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BY
Knut K. Aase

Norges Handelshøyskole
NORWEGIAN SCHOOL OF ECONOMICS

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Knut K. Aase *

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Abstract

We derive the equilibrium interest rate and risk premiums using recursive utility for jump-diffusions. Compared to to the continuous version, including jumps allows for a separate risk aversion related to jump size risk in addition to risk aversion related to the continuous part. We also consider a version that allows marginal utility to depend on past consumption. The models with jumps are shown to have a potential to give better explanation of empirical regularities than the recursive models based on merely continuous dynamics.

KEYWORDS: recursive utility, jump dynamics, the stochastic maximum principle, early resolution, utility gradients
JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

Rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time. Questions like the following are sometimes asked: Are asset prices too volatile relative to the information arriving in the market? Is the mean risk premium on equities over the riskless rate too large? Is the real interest rate too low? Is the market's risk aversion too high?

The results of Mehra and Prescott (1985) gave rise to some of these questions in their well-known paper, using a variation of Lucas's (1978) pure exchange economy with a Kydland and Prescott (1982) "calibration" exercise. They chose the parameters of the endowment process to match the

*The Norwegian School of Economics, 5045 Bergen Norway and Centre of Mathematics for Applications (CMA), University of Oslo, Norway. Telephone: (+47) 55959249. E-mail: Knut.Aase@NHH.NO. Special thanks to Bernt Øksendal for valuable comments

sample mean, variance and the annual growth rate of per capita consumption in the years 1889-1978. The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

The puzzle has been verified by many others, e.g., Hansen and Singleton (1983), Ferson (1983), Grossman, Melino, and Shiller (1987). Many theories have been suggested during the years to explain the puzzle, but to date there does not seem to be any consensus that the puzzles have been fully resolved by any single of the proposed explanations [1].

In the present paper we reconsider recursive utility in a continuous-time model including jump dynamics along the lines of Øksendal and Sulem (2013). This is an extension of the model developed by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994) which elaborate the foundational work by Kreps and Porteus (1978) and Epstein and Zin (1989) of recursive utility in dynamic models. The data set we consider is the same as that used by Mehra and Prescott (1985) in their seminal paper on this subject[2].

The state price deflator (the state prices in units of probability) depends on past values of consumption and utility, which invites us to consider a version of recursive utility where the marginal utility depends on past consumption. This seems like a modest extension of the standard model, and was carried out in the model with continuous dynamics in Aase (2014a) and in a discrete-time model in Aase (2014b). In the present model this interpretation also leads to a new solution via the stochastic maximum principle and the associated forward/backward system of stochastic differential equations.

While jump dynamics has been introduced in the conventional model, among other things to throw some light on the puzzles (see Aase (1993a-b),

[1]Constantinides (1990) introduced habit persistence in the preferences of the agents. Also Campbell and Cochrane (1999) used habit formation. Rietz (1988) introduced financial catastrophes, Barro (2005) developed this further, Weil (1992) introduced non-diversifiable background risk, and Heaton and Lucas (1996) introduce transaction costs. There is a rather long list of other approaches aimed to solve the puzzles, among them are borrowing constraints (Constantinides et al. (2001)), taxes (Mc Grattan and Prescott (2003)), loss aversion (Benartzi and Thaler (1995)), survivorship bias (Brown, Goetzmann and Ross (1995)), and heavy tails and parameter uncertainty (Weitzmann (2007)).

[2]There is by now a long standing literature that has been utilizing recursive preferences. We mention Avramov and Hore (2007), Avramov et al. (2010), Eraker and Shaliastovich (2009), Hansen, Heaton, Lee, Roussanov (2007), Hansen and Scheinkman (2009), Wacther (2012), Bansal and Yaron (2004), Campbell (1996), Bansal and Yaron (2004), Kocherlakota (1990 b), and Ai (2012) to name some important contributions. Related work is also in Browning et. al. (1999), and on consumption see Attanasio (1999). Bansal and Yaron (2004) study a richer economic environment than we employ.

in the recursive models that we analyze in this paper, jump dynamics may play an even more interesting role. This is particularly so for the model where marginal utility is allowed to depend on past consumption. The reason for this is that this recursive model has already changed matters so much in the right direction, that second order effects may be enough to get satisfactory results. Also jump dynamics in the recursive models allow for one new preference parameter related to relative risk aversion for jump size risk, which gives the model added flexibility. This may also throw some light on the behavioral puzzle of 'loss aversion'.

In the calibrations we have assumed that all income is investment income. This may be justified in the present paper, since the goal is here to compare two different models. One can view exogenous income streams as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include the new asset. However, if the latter is not traded, then the return to the wealth portfolio is not readily observable or estimable from available data. Under various assumptions this has been examined in the continuous model by Aase (2014a), and the conclusion is that the model with past dependence yields more stable results in terms of the preference parameters, than the standard recursive model. This is reexamined in the present models with jump dynamics.

It has been a goal in the modern theory of asset pricing to internalize probability distributions. To a large extent this has been achieved in our approach. The system of forward/backward stochastic differential equations leaves parameters in the probability distributions of utility to be determined in equilibrium.

The paper is organized as follows: In Section 2 we explain the problems with the conventional, time additive model including jump dynamics. Section 3 contains a preview of results for both the models we consider. Section 4 starts with a brief introduction to recursive utility in continuous time including jump dynamics, Section 5 derives the first order conditions, Section 6 details the financial market, and Section 7 presents the analysis relevant for recursive utility with jumps. Section 8 discusses the situation when the market portfolio is not a proxy for the wealth portfolio, and Section 9 concludes.

2 The problems with the conventional model

The conventional asset pricing model in financial economics, the consumption-based capital asset pricing model (CCAPM) of Lucas (1978) and Breeden (1979), assumes a representative agent with a utility function of consumption that is the expectation of a sum, or a time integral, of future discounted utility functions. The model has been criticized for several reasons. First, it does not perform well empirically. Second, the usual specification of utility can not separate the risk aversion from the elasticity of intertemporal substitution, while it would clearly be advantageous to disentangle these two conceptually different aspects of preference. Third, while this representation seems to function well in deterministic settings, and for *timeless* situations, it is not well founded for *temporal* problems (derived preferences do not in general satisfy the substitution axiom, e.g., Mossin (1969)).

In the conventional model the utility $U(c)$ of a consumption stream c_t is given by $U(c) = E\{\int_0^T u(c_t, t)\, dt\}$, where the felicity index u has the separable form $u(c,t) = \frac{1}{1-\gamma} c^{1-\gamma}\, e^{-\delta\, t}$. The parameter γ is the representative agent's relative risk aversion and δ is the utility discount rate, or the impatience rate, and T is the time horizon. These parameters are assumed to satisfy $\gamma > 0$, $\delta \geq 0$, and $T < \infty$.

When jumps are included the risk premium $(\mu_R - r)$ of any risky security labeled R (for "risky") is given by

$$\mu_R(t) - r_t = \gamma\, \sigma_{Rc}(t) - \int_{\mathcal{Z}} \big((1 + \gamma_c(t,\zeta))^{-\gamma} - 1\big)\gamma_R(t,\zeta)\nu(d\zeta). \tag{1}$$

Here r_t is the equilibrium real interest rate at time t, and the term $\sigma_{Rc}(t) = \sum_{i=1}^d \sigma_{R,i}(t)\sigma_{c,i}(t)$ is the covariance rate between returns of the risky asset and the growth rate of aggregate consumption at time t, a measurable and adaptive process satisfying standard conditions. The dimension of the Brownian motion is $d > 1$. Underlying the jump dynamics we have $\{N_j\}$, $j = 1, 2, \cdots, l$ independent Poisson random measures with Levy measures ν_j coming from l independent (1-dimensional) Levy processes. The possible time inhomogeneity in the jump processes is expressed through the terms denoted $\gamma_{R,j}(t, \zeta_j)$ for the risky asset under consideration, and $\gamma_{c,j}(t, \zeta_j)$ for the aggregate consumption process, both measuring the jump sizes. Here also jump frequencies at time t are embedded. The "mark space" $\mathcal{Z} = \mathbb{R}^l$ in this paper, where $\mathbb{R} = (-\infty, \infty)$. Thus the above term in (1) is short-hand notation for the following

$$\int_{\mathcal{Z}} \big((1 + \gamma_c(t,\zeta))^{-\gamma} - 1\big)\gamma_R(t,\zeta)\nu(d\zeta) \\ = \sum_{j=1}^{l} \int_{\mathbb{R}} \big((1 + \gamma_{c,j}(t,\zeta_j))^{-\gamma} - 1\big)\gamma_{R,j}(t,\zeta_j)\nu(d\zeta_j). \tag{2}$$

This is a continuous-time version of the consumption-based CAPM, allowing for jumps at random time points. Similarly the expression for the risk-free, real interest rate is

$$r_t = \delta + \gamma\,\mu_c(t) - \frac{1}{2}\gamma\,(\gamma+1)\,\sigma_c'(t)\sigma_c(t)$$
$$-\Big(\gamma \int_{\mathcal{Z}} \gamma_c(t,\zeta)\,\nu(d\zeta) + \int_{\mathcal{Z}} \big((1+\gamma_c(t,\zeta))^{-\gamma} - 1\big)\nu(d\zeta)\Big). \quad (3)$$

In the risk premium (1) the last term stems from the jump dynamics of the risky asset and aggregate consumption, while in (3) the last two terms have this origin. These results follow from Aase (1993a,b).

If the consumption process were as volatile as the stock market index, the jump dynamics could potentially contribute to giving a better explanation of empirical regularities than the continuous model can alone. However, because of the relatively small sizes of the potential jumps in the consumption process, it is unlikely that the last terms in these two relationships move these quantities enough in the right direction. As with the continuous model, the main problem stems from the low covariance rate between consumption and the market index.

The process $\mu_c(t)$ is the annual growth rate of aggregate consumption and $(\sigma_c'(t)\sigma_c(t))$ is the annual variance rate of the consumption growth rate, both at time t, again dictated by the Ito-isometry. Both these quantities are measurable and adaptive stochastic processes, satisfying usual conditions. The return processes as well as the consumption growth rate process in this paper are also assumed to be ergodic processes, implying that statistical estimation makes sense.

Notice that in the model is the instantaneous correlation coefficient between returns and the consumption growth rate given by

$$\kappa_{Rc}(t) = \frac{\sigma_{Rc}(t)}{||\sigma_R(t)|| \cdot ||\sigma_c(t)||} = \frac{\sum_{i=1}^{d} \sigma_{R,i}(t)\sigma_{c,i}(t)}{\sqrt{\sum_{i=1}^{d} \sigma_{R,i}(t)^2}\sqrt{\sum_{i=1}^{d} \sigma_{c,i}(t)^2}},$$

and similarly for other correlations given in this model. Here $-1 \leq \kappa_{Rc}(t) \leq 1$ for all t. With this convention we can equally well write $\sigma_R'(t)\sigma_c(t)$ for $\sigma_{Rc}(t)$, and the former does *not* imply that the instantaneous correlation coefficient between returns and the consumption growth rate is equal to one. Prime means transpose.

Similarly the term $\sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{R,j}(t,\zeta_j)\gamma_{c,j}(t,\zeta_j)\nu(d\zeta_j)$ is the covariance rate at time t between returns of the risky asset and the growth rate of aggregate consumption stemming from the discontinuous dynamics. We use the short-

	Expectation	Standard dev.	Covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 1: Key US-data for the time period 1889-1978. Continuous-time compounding.

hand notation $\int_{\mathcal{Z}} \gamma_R(t,\zeta)\gamma_c(t,\zeta)\nu(d\zeta)$ for this term as well.

Using a Taylor series expansion, the risk premium is approximately

$$\mu_R(t) - r_t = \gamma\left(\sigma_{Rc}(t) + \int_{\mathcal{Z}} \gamma_R(t,\zeta)\gamma_c(t,\zeta)\,\nu(d\zeta)\right) - \frac{1}{2}\gamma(\gamma+1)\int_{\mathcal{Z}} \gamma_R(t,\zeta)\gamma_c^2(t,\zeta)\,\nu(d\zeta) + \cdots \quad (4)$$

and an approximation for the interest rate is

$$r_t = \delta + \gamma\mu_c(t) - \frac{1}{2}\gamma(1+\gamma)\left(\sigma_c'(t)\sigma_c(t) + \int_{\mathcal{Z}} \gamma_c^2(t,\zeta)\nu(d\zeta)\right) + \frac{1}{6}\gamma(\gamma+1)(\gamma+2)\int_{\mathcal{Z}} \gamma_c^3(t,\zeta)\nu(d\zeta) - \cdots \quad (5)$$

Here the term $\int_{\mathcal{Z}} \gamma_c^2(t,\zeta)\nu(d\zeta)$ is the variance rate of the consumption growth rate at time t, stemming from the discontinuous dynamics, so that the total consumption variance rate is $(\sigma_c'(t)\sigma_c(t) + \int_{\mathcal{Z}} \gamma_c^2(t,\zeta)\nu(d\zeta))$ at time t. Similarly the total covariance rate between returns of the risky asset and the consumption growth rate is $(\sigma_{Rc}(t) + \int_{\mathcal{Z}} \gamma_R(t,\zeta)\gamma_c(t,\zeta)\nu(d\zeta))$.

The summary statistics for the US-economy for the period 1889-1978 is presented in Table 1. The table is based on the data used by Mehra and Prescott (1985). By $\hat{\sigma}_{c,M}(t)$ we mean the estimate of the covariance rate between the return on the index S&P-500 and the consumption growth rate, and likewise for the other quantities in the table. We have used the raw data, and adjusted for continuous compounding. This gives, for example, the estimate $\hat{\kappa}_{Mc} = .4033$ for the instantaneous correlation coefficient $\kappa_{M,c}(t)$.

Interpreting the risky asset R as the value weighted market portfolio M corresponding to the S&P-500 index, equations (4) and (5) are two equations in two unknowns that can provide estimates of the two preference parameters by the "method of moments". Ignoring the higher order terms in each of these equations, the result is $\gamma = 26.3$ and $\delta = -.015$, i.e., a relative risk aversion of about 26 and an impatience rate of minus 1.5%.

The jump terms might mitigate these numbers somewhat, since the jump model can, under certain distributional assumptions, produce a larger equity premium than the continuous model can alone. As an example, suppose the cross-moment term $\int_{\mathcal{Z}} \gamma_R(t,\zeta)\gamma_c^2(t,\zeta)\,\nu(d\zeta)$ is of the order -1.3 $\cdot\, 10^{-3}$ and the third moment term $\int_{\mathcal{Z}} \gamma_c^3(t,\zeta)\nu(d\zeta)$ is of the order -1.6 $\cdot\, 10^{-3}$. Then the model produces results of the order $\delta = .08$ and $\gamma = 7.7$. It is an empirical question to estimate these quantities (e.g., Ait Sahalia and Jacod (2009-11)). As we demonstrate below, jump dynamics may be more useful when combined with recursive utility.

3 Preview of results

3.1 A continuous-time recursive model with jump dynamics

Turning to recursive utility, one more parameter occurs in its most basic form. It is the time preference denoted by ρ. In the form we consider, the parameter $\psi = 1/\rho$ is the elasticity of intertemporal substitution in consumption (EIS), which we refer to as the EIS-parameter. In the conventional Eu-model $\gamma = \rho$, but relative risk tolerance $(1/\gamma)$ is something quite different from EIS.

We show that the standard recursive model extended to include jump dynamics takes the following form: For $\rho \neq 1$ and with the same notation as above

$$\mu_R(t) - r_t = \rho\,\sigma_c(t)'\sigma_R(t) + (\gamma-\rho)\,\sigma_V(t)'\sigma_R(t) + \int_{\mathcal{Z}} \left\{ \frac{\gamma_M(t,\zeta) - K_V(t,\zeta) - \frac{\gamma_0 K_V(t,\zeta)(1+\gamma_M(t,\zeta))}{1+\gamma_0 K_V(t,\zeta)}}{1+\gamma_M(t,\zeta) - \frac{\gamma_0 K_V(t,\zeta)}{1+\gamma_0 K_V(t,\zeta)}} \right\} \gamma_R(t,\zeta)\nu(d\zeta). \quad (6)$$

Here the term $K_V(t,\cdot)$ signify the jump sizes in the future utility process V, which is internalized in equilibrium as follows

$$\frac{\gamma_M(t,\zeta) - K_V(t,\zeta) - \gamma_0 K_V(t,\zeta)(1+\gamma_M(t,\zeta))/(1+\gamma_0 K_V(t,\zeta))}{1+\gamma_M(t,\zeta) - \gamma_0 K_V(t,\zeta)/(1+\gamma_0 K_V(t,\zeta))} = \gamma_0 K_V(t,\zeta) - \Big(\big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\big)^{\rho} - 1 \Big) \Big(1 + \gamma_0 K_V(t,\zeta) \Big), \quad (7)$$

where the equality holds $\nu(\cdot)$ a.e. Also the volatility of the utility process V

is given by

$$\sigma_V(t) = \frac{1}{1-\rho}\big(\sigma_M(t) - \rho\sigma_c(t)\big). \tag{8}$$

The jump term in (6) reduces to the jump term in (1) when $K_V(t,\cdot) = \sigma_V(t) = 0$ a.e., so K_V and σ_V have strictly to do with recursive utility. The short term real interest rate is given by

$$\begin{aligned} r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\rho(\rho+1)\,\sigma_c'(t)\sigma_c(t) - \rho(\gamma-\rho)\sigma_c(t)'\sigma_V(t) \\ - \frac{1}{2}\,(\gamma-\rho)(1-\rho)\,\sigma_V'(t)\sigma_V(t) - \frac{1}{2}(1+\rho)\gamma_0\int_{\mathcal{Z}} K_V' K_V\,\nu(d\zeta) \\ - \int_{\mathcal{Z}} \gamma_0 K_V(t,\zeta)\Big\{\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^{\rho} - 1\Big\}\,\nu(d\zeta) \\ - \int_{\mathcal{Z}} \Big\{\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^{\rho} - 1 + \rho\gamma_c(t,\zeta) - \rho\,K_V(t,\zeta)\Big\}\,\nu(d\zeta), \end{aligned} \tag{9}$$

Here $\sigma_M(t)$ signifies the volatility of the return on the market portfolio of the risky securities, $\sigma_R'(t)\sigma_M(t) = \sigma_{RM}(t)$ is the instantaneous covariance rate of the returns on any risky asset, with the return of the market portfolio. In the model these quantities are assumed to be measurable, adaptive, ergodic stochastic processes satisfying standard conditions. The parameter γ_0 is the agent's relative risk aversion related to jump size risk. When there are no jumps, we obtain what we call the standard recursive model. When $\rho = \gamma = \gamma_0$, the model reduces to the conventional, additive Eu-model with jumps presented in the previous section (same as when $K_V = \sigma_V = 0$).

The term $\gamma_M(t,\zeta)$ models the jump sizes and frequency for the market portfolio, and the term $K_V(t,\zeta)$ plays the same role for the utility process V_t, to be explained in the next section.

Calibrating the standard recursive model with only continuous dynamics we obtain Table 2. This model is based on the aggregator (18) in Section 4, and the risk premium was first derived by Duffie and Epstein (1992a)[3]. The interest rate was first derived in Aase (2014a), and follows from our approach in the present paper. Here we have fixed the time impatience rate δ and solved the two equations (6) and (9) in the two remaining unknowns γ and ρ, for values of δ between zero and 11 per cent.

From this table we notice that there is a fairly narrow band of values of the impatience rate δ that give reasonable values for the parameters[4], and

[3]The coefficients were all constants, since dynamic programming was used. This is not necessary in our approach

[4]The recursive model has also another solution where γ varies from 27.20 to 68.13, and

	γ	ρ	EIS
Conventional Eu-Model			
$\delta = -.015$	26.37	26.37	.037
Standard recursive model with no jumps			
$\delta = .01$	.005	1.61	.62
$\delta = .02$	.90	1.06	.94
$\delta = .03$	1.74	.49	2.04
$\delta = .04$	2.58	- .13	- 7.69
$\delta = .05$	3.31	- .79	-1.26
$\delta = .06$	4.02	-1.47	- .68
$\delta = .07$	4.69	-2.20	- .45
$\delta = .08$	5.32	- 2.97	- .34
$\delta = .09$	5.91	- 3.76	- .27
$\delta = .10$	6.46	- 4.59	- .22
$\delta = .11$	6.98	-5.45	- .18

Table 2: Calibrations of the standard continuous model

small changes in one parameter may easily lead one or more of the other two parameter out of the plausible region.

In applied economics values of the impatience rate between 1 and 2 per cent seem common. One reason for this is of course that the conventional, additive Eu-model is often taken for granted, and from the expression for the interest rate in (3) one simply does not obtain reasonable values for the short rate unless δ is in this range, or smaller.

In this connection it may be of interest to consider the study of Andersen et. al. (2008). They use controlled experiments with field subjects in Denmark to elicit the impatience rate and risk preference, ignoring the subject of time preferences. First, an estimate of δ around 25% is reached assuming risk neutrality, second, a new estimate of δ around 10% is obtained assuming risk aversion, with an associated estimate of γ around .74, both based on arithmetic averaging. Notice that a value of about ten per cent does not fit well with the standard recursive model.

With the jump terms added, this may change. The above continuous model gives some interesting results, albeit in a rather narrow band of parameter values. One might conjecture that this requires minor adjustments to the model, which the discontinuous part could provide.

As an illustration, of the total annual variation of .0250 in the stock

ρ varies from 25.19 to 14.03. This is not better than the conventional, additive Eu-model.

	γ	ρ	γ_0	EIS
Standard recursive model including jump dynamics				
$\delta = .01$	-1.15	1.60	6.00	.63
$\delta = .02$	.71	1.09	4.00	.92
$\delta = .03$	2.59	.45	3.00	2.22
$\delta = .04$	31.16	10.00	2.00	.10
$\delta = .05$	32.77	10.52	2.00	.09
$\delta = .10$	42.27	8.96	2.00	.11
$\delta = .01$	2.0	1.07	.01	.93
$\delta = .02$	2.5	.88	-.02	1.14
$\delta = .03$	1.5	1.39	.09	.72
$\delta = .04$	2.5	.75	-.02	1.33
$\delta = .05$	2.0	1.20	.05	.83
$\delta = .10$	1.78	.93	.01	1.19

Table 3: Calibrations of the standard model including jump dynamics

market, measured as variance, suppose we allocate .010 to jumps. Further we suppose there is no significant jump activity in consumption. Ignoring higher order terms, the resulting jump-diffusion of this section calibrates to $\delta = .040$, $\gamma = \gamma_0 = .47$ and $\rho = 1.13$. When Table 2 starts giving implausible values, the jump part may adjust for this.

As Table 3 illustrates, however, there is still some variation in the values obtained. The upper half of the table indicates that the model does not explain well larger values of γ_0.

The values in the lower half of the table are seen to be quite stable as δ varies, for reasonable values of γ, while the values of γ_0 are then small, and sometimes negative. We return to a discussion in the next section.

More numerical work is needed here, combined with statistical work, aimed at separating the discontinuous dynamics from the continuous part (e.g., Ait Sahalia and Jacod (2009-11)). However, the general picture seems to be that jumps may be of particular interest in the recursive models.

3.2 A recursive model with past dependence

Based on the analysis to be presented later, where we relax the assumption that past consumption does not matter for current marginal utility, the relationships corresponding to the above are given by

$$\mu_R(t) - r_t = \rho\,\sigma_c(t)'\sigma_R(t) + (\gamma - \rho)\,\sigma_V(t)'\sigma_R(t).$$
$$+ \int_{\mathcal{Z}} \Big(\frac{\gamma_M(t,\zeta) - K_V(t,\zeta)}{1+\gamma_M(t,\zeta)}\Big)\gamma_R(t,\zeta)\nu(d\zeta). \quad (10)$$

Here the term $K_V(\cdot,\cdot)$ also signify the jump sizes in the utility process V, which is internalized in equilibrium as follows

$$\frac{\gamma_M(t,\zeta) - K_V(t,\zeta)}{1+\gamma_M(t,\zeta)} = \gamma_0\, K_V(t,\zeta)$$
$$- \Big(\frac{(1+K_V(t,\zeta))^{\rho}}{(1+\gamma_c(t,\zeta))^{\rho}} - 1\Big)\Big(1 + \gamma_0\, K_V(t,\zeta)\Big), \quad (11)$$

where the equality holds $\nu(\cdot)$ a.e., and

$$\sigma_V(t) = \frac{1}{1+\gamma-\rho}\Big(\sigma_M(t) - \rho\sigma_c(t)\Big). \quad (12)$$

The right-hand side of (11) is the same as the right-hand side of (7). The equilibrium short rate with jumps for this model is

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\rho(\rho+1)\,\sigma_c'(t)\sigma_c(t) - \rho(\gamma-\rho)\sigma_c(t)'\sigma_V(t)$$
$$- \frac{1}{2}\,(\gamma-\rho)(1-\rho)\,\sigma_V'(t)\sigma_V(t) - \frac{1}{2}(1+\rho)\gamma_0 \int_{\mathcal{Z}} K_V' K_V \nu(d\zeta)$$
$$- \int_{\mathcal{Z}} \gamma_0 K_V(t,\zeta)\Big\{\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^{\rho} - 1\Big\}\,\nu(d\zeta)$$
$$- \int_{\mathcal{Z}} \Big\{\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^{\rho} - 1 + \rho\gamma_c(t,\zeta) - \rho\, K_V(t,\zeta)\Big\}\,\nu(d\zeta). \quad (13)$$

which is the same as (9), but with K given by (11). When $\gamma = \gamma_0 = \rho$ the above results also reduce to the ones of the conventional model of Section 2. The same happens when $K_V(t,\cdot) = \sigma_V(t) = 0$ a.e.

Calibrations of the version including jumps are presented in Table 4, under the same assumptions as for Table 3. Parameter values are seen to be more stable throughout the whole range of values of δ than for the standard model. This model explains well this range of values for the impatience rate δ, including the high values reported by Andersen et. al. (2008). This is also true for the model without jumps for the present version, as demonstrated in Aase (2014a,b).

Parameters	γ	ρ	γ_0	EIS
The model (10) and (13) including jumps				
$\delta = .01$	-.12	1.21	6.00	.83
$\delta = .02$	.01	1.32	4.00	.76
$\delta = .03$	.09	1.40	3.00	.71
$\delta = .04$	.09	1.37	2.00	.73
$\delta = .05$	.18	1.46	2.00	.68
$\delta = .10$	.50	1.77	2.00	.56
$\delta = .01$	2.0	1.26	.02	.79
$\delta = .02$	2.0	1.27	.03	.79
$\delta = .03$	2.0	1.28	.04	.78
$\delta = .04$	2.5	1.30	.05	.77
$\delta = .05$	1.5	1.27	.05	.79
$\delta = .10$	.57	1.23	.06	.81

Table 4: Calibrations of the model with jumps where past consumption matters.

The upper half of Table 4 should be contrasted with the same part of Table 3. The values in Table 4 are much more stable. Here the weighted average risk aversion is larger than the time preference. From Table 2 we notice that the standard recursive model calibrates to $\gamma > \rho$ when $\delta = .03$, but aside from this, the other values have negative time preference, or $\gamma < \rho$.

The lower part of Table 4 also points in the direction of preference for early resolution for the US-data when γ takes on values around 2, in which case the risk aversion γ_0 for jump size risk is small. Since jumps in the market index are primarily negative, a different risk aversion γ_0 for jump-size risk can be a utility-based explanation of "loss aversion" (see Kahneman and Tversky (1979)), in which case risk proclivity is observed in experimental situations. Much the same conclusions can be drawn from the lower part of Table 3.

The special situation with $\gamma = \rho \neq \gamma_0$ is also calibrated, and give plausible values for the parameters, with $\gamma_0 < \rho$. In this calibration all the variance in the stock market was attributed to the jump part. More generally, we can consider the situation where the dynamics only involve jumps. Since the data we consider are annual observations, this would correspond to an average jump frequency of one per year. This yields plausible calibrations as well.

4 Recursive Stochastic Differentiable Utility

In this section we give a brief introduction to recursive, stochastic differential utility in the continuous-time model including jumps, along the lines of Øksendal and Sulem (2013). The starting point for this theory for the continuous model is Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994). Our approach based on Øksendal and Sulem (2013) is more general, and in particular does not require any Markov structure.

We are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0,T], P)$ satisfying the 'usual' conditions, and a standard model for the stock market with Brownian motion driven uncertainty, N risky securities and one riskless asset (Section 5 provides more details). Consumption processes are chosen form the space L of square integrable progressively measurable processes with values in R_+.

The stochastic differential utility $U : L \to R$ is defined as follows by three primitive functions: $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $A : \mathbb{R} \to \mathbb{R}$ and $A_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The function $f(t, c_t, V_t, \omega)$ is a felicity index at time t, A is a measure of absolute risk aversion related to the continuous dynamics, while A_0 measures risk aversion related to jumps. Both the latter two terms may also depend on t. In addition to current consumption c_t, the felicity index also depends utility V_t.

The utility process V for a given consumption process c that we consider, satisfying $V_T = 0$, is given by the representation

$$V_t = E_t\Big\{ \int_t^T \big(f(s, c_s, V_s) - \frac{1}{2} A(V_s)\, Z(s)' Z(s) \\ - \frac{1}{2} \int_{\mathcal{Z}} A_0(V_s, \zeta) K'(s,\zeta) K(s,\zeta) \nu(d\zeta)\big) ds \Big\}, \quad t \in [0,T] \quad (14)$$

where $E_t(\cdot)$ denotes conditional expectation given $\mathcal{F}_t$ and $Z(t)$ as well as $K(t,\cdot)$ are square-integrable progressively measurable processes, to be determined in our analysis. Here d is the dimension of the Brownian motion B_t, and $K(t,\cdot)$ is an l dimensional vector. We think of V_t as the utility for c at time t, conditional on current information $\mathcal{F}_t$. The term $A(V_t)$ is penalizing for risk in the continuous model, while the term $A_0(V_t, \cdot)$ penalize for jump size risk.

If, for each consumption process c_t, there is a well-defined utility process V, the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The triplet (f, A, A_0) generating V is called an aggregator.

Since $V_T = 0$ and $\int Z(t) dB_t$ and $\int \int_{\mathcal{Z}} K(t,\zeta) \tilde{N}(dt, d\zeta)$ are martingales,

(14) has the stochastic differential equation representation

$$dV_t = \Big(- f(t, c_t, V_t) + \frac{1}{2} A(V_t)\, Z(t)' Z(t) +$$
$$\frac{1}{2} \int_{\mathcal{Z}} A_0(V_t, \zeta) K'(t, \zeta) K(s, \zeta) \nu(d\zeta) \Big)\, dt + Z(t)\, dB_t + \int_{\mathcal{Z}} K(t, \zeta) \tilde{N}(dt, d\zeta). \tag{15}$$

Here $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$ is an l-dimensional compensated Poisson random measure of the underlying l-dimensional Levy process, and $B(t)$ is an independent d dimensional, standard Brownian motion.

If terminal utility different from zero is of interest, like for applications to life insurance, then V_T may be different from zero. We think of A and A_0 as associated with functions $h, h_0 : \mathbb{R} \to \mathbb{R}$ such that $A(v) = -\frac{h''(v)}{h'(v)}$, where h is two times continuously differentiable, and similarly for h_0. U is monotonic and risk averse if $A(\cdot) \geq 0$, $A_0(\cdot, \cdot) \geq 0$ and f is jointly concave and increasing in consumption.

The preference ordering represented by recursive utility is usually assumed to satisfy A1: Dynamic consistency (in the sense of Johnsen and Donaldson (1985)), A2: Independence of past consumption, and A3: State independence of time preference (see Skiadas (2009a)).

One of the advantages with the recursive model is that utility may depend on the past. This we make use of in the present paper. Below we relax assumption A2 related to marginal utility at any time $t > 0$: In the recursion (14), if V_s depends on past consumption for $s \geq t > 0$, so will V_t.

The version we consider has the Kreps-Porteus utility representation, which corresponds to the aggregator with a CES specification

$$f(c, v) = \frac{\delta}{1-\rho} \frac{c^{1-\rho} - v^{1-\rho}}{v^{-\rho}}, \quad A(v) = \frac{\gamma}{v} \text{ and } A_0(v, \zeta) = \frac{\gamma_0}{v}, \forall \zeta \in R \tag{16}$$

corresponding to functions $u(c) = \frac{c^{1-\rho}}{1-\rho}$ and $h(v) = \frac{v^{1-\gamma}}{1-\gamma}$, and $h_0(x) = \frac{x^{1-\gamma_0}}{1-\gamma_0}$. If, for example, $A(v) = A_0(v) = 0$ for all v, this means that the recursive utility agent is risk neutral.

Here $\rho \geq 0, \rho \neq 1, \delta \geq 0, \gamma \geq 0, \gamma \neq 1, \gamma_0 \geq 0, \gamma_0 \neq 1$ (when $\rho = 1$, $\gamma = 1$ or $\gamma_0 = 1$ it is the logarithms that apply). The elasticity of intertemporal substitution in consumption $\psi = 1/\rho$. The parameter ρ is the time preference parameter. Here $u(\cdot)$, $h(\cdot)$ and $h_0(\cdot)$ can all be different functions, resulting in the desired disentangling of γ or γ_0 from ρ.

For the model with continuous dynamics only, an ordinally equivalent specification can be derived as follows. When an aggregator (f_1, A_1) is given corresponding to the utility function U_1, there exists a strictly increasing and smooth function $\varphi(\cdot)$ such that the ordinally equivalent $U_2 = \varphi \circ U_1$ has the aggregator (f_2, A_2) where

$$f_2(c,v) = ((1-\gamma)v)^{-\frac{\gamma}{1-\gamma}} f_1(c, ((1-\gamma)v)^{\frac{1}{1-\gamma}}), \qquad A_2 = 0.$$

The function φ is given by

$$U_2 = \frac{1}{1-\gamma} U_1^{1-\gamma}, \tag{17}$$

for the Kreps-Porteus specification. It has has the CES-form

$$f_2(c,v) = \frac{\delta}{1-\rho} \frac{c^{1-\rho} - ((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \qquad A_2(v) = 0. \tag{18}$$

The reduction to a normalized aggregator $(f_2, 0)$ does not mean that intertemporal utility is risk neutral, or that the representation has lost the ability to separate risk aversion from substitution (see Duffie and Epstein(1992a)). The corresponding utility U_2 retains the essential features, namely that of (partly) disentangling intertemporal elasticity of substitution from risk aversion. This version is not as natural with jumps, and will not be used in this paper.

In Aase (2013a) it is shown that these two versions have the same risk premiums and the same short term interest rate in standard recursive model with no jump dynamics. In the model where marginal utility is allowed to depend on past consumption, these quantities are different. However, the ordinally equivalent specification in the latter framework has the same risk premiums and interest rate as the standard recursive model based on (18).

It is instructive to recall the that the conventional additive and separable utility has aggregator

$$f(c,v) = u(c) - \delta v, \quad A = 0. \tag{19}$$

in the present framework (an ordinally equivalent one). As can be seen, even if $A = 0$, the agent of the conventional model is not risk neutral.

Applying this last observation to the conventional model with jumps presented in Section 2, if $A = A_0 = 0$ (since there is only one type of risk aversion in the conventional model), this means that $Z = K = 0$, in which case the risk premiums and the interest rate in Section 3 is seen to reduce to the ones in Section 2 when $\rho = \gamma = \gamma_0$.

4.1 Homogeniety

The following result will be made use of in sections 7.3-4. For a given consumption process c_t we let $(V_t^{(c)}, Z_t^{(c)}, K_t(\zeta)^{(c)})$ be the solution of the BSDE

$$\begin{cases} dV_t^{(c)} = \Big(-f(t,c_t,V_t^{(c)}) + \frac{1}{2}A(V_t^{(c)})\, Z(t)'^{(c)} Z(t)^{(c)} + \\ \frac{1}{2}\int_{\mathcal{Z}} A_0(V_t^{(c)},\zeta) K'(t,\zeta)^{(c)} K(s,\zeta)^{(c)} \nu(d\zeta)\Big)\, dt + Z(t)^{(c)}\, dB_t \\ + \int_{\mathcal{Z}} K(t,\zeta)^{(c)} \tilde{N}(dt,d\zeta); \qquad 0 \le t \le T \\ V_T^{(c)} = 0 \end{cases} \tag{20}$$

Theorem 1 *Assume that, for all* $\lambda > 0$,
(i) $\lambda\, f(t,\ c,\ v) = f(t,\ \lambda c,\ \lambda v);\ \forall\ t,\ c,\ v,\ \omega$
(ii) $A(\lambda v) = \frac{1}{\lambda} A(v);\ \forall\ v$
(iii) $A_0(\lambda v) = \frac{1}{\lambda} A_0(v);\ \forall\ v$
Then

$$V_t^{(\lambda c)} = \lambda V_t^{(c)},\ Z_t^{(\lambda c)} = \lambda Z_t^{(c)} \text{ and } K_t^{(\lambda c)}(\zeta) = \lambda K_t^{(c)}(\zeta);\ \forall \zeta,\ t \in [0,T]. \tag{21}$$

Proof By uniqueness of the solution of the BSDEs of the type (20), all we need to do is to verify that the triple $(\lambda V_t^{(c)}, \lambda Z_t^{(c)}, \lambda K_t(\cdot)^{(c)})$ is a solution of the BSDE (20) with c_t replaced by λc_t, i.e. that

$$\begin{cases} d(\lambda V_t^{(c)}) = \Big(-f(t,\lambda c_t,\lambda V_t^{(c)}) + \frac{1}{2}A(\lambda V_t^{(c)})\, \lambda Z(t)'^{(c)} \lambda Z(t)^{(c)} + \\ \frac{1}{2}\int_{\mathcal{Z}} A_0(\lambda V_t^{(c)},\zeta) \lambda K'(t,\zeta)^{(c)} \lambda K(s,\zeta)^{(c)} \nu(d\zeta)\Big)\, dt + \lambda Z(t)^{(c)}\, dB_t \\ + \int_{\mathcal{Z}} \lambda K(t,\zeta)^{(c)} \tilde{N}(dt,d\zeta); \qquad 0 \le t \le T \\ \lambda V_T^{(c)} = 0 \end{cases} \tag{22}$$

By (i), (ii) and (iii) the BSDE (22) can be written

$$\begin{cases} \lambda dV_t^{(c)} = \Big(-\lambda f(t,c_t,V_t^{(c)}) + \frac{1}{2}\frac{1}{\lambda}A(V_t^{(c)})\, \lambda^2 Z(t)'^{(c)} Z(t)^{(c)} + \\ \frac{1}{2}\int_{\mathcal{Z}} \frac{1}{\lambda} A_0(V_t^{(c)},\zeta) \lambda^2 K'(t,\zeta)^{(c)} K(s,\zeta)^{(c)} \nu(d\zeta)\Big)\, dt + \lambda Z(t)^{(c)}\, dB_t \\ + \lambda \int_{\mathcal{Z}} K(t,\zeta)^{(c)} \tilde{N}(dt,d\zeta); \qquad 0 \le t \le T \\ \lambda V_T^{(c)} = 0 \end{cases} \tag{23}$$

But this is exactly the equation (20) multiplied by the constant λ. Hence (23) holds and the proof is complete. $\square$

Remarks 1) Note that the system need not be Markovian in general, since we allow

$$f(t,c,v,\omega);\ \ (t,\omega)\in[0,T]\times\Omega$$

to be an adapted process, for each fixed c, v.
2) Similarly, we can allow A_0 and A to depend on t as well[5].

Corollary 1 *Define $U(c) = V_0^{(c)}$. Then $U(\lambda c) = \lambda U(c)$ for all $\lambda > 0$.*

Notice that the aggregator in (16) satisfies the assumptions of the theorem.

5 The First Order Conditions

In the following we solve the consumer's optimization problem, where the assumption A2 plays no role, using the stochastic maximum principle and forward/backward stochastic differential equations. We return to the issue of relaxing A2 later. We have the specification in (15) and (16) in mind, formulated in the previous section, where the $\tilde{f}$ to appear below is the drift term in (15). However, in principle the analysis is valid for any f, A and A_0 satisfying the stated conditions. The representative agent's problem is to solve

$$\sup_{\tilde{c}\in L} U(\tilde{c})$$

subject to

$$E\Big\{\int_0^T \tilde{c}_t\pi_t dt\Big\}\leq E\Big\{\int_0^T c_t\pi_t dt\Big\}.$$

Here $V_t = V_t^{\tilde{c}}$, and $(V_t, Z(t), K(t,\cdot))$ is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dV_t = -\tilde{f}(t,\tilde{c}_t,V_t,Z(t),K(t,\zeta))\,dt + Z(t)\,dB_t + \int_{\mathcal{Z}} K(t,\zeta)\tilde{N}(dt,d\zeta)\\ V_T = 0.\end{cases} \tag{24}$$

For $\alpha > 0$ we define the Lagrangian

$$\mathcal{L}(\tilde{c};\lambda) = U(\tilde{c}) - \alpha\, E\Big(\int_0^T \pi_t(\tilde{c}_t - c_t)dt\Big).$$

Important is here that the volatility $Z(t)$ as well as the jump size quantity $K(t,\zeta)$ are both to be determined, together with the dynamics of utility V.

[5] not common in economics

Market clearing combined with properties of recursive utility in Theorem 1 will be used to internalize these quantities.

In order to set down the first order condition for the representative consumer's problem, we use Kuhn-Tucker and either directional derivatives in function space, or the stochastic maximum principle. Neither of these principles require any Markovian structure of the economy. The problem is well posed since U is increasing and concave and the constraint is convex.

Because of the generality of the problem, we utilize the stochastic maximum principle (see Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Øksendal and Sulem (2013), or Peng (1990)): We then have a forward backward stochastic differential equation (FBSDE) system consisting of the simple FSDE $dX(t) = 0; X(0) = 0$ and the BSDE (24). The Hamiltonian for this problem is

$$H(t, \tilde{c}, v, z, k, y) = y_t\, \tilde{f}(t, \tilde{c}_t, v_t, z_t, k_t) - \alpha\, \pi_t(\tilde{c}_t - c_t), \tag{25}$$

where

$$\tilde{f}(t, c, v, z, k) = f(c, v) - \frac{1}{2} A(v) z' z - \frac{1}{2} \int_{\mathcal{Z}} A_0(v, \zeta) k'(t, \zeta) k(t, \zeta) \nu(d\zeta) \tag{26}$$

with A and A_0 given in (16). Let $\bigtriangledown_k \tilde{f}$ denote the Frechet derivative of $\tilde{f}$ with respect to k, and $\frac{d \bigtriangledown_k \tilde{f}}{d\nu}(\zeta)$ denote its Radon-Nikodym derivative with respect to ν. From the general theory, the adjoint equation is then

$$\begin{cases} dY_t = Y(t-) \big\{ \big(\frac{\partial f}{\partial v}(t, \tilde{c}_t) - \frac{1}{2} (\frac{\partial}{\partial v} A(V_t))\, Z'(t) Z(t) \\ -\frac{1}{2} \int_{\mathcal{Z}} (\frac{\partial}{\partial v} A_0(V_t, \zeta))\, K'(t, \zeta) K(t, \zeta) \nu(d\zeta) \big) dt \\ -\frac{1}{2} \frac{\partial}{\partial z} \big(A(V_t) Z_t' Z_t \big) dB_t + \int_{\mathcal{Z}} \frac{d \bigtriangledown_k \tilde{f}}{d\nu}(t, \tilde{c}_t, V_t, Z_t, K(t, \cdot))(\zeta) \tilde{N}(dt, d\zeta) \big\}, \\ Y_0 = 1. \end{cases}$$

With A_0 as in (16), we see that $\bigtriangledown_k \tilde{f}$ is the linear operator

$$h \to (\bigtriangledown_k \tilde{f})(h) = -\int_{\mathcal{Z}} A_0(v, \zeta) k'(\zeta) h(\zeta) \nu(d\zeta); \;\; h \in L^2(\nu).$$

Therefore, as a random measure we have that $\bigtriangledown_k \tilde{f} << \nu$, with Radon-Nikodym derivative

$$\frac{d \bigtriangledown_k \tilde{f}}{d\nu}(\zeta) = -A_0(v, \zeta) k(\zeta).$$

Based on this, the adjoint equation can be written

$$\begin{cases} dY_t = Y(t-)\{\big(\frac{\partial f}{\partial v}(t,\tilde{c}_t) + \frac{1}{2}\frac{\gamma}{V_t^2}Z'(t)Z(t) \\ +\frac{1}{2}\int_{\mathcal{Z}} \frac{\gamma_0}{V_{t-}^2}K'(t,\zeta)K(t,\zeta)\nu(d\zeta)\big)dt \\ -\frac{\gamma}{V_t}Z(t)dB_t - \int_{\mathcal{Z}} \frac{\gamma_0}{V_{t-}}K(t,\zeta)\tilde{N}(dt,d\zeta)\}, \\ Y(0) = 1, \end{cases} \tag{27}$$

which has the solution

$$\begin{aligned} Y_t = \exp\Big(\int_0^t \big(\frac{\partial f}{\partial v}(s,\tilde{c}_s) + \frac{1}{2}\frac{\gamma(1-\gamma)}{V_s^2}Z'(s)Z(s) \\ + \frac{1}{2}\int_{\mathcal{Z}} \frac{\gamma_0}{V_{s-}^2}K'(s,\zeta)K(s,\zeta)\nu(d\zeta)\big)ds - \int_0^t \frac{\gamma}{V_s}Z(s)dB_s \\ + \int_0^t \int_{\mathcal{Z}} \{\ln(1 - \frac{\gamma_0}{V_{s-}}K(s,\zeta)) + \frac{\gamma_0}{V_{s-}}K(s,\zeta)\}\nu(d\zeta)ds \\ + \int_0^t \int_{\mathcal{Z}} \ln(1 - \frac{\gamma_0}{V_{s-}}K(s,\zeta))\tilde{N}(ds,d\zeta)\Big). \end{aligned} \tag{28}$$

The adjoint equation is now reduced to primitives of the economy, in addition to the two unknowns K and Z. Maximizing the Hamiltonian with respect to $\tilde{c}$ gives the first order equation

$$y\,\frac{\partial \tilde{f}}{\partial \tilde{c}}(t,c^*,v,z,k) - \alpha\,\pi = 0$$

or

$$\alpha\,\pi_t = Y(t)\frac{\partial \tilde{f}}{\partial \tilde{c}}(t,c_t^*,V(t),Z(t),K(t,\cdot)) \quad \text{a.s. for all } t \in [0,T]. \tag{29}$$

where c^* is optimal. The state price deflator π_t at time t depends, through the adjoint variable Y_t, on the entire optimal paths $(c_s^*, V_s, Z(s), K(s,\cdot))$ for $0 \le s \le t$, which means that marginal utility at time t depends on the consumption history.

When $\gamma = \gamma_0 = \rho$ then $Y_t = e^{-\delta t}$ for the aggregator (19) of the conventional model, so the state price deflator is a Markov process, the utility is additive and dynamic programming works well.

For the representative agent equilibrium the optimal consumption process is the given aggregate consumption c in society, and for this consumption process the utility V_t at time t is optimal.

We now have the first order conditions for recursive utility. Before we proceed to a solution of the problem, we need to specify the financial market model.

6 The financial market

Having established the general recursive utility form of interest, in his section we specify our model for the financial market. The model is much like the one used by Duffie and Epstein (1992a), except that we do not assume any unspecified factors in our model.

Let $\nu(t) \in R^N$ denote the vector of expected rates of return of the N given risky securities in excess of the riskless instantaneous return r_t, and let $\sigma(t)$ denote the $N \times d$-matrix of diffusion coefficients of the risky asset prices, normalized by the asset prices, so that $\sigma(t)\sigma(t)'$ is the instantaneous covariance matrix for the continuous part of asset returns. The jumps in the various assets are captured by the $N \times l$-matrix $\gamma(t,\zeta)$ and a vector valued, compensated random measure

$$\begin{aligned}\tilde{N}(dt, d\zeta)' = (\tilde{N}_1(dt, d\zeta_1), \cdots, \tilde{N}_l(dt, d\zeta_l)) = \\ (N_1(dt, d\zeta_1) - \nu_1(d\zeta_1)dt, \cdots, N_l(dt, d\zeta_l) - \nu_l(d\zeta_l)dt),\end{aligned}$$

where $\{N_j\}$ are independent Poisson random measures with Levy measures ν_j coming from l independent (1-dimensional) Levy processes.

The representative consumer's problem is, for each initial level w of wealth to solve

$$\sup_{(c,\varphi)} U(c) \tag{30}$$

subject to the intertemporal budget constraint

$$\begin{aligned}dW_t = \big(W_t(\varphi_t' \cdot \nu(t)) + r_t) - c_t\big)dt + W_t\varphi_t' \cdot \sigma(t)dB_t \\ + W_t\varphi_t' \cdot \int_{R^l} \gamma(t,\zeta)\tilde{N}(dt, d\zeta).\end{aligned} \tag{31}$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ are the fractions of total wealth W_t held in the risky securities. The processes $\nu(t)$, $\sigma(t)$ and $\gamma(t)$ are progressively measurable, ergodic processes.

Market clearing requires that $\varphi_t'\sigma(t) = (\delta_t^M)'\sigma(t) = \sigma_M(t)$ and $\varphi_t'\gamma(t,\cdot) = (\delta_t^M)'\gamma(t,\cdot) = \gamma_M(t,\cdot)$ in equilibrium, where $\sigma_M(t)$ is the volatility of the return on the market portfolio, $\gamma_M(t,\cdot)$ is the corresponding jump size function, and δ_t^M are the fractions of the different securities, $j = 1, \cdots, N$ held in the value-weighted market portfolio. That is, the representative agent must hold the market portfolio in equilibrium, by construction.

7 The consequences of the recursive models

We now turn our attention to pricing restrictions relative to the given optimal consumption plan. Recall the first order conditions are given in (29).

It is convenient to use the notation $Z(t)/V_t := \sigma_V(t)$ and $K(t,\cdot)/V(t-) := K_V(t,\cdot)$, where V_{t-} means the value of V just before a possible jump at time t, assuming $V \neq 0$. By Theorem 1, $\sigma_V(t)$ and $K_V(t,\cdot)$ are both homogeneous of degree zero in c. With this convention the utility process V_t satisfies the following backward equation

$$\frac{dV_t}{V_{t-}} = \Big(- \frac{\delta}{1-\rho} \frac{c_t^{1-\rho} - V_t^{1-\rho}}{V_t^{-\rho+1}} + \frac{1}{2}\gamma \sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 \, K_V'(t,\zeta)K_V(t,\zeta)\nu(d\zeta)\Big) dt + \sigma_V(t)dB_t + \int_{\mathcal{Z}} K_V(t,\zeta)\tilde{N}(dt,d\zeta), \quad (32)$$

where $V(T) = 0$. The short-hand notation for the integrals with jump dynamics is as explained in Section 2. Since the jump times have Lebesgue measure zero, $V_t = V_{t-}$ a.e. on $[0,T]$.

Aggregate consumption is exogenous, with dynamics on of the form

$$\frac{dc_t}{c_{t-}} = \mu_c(t)\, dt + \sigma_c(t)\, dB_t + \int_{\mathcal{Z}} \gamma_c(t,\zeta)\tilde{N}(dt,d\zeta), \quad (33)$$

where $\mu_c(t)$, $\sigma_c(t)$ and $\gamma_c(t,\cdot)$ are measurable, $\mathcal{F}_t$ adapted stochastic processes, satisfying appropriate integrability conditions. We assume these processes to be ergodic, so that they can be estimated.

Under these conditions the adjoint variable Y has dynamics

$$dY_t = Y_{t-}\Big(\big\{ f_v(c_t,V_t) + \frac{1}{2}\gamma\, \sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 K_V'(t,\zeta)K_V(t,\zeta)\nu(d\zeta)\big\}\, dt - \gamma\sigma_V(t)\, dB_t - \int_{\mathcal{Z}} \gamma_0 K_V(t,\zeta)\tilde{N}(dt,d\zeta)\Big), \quad (34)$$

where $Y(0) = 1$.

From the FOC in equation (29) we derive the dynamics of the state price deflator. We then seek the joint determination of V_t, $\sigma_V(t)$ and $K_V(t,\cdot)$. By Ito's generalized lemma, normalizing to $\alpha = 1$, we get

$$d\pi_t = f_c(c_t, V_t)\, dY_t + Y_t\, df_c(c_t, V_t) + d[Y, f_c(c, V)](t), \tag{35}$$

since $\tilde{f}_c = f_c$, where $[X, Y](t)$ is the quadratic covariation of the processes X and Y given by

$$[X,Y](t) = \int_0^t \big(\sigma_X(s)\sigma_Y(s) + \int_{\mathcal{Z}} \gamma_X(s,\zeta)\gamma_Y(s,\zeta)\nu(d\zeta)\big)ds \\ + \int_0^t \int_{\mathcal{Z}} \gamma_X(s,\zeta)\gamma_Y(s,\zeta)\tilde{N}(ds,d\zeta).$$

By the dynamics of the adjoint and the backward equations, this can be written, using Ito's multi-dimensional formula

$$\begin{aligned} d\pi_t &= Y_t\, f_c(c_t, V_t)\Big(\{f_v(c_t, V_t) + \frac{1}{2}\gamma\sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 K_V' K_V \nu(d\zeta)\}dt \\ &-\gamma\sigma_V(t)dB_t - \int_{\mathcal{Z}} \gamma_0 K_V(t,\zeta)\tilde{N}(dt,d\zeta)\Big) + Y_t\,\frac{\partial f_c}{\partial c}(c_t, V_t)(c_t\mu_c(t)dt + c_t\sigma_c(t)dB_t) \\ &+ Y_t\,\frac{\partial f_c}{\partial v}(c_t, V_t)\Big(\{-f(c_t, V_t) + \frac{1}{2}\gamma V_t\sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} V_{t-}\gamma_0 K_V' K_V \nu(d\zeta)\}dt \\ &V_t\sigma_V(t)dB_t\Big) + Y_t\Big(\frac{1}{2}\,\frac{\partial^2 f_c}{\partial c^2}(c_t, V_t)\, c_t^2\sigma_c'(t)\sigma_c(t) + \frac{\partial^2 f_c}{\partial c\partial v}(c_t, V_t)\sigma_c'(t)\sigma_V(t) \\ &+\frac{1}{2}\frac{\partial^2 f_c}{\partial v^2}(c_t, V_t)\, V_t^2\sigma_V'(t)\sigma_V(t)\Big)dt + Y_t\Big(\int_{\mathcal{Z}} \{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) \\ &- f_c(c_{t-}, V_{t-}) - \gamma_c(t,\zeta)c_{t-}\frac{\partial f_c}{\partial c}(c_t, V_t) - K_V(t,\zeta)V_{t-}\frac{\partial f_c}{\partial v}(c_t, V_t)\}\nu(d\zeta)dt \\ &+ \int_{\mathcal{Z}} \{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-}, V_{t-})\}\tilde{N}(dt,d\zeta)\Big) \\ &- \gamma\sigma_V(t)Y_t\{c_t\sigma_c(t)\frac{\partial f_c}{\partial c}(c_t, V_t) + V_t\sigma_V(t)\frac{\partial f_c}{\partial v}(c_t, V_t)\}dt \\ &+Y_t \int_{\mathcal{Z}} \gamma_0 K_V(t,\zeta)\{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-}, V_{t-})\}\nu(d\zeta)dt \\ &+Y_t \int_{\mathcal{Z}} \gamma_0 K_V(t,\zeta)\{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-}, V_{t-})\}\tilde{N}(dt,d\zeta). \end{aligned} \tag{36}$$

Here

$$f_c(c,v) := \frac{\partial f(c,v)}{\partial c} = \delta c^{-\rho} v^{\rho},\ f_v(c,v) := \frac{\partial f(c,v)}{\partial v} = -\frac{\delta}{1-\rho}(1-\rho c^{1-\rho} v^{\rho-1}),$$

$$\frac{\partial f_c(c,v)}{\partial c} = -\delta\rho c^{-(1+\rho)} v^{\rho}, \quad \frac{\partial f_c(c,v)}{\partial v} = \delta\rho v^{\rho-1} c^{-\rho},$$

$$\frac{\partial^2 f_c}{\partial c^2}(c,v) = \delta\rho(\rho+1) v^{\rho} c^{-(\rho+2)}, \quad \frac{\partial^2 f_c}{\partial c\,\partial v}(c,v) = -\delta\rho^2 v^{\rho-1} c^{-(\rho+1)},$$

and

$$\frac{\partial^2 f_c}{\partial v^2}(c,v) = \delta\rho(\rho-1) v^{\rho-2} c^{-\rho}.$$

From the cannonical representation of the state price deflator

$$d\pi_t = \mu_\pi(t)dt + \sigma_\pi(t)dB_t + \int_{\mathcal{Z}} \gamma_\pi(t,\zeta)\tilde{N}(dt,d\zeta),$$

from (36) we find the key characteristics of π. They are

$$\begin{aligned}
\mu_\pi(t) = Y_t\Big(& f_c(c_t,V_t)\,\big(f_v(c_t,V_t) + \frac{1}{2}\gamma\sigma_V'(t)\sigma_V(t) \\
& + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 K_V' K_V \nu(d\zeta)\big) + \frac{\partial f_c}{\partial c}(c_t,V_t) c_t \mu_c(t) \\
& + \frac{\partial f_c}{\partial v}(c_t,V_t)\{-f(c_t,V_t) + \frac{1}{2}\gamma V_t \sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} V_{t-}\gamma_0 K_V' K_V \nu(d\zeta)\} \\
& + \frac{1}{2}\frac{\partial^2 f_c}{\partial c^2}(c_t,V_t)\, c_t^2 \sigma_c'(t)\sigma_c(t) + \frac{\partial^2 f_c}{\partial c \partial v}(c_t,V_t)\sigma_c'(t)\sigma_V(t) \\
& + \frac{1}{2}\frac{\partial^2 f_c}{\partial v^2}(c_t,V_t)\, V_t^2 \sigma_V'(t)\sigma_V(t) + \int_{\mathcal{Z}} \{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) \\
& - f_c(c_{t-},V_{t-}) - \gamma_c(t,\zeta)c_{t-}\frac{\partial f_c}{\partial c}(c_t,V_t) - K_V(t,\zeta)V_{t-}\frac{\partial f_c}{\partial v}(c_t,V_t)\}\nu(d\zeta) \\
& - \gamma\sigma_V(t)\{c_t\sigma_c(t)\frac{\partial f_c}{\partial c}(c_t,V_t) + V_t\sigma_V(t)\frac{\partial f_c}{\partial v}(c_t,V_t)\} \\
& + \int_{\mathcal{Z}} \gamma_0 K_V(t,\zeta)\{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-},V_{t-})\}\nu(d\zeta)\Big),
\end{aligned} \tag{37}$$

$$\sigma_\pi(t) = Y_t\Big(-f_c(c_t,V_t)\gamma\sigma_V(t) + \frac{\partial f_c}{\partial c}(c_t,V_t)c_t\sigma_c(t) + V_t\sigma_V(t)\frac{\partial f_c}{\partial v}(c_t,V_t)\Big) \tag{38}$$

and

$$\gamma_\pi(t,\zeta) = Y_t\Big(f_c(c_t,V_t)(-\gamma_0 K_V(t,\zeta)) \\ + \{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-},V_{t-})\} \\ + \gamma_0 K_V(t,\zeta)\{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-},V_{t-})\}\Big). \quad (39)$$

7.1 The risk premiums

The risk premium of any risky security with return process R is given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t}\,\sigma'_\pi(t)\,\sigma_R(t) - \frac{1}{\pi_t}\int_{\mathcal{Z}} \gamma_\pi(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) \quad (40)$$

where the last term follows from Aase (1993a,b). Since $\pi_t = Y_t f_c(c_t, V_t)$, it is a consequence of the expressions in (38) and (40) that the risk premium of any risky security is given by

$$\mu_R(t) - r_t = \Big(-\frac{\frac{\partial f_c}{\partial c}(c_t,V_t)}{f_c(c_t,V_t)} c_t \sigma'_c(t)\,\sigma_R(t) + \Big(\gamma - \frac{\frac{\partial f_c}{\partial v}(c_t,V_t)}{f_c(c_t,V_t)} V_t\Big)\sigma'_V(t)\,\sigma_R(t)\Big) \\ + \int_{\mathcal{Z}} \Big(\gamma_0 K_V(t,\zeta) - \frac{1}{f_c(c_t,V_t)}\big(f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-},V_{t-})\big)\cdot \\ \Big(1+\gamma_0 K_V(t,\zeta)\Big)\Big)\gamma_R(t,\zeta)\nu(d\zeta). \quad (41)$$

This is our basic result for risk premiums. We now substitute in for f given in (16) and the various partial derivatives derived above. This gives

$$\mu_R(t) - r_t = \rho\,\sigma_c(t)'\sigma_R(t) + (\gamma-\rho)\,\sigma_V(t)'\sigma_R(t) \\ + \int_{\mathcal{Z}} \Big(\gamma_0\, K_V(t,\zeta) - \Big(\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^\rho - 1\Big)\Big(1+\gamma_0\, K_V(t,\zeta)\Big)\Big)\gamma_R(t,\zeta)\nu(d\zeta). \quad (42)$$

It remains to determine σ_V and K_V, which we do below. Before that we turn to the interest rate.

7.2 The equilibrium interest rate

The equilibrium short-term, real interest rate r_t is given by the formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \quad (43)$$

The real interest rate at time t can be thought of as the expected exponential rate of decline of the representative agent's marginal utility, which is π_t in equilibrium.

In order to find an expression for r_t in terms of the primitives of the model, we use (37). Using the expression for f and its various partial derivatives, we obtain the expression given in (9) of Section 2, which is our basic result for the equilibrium short rate.

7.3 The determination of the volatility and jump characteristics of utility: The standard model

In order to determine $\sigma_V(t)$, and $K_V(t,\cdot)$, i.e., to solve the adjoint equation, first notice that the wealth at any time t is given by

$$W_t = \frac{1}{\pi_t} E_t\Big(\int_t^T \pi_s c_s^* \, ds \Big). \tag{44}$$

By the definition of directional derivatives (the Frechet derivative) we have that

$$\begin{aligned} \bigtriangledown U_{c^*}(c^*) &= \lim_{\alpha \downarrow 0} \frac{U(c^* + \alpha c^*) - U(c^*)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{U(c^*(1+\alpha)) - U(c^*)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{(1+\alpha)U(c^*) - U(c^*)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{\alpha U(c^*)}{\alpha} = U(c^*), \end{aligned}$$

where the third equality uses that U is homogeneous of degree one as shown in Theorem 1. By the Riesz representation theorem and dominated convergence theorem it follows from the linearity and continuity of the directional derivative that

$$\bigtriangledown U_{c^*}(c^*) = E\Big(\int_0^T \pi_t c_t^* \, dt \Big) = W_0 \pi_0 \tag{45}$$

where W_0 is the wealth of the representative agent at time zero, and the last equality follows from (44) for $t = 0$. Thus $U(c^*) = \pi_0 W_0$.

Let $V_t(c_t^*)$ denote future utility at the optimal consumption for our representation. Since this function is also homogeneous of degree one and is continuously differentiable, by Riesz' representation theorem and the dominated convergence theorem, the same type of basic relationship holds here for the associated directional derivatives at any time t, i.e.,

$$\bigtriangledown V_t(c^*; c^*) = E_t\Big(\int_t^T \pi_s^{(t)} \, c_s^* \, ds \Big) = V_t(c^*)$$

where the Riesz representation $\pi_s^{(t)}$ for $s \geq t$ is the state price deflator at time $s \geq t$, conditional on time t information. As for the discrete time model, it follows by results in Skiadas (2009a) that with assumption A2, implying that marginal utility at any time t is independent of past consumption, the consumption history in the adjoint variable Y_t is removed from the state price deflator π_t, so that $\pi_s^{(t)} = \pi_s / Y_t$ for all $t \leq s \leq T$. By this it follows that

$$V_t = \frac{1}{Y_t} \pi_t W_t. \tag{46}$$

This gives us the dynamics of V in terms of the primitives of the model. By the product rule,

$$dV_t = d\big(Y_t^{-1}\big)(\pi_t W_t) + Y_t^{-1} d(\pi_t W_t) + d[Y_t^{-1}, (\pi_t W_t)](t) \tag{47}$$

where

$$d(\pi_t W_t) = W_t d\pi_t + \pi_t dW_t + d[\pi_t, W_t](t). \tag{48}$$

Ito's lemma gives

$$\begin{aligned} d\Big(\frac{1}{Y_t}\Big) = \Big(-\Big(\frac{1}{Y_t}\Big)\big(f_v(c_t, V_t) + \frac{1}{2}\gamma \sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 K_V'(t,\zeta) K_V(t,\zeta)\nu(d\zeta)\big) \\ + \frac{\gamma^2}{Y_t}\sigma_V'(t)\sigma_V(t)\Big)dt + \frac{1}{Y_t}\gamma\sigma_V(t) dB_t \\ + \int_{\mathcal{Z}} \Big\{\frac{1}{Y_{t-} + Y_t A_0(V_t) K(t,\zeta)} - \frac{1}{Y_{t-}} + \frac{Y_t}{Y_{t-}^2} A_0(V_t,\zeta) K(t,\zeta)\Big\}\nu(d\zeta)dt \\ + \int_{\mathcal{Z}} \Big\{\frac{1}{Y_{t-} + Y_t A_0(V_t) K(t,\zeta)} - \frac{1}{Y_{t-}}\Big\}\tilde{N}(dt, d\zeta) \end{aligned} \tag{49}$$

From the equations (47)-(49) it follows by the market clearing condition $\varphi_t' \cdot \sigma(t) = \sigma_M(t)$ that

$$V_t \sigma_V(t) = \frac{1}{Y_t}\Big(\pi_t W_t \gamma \sigma_V + \pi_t W_t \sigma_M(t) - \pi_t W_t\big(\rho \sigma_c(t) + (\gamma - \rho)\sigma_V(t)\big)\Big) \tag{50}$$

From the expression (46) for V_t we obtain the following equation for σ_V

$$\sigma_V(t) = \gamma \sigma_V(t) + \sigma_M(t) - (\rho \sigma_c(t) + (\gamma - \rho)\sigma_V(t)$$

from which it follows that

$$\sigma_V(t) = \frac{1}{1-\rho}(\sigma_M(t) - \rho \sigma_c(t)). \tag{51}$$

Inserting this expression into (42) and (9) the standard version of recursive utility given in Section 2.2 results for the continuous dynamics part. The version treated by Duffie and Epstein (1992a) is the ordinally equivalent one based on (18), which was claimed to be better suited for dynamic programming, the solution method used by them. This shows that under the standard assumptions, the two ordinally equivalent versions give the same expressions for the risk premiums and the real interest rate in the model with *continuous* dynamics only.

We turn to the determination of $K_V(t,\cdot)$. From the equations (46)-(49), using the market clearing condition $\varphi_t'\gamma(t,\cdot) = \gamma_M(t,\cdot)$, we have that

$$\int_{\mathcal{Z}} V_t\, K_V(t,\zeta)\tilde{N}(dt,d\zeta) = \frac{1}{Y_t}\pi_t W_t \int_{\mathcal{Z}} \gamma_M(t,\zeta)\tilde{N}(dt,d\zeta) +$$
$$\frac{1}{Y_t} W_t \int_{\mathcal{Z}} \Big(Y_t f_c(c_t,V_t)(-\gamma_0)K_V(t,\zeta) + Y_t(f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) -$$
$$f_c(c_{t-},V_{t-})) + Y_t\gamma_0 K_V(t,\zeta)(f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) -$$
$$f_c(c_{t-},V_{t-})\Big)\tilde{N}(dt,d\zeta) + \int_{\mathcal{Z}} \Big\{ \frac{1}{Y_{t-} + Y_t A_0(V_t)K(t,\zeta)} - \frac{1}{Y_{t-}} \Big\} \pi_t W_t \tilde{N}(dt,d\zeta)$$
$$+ \frac{1}{Y_t}\int_{\mathcal{Z}} W_t\gamma_W(t,\zeta)\gamma_\pi(t,\zeta)\tilde{N}(dt,d\zeta) + \int_{\mathcal{Z}} \Big\{ \frac{1}{Y_{t-} + Y_t\frac{\gamma_0}{V_t}K(t,\zeta)} - \frac{1}{Y_t} \Big\} \cdot$$
$$\Big\{ W_t\gamma_\pi(t,\zeta) + \pi_t W_t\gamma_M(t,\zeta) + W_t\gamma_W(t,\zeta)\gamma_\pi(t,\zeta) \Big\} \tilde{N}(dt,d\zeta). \quad (52)$$

We now use the expression for $\gamma_\pi(t,\cdot)$ found in (40). This leads to the following equation for $K_V(t,\cdot)$:

$$\gamma_0 K_V(t,\zeta) - \Big(\big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\big)^\rho - 1 \Big)\Big(1+\gamma_0 K_V(t,\zeta)\Big) =$$
$$\frac{\gamma_M(t,\zeta) - K_V(t,\zeta) - \gamma_0 K_V(t,\zeta)(1+\gamma_M(t,\zeta))/(1+\gamma_0 K_V(t,\zeta))}{1+\gamma_M(t,\zeta) - \gamma_0 K_V(t,\zeta)/(1+\gamma_0 K_V(t,\zeta))}, \quad (53)$$

ν a.e. This proves the results of Section 3.1, which we formulate as:

Theorem 2 *For the standard recursive model with jump dynamics included, in equilibrium the risk premium of any risky asset R is given by (42), and the real interest rate by (9), where $\sigma_V(t)$ is given in (51) and $K_V(t,\cdot)$ satisfies the equation (53).*

7.4 The determination of the volatility and jump characteristics of utility: The model with past dependence

It seems reasonable that an individual's current marginal utility is affected by the individual's consumption history, not only of current consumption. For additional utility models in which past consumption plays a role in determining utility, see e.g., Sundaresan (1989). In our model the natural way to achieve this is as follows: Simply keep the first order conditions in (29) at time t. This amounts to letting marginal utility be dependent on past consumption. By relaxing assumption A2 it no longer follows that $\pi_s^{(t)}$ has the form given above. It is important that A1 holds. In addition homogeneity in c must hold for consistency, and finally the relationship $V_0 = \pi_0 W_0$ in (44) must result at $t = 0$, for any such extension. To this end, consider

$$\pi_s^{(t)} = \pi_s \quad \text{for all } t \leq s \leq T \tag{54}$$

We must examine the expression for $\pi_t = Y_t \frac{\partial f}{\partial c}(c_t, V_t)$. By the above results and (28) this can be written

$$\begin{aligned}
\pi_t = \Big\{ Y_0 \exp\Big(\int_0^t \big(-\frac{\delta}{1-\rho} + \rho \frac{\delta}{1-\rho} c_s^{1-\rho} V_s^{\rho-1} \\
+ \frac{1}{2}\gamma(1-\gamma)\sigma_V'(s)\sigma_V(s)\big) ds - \gamma \int_0^t \sigma_V(s) dB_s \\
+ \int_0^t \int_{\mathcal{Z}} \Big\{ \ln\Big(1 - \gamma_0 K_V(s,\zeta)\Big) + \gamma_0 K_V(s,\zeta) \Big\} \nu(d\zeta) ds \\
+ \int_0^t \int_{\mathcal{Z}} \ln\Big(1 - \gamma_0 K_V(s,\zeta)\Big) \tilde{N}(ds, d\zeta) \Big) \Big\} \, \delta c_t^{-\rho} V_t^{\rho}
\end{aligned} \tag{55}$$

From (55) we notice that π_t is homogeneous of degree zero in c for all t. With this choice we obtain the same homogeneity results as the standard solution. Dynamic consistency holds by symmetry, when the observer stands at time t and looks back at the consumption history. With this choice of $\pi_s^{(t)}$, since V is homogeneous of degree one, it follows that

$$\nabla V_t(c^*; c^*) = E_t \Big(\int_t^T \pi_s \, c_s^* \, ds \Big) = V_t(c^*).$$

From (44) we get that $V_t(c^*) = W_t \pi_t$. This shows that

$$V_t = \pi_t W_t \tag{56}$$

at the optimal consumption path c^*, so that for our version of recursive utility with dependence on history, the optimal utility at time t is the deflated wealth at this time. This V is not a Markov process, allowed by our approach.

Since the process V_t is a function of the agent's wealth and the state price deflator, it is a consequence of Ito's generalized lemma

$$
\begin{aligned}
dV_t = \pi_t dW_t + W_t d\pi_t + d[\pi_t, W_t](t) = \\
\pi_t\big(\mu_W(t)dt + W_t\sigma_M(t)dB_t + W_t\int_{\mathcal{Z}} \gamma_M(t,\zeta)\tilde{N}(dt,d\zeta)\big)+ \\
W_t\big(\mu_\pi(t)dt + \sigma_\pi(t)dB_t + \int_{\mathcal{Z}} \gamma_\pi(t,\zeta)\tilde{N}(dt,d\zeta)\big)+ \\
\sigma_\pi(t)\sigma_W(t)dt + \int_{\mathcal{Z}} \gamma_\pi(t,\zeta)\gamma_W(t,\zeta)\nu(d\zeta)dt + \int_{\mathcal{Z}} \gamma_\pi(t,\zeta)\gamma_W(t,\zeta)\tilde{N}(dt,d\zeta) = \\
\Big(-\frac{\delta}{1-\rho}\frac{c_t^{1-\rho} - V_t^{1-\rho}}{V_t^{-\rho}} + \frac{1}{2}\gamma V_t\sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 V_t K_V'(t,\zeta)K_V(t,\zeta)\nu(d\zeta)\Big)dt \\
+ V_t\sigma_V(t)dB_t + \int_{\mathcal{Z}} V_t K_V(t,\zeta)\tilde{N}(dt,d\zeta). \quad (57)
\end{aligned}
$$

First, regarding the continuous dynamics this shows that

$$V_t\sigma_V(t) = \sigma_\pi(t)W_t + \sigma_W(t)\pi_t. \quad (58)$$

We now use the equation for the optimal wealth, and observe that in equilibrium $\varphi_t' \cdot \sigma(t) = \sigma_M(t)$, so that by (31), $\sigma_W(t) = W_t\sigma_M(t)$. This gives

$$V_t\sigma_V(t) = -\pi_t W_t(\rho\sigma_c(t) + (\gamma-\rho)\sigma_V(t)) + W_t\sigma_M(t)\,\pi_t.$$

This leads to the following equation for $\sigma_V(t)$

$$\sigma_V(t) = \sigma_M(t) - \rho\sigma_c(t) - (\gamma-\rho)\sigma_V(t),$$

from which it follows that

$$\sigma_V(t) = \frac{1}{1+\gamma-\rho}\,\Big(\sigma_M(t) - \rho\sigma_c(t)\Big). \quad (59)$$

By comparing with (51), this shows our point of departure from the standard recursive model for the diffusion part. With past dependence on marginal utility at any time $t > 0$, we obtain different results. Finally we turn to the restriction on $K_V(t,\cdot)$.

$$\int_{\mathcal{Z}} V_t\, K_V(t,\zeta)\tilde{N}(dt,d\zeta) = \pi_t W_t \int_{\mathcal{Z}} \gamma_M(t,\zeta)\tilde{N}(dt,d\zeta)+$$
$$W_t\pi_t \int_{\mathcal{Z}} \Big(-\gamma_0 K_V(t,\zeta) + \Big((\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)})^{\rho} - 1\Big)\Big(1+\gamma_0 K_V(t,\zeta)\Big)\tilde{N}(dt,d\zeta)$$
$$+\pi_t W_t \int_{\mathcal{Z}} \gamma_M(t)\Big\{-\gamma_0 K_V(t,\zeta)+\Big((\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)})^{\rho} - 1\Big)\Big(1+\gamma_0 K_V(t,\zeta)\Big)\Big\}\tilde{N}(dt,d\zeta). \tag{60}$$

Since $V_t = \pi_t W_t$, this leads directly to

$$\frac{\gamma_M(t,\zeta) - K_V(t,\zeta)}{1+\gamma_M(t,\zeta)} = \gamma_0\, K_V(t,\zeta) - \Big(\frac{(1+K_V(t,\zeta))^{\rho}}{(1+\gamma_c(t,\zeta))^{\rho}} - 1\Big)\Big(1+\gamma_0\, K_V(t,\zeta)\Big), \tag{61}$$

ν a.e., which is (11) of Section 3.2. This shows our point of departure for the jump component.

With the new interpretation we have obtained a new solution of the system of forward/backward stochastic differential equations, where $\sigma_V(t)$ in (59) and $K_V(t,\cdot)$ in (61) represent this new solution with consumption dependence [6].

In the expressions for the equilibrium risk premiums and the real interest rate $\sigma_V(t)$ and $K_V(t,\cdot)$ were the only undetermined quantities. Inserting (59) into (42) and using the above result, we obtain

$$\mu_R(t) - r_t = \frac{\rho}{1+\gamma-\rho}\,\sigma_R'(t)\sigma_c(t) + \frac{\gamma-\rho}{1+\gamma-\rho}\,\sigma_R'(t)\sigma_M(t) + \int_{\mathcal{Z}} \Big(\frac{\gamma_M(t,\zeta) - K_V(t,\zeta)}{1+\gamma_M(t,\zeta)}\Big)\gamma_R(t,\zeta)\nu(d\zeta), \tag{62}$$

[6] Notice that we have not imposed an exogenous consumption history on marginal utility, only the one inherent in recursive utility is utilized.

and

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho\big(1+\gamma+(\gamma-\rho)(1+\gamma-\rho\gamma)\big)}{(1+\gamma-\rho)^2}\,\sigma_c(t)'\sigma_c(t)$$
$$+\frac{\gamma\rho(\rho-\gamma)}{(1+\gamma-\rho)^2}\,\sigma_c'(t)\sigma_M(t) - \frac{1}{2}\frac{(\gamma-\rho)(1-\rho)}{(1+\gamma-\rho)^2}\,\sigma_M'(t)\sigma_M(t)$$
$$-\frac{1}{2}(1+\rho)\gamma_0\int_{\mathcal{Z}} K_V' K_V \nu(d\zeta) - \int_{\mathcal{Z}} \gamma_0 K_V(t,\zeta)\Big\{\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^{\rho} - 1\Big\}\,\nu(d\zeta)$$
$$-\int_{\mathcal{Z}}\Big\{\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^{\rho} - 1 + \rho\gamma_c(t,\zeta) - \rho\,K_V(t,\zeta)\Big\}\,\nu(d\zeta). \quad (63)$$

Taking existence of equilibrium as given, the main results in this section are then summarized as;

Theorem 3 *For the model with marginal utility depending on the consumption history, in equilibrium the risk premium of any risky asset is given by (62) and the real interest rate by (63), where $K_V(t,\cdot)$ satisfies (61).*

7.5 Discussion

The resulting risk premiums in Theorem 3 are linear combinations of the consumption-based CAPM and the market-based CAPM for the continuous part, with different coefficients from the standard version in Theorem 2, and the jump terms also differ. In Theorem 3 the latter is seen to have a simpler and more intuitive form than the corresponding one in Theorem 2.

The calibrations to the US-data summarized in Table 1, reported in Table 4, correspond to plausible values of the various parameters. Also for the standard version of recursive utility the jump version indicates more stable results than the corresponding model without jump dynamics, as demonstrated in the Table 2 and 3.

Without jump dynamics present, the model with consumption history dependence calibrates to $\gamma < \rho$ for this set of data. In Aase (2014a) it was demonstrated that for other sets of data this may be different.

An example of this was shown for Norwegian data, where this model calibrated to $\gamma > \rho$. With jumps included this still holds true with a value of γ_0 of around three.

Tables 3 and 4 indicate that with jumps allowed in the model, only a small risk aversion related to jump size risk in the stock market is needed in order to explain preference for early resolution of uncertainty for the US-data.

8 The market portfolio is not a proxy for the wealth portfolio

In the paper we have focused on comparing two models, assuming the market portfolio can be used as a proxy for the wealth portfolio. Suppose we can view exogenous income streams as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include the new asset. However, if the latter is not traded, then the return to the wealth portfolio is not readily observable or estimable from available data. Still we should be able to get a pretty good impression of how the two models compare, which we now attempt.

In the conventional model with constant coefficients the growth rate of the wealth portfolio has the same volatility as the growth rate of aggregate consumption. Taking this quantity as the lower bound for this volatility, we indicate how the models compare when the market portfolio can not be taken as a proxy for the wealth portfolio. Below we first set the value of $\sigma_W(t)$ equal to the value of $\sigma_c(t)$, $\kappa_{c,W} = .40$ as before, and $\kappa_{W,M} = .70$. The model with a past dependence structure can be written

$$\mu_M(t) - r_t = \frac{\rho}{1+\gamma-\rho}\,\sigma_M'(t)\sigma_c(t) + \frac{\gamma-\rho}{1+\gamma-\rho}\,\sigma_M'(t)\sigma_W(t) + \int_{\mathcal{Z}} \Big(\frac{\gamma_W(t,\zeta) - K_V(t,\zeta)}{1+\gamma_W(t,\zeta)}\Big)\gamma_M(t,\zeta)\nu(d\zeta), \quad (64)$$

with a corresponding adjustment for the interest rate. Here M stands for the market portfolio and W for the wealth portfolio, so that (64) is the equity premium. Similar adjustments apply for the standard recursive model with jumps.

For the model with past dependence some calibrations are presented in Table 5. The assumptions about the jump dynamics are as before. Results for the standard model are presented in Table 6. With jumps included the difference between these two models seems to have diminished, at least for the calibrations in these two tables. Both models are seen to give plausible results. In particular is the weighted average risk aversion larger than the time preference for most of the calibrations, for both models.

The value for the variance rate of the wealth portfolio may be somewhat low. A more reasonable quantity is likely to be somewhere between $\sigma_c(t)$ and $\sigma_M(t)$; we suggest $\sigma_W(t) = .10$. We set the correlation coefficient $\kappa_{W,M} = .80$, and maintain $\kappa_{c,W} = .40$. Calibrations under these assumptions are given in Table 7 and 8. Again the two model produce similar results, with low values

Parameters	γ	ρ	γ_0	EIS
The model with past dependence				
$\delta = .01$	2.0	1.04	.01	.96
$\delta = .02$	2.5	1.05	.01	.95
$\delta = .03$	1.5	1.05	.01	.95
$\delta = .04$	3.5	1.05	.02	.95
$\delta = .05$	2.5	1.05	.02	.95
$\delta = .10$	2.5	1.07	.03	.93
$\delta = .25$	2.5	1.11	.08	.90

Table 5: Calibrations of the model with past dependence and jumps when $\sigma_W(t) = .0355$, $\kappa_{W,M} = .70$ and $\kappa_{c,W} = .40$.

Parameters	γ	ρ	γ_0	EIS
Standard recursive model				
$\delta = .01$	2.0	1.06	.01	.94
$\delta = .02$	2.5	1.07	.01	.93
$\delta = .03$	1.5	1.07	.01	.93
$\delta = .04$	3.5	1.10	.03	.91
$\delta = .05$	2.5	1.09	.03	.92
$\delta = .10$	2.5	1.13	.04	.88
$\delta = .25$	2.5	1.25	.10	.80

Table 6: Calibrations of the standard recursive model with jumps.

Parameters	γ	ρ	γ_0	EIS
The model with past dependence				
$\delta = .01$	2.0	1.14	.03	.88
$\delta = .02$	2.5	1.17	.04	.86
$\delta = .03$	1.5	1.17	.05	.86
$\delta = .04$	3.5	1.19	.06	.84
$\delta = .05$	2.5	1.20	.07	.83
$\delta = .10$	2.5	1.26	.13	.79
$\delta = .25$	2.5	1.40	.27	.71

Table 7: Calibrations of the model with past dependence and jumps when $\sigma_W(t) = .10$, $\kappa_{W,R} = .80$ and $\kappa_{c,W} = .40$.

Parameters	γ	ρ	γ_0	EIS
Standard recursive model				
$\delta = .01$	2.0	1.15	.03	.87
$\delta = .02$	2.5	1.13	.03	.89
$\delta = .03$	1.5	1.17	.05	.86
$\delta = .04$	3.5	1.07	.02	.93
$\delta = .05$	2.5	1.18	.05	.85
$\delta = .10$	2.5	1.31	.10	.76
$\delta = .25$	2.5	1.96	.39	.51

Table 8: Calibrations of the standard model with jumps

of γ_0.

Alternatively the model with past consumption history can calibrate to $\delta = .06$, $\gamma_0 = 2.5$, $\gamma = .01$, and $\rho = 1.08$, while the standard recursive version then gives $\delta = .06$, $\gamma_0 = 2.5$, $\gamma = 29.2$, and $\rho = 7.07$ for the best of several solutions. A relatively high risk aversion for jump size risk can only be explained by the model with past dependence. In this situation the risk aversion γ for the continuous part is low. These two risk aversions seem to complement each other.

The illustrations in this section only give an indication of how these models do when the market portfolio is not a proxy for the wealth portfolio. Many additional examples could of course be given, and the models could have been extended and moved in a different directions. However, the examples presented are fairly typical, and give an illustration of how the recursive models behave. Compared to the conventional model the difference is dramatic. With only continuous dynamics, the model with marginal utility depending on past consumption history tend to give more plausible and stable results than the standard recursive model (see Aase (2014a)). When jumps are included, the models can still give very different results, but can also agree on reasonable parameter values, as demonstrated above.

In both situations presented, the risk aversion on jump size risk were low for both models, giving a utility based support for loss aversion.

9 Conclusions

We have addressed the well-known empirical deficiencies of the conventional asset pricing model in financial- and macro economics. Although the standard recursive model gives better results than the conventional Eu-model, the results suffer from lack of stability in the parameters. Our approach is to relax the condition that past consumption does not matter for the marginal utility. This leads to a version of recursive utility that is more stable in the parameters than the ordinary version. In this setting we introduce jump dynamics in addition to the continuous components in both the standard, and our new version of recursive utility.

We use a general method of optimization, the stochastic maximum principle, together with the theory of forward/backward stochastic differential equations, which allows for the extension to jump dynamics. This method does not require any Markov structure.

For the US-data our extended model may calibrate, with a few simplifications regarding the jump dynamics, to reasonable values of the preference parameters. Here we lack an exact statistical analysis, so our results are only suggestive at this point. The standard model with jump dynamics included may also calibrate to more reasonable values of the parameters than without jumps.

When the market portfolio is not a proxy for the wealth portfolio, calibrations naturally change, but still yield plausible parameter values. The "stability" of the results seem good for both models studied in these situations, although the two models can also provide very different results.

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