thus, it is no surprise that the CAPM is one of the common models in financial theory and it is hard to find a standard textbook which does not spend a whole chapter to it. Additionally, the related financial ratios of the CAPM are updated constantly in the financial press and are not only the foundation of investment decision of some investors but also the foundation of the evaluation of assets, securities or whole companies.

A special circumstance of the CAPM is the multitude of contributed scholars to this model. However, the CAPM is not only based on different working papers but it can also be derived in different ways. In general, these approaches can be differentiated into two methods. The first is based on a graphical analysis and displays the qualitative results predominantly in graphs. The second is more based on a deeper quantitative analysis and describes discussions of equilibrium models. The latter method has been used by John Lintner (1965) and Jan Mossin (1966). Remarkably is the fact that Mossin developed his equilibrium model from a complete microeconomic perspective.

However, this thesis focuses on the first method of a more graphical analysis. This method had been used by Markowitz in his original publication about portfolio selection and in the publication about capital asset prices by William Sharpe (1964). It is not only the most preferred method in economic textbooks but also a very intuitive approach. The outline of this thesis is to reconsider the assumptions of the model followed by presenting the major steps of its derivation. Subsequently, the graphical results are discussed followed by a geometric analysis.

2. Assumptions

To put the derivation of a model on a solid foundation it is mandatory to be clear on the assumptions it is based on. A simple first step would be to assume a market with *ideal* attributes where investors can buy or sell certain tradeable assets we now refer as securities. The first assumptions concern how the investor proceeds during the investment decision. The properties of the idealized market and the attributes of the securities are discussed later on.

2.1. Homogeneous Investor

To have an idea of the demand of an investor for a security we need to know more about its decision process. The process of portfolio selecting can be complex but in general we can differentiate it into two stages. In the first stage the investor generates its belief about the future performances of every security. It can be argued, that this belief depends on the information the investor can extract out of the market and his personal skills or experience. Thus every investor would be distinguishable from each other. To avoid this we assume the following homogeneous condition:

> ASSUMPTION 1: Every investor has the same belief about future performances of all securities.

This is reasonable if we assume that every investor has the same access to information and the same method to evaluate it.

The second stage contains the actual portfolio selection based on the beliefs of the first stage. We assume here:

> ASSUMPTION 2: Every investor concerns only about return and risk of a security. (μ-σ-rule)

With this assumption not the entire information the investor can extract out of the market is needed for the decision process. Moreover, the problem can be simplified by reducing the information to two attributes of the securities we refer as risk and return.

2.2. Securities: Risk and Return

Evidently a definition of risk and return is needed. The term *return* follows the maxim that an investor aims to maximize his future income generated by the security either by future cash-flows or by its value-added. The problem is therefore to find a figure which represents these future incomes. In the case of a single period consideration the yield (1) or continuously compounded return (2) is often used to estimate the future incomes by its expected value (3).

$$r_Y = \frac{V_{i+1} - V_i}{V_i} \tag{1}$$

$$r_{CP} = \ln\left(\frac{V_{i+1}}{V_i}\right) \tag{2}$$

$$\mu = \mathrm{E}\left[r\right] \tag{3}$$

This simple approach neglects the fact that the future and therefore also the return is uncertain or *risky*. Since, there are different definitions of uncertainty we follow the definition commonly used in modern decision theory made by Tversky and Kahneman (1992). In this case a decision under *risk* is a form of uncertainty where the future incomes occur in certain states with respective probabilities. In contrast to that, we define *uncertainty* if these probabilities are not precisely given. If the uncertainty cannot be described with discrete certain states we refer to this as *ambiguity*. In the following, we assume that the decision of the investor is under risk so that probabilities of certain states exists or at least beliefs about these probabilities. An often used figure for risk is the volatility calculated by the variance (4) or standard deviation (5).

$$\begin{aligned} \sigma^2 &= \mathrm{VAR}[r] \\ &= \mathrm{E}\left[(r-\mu)^2\right] \end{aligned} \tag{4}$$

$$\begin{aligned} \sigma &= \sqrt{\mathrm{VAR}[r]} \\ &= \sqrt{\mathrm{E}\left[(r-\mu)^2\right]} \end{aligned} \tag{5}$$

Note that the equations (1) to (5) only represent indicators for the two attributes of a security *return* and *risk*. Moreover, these indicators represent future states and must therefore be estimated by prior data. There are also more complicated statistical ways in estimating the volatility by using the autoregressive conditional heteroskedasticity (ARCH) model or generalized autoregressive conditional heteroscedasticity (GARCH). They take into account that volatility tends to cluster in time series.

A different explanation is to describe a security with its probability density function since the probabilities are known or beliefs exist (decision under risk). Figure (1) shows some possible probability density functions with the same variance, standard deviation and expected return.

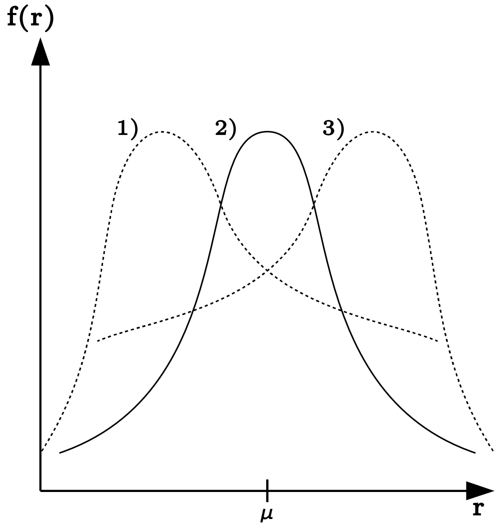

Figure 1: Exemplary probability density functions with identical σ^2, σ and μ

Each curve represents an investment and with assumption 2 an investor would be indifferent in choosing a particular investment. This indicates that the reduction of information to two figures can be problematic. Even if figure (1) makes the impression that the density function is unimodal or sometimes even a Gaussian distribution we made no further assumptions on the particular shape.

2.3. Foundation of Investment Decision

The hypothesis that the investor tries only to maximize the (discounted) return must be rejected with assumption 2 since the investor also concerns about risk. Instead we assume:

Assumption 3: Every investor tries to maximize its utility U.

This utility must then of course depend on risk and return. Since an investor simply tries to maximize the return in a risk-less decision problem return must be desirable. Therefore we can assume for the utility function:

Assumption 3.a: The utility increases if the return of the security increases. $\left(\frac{\partial U}{\partial \mu} > 0\right)$

Contrariwise, we assume that the investor tries to avoid risk. Hence for the utility applies:

Assumption 3.b: The utility decreases if the risk of the security increases. $\left(\frac{\partial U}{\partial \sigma} < 0\right)$

Note that risk is a form of uncertainty as described above and hence does not exclusively include a probability for a security to fall below its mean but also a chance to exceed. Therefore with assumption 3.b the investor also avoids to exceed if he avoids every uncertainty. This can lead to unsatisfactory conclusions as mentioned above (see figure (1)).

In the following, only the given assumptions 1-3 are important and not the given examples how to measure them.

2.4. The Idealized Market

The last step is to find assumptions for the above mention idealized market. It is commonly assumed that every investor has the same access to securities and hence the same information about all securities. There are no taxes, transaction costs or other barriers for the investors so that the market can also be described as frictionless.

We assume further, that the market contains m risky securities, i.e. $\sigma_i^2 > 0$ and $\sigma_i > 0,\ i = 1, \ldots, m$, with the expected return $\mu_i > 0,\ i = 1, \ldots, m$. For calculatory reasons we assume that no security can be represented as a linear combination of the other $(m-1)$ securities. Additionally, we neglect short sells in the beginning.

3. Portfolio Selection

Under the given assumptions of the last section we can now prove that in general not all securities are efficient. This was first published by Markowitz. For a stepwise derivation we start with the simple cases with a market dimension of 2 and 3 securities before we expand to markets with arbitrary dimensions. Since all proofs of the $m > 3$ case apply also here not all proofs are explicitly derived and we refer to the later sections.

To maximize its utility the investor can pick any security on the market he likes to invest. If we assume further, that every security is divisible the investor has not only the opportunity to invest in m different securities but also in every linear combination, we refer as portfolio. The composition of the portfolio is based on the investment decision. With assumption 2 every investor concerns only about the μ and σ of a portfolio.

3.1. The $m = 2$ Security Case

As indicated above in the $m = 2$ security case the investor does not only have the choice to invest exclusively in one of the securities but also in every linear combination. The μ_p and σ_p of the resulting portfolio depends on five parameters: μ_1, μ_2, σ_1, σ_2 and $\sigma_{1,2}$. The investor can choose the portion x he likes to invest in security 1. Obviously, the portion of security 2 is then $(1 - x)$.

PROPOSITION 1: The linear combination of 2 securities form a straight line in the $\rho = 1$ case, two lines in the $\rho = -1$ case and a hyperbola in the $-1 < \rho < 1$ case.

Here, ρ stands for the correlation coefficient.

PROOF:
$$\mu_p = x\mu_1 + (1 - x)\mu_2 \qquad \Rightarrow x = \frac{\mu_p - \mu_2}{\mu_1 - \mu_2} \tag{6}$$
$$\sigma_p^2 = x^2\sigma_1^2 + (1 - x)^2\sigma_2^2 + 2x(1 - x)\sigma_1\sigma_2\rho_2 \tag{7}$$

For:

$$\rho = 1: \ \sigma_p = \left| (\sigma_1 - \sigma_2) \underbrace{\frac{\mu_p - \mu_2}{\mu_1 - \mu_2}}_{x} - \sigma_2 \right| = \left| \frac{\sigma_2 - \sigma_1}{\mu_2 - \mu_1}\mu_p - \mu_2 \frac{\sigma_2 - \sigma_1}{\mu_2 - \mu_1 + \sigma_2} \right| \tag{8}$$

$$\rho = -1: \ \sigma_p = \left| (\sigma_1 + \sigma_2) \underbrace{\frac{\mu_p - \mu_2}{\mu_1 - \mu_2}}_{x} - \sigma_2 \right| = \left| \frac{\sigma_1 + \sigma_2}{\mu_1 - \mu_2}\mu_p - \frac{\sigma_1 + \sigma_2}{\mu_1 - \mu_2}\mu_2 - \sigma_2 \right| \tag{9}$$

$$\text{Else: } \sigma_p = \left(\frac{\left(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho\right)\mu_p^2 - 2\left(\sigma_1^2\mu_2 + \sigma_2^2\mu_1 - \sigma_1\sigma_2\rho\left(\mu_1 + \mu_2\right)\right)\mu_p}{(\mu_1 - \mu_2)} \right.$$
$$\left. \frac{+\left(\sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2 - 2\sigma_1\sigma_2\rho\mu_1\mu_2\right)}{(\mu_1 - \mu_2)} \right)^{\frac{1}{2}} \tag{10}$$

The results of the three cases can be seen in figure 2. The line in the $\rho = 1$ case has the slope $\frac{\partial \mu_p}{\partial \sigma_p}\Big|_{\rho=1} = \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}$ and connects both securities. It is possible to continue the line if we allow short sales. If we do so, only for the cases were $\frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} < \frac{\mu_1}{\sigma_1}$ we gain an intersection point with the ordinate an hence a risk-free portfolio with $\sigma_p = 0$ and a positive μ_p. Otherwise the Intersection point of the line is the origin or below.

However, even if we neglect short sales we can gain a risk-free portfolio with $\mu_0 = \mu_p = \frac{\mu_1\sigma_2 + \mu_2\sigma_1}{\sigma_1 + \sigma_2}$ in the $\rho = -1$ case. The portion of security 1 is at this point $x = \frac{\sigma_2}{\sigma_1 + \sigma_2}$.
In general, both securities do not have an absolute (negative) correlation. We can calculate the minimum variance point $M = (\mu^*, \sigma^*)$:

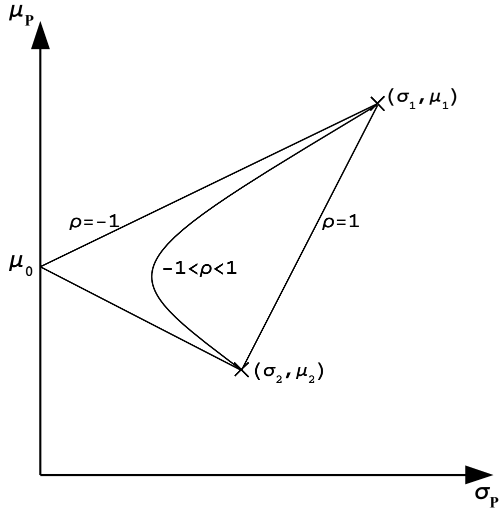

Figure 2: Possible portfolio combinations in the $m = 2$ security case for different correlation values

$$\frac{\partial \sigma_p}{\partial x} = 0 \Rightarrow x^* = \frac{\sigma_2^2 - \sigma_1 \sigma_2 \rho}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho} \tag{11}$$

$$\mu_p^* = \frac{\mu_1 \sigma_2^2 + \mu_2 \sigma_1^2 - (\mu_1 + \mu_2)\,\rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{12}$$

$$\sigma_p^* = \sqrt{\frac{(1-\rho^2)\,\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}} \tag{13}$$

Note that we gain the same results as above by setting $\rho = -1$ or $\rho = 1$. If we assume a zero correlation coefficient, the minimum variance point would be

$$M_{\rho=0} = \left(\mu_{\rho=0}^* = \frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}, \sigma_{\rho=0}^* = \sqrt{\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \right) \tag{14}$$

This is the case where both securities are completely uncorrelated.

3.2. The $m = 3$ Security Case

If we add a third available security to the market the investor decision changes to the choice of the portion in security 1 (x_1) and security 2 (x_2). The portion of Security 3 is then $(1 - x_1 - x_2)$. This yields:

PROPOSITION 2: The combination of 3 securities form a plane in the μ-σ-space.

Additionally, we assume here that $\mu_1 < \mu_2 < \mu_3$ and $\sigma_1 < \sigma_2 < \sigma_3$ and that none of two securities are completely (negative) correlated or uncorrelated.

PROOF:
$$\mu_p = x_1\mu_1 + x_2\mu_2 + (1 - x_1 - x_2)\mu_3 \tag{15}$$
$$\Rightarrow x_2 = \frac{\mu_p - \mu_3}{\mu_2 - \mu_3} - \frac{\mu_1 - \mu_3}{\mu_2 - \mu_3}x_1 \tag{16}$$
$$\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + (1 - x_1 - x_2)\sigma_3^2 + x_1x_2\sigma_{1,2} + x_1(1 - x_1 - x_2)\sigma_{1,3} + x_2(1 - x_1 - x_2)\sigma_{2,3} \tag{17}$$
$$\sigma_p^2 = (\sigma_1^2 - 2\sigma_{13} + \sigma_3^2)x_1^2 + (\sigma_2^2 - 2\sigma_{23} + \sigma_3^2)x_2^2 + 2x_1x_2(\sigma_{12} - \sigma_{13} - \sigma_{23} + \sigma_{33}) + 2x_1(\sigma_{13} - \sigma_3^2) + 2x_2(\sigma_{23} - \sigma_3^2) + \sigma_3^2 \tag{18}$$

Figure 3 shows an example of the plane of available μ-σ-combinations of the portfolio. The boundary or frontier of the plane is formed by hyperbolas. Since we already know that an investor tries to maximize its utility only the bold line of the frontier satisfies assumption 3.a. and 3.b. sufficiently. This line is therefore also known as the efficient frontier of all attainable portfolio combinations.

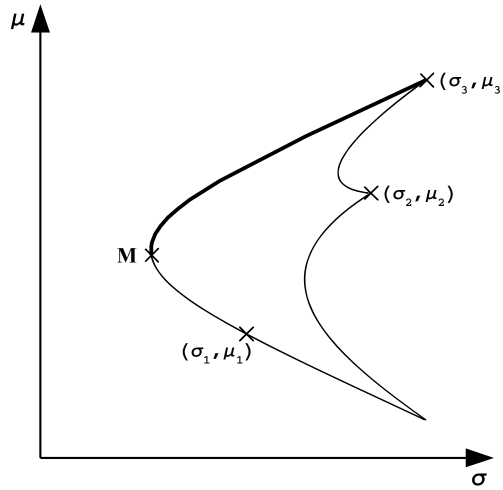

Figure 3: Exemplary possible portfolio combinations (plane) in the $m = 3$ security case for random values of σ_i and μ_i. Only a minor set is efficient (bold line).

We neglect all further geometric analysis here and refer to the m>3 case in section (4).

3.3. Alternative Visualization: The x_n-Space

A different way to display the results is to use the x_2-x_1 allocation space. Hence, we reverse the question from *How much is μ_p and σ_p if we know the portions x_i?* to *What portions are necessary to gain a certain μ_p and σ_p?*.

It is therefore possible to see μ_p and σ_p as parameters. Since not all combinations are efficient, we have to find first the exact form of efficient combinations in the x_2-x_1 space.

Proposition 3: All portfolios with a given expected return $\mu_p = \mu_{p,j}$ form a straight line in the x_2-x_1 space (*isomean*-curve).

We already derived the proof with equation (16). The slope of the isomean-curve is determined by $-\frac{\mu_1 - \mu_3}{\mu_2 - \mu_3}$ and therefore independent to the given portfolio return μ_p. The intersection point with the x_2-axes is determined with $\frac{\mu_p - \mu_3}{\mu_2 - \mu_3}$ and depends on μ_p.

Proposition 4: All portfolios with a given variance $\sigma_p = \sigma_{p,j}$ form an ellipse in the x_2-x_1 space (*isovariance*-curve).

To prove this we rearrange equation (18) to

Proof:

$$\begin{aligned} x_2 = &\frac{1}{\sigma_2^2 - 2\sigma_{2,3} + \sigma_3^2}\sigma_3^2 + \sigma_{2,3}(x_1 - 1) - \sigma_{1,2}x_1 + \sigma_{1,3}x_1 - \sigma_3^2 x_1 \\ &+ \frac{1}{2}\sqrt{4}\left(\sigma_3^2 + \sigma_{2,3}(x_1 - 1) - \sigma_{1,2} - \sigma_{1,3} + \sigma_3^2)x_1\right)^2 \\ &- 4(\sigma_2^2 - 2\sigma_{2,3} + \sigma_3^2)(\sigma_3^2(x_1 - 1)^2 - \sigma_p + x_1(\sigma_1^2 x_1 - 2\sigma_{1,3}(x_1 - 1))) \end{aligned} \tag{19}$$

Exemplary the isomean and isovariance curves can be seen in figure (4) and figure (5). The attainable set of portfolios lies inside of the *abc* triangle.

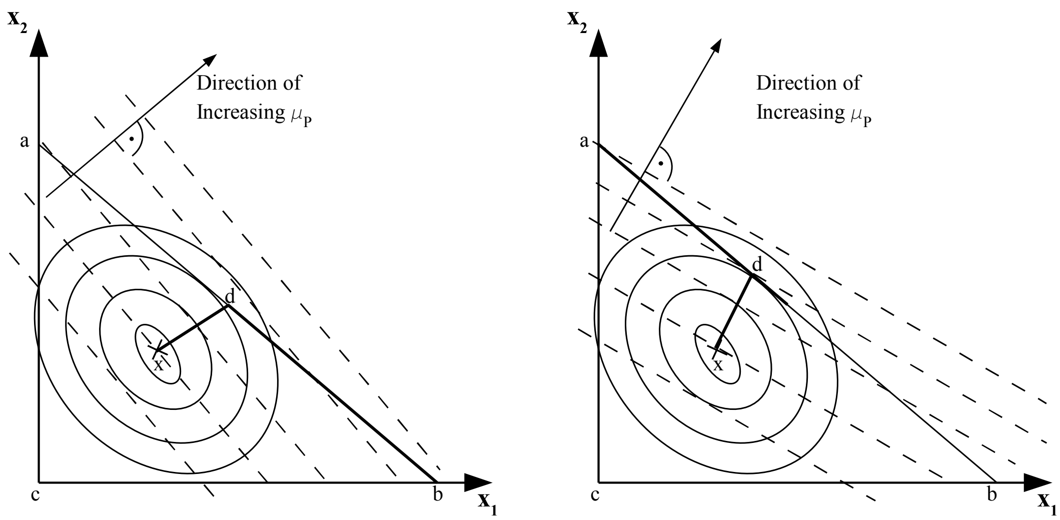

Figure 4: x_2-x_1 allocation space for different isomean slopes; Isomean (dashed line); isovariance (ellipses); Efficient portfolios (bold line); Minimum variance point x inside the attainable set

The minimum variance point x is the center of the concentric ellipses. This point represents always an efficient portfolio. If x lies inside the attainable set (see figure (4)) we can form an outgoing straight line with increasing μ_p to find other efficient portfolios. The line $\overline{xd}$ represents all tangent points of the isovariance curves to the isomean curves. Since it is the locus of the maximum μ_p for a given variance σ_p the line contains efficient portfolios. From point d it is still possible to increase μ_p by increasing σ_p if the slope of the isomean curve is not identical to the $\overline{ab}$ line.

Respectively, to the isomean slope we can continue the line of efficient portfolios. All efficient portfolios lie therefore on the $\overline{xdb}$ line in figure (4, left) since the slope of the isomean is

$$\frac{\partial x_2}{\partial x_1} = -\frac{\mu_1 - \mu_3}{\mu_2 - \mu_3} < -1$$

Contrariwise, the efficient portfolio line is represented by $\overline{xda}$ (see, figure (4, right)) if the slope is

$$\frac{\partial x_2}{\partial x_1} = -\frac{\mu_1 - \mu_3}{\mu_2 - \mu_3} > -1$$

and ends at the point d if

$$\frac{\partial x_2}{\partial x_1} = -\frac{\mu_1 - \mu_3}{\mu_2 - \mu_3} = -1$$

In contrast to figure (4) is the center in figure (5) outside of the attainable set.

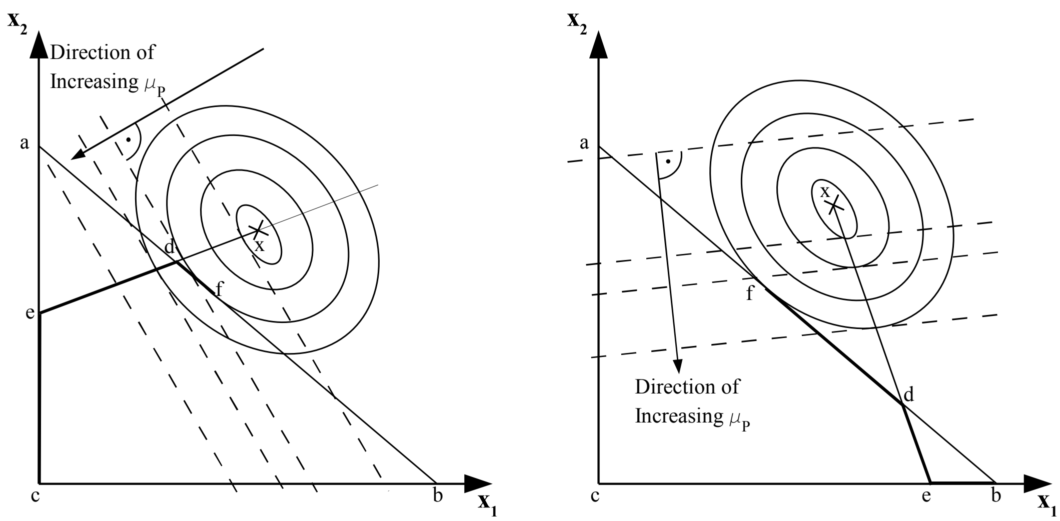

Figure 5: x_2-x_1 allocation space for different isomean slopes; Isomean (dashed line); isovariance (ellipses); Efficient portfolios (bold line); Minimum variance point x outside the attainable set

To find all attainable efficient portfolios we start again at the minimum variance point x and drag out a line built by the tangent points of the isomean curve and the isovariance ellipses. Point d of the $\overline{de}$ line represents then the first attainable efficient portfolio but it is neither the attainable efficient portfolio with the lowest nor the highest variance. At point f we find the minimum attainable variance portfolio which is also an efficient portfolio since also the return decreases compared to d. Geometrically, point f is the tangent point of the isovariance ellipses to the $\overline{ab}$ line. It is easy to see that d and f are only identical if the isomean curves have a slope of -1 and are therefore parallel to the $\overline{ab}$ line.

In point e we can still increase the return by increasing variance and find more efficient portfolios on the $\overline{ec}$ line. If the isomean curves have an other slope the last point could be b (see figure (5), right). In our example all attainable efficient portfolios lie therefore on the $\overline{fdec}$ line in figure (5,left) or on the $\overline{fdeb}$ line figure (5, right)

It is possible to apply the same algorithm to find all attainable efficient portfolios in case of m risky assets. As equation (16) indicates the solution would be a line in the $n = (m-1)$ space. Since the base of this space would be the portions x of n securities we refer to this space as x_n space.

What is now the exact advantage of the x_n space compared to the μ-σ space? It is very easy to add securities into the μ-σ space but impossible to compute the efficient frontier geometrically. It was necessary to compute the frontier mathematically then to determine the form before we could add the hyperbola into the μ-σ space as a solution. The x_n space changed this, so that it was possible to determine the efficient portfolios geometrically. Since there is no free lunch the mathematical work was to determine the forms of the isovariance and isomean curves. Unfortunately, we have some additional work to do if we would like to transform the results from the x_n space into the μ-σ space and vice versa. Therefore the x_n space is not a more convenient way to find the efficient frontier but can give a better insight into the hyperbola of figure (3). By comparing both spaces we find the following:

PROPOSITION 5: The minimum variance point M in the μ-σ space equals point x in the x_n space if x lies inside the attainable set. Otherwise M equals point f.

PROPOSITION 6: In the majority of cases the minimum variance portfolio M consists of several securities.

Since M equals x or f this is only true if respectively x or f is not equal to one of the corners of the abc triangle.

PROPOSITION 7: The end point of the hyperbola (if we neglect short sales) with the highest variance and highest return consists always of one security.

This is very easy to see in the x_n space since the line of the attainable efficient portfolios ends always at point a, b, or c. If we take another look on figure (3) we already considered these circumstances. Security with the highest risk and highest return lies at the end of the efficient frontier. Analogically, security 2 lies at the upper end of the right small frontier hyperbola.

4. Capital Asset Pricing Model

4.1. Derivation of the Efficient Frontier

As mentioned in section (3.2) any investor would only invest in a portfolio combination which lies on the efficient frontier, since only these combinations have the lowest variance for a given return. It is therefore not necessary to compute the complete plane of attainable combinations. To compute the efficient frontier we have to find the minimum variance point for every return. The efficient frontier is then the line above the absolute minimum variance point M.
Since the derivation has been already shown in many publications (for example see Merton (1972)) we present only an outline of the main points.

Since the task is to find all minimum variance points we gain the following constrained optimization problem

$$\min \frac{1}{2}\sigma_p^2 \tag{20}$$

$$\text{with subject to:} \qquad \sigma_p^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \sigma_{ij} \tag{21}$$

$$\mu_p = \sum_{i=1}^{m} x_i \mu_i \tag{22}$$

$$1 = \sum_{i=1}^{m} x_i \tag{23}$$

Equation (21) represents the definition of the portfolio variance and can be calculated by the covariance of all two security combinations σ_{ij} weighted by their percentage value x_i in the portfolio. We define further that $\sigma_{ii} = \sigma_i^2$ is the variance of the i^{th} security. The return of the portfolio (equation (22)) is then the linear combination of the returns of its containing securities. This time we do not exclude short sales and hence some x_i can be negative as long they sum to unity according to equation (23).

A different way is to present the constraints in form of vectors and matrices. In this case the vector $\mathbf{x} = \sum_{i=1}^{m} x_i \cdot \mathbf{e}_i$ contains all allocated shares of the portfolio. We can define the variance-covariance matrix as $\mathbf{\Omega} = [\sigma_{ij}]$. By the definition of the covariance this matrix has to be symmetric since $\sigma_{ij} = \sigma_{ji}$. The main diagonal contains all variances. Since the variance-covariance matrix is symmetric and positive definite it is nonsingular and hence invertible. We define the inverse variance-covariance matrix as

$$\mathbf{\Omega^{-1}} = \mathbf{V} = [v_{ij}]$$

In the matrix notation we can rewrite the constrains (21-22) as

$$\sigma_p^2 = \mathbf{x}^T \mathbf{\Omega} \mathbf{x} \tag{24}$$

$$\mu_p = \mathbf{x}\boldsymbol{\mu} \tag{25}$$

$$1 = \mathbf{x}\mathbf{1} \tag{26}$$

In equation (26) the vector **1** represents an m-dimensional vector with only ones as components.

By using Langrangian multipliers λ_1 and λ_2 we can rewrite equation (20) as

$$\min \left[\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j \sigma_{ij} + \lambda_1 \left(\mu_p - \sum_{i=1}^{m} x_i \mu_i \right) + \lambda_2 \left(1 - \sum_{i=1}^{m} x_i \right) \right] \tag{27}$$

Since this method is a common approach in the optimization theory we can skip the detailed calculation. By calculating the partial derivatives $\frac{\partial L}{\partial x_i}$, $\frac{\partial L}{\partial \lambda_1}$, $\frac{\partial L}{\partial \lambda_2}$ and substitution we gain the following results

$$\mu_p = B\lambda_1 + A\lambda_2 \qquad 1 = A\lambda_1 + C\lambda_2 \tag{28}$$

$$\lambda_1 = \frac{C\mu - A}{D} \qquad \lambda_2 = \frac{(B - A\mu)}{D} \tag{29}$$

To shorten the results we used the following definitions

$$A = \sum_{j=1}^{m} \sum_{k=1}^{m} V_{kj} \mu_j = \boldsymbol{\mu}^T \mathbf{V} \mathbf{1} \qquad B = \sum_{j=1}^{m} \sum_{k=1}^{m} V_{kj} \mu_j \mu_k = \boldsymbol{\mu}^T \mathbf{V} \boldsymbol{\mu} \tag{30}$$

$$C = \sum_{j=1}^{m} \sum_{k=1}^{m} V_{kj} = \mathbf{1}^T \mathbf{V} \mathbf{1} \qquad D = BC - A^2 \tag{31}$$

Due to the symmetry of the variance-covariance matrix we find that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} v_{ij} \mu_j = \sum_{i=1}^{m} \sum_{j=1}^{m} v_{ij} \mu_i$$

and that $B > 0$ and $C > 0$. Hence also $D > 0$.

By solving the optimization problem we can thus find the efficient frontier in the σ^2-μ space:

$$\sigma_p^2 = \lambda_1 \mu_p + \lambda_2 = \frac{C\mu_p^2 - 2A\mu_p + B}{D} \tag{32}$$

and formulate the next proposition.

PROPOSITION 8: The left frontier in the σ^2-μ space is a parabola.

The form of the efficient frontier can be seen in figure (6).

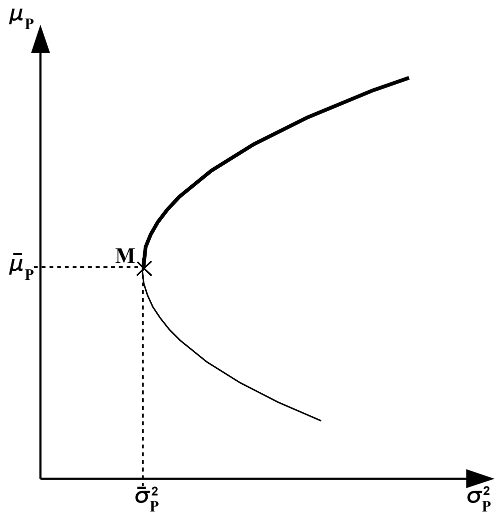

Figure 6: The efficient frontier in the σ^2-μ space is the upper part of a parabola

For the minimum variance point we can find by differentiating equation (32)

$$\frac{\mathrm{d}\sigma_p^2}{\mathrm{d}\mu_p} = \frac{2(C\mu - A)}{D} = 0 \tag{33}$$

$$\bar{\mu}_p = \frac{A}{C} \quad \text{and} \quad \bar{\sigma}_p^2 = \frac{1}{C} \tag{34}$$

As can be seen here the minimum variance point depends only on A and C.

In the σ-μ space we have the same minimum variance point only the shape of the frontier is different.

$$\sigma_p = \sqrt{\frac{C\mu_p^2 - 2A\mu_p + B}{D}} \tag{35}$$

PROPOSITION 9: The left frontier in the σ-μ space is a hyperbola.

Finally, we can solve equation (35) for μ_p

$$\mu_p = \bar{\mu}_p \pm \sqrt{\frac{D\left(C\sigma_p^2 - 1\right)}{C^2}} \tag{36}$$

$$\mu_p = \bar{\mu}_p + \sqrt{\frac{D\left(C\sigma_p^2 - 1\right)}{C^2}} \tag{37}$$

Note that only equation (37) represents the efficient frontier.

We presumed the form in figure (3). Now we have the proof for arbitrary m securities (see figure (7)). To indicate the difference between a hyperbola and a parabola figure (7) contains also the parabola of figure (6) as a dashed line.

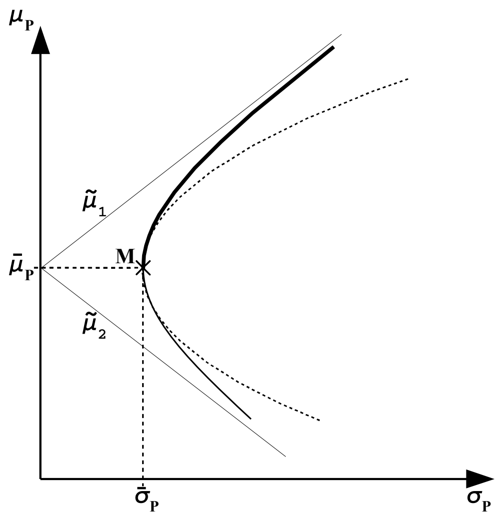

Figure 7: The efficient Frontier in the σ-μ space is the upper part of a hyperbola with the asymptotes $\tilde{\mu}_1$ and $\tilde{\mu}_2$

Figure (7) also shows that the frontier converge to a straight line for large σ_p. Hence we can calculate for the asymptotes:

$$\text{For } \sigma_p^2 \gg 1: \qquad \tilde{\mu}_1 = \bar{\mu}_p + \sqrt{\frac{D}{C}}\sigma_p \tag{38}$$

$$\tilde{\mu}_2 = \bar{\mu}_p - \sqrt{\frac{D}{C}}\sigma_p \tag{39}$$

In some financial textbooks (e.g. Brealey, Myers and Allen (2006)) or publications (e.g. Sharpe (1964)) the shape of the frontier is presented very sketchy and does not look like a hyperbola since the shape does not converge against any asymptotes. This kind of simplification can yield in wrong conclusions as we can see in the next sections.

In contrast to the detailed Lagrangian approach it is also possible to show geometrically that the frontier has to be a hyperbola in the $m > 2$ case. To do so, we first refer again to proposition 1 in subsection 3.1. It describes that the combination of two securities (with $-1 < \rho < 1$) is a hyperbola. This means that only two securities are needed to determine a hyperbola definitely. This is possible, since two securities together with their correlation coefficient contains the information of actually three points. The third point is M the minimum variance point, calculated by equation (12) and (13). If the the minimum variance point M of a hyperbola is known, only another frontier point is needed to determine the hyperbola definitely.

We can distinguish two hyperbolas in the following way (see figure (8)). If two hyperbolas have no intersection point at all (h_1 and h_2), then one of them lies completely on the right side of the other. If both hyperbolas have only one intersection point (h_1 and h_3), then both hyperbolas are tangent at this point. Besides the tangent point all other points of one of the hyperbolas lies completely on the right side of the other hyperbola. In this case the hyperbola on the left has a greater arm-spread. If both have two intersection points (h_1 and h_4), then none of the hyperbolas lies completely in the right side of the other hyperbola. If both hyperbolas have more than two intersection points, they overlap.

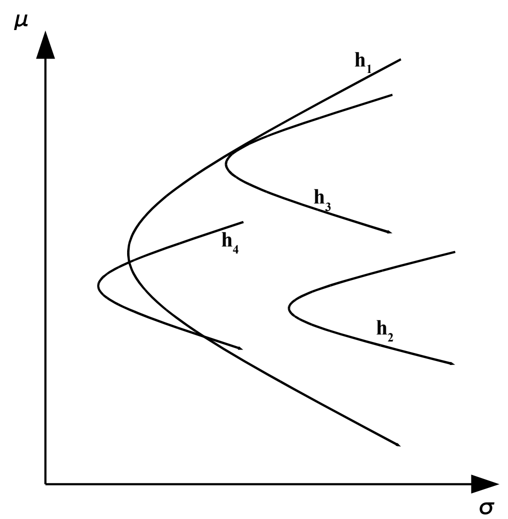

Figure 8: Possible differentiation of two hyperbolas in the μ-σ-space

Hence in the $m > 3$ security case we can form always a hyperbola with the minimum variance point M and another portfolio P_1 (see figure (9)).
This hyperbola h_1 is then described definitely by P_1 and M. Obviously, P_1 would be a frontier portfolio of this hyperbola. If the hyperbola h_1 describes not the frontier then there must lie another portfolio P_2 on the left side of P_1. The new hyperbola h_2 described by M and P_2 has only one intersection point with h_1, the tangent point M. Therefore h_2 lies on the left side of h_1 (or is the hyperbola with the greater arm-spread) and P_1 is

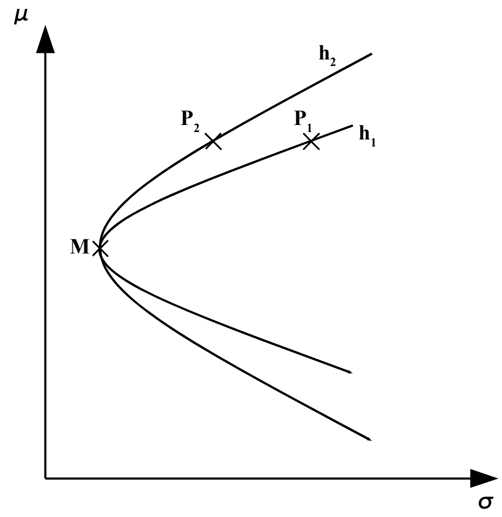

Figure 9: Forming a hyperbola by M and a portfolio P. If there is no other portfolio on the left side of portfolio P then the hyperbola is the frontier.

not a frontier portfolio. If there is no other portfolio on the left side of P_2, then h_2 is the hyperbola with the greatest arm-spread and hence the frontier.

4.2. Mutual Fund Theorem

In contrast to the Tobin Separation Theorem in section (4.3) which is sometimes also referred as the mutual fund theorem we can formulate a different mutual fund theorem.

Proposition 10: There is a set of two portfolios constructed from the original market securities such that all investors are indifferent in choosing between the original securities or these two portfolios.

This finding can also be seen in Merton (1972). To prove this we start again from the original optimization problem (equation (20)) we find the portion x_i an investor has to invest in the i^{th} security to be on the frontier of the attainable plane.

$$x_i = \frac{\mu_p \sum_{j=1}^{m} V_{ij} \left(C\mu_j - A \right) + \sum_{j=1}^{m} V_{ij} \left(B - A\mu_j \right)}{D} \qquad i = 1, \ldots, m \tag{40}$$

To shorten this expression we use the following substitution.

$$g_i = \frac{\sum_{j=1}^{m} V_{ij} \left(C\mu_j - A \right)}{D} \tag{41}$$

$$h_i = \sum_{j=1}^{m} V_{ij} \left(B - A\mu_j \right) D \tag{42}$$

$$x_i = \mu_p g_i + h_i \qquad i = 1, \ldots, m \tag{43}$$

Note that we gain for the sum of all securities $\sum_{i=1}^{m} g_i = 0$ and $\sum_{i=1}^{m} h_i = 1$ by their definition.

Obviously, the exact portion x_i changes if we move the portfolio along the frontier. Since we assumed that all securities has to be risky and non of them can be described as a linear combination of the others the frontier of the plane consists of all market portfolios and it is therefore impossible to set x_i constantly to zero for some securities. Hence, two portfolios a and b which satisfy proposition 10 have to contain every security of the market. We denote the portions a_i and b_i respectively. Finally, Both portfolios combined have to have the same portion x_i of the i^{th} security we determined in equation (43). This has to be true not only for one security but for all.

$$x_i = g_i \mu_p + h_i = \gamma a_i + (1 - \gamma) b_i \qquad i = 1, \ldots, m \tag{44}$$

The portion x_i depends on μ_p and changes with the investor decision. Since proposition 10 has to be true for all investors the fraction of the i^{th} security in the portfolios a and b has to be independent of μ_p. Therefore, a_i and b_i for all $i = 1, \ldots, m$ has to be the same for every investor. This makes sense since a and b simply represent the market.

Contrary, the mix parameter γ in equation (44) depends on μ_p so that we can move along the frontier by changing the portion of the two funds a and b in the portfolio with:

$$\gamma = \delta \mu_p - \alpha \quad (\delta \neq 0) \tag{45}$$

After substitution of equation (45) into (44) we can solve for a_i and b_i.

$$a_i = b_i + g_i/\delta \qquad b_i = h_i + \alpha g_i/\delta \qquad i = 1, \ldots, m. \tag{46}$$

Since the portfolios a and b are independent of the investor decision and also independent of each other they are commonly also referred as two m-vectors forming a basis for the vector space of frontier portfolios. Both portfolios a and b are frontier portfolios itself. This is very easy to see, since we already proved, that the frontier is a hyperbola in proposition 9. If we compare this with proposition 1, then we can see, that also two securities can form a hyperbola. The task from proposition 10 is hence only to find these two portfolios which form the same hyperbola than the original set of m securities. As can be seen in figure (2) both portfolios lie then on the hyperbola.

The parameters δ and α depend on the return of the two portfolios μ_a and μ_b. To satisfy proposition 10 we can use any two frontier portfolios and determine the two parameters δ and α by:

$$\delta = \frac{1}{\mu_a - \mu_b} \tag{47}$$

$$\alpha = \frac{\mu_b}{\mu_a - \mu_b} \tag{48}$$

Since both portfolios lie on the frontier we already know how to calculate their variances and covariance to each other. The variances are determined by equation (32) and we find:

$$\sigma_b^2 = \frac{C\alpha^2 - 2A\alpha\delta + B\delta^2}{D\delta^2} \tag{49}$$

$$\sigma_a^2 = \sigma_a^2 + \frac{C + 2\left(\alpha C - A\delta\right)}{D\delta^2} \tag{50}$$

Hence we can calculate the covariance σ_{ab}:

$$\sigma_{ab} = \sum_{i=1}^{m}\sum_{j=1}^{m} a_i b_j \sigma_{ij} \tag{51}$$

$$= \sigma_b^2 - \left(\frac{A\delta}{C} - \alpha\right)\left(\frac{C}{D\delta^2}\right) > 0 \tag{52}$$

If we claim that both portfolios are uncorrelated then equation (52) has to be zero. Hence by substitution of equation (49) into (52) and $\delta \neq 0$ we gain for $\sigma_{ab} = 0$:

$$C\alpha^2 + B\delta^2 - 2A\alpha\delta + C\alpha - A\delta = 0 \tag{53}$$

With the definitions of A, B, C and D by the equations (30) and (31) the condition of equation (53) is an ellipse in the α-δ-space.

If we claim further that both portfolios are not only uncorrelated frontier portfolios but also efficient portfolios with $\sigma_a^2 > \sigma_b^2$ and $\mu_a > \mu_a > \mu_p$ we find with equation (47) and equation (48)

$$\alpha \geq \frac{A}{C}\delta = \mu_p\delta \tag{54}$$

This is a straight line with the minimum slope of μ_p and the intersection point at the origin for $\lim_{\delta \to 0} \mu_p \delta = 0$. The straight line and the ellipses can be seen in figure (10).

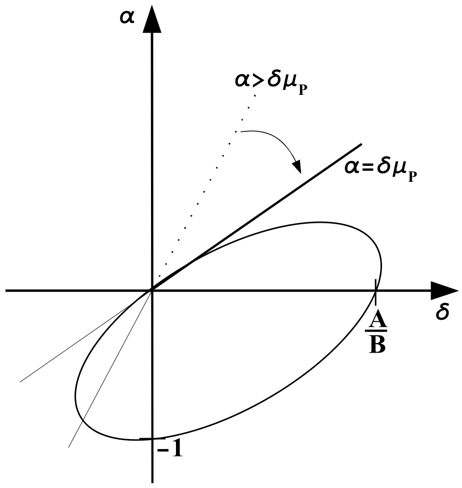

Figure 10: Dependency of the parameters α and β. The configuration of uncorrelated portfolios is an ellipse and the configuration of efficient portfolios a straight line.

As figure (10) indicates there is only a tangent point between the line and the ellipse at $(\delta = 0,\ \alpha = 0)$. Hence two uncorrelated efficient portfolios do not exist. Moreover we can say from equation (52) that all efficient portfolios have to be positive correlated.

Furthermore, we can find that $\sigma_{ab} = \sigma_b^2$ if $\alpha = \frac{A}{C}\delta = \mu_p \delta$. This is the case of the bold line in figure (10) where the b portfolio is the minimum-variance portfolio and we can find for both portfolios

$$\mu_b = \frac{A}{C} \qquad \mu_a = \frac{1}{\delta} + \frac{A}{C} \qquad \Rightarrow \mu_a = \frac{1}{\delta} + \mu_b \tag{55}$$

$$\sigma_b^2 = \frac{1}{C} \qquad \sigma_a^2 = \frac{1}{C} + \frac{C}{D\delta^2} \qquad \Rightarrow \sigma_a^2 = \sigma_b^2 + \frac{C}{D\delta^2} \tag{56}$$

where the value of δ can be choose arbitrary.

4.3. Tobin Separation

Another mutual fund theorem has been formulated by Tobin (1958) also known as Tobin Separation. Before its derivation we have to extend our assumptions and introduce a risk-free security on the market with $\mu_r = r_f$ and $\sigma_r = 0$. Then we can formulate the next

Proposition 11: If the market contains a risk-free security with $r_f < \bar{\mu}_p$ then all efficient portfolios lie on a straight line. Moreover, these portfolios contain only a different mix of the risk-free security and the market portfolio.

This problem is a special case of the $m = 2$ security case and to prove the proposition we can rewrite equation (6) to

$$\mu_p = \mu_2 + x\left(\mu_1 - \mu_2\right) \tag{57}$$

and equation (7) to

$$x = \frac{\sigma_p}{\sigma_1} \tag{58}$$

With the substitution $\mu_1 = \mu_{m^*}$, $\mu_2 = r_f$ and $\sigma_1 = \sigma_{m^*}$ we gain a straight line for the efficient portfolios

$$\mu_p = r_f + \frac{\sigma_p}{\sigma_{m^*}}\left(\mu_{m^*} - r_f\right) \tag{59}$$

Equation (59) is also known as the capital market line and the slope of this straight line is also known as the Sharpe ratio $S = \frac{\mu_{m^*} - r_f}{\sigma_{m^*}}$. As can be seen in figure (11) the tangent point of the line at the hyperbola is the so-called market portfolio.

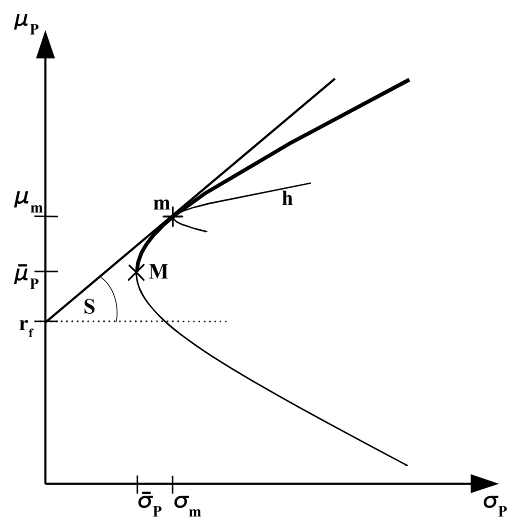

Figure 11: Tobin Separation. By adding a risk-free security to the market we gain the capital market line as locus of efficient portfolios.

Only at this point the slope of the line and the hyperbola are equal to S

$$\frac{\mathrm{d}\mu_p}{\mathrm{d}\sigma_p} = \frac{\mu_{m^*} - r_f}{\sigma_{m^*}} = S \tag{60}$$

The assumption $r_f < \bar{\mu}_p$ in proposition 11 is a necessary condition for a tangent point. To prove this we determine the δ that satisfies equation (60)

$$\delta = \frac{C\left(A - C \cdot r_f\right)}{D} \tag{61}$$

This δ is only positive (and hence the portfolio efficient) if

$$r_f < \frac{A}{C} = \bar{\mu}_p \tag{62}$$

otherwise δ is negative (and hence the portfolio inefficient) or zero in case where $r_f = \bar{\mu}_p$.

Geometrically, this is also easy to see in figure (12).

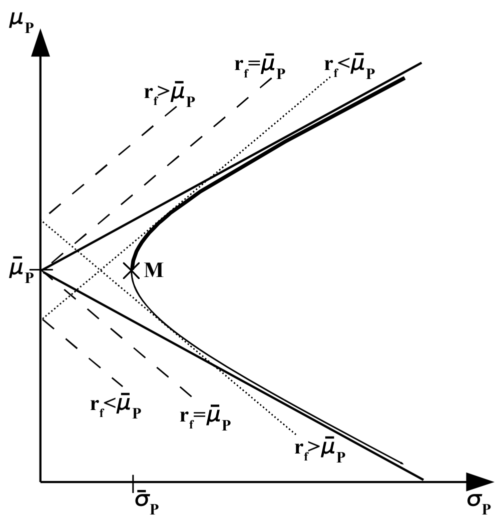

Figure 12: The tangent point between the capital market line and the hyperbola depends on the level of the return r_f of the risk-free security.

Since the hyperbola converge against its asymptotes (bold lines) a tangent point for $r_f = \bar{\mu}_p$ do not exist. In the case where a tangent point exists (dotted line) it lies below the minimum variance point M in the case if $r_f > \bar{\mu}_p$ and above M if $r_f < \bar{\mu}_p$. Only in the latter case the tangent point is an efficient portfolio. Therefore the assumption for $r_f < \bar{\mu}_p$ is mandatory.

As mentioned above some textbooks or publications contain imprecise shapes for the frontier and therefore also present an efficient tangency for $r_f > \bar{\mu}_p$ (e.g. Fama (1971)) which is actually wrong. It would also be possible to find more than one tangency if the exact shape is imprecise. Sharpe (1964) presents for instance a double tangency for $r_f = \bar{\mu}_p$.

The last step is to derive another fundamental result of the CAPM, the security market line. We can find it by calculating the covariance of every security to the market portfolio and the variance of the market portfolio

$$\sigma_{im^*} = \frac{\mu_i - r_f}{A - RC} \qquad \sigma^2_{m^*} = \frac{\mu^*_m - r_f}{A - RC} \tag{63}$$

$$\mu_i = (\mu^*_m - r_f)\,\beta_i + r_f \qquad \text{with } \beta_i = \frac{\sigma_{im^*}}{\sigma^2_{m^*}} \tag{64}$$

However, the most used method in textbooks is again more graphically. If we build up a new portfolio $\tilde{p}$ consisting the market portfolio and the i^{th} security its return and variance can be calculated by:

$$\mu_{\tilde{p}} = \alpha\mu_i + (1-\alpha)\mu_{m^*} \tag{65}$$

$$\sigma^2_{\tilde{p}} = \alpha^2\sigma^2_i + (1-\alpha)^2\sigma^2_{m^*} + 2\alpha(1-\alpha)\sigma_{im^*} \tag{66}$$

This new portfolio is presented in figure (11) by hyperbola h. Note that α is not the portion of the i^{th} security held in the portfolio $\tilde{p}$ since also the market portfolio already contains a portion of the i^{th} security.
The next step is to calculate the slope of the hyperbola h with

$$\frac{\partial\mu_{\tilde{p}}}{\partial\sigma_{\tilde{p}}} = \frac{\partial\mu_{\tilde{p}}/\partial\alpha}{\partial\sigma_{\tilde{p}}/\partial\alpha} \tag{67}$$

In particular we are only interested in the slope at the point of market portfolio hence where $\alpha = 0$. At this point the slope of the hyperbola has to be equal to the slope of the straight line and hence equal to the Sharpe ratio

$$\left.\frac{\partial\mu_{\tilde{p}}/\partial\alpha}{\partial\sigma_{\tilde{p}}/\partial\alpha}\right|_{\alpha=0} = \frac{\mu_{m^*} - r_f}{\sigma_{m^*}} = S \tag{68}$$

$$\frac{\mu_i - \mu_{m^*}}{\left(\sigma_{im^*} - \sigma^2_{m^*}\right)/\sigma_{m^*}} = \frac{\mu_{m^*} - r_f}{\sigma_{m^*}} \tag{69}$$

If we rewrite equation (69) we gain again the security market line described by equation (64).

4.4. Zero-CAPM

In the last section we assumed a risk-free security with $\sigma_r = 0$. This made it very easy to calculate the standard deviation of the portfolio containing only the risk-free and market portfolio (see equation (58)). In general the calculation of the variance of a two security portfolio has three terms (see equation (7)). Nonetheless, there is another simplification without the assumption of a risk-free security if we assume a security z with zero covariance to the market portfolio

$$\sigma_{zm^*} = 0 \quad \Rightarrow \beta_z = 0 \tag{70}$$

This idea was first published by Black (1972) and is commonly referred as Zero-Beta CAPM.

Actually, the formulation in equation (70) can be irritating since we do not have a capital market line. Hence without a risk-free security and without the capital market line we cannot determine a market portfolio. Therefore a new definition of the market portfolio is necessary and we define it as the tangent point between the efficient frontier to the straight line with the intersection point at μ_z. This is the dotted line in figure (13).

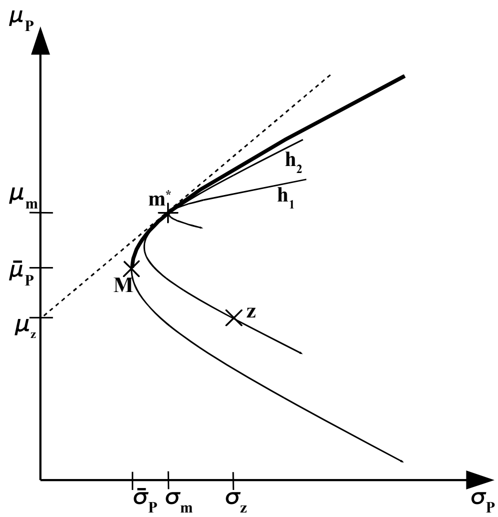

Figure 13: Zero-Beta CAPM without a risk-free security but a security with zero covariance and hence zero beta to the market portfolio.

This definition seems to be completely arbitrary but it is reasonable as we can see later. Now we can formulate

Proposition 12: If the market contains a zero-beta security z with $\mu_z < \bar{\mu}_p$ then we can formulate an equilibrium condition analogically to the security market line of the risk-free case.

To prove this we can start as in the last section by comparing both slopes but this time we have two hyperbolas h_1 and h_2. We get h_1 by building a portfolio with the market portfolio and the i^{th} portfolio and h_2 by a portfolio with the market portfolio and security z. We already know the slope of h_1 at the point of the market portfolio (see equation (69), left).

With the same method we get the slope of h_2 at the market portfolio.

$$\left.\frac{\partial\mu_{\tilde{p_2}}/\partial\alpha_2}{\partial\sigma_{\tilde{p_2}}/\partial\alpha_2}\right|_{\alpha_2=0} = \frac{\mu_{m^*}-\mu_z}{\sigma_{m^*}} \tag{71}$$

This looks almost identical to the slope of the capital market line and hence the new definition of the market portfolio seems to be reasonable.

Since the slope of both hyperbolas has to be equal at the point of the market portfolio (i.e. at $\alpha_1 = \alpha_2 = 0$) we gain the following

$$\frac{\mu_i-\mu_{m^*}}{\left(\sigma_{im^*}-\sigma_{m^*}^2\right)/\sigma_{m^*}} = \frac{\mu_{m^*}-\mu_z}{\sigma_{m^*}} \tag{72}$$

$$\mu_i = \left(\mu_m^* - \mu_z\right)\beta_i + \mu_z \quad \text{with } \beta_i = \frac{\sigma_{im^*}}{\sigma_{m^*}^2} \tag{73}$$

This looks almost identical to the security market line in equation (64).

5. Conclusion

Before we derived the CAPM we introduced the main assumptions without any deeper discussion. These assumptions and in particular the μ-σ principle is often target of critic. We can even find in Lintner (1965, pg. 15): *It is emphasized that the results of this publication are not being presented as directly applicable to practical decision, because many of the factors which matter very significantly in practice have had to be ignored or assumed away.* Despite this, the CAPM is a very well established model for practical problems. To defend this change we can cite also Sharpe (1964, pg. 434): *Needless to say, these are highly restrictive and undoubtedly unrealistic assumptions. However, since the proper test of a theory is not the realism of its assumptions but the acceptability of its implications,...*

To have a better understanding for the derivation of the CAPM it is helpful to start with the $m = 2$ and $m = 3$ case before the expanding to arbitrary m. The Mutual Fund Theorem discussed in section (4.2) shows that the problem of $m > 3$ securities is basically the same as in the 2 security case (e.g. the shape of the frontier). Additionally, to the common illustration within the μ-σ-space we presented in section (3.3) an alternative visualization which resulted in supplementary conclusions about the efficient frontier.

In the last sections we discussed in particular the Tobin Separation and derived the fundamental equation of the CAPM, the security market line. We derived an analogical equation if we add a security with zero beta instead of a risk-free security. This was called zero beta CAPM. We attached great importance for the specific shape of the efficient frontier and concluded that some of the figures in textbooks or publications are imprecise. Especially for the tangent point between the frontier an the capital market line results this simplification wrong conclusions if the risk-free rate is higher than the minimum-variance point of the frontier.

A. Appendix

List of Figures

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