INSTITUTT FOR FORETAKSØKONOMI
DEPARTMENT OF BUSINESS AND MANAGEMENT SCIENCE

FOR 15 2014
ISSN: 1500-4066
April 2014

Discussion paper

Common Mistakes in Computing the Nucleolus

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COMMON MISTAKES IN COMPUTING THE NUCLEOLUS *

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Abstract

Despite linear programming and duality have correctly been incorporated in algorithms to compute the nucleolus, we have found mistakes in how these have been used in a broad range of applications. Overlooking the fact that a linear program can have multiple optimal solutions and neglecting the relevance of duality appear to be crucial sources of mistakes in computing the nucleolus. We discuss these issues and illustrate them in mistaken examples collected from a variety of literature sources. The purpose of this note is to prevent these mistakes propagate longer by clarifying how linear programming and duality can be correctly used for computing the nucleolus.

Keywords: Game theory; Nucleolus; Cost allocation; Linear programming; Duality.

*Preprint for the *Discussion Series* of the Department of Business and Management Science, NHH Norwegian School of Economics, March 2014. For an update on this article please contact the authors.

1 Introduction

One of the main solution concepts in cooperative game theory is the nucleolus, proposed by Schmeidler (1969). A number of approaches have been developed in order to compute it, as reviewed by Leng and Parlar (2010) and Çetiner (2013). Although linear programming and duality have been correctly used in several approaches (e.g. Fromen (1997); Hallefjord et al. (1995); Kimms and Çetiner (2012)), we have found that the nucleolus has been wrongly computed over the years in a wide variety of contexts. The mistakes appear to be caused by overlooking the possibility that a linear program can have multiple solutions, and by neglecting the use of the dual solution as a valuable source of information in such cases. In this note, we discuss these issues and illustrate them in six examples taken from a variety of literature sources. The examples correspond to applications of cooperative game theory in insurances (Lemaire, 1991), joint development of projects (Kruś and Bronisz, 2000), production and transportation planning (Sakawa et al., 2001), electricity markets (SatyaRamesh and Radhakrishna, 2009), mobiles in broadcast transmission (Hasan et al., 2011), and manufacturing (Oh and Shin, 2012). It came to our attention that similar errors have appeared in such a wide range of applications. Our purpose in this note is to clarify how linear programming and duality can be used to correctly calculate the nucleolus, thus to prevent an even larger propagation of these errors.

2 Cooperative games and linear programming

Let $N = \{1, \ldots, n\}$ be the set of players and K the set of all non-empty subsets of N. The characteristic function $v : K \to \mathbb{R}$ assigns to each coalition S in K the *cost* of coalition S. A preimputation or cost allocation vector $x = (x_1, \ldots, x_n)$ assigns to each player j in N a quantity x_j such that $\sum_{j \in N} x_j = v(N)$; that is, the cost of the grand coalition N is split among its members according to the allocation x ($x_j \in \mathbb{R}\ \forall j \in N$). An allocation vector x satisfies rationality if $\sum_{j \in S} x_j \leq v(S)\ \forall S \in K$. The core of the game is the set of preimputations that satisfy the rationality conditions.

Define the excess of coalition S at x as $\varepsilon(x, S) = v(S) - \sum_{j \in S} x_j$. The excess is a measure of how *satisfied* a coalition S is with the cost allocation x. The larger the excess of S, the more satisfied coalition S is. Define the excess vector at x as $e(x) = (\varepsilon(x, S_1), \ldots, \varepsilon(x, S_m))$, where the sets S_i represent the coalitions in $K \setminus N$ and $m = 2^n - 2$. For an excess vector $e \in \mathbb{R}^m$, define a mapping θ such that $\theta(e) = y$, where $y \in \mathbb{R}^m$ is the vector which results from arranging the components of e in a non-decreasing order. A vector $y = (y_1, \ldots, y_m)$ is said to be lexicographically greater than another vector $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m)$ if either $y = \bar{y}$ or there exists $h \in \{1, \ldots, m\}$ such that $y_h > \bar{y}_h$ and $y_i = \bar{y}_i\ \forall i < h$ (if $h = 1$, it is enough that $y_h > \bar{y}_h$). We annotate $y \succeq \bar{y}$.

Note in some contexts the characteristic function v is defined as a benefit instead of cost and the excess as a measure of dissatisfaction instead of satisfaction. Both perspectives can be approached in equivalent ways. We rather adopt the cost perspective, since most of the recent interest for cooperative games in Operations Research comes from cost sharing problems in collaborative logistics. Also, our attention focus in games with a non-empty core. A main question in these games is how the players should share the cost $v(N)$ when collaborating in the grand coalition N. The nucleolus is one of the most used solution concepts for this problem. The prenucleolus is a related concept which for games with non-empty core coincide with the nucleolus.

2.1 The prenucleolus

The prenucleolus of a game with non-empty core is the preimputation x which lexicographically maximizes the excess vector, that is, $\theta(e(x)) \succeq \theta(e(\bar{x}))$ for all preimputation $\bar{x}$. In order to compute the prenucleolus, let us first consider the following linear programming model, which looks for a preimputation $x = (x_1, \ldots, x_n)$ that maximizes the minimum excess ε among all the coalitions.

$$\max \quad \varepsilon \tag{1}$$

$$\text{s.t.} \quad \varepsilon + \sum_{j \in S} x_j \leq v(S) \quad \forall S \subset N, S \neq \emptyset \tag{2}$$

$$\sum_{j \in N} x_j = v(N) \tag{3}$$

$$\varepsilon \in \mathbb{R}, \quad x_j \in \mathbb{R} \;\; \forall j \in N \tag{4}$$

Objective function (1) maximizes ε. Constraints (2) impose that such ε cannot be greater than the excess of any coalition. Thus, (1) and (2) together provide that ε is exactly equal to the minimum excess. Constraint (3) is the efficiency condition, which provides that the the cost of the grand coalition $v(N)$ is split among its players according to the allocation x. Constraints (4) state the nature of the variables. We refer to this model simply as P.

The solution to P is not necessarily unique. As we will illustrate in the numerical examples, it may occur that more than one allocation x leads to the optimal objective value. In addition, a solution of P provides an allocation that maximizes the lowest excess, but not necessarily the second or any subsequent lowest excess.

The prenucleolus can be found by solving a sequence of linear programs (LPs), as in the algorithm by Fromen (1997) which we briefly outline below. The first LP in the sequence corresponds to P. Let ε_1 be the optimal objective value of P. The k-th LP

$(k > 1)$ in the sequence is formulated as follows:

$$
\begin{aligned}
& \max \quad \varepsilon_k && && (5)\\
\text{s.t.} \quad & \varepsilon_k + \sum_{j \in S} x_j \leq v(S) && \forall S \subset N : S \notin \mathcal{F}_k && (6)\\
& \varepsilon_i + \sum_{j \in S} x_j = v(S) && \forall S \in F_i,\ i \in \{1, \ldots, k-1\} && (7)\\
& \sum_{j \in N} x_j = v(N) && && (8)\\
& \varepsilon_k \in \mathbb{R},\quad x_j \in \mathbb{R} \ \ \forall j \in N && && (9)
\end{aligned}
$$

In this k-th LP, objective function (5) and constraints (6) provide that the k-th minimum excess ε_k is maximized. Constraints (7) state that the excess of the coalitions contained in set F_i must be equal to the optimal objective value ε_i to the i-th LP. Constraints (8) and (9) state conditions for the efficiency and nature of the variables, respectively. The set F_i is the set of all coalitions for which the excess constraint (6) is satisfied with equality sign for all the solutions to the i-th LP. Thus, the excess of the coalitions in F_i must be fixed to ε_i in the k-th LP in the series for all $k > i$, as expressed in constraint (7). The set $\mathcal{F}_k$ is simply the union of all the coalitions for which its excess has been fixed in a previous LP in the sequence, that is, $\mathcal{F}_k = \bigcup_{i<k} F_i$. Note by defining $\mathcal{F}_1 = \emptyset$ and omitting constraints (7) for $k = 1$, one recovers the first problem P in the sequence. A key issue is how to find the set F_i, and here is where dual linear programming plays a relevant role. The dual of P, which we will refer as model D, can be formulated as follows:

$$
\begin{aligned}
\max \quad & \sum_{S \in K} v(S) \cdot y_S && (10)\\
\text{s.t.} \quad & \sum_{S \in K \setminus N} y_S = 1 && (11)\\
& \sum_{S \in K : j \in S} y_S = 0 \ \ \forall j \in N && (12)\\
& y_S \geq 0 \ \forall S \in K \setminus N,\quad y_N \in \mathbb{R} && (13)
\end{aligned}
$$

From duality theory, when the optimal value of a dual variable is positive, the inequality constraint associated to this variable must hold with equality at any optimal solution of P. Therefore, given a solution to P the set F_1 can be formed by all the coalitions S for which y_S is positive in the corresponding solution to D. Analogously, for a general k, the set F_k can be formed by all the coalitions such that the dual variable associated to constraint (6) is positive in the corresponding optimal solution to the dual problem of the k-th LP in the sequence. In order to find the prenucleolus, the solution process proceeds until a k where the LP has a unique solution. At the latest, such unique solution will be obtained when constraints (7) and (8) define a system of n independent linear equations.

2.2 The nucleolus

Define the set X by all the allocation vectors which satisfy the efficiency condition and also the individual rationality constraint $x_j \leq v(\{j\})\ \forall j \in N$. The nucleolus is an allocation vector $x \in X$ whose excess vector is lexicographically greatest, that is, $\theta(e(x)) \succeq \theta(e(\bar{x}))$ for all $\bar{x} \in X$. Incorporating the individual rationality constraint in the LPs defined in Section 2.1, conduces to the nucleolus. The corresponding dual problem is formulated similarly as problem D, but needs to add a decision variable $\bar{y}_j \geq 0$ associated to the rationality condition for all $j \in N$. Consequently, the term $\sum_{j \in N} v(\{j\}) \cdot \bar{y}_{\{j\}}$ must be added in objective function (10), and also $\bar{y}_j$ must be added in the left-hand side of constraint (12) for all $j \in N$. The nucleolus can be found by solving the corresponding sequence of LP in analogous way as for the prenucleolus. Schmeidler (1969) proves that the nucleolus consists of a single point. As we focus in games with non-empty core, the nucleolus and the prenucleolus coincide. (In other type of games, these concepts may be defined as a set instead of a unique point.)

3 Numerical examples

In this section we present six examples taken from a variety of contexts in the literature, where the nucleolus has been wrongly calculated. We identify two main sources of error. First, overlooking the fact that the solution to model P is not unique. Second, given that a particular solution to the i-th LP in the sequence, the set F_i has been wrongly computed as the set of all coalitions whose excess is equal to ε_i at such particular solution.

We use the notation $\hat{v}(S)$ for referring to the characteristic function of games where the players share benefits instead of costs (the LP models for these games remain the same as in Section 2.1 by defining $v(S) = -\hat{v}(S)$).

3.1 Insurances

Lemaire (1991) presents several examples on how cooperative game theory can be used in the context of insurance companies. The *Example 3* on his article illustrate a problem where different associations can collaborate by investing in common funds. The data and results for this example are shown in Table 1.

The allocation $\bar{x}$ solves the first LP of the sequence, but not the second one. This same example was used in an earlier article (Lemaire, 1984), where the same author states that in order to compute the nucleolus one has to solve a linear program which is equivalent to P. A fact that is omitted by the author is that this model may have multiple solutions. In the solution we obtain for the first LP in the series, there are three coalitions with the lowest excess $\varepsilon_1 = 6562.5$, but only two optimal dual values are positive (y_1 and y_6). By defining $F_1 = \{1, 6\}$ and running the second LP, we obtain the solution x which is the correct nucleolus for this game. Note from the excess vectors in non-decreasing order shown in Table 1, the excess vector at x is lexicographically greater than the excess vector

Table 1: Data, correct and wrong results for the game on insurances by Lemaire (1991)

			Correct			Wrong		
c	S	$\hat{v}(S)$	x	c	ε	$\bar{x}$	$\bar{c}$	$\bar{\varepsilon}$
1	{1}	46 125.0	52 687.5	1	6 562.5	52 687.5	1	6 562.5
2	{2}	17 437.5	24 468.8	6	6 562.5	24 937.5	3	6 562.5
3	{3}	5 812.5	12 843.8	2	7 031.3	12 375.0	6	6 562.5
4	{1,2}	69 187.5		3	7 031.3		2	7 500.0
5	{1,3}	53 812.5		4	7 968.8		4	8 437.5
6	{2,3}	30 750.0		5	11 718.8		5	11 250.0
7	{1,2,3}	90 000.0						

at $\bar{x}$, since their two first components are equal but the third component of the former is greater than the third component of the latter ($7031.3 > 6562.5$).

3.2 Joint projects

Kruś and Bronisz (2000) consider a cooperative game where different agents are interested in the implementation of a project. The authors outline an algorithm for calculating the nucleolus (and other nucleoli variants). Although the algorithm is correct and the authors acknowledge that the solution to a model in the sequence of LP may not have a unique solution, how the dual values can be used in the the definition of the sets F_i is not detailed. Instead, they refer the reader to Christensen et al. (1996), who correctly incorporates the information on the dual values in the solution process.

The characteristic function of the example in Kruś and Bronisz (2000) is shown in the third column of Table 2. The first and second columns of the table show an index $c \in \{1, \ldots, 2^n - 1\}$ that we use to refer to each coalition and the players who conform them, respectively. The next three columns show the correct nucleolus solution x we have computed for this example, and the excess vector in non-decreasing order together with the index of each coalition in this vector. The last three columns show the solution $\bar{x}$ given by Kruś and Bronisz (2000), and the corresponding excess vector.

The allocation $\bar{x} = (0.96, 0.26, 0.18, 0.49)$ is one of the multiple optimal solutions to model P. The optimal objective value to this model is $\varepsilon_1 = 0.18$. For the allocation $\bar{x}$, the excess of four coalitions (3, 8, 12 and 14) equals ε_1. By defining $F_1 = \{3, 8, 12, 14\}$, constraints (7) and (8) conform a system of linear equations whose unique solution is $\bar{x}$, so there is no need to solve more LPs in the sequence.

The optimal solution we obtain for the dual problem D in this example is $y_3 = y_{12} = 0.5$ and $y_c = 0\ \forall c \in K \setminus \{3, 12\}$. Then, we define $F_1 = \{3, 12\}$, which determines a unique value for x_3. By solving the corresponding second LP, we obtain $\varepsilon_2 = 0.38667$ and positive optimal dual values for y_{11}, y_{13} and y_{14}. By fixing the excess of these three coalitions to ε_2 and using the efficiency condition and the allocation for x_3 previously obtained, the unique allocation $x = (0.75, 0.48, 0.18, 0.47)$ is found, which is the nucleolus of this game.

Table 2: Data, correct and wrong results for the game on joint projects by Kruś and Bronisz (2000)

			Correct			Wrong		
c	S	$\hat{v}(S)$	x	c	ε	$\bar{x}$	$\bar{c}$	$\bar{\varepsilon}$
1	{1}	0.00	0.75	3	0.18	0.96	3	0.18
2	{2}	0.00	0.48	12	0.18	0.26	8	0.18
3	{3}	0.00	0.18	11	0.39	0.18	12	0.18
4	{4}	0.00	0.47	13	0.39	0.49	14	0.18
5	{1,2}	0.68		14	0.39		9	0.24
6	{1,3}	0.24		8	0.40		2	0.26
7	{1,4}	0.75		9	0.45		11	0.37
8	{2,3}	0.26		4	0.47		4	0.49
9	{2,4}	0.51		7	0.48		5	0.54
10	{3,4}	0.07		2	0.48		10	0.60
11	{1,2,3}	1.03		5	0.56		13	0.61
12	{1,2,4}	1.53		10	0.58		7	0.70
13	{1,3,4}	1.02		6	0.69		6	0.90
14	{2,3,4}	0.75		1	0.75		1	0.96
15	{1,2,3,4}	1.89						

Note in Table 2, the first and second lowest excesses are the same for both solutions x and $\bar{x}$, but the third lowest excess $\varepsilon = 0.39$ at x is greater (and thus better regarding the nucleolus notion of fairness) than the third lowest excess $\bar{\varepsilon} = 0.18$ at $\bar{x}$.

3.3 Production and transportation planning

Sakawa et al. (2001) deal with a problem on production and transportation planning based on a real case of a housing material manufacturer. The authors acknowledge the usefulness of solving a sequence of linear programs for calculating the nucleolus, and also mention that by examining the optimal solution of the dual problem one can identify which constraints must hold with the equality when solving such LPs. However, there is no explicit mention to what this examination consists on. They present data for a 5-player game, where each player represents one city or sale base in the network of the manufacturer. The characteristic function of this game, as well as our solution x and their solution $\bar{x}$, are shown in Table 3.

In order to compute the nucleolus x, we solve the first LP in the sequence and find a solution where five coalitions are left with the lowest excess $\varepsilon_1 = 0.034$, but only three optimal dual variables are positive (y_{16}, y_{21} and y_{30}). Then, by defining $F_1 = \{16, 21, 30\}$ and running the second LP, the solution we obtain has two positive optimal dual variables (y_9 and y_{26}). Fixing the excess of these two coalitions, together with the excess of the

Table 3: Data, correct and wrong results for the game on transportation and production planning by Sakawa et al. (2001)

			Correct			Wrong		
c	S	$\hat{v}(S)$	x	c	ε	$\bar{x}$	$\bar{c}$	$\bar{\varepsilon}$
1	{1}	0.060	0.165000	16	0.034	0.165239	9	0.034
2	{2}	0.168	0.320500	21	0.034	0.327849	30	0.034
3	{3}	0.030	0.084500	30	0.034	0.077235	25	0.034
4	{4}	0.249	0.374500	9	0.039	0.379218	21	0.034
5	{5}	0.000	0.055500	26	0.039	0.050459	16	0.034
6	{1,2}	0.378		25	0.042		20	0.038
7	{1,3}	0.144		28	0.042		29	0.038
8	{1,4}	0.408		15	0.044		26	0.044
9	{1,5}	0.182		29	0.046		15	0.044
10	{2,3}	0.337		20	0.050		14	0.045
11	{2,4}	0.538		27	0.053		3	0.047
12	{2,5}	0.279		3	0.055		27	0.048
13	{3,4}	0.383		5	0.056		28	0.049
14	{3,5}	0.083		18	0.056		5	0.050
15	{4,5}	0.386		14	0.057		18	0.059
16	{1,2,3}	0.536		10	0.068		10	0.068
17	{1,2,4}	0.747		22	0.074		13	0.073
18	{1,2,5}	0.485		13	0.076		19	0.076
19	{1,3,4}	0.546		19	0.078		23	0.077
20	{1,3,5}	0.255		23	0.082		22	0.078
21	{1,4,5}	0.561		24	0.083		24	0.090
22	{2,3,4}	0.706		12	0.097		7	0.098
23	{2,3,5}	0.379		1	0.105		12	0.099
24	{2,4,5}	0.668		7	0.106		1	0.105
25	{3,4,5}	0.473		6	0.108		6	0.115
26	{1,2,3,4}	0.906		17	0.113		17	0.125
27	{1,2,3,5}	0.573		4	0.126		4	0.130
28	{1,2,4,5}	0.874		8	0.132		8	0.136
29	{1,3,4,5}	0.634		2	0.153		2	0.160
30	{2,3,4,5}	0.801		11	0.157		11	0.169
31	{1,2,3,4,5}	1.000						

three previous coalitions and the efficiency constraint, define a system of equations whose unique solution is the nucleolus x. Note that the excess vector at x is lexicographically greater than the excess vector at $\bar{x}$. While the absolute difference between the components of x and $\bar{x}$ appear to be small, their relative differences amount up to 9.1%, which in our view is significant (specially since the allocations in this problem represent percentages applied to a profit in the order of millions). Since the characteristic function of this game is given with three decimal digits and the allocations with six decimal digits, one may interpret that the differences in x and $\bar{x}$ are merely due to numerical issues. However, we discard this interpretation by an exhaustive exploration where we verified that small perturbations of $v(S)$ have relatively low effects in the nucleolus of the game.

3.4 Electricity markets

SatyaRamesh and Radhakrishna (2009) present a cooperative game to allocate the transactional transmission losses in a problem on electricity markets. In their first case study, they use a dataset from the *IEEE 14-bus test system*. The characteristic function for this system is shown in Table 4. The allocation $\bar{x}$ they report as the nucleolus is a solution to

Table 4: Data, correct and wrong results for the game on electricity by SatyaRamesh and Radhakrishna (2009)

			Correct			Wrong		
c	S	$\hat{v}(S)$	x	c	ε	$\bar{x}$	$\bar{c}$	$\bar{\varepsilon}$
1	{1}	1.275	3.0893	3	1.3695	3.1764	3	1.3695
2	{2}	3.471	5.2853	4	1.3695	5.1981	4	1.3695
3	{3}	1.466	2.8355	1	1.8143	2.8355	2	1.7271
4	{1,2}	7.005		2	1.8143		1	1.9014
5	{1,3}	4.081		5	1.8437		5	1.9309
6	{2,3}	5.672		6	2.4488		6	2.3616
7	{1,2,3}	11.210						

the first LP in the sequence, but not for the second one. In the solution we obtain for the first LP, the dual variables with positive values are y_3 and y_4. By defining $F_1 = \{3, 4\}$, the value of x_3 is fixed and the second LP gives as solution the allocation x, which is the correct nucleolus for this game.

3.5 Mobiles in broadcast transmission

Hasan et al. (2011) consider a game where the players represent mobiles which are interested on receiving the same information from a base station. The information could be, for example, the streaming transmission of a sport or cultural event. The cost for broadcast transmission must be shared among the mobiles. Table 5 shows the data and results for this game.

Table 5: Data, correct and wrong results for the game on mobiles by Hasan et al. (2011)

			Correct			Wrong		
c	S	$v(S)$	x	c	ε	$\bar{x}$	$\bar{c}$	$\bar{\varepsilon}$
1	{1}	8	7.5	5	0	8	1	0
2	{2}	1	0.5	10	0	0	5	0
3	{3}	10	1.5	1	0.5	2	6	0
4	{4}	11	10.5	2	0.5	10	10	0
5	{1,2}	8		4	0.5		11	0
6	{1,3}	10		11	0.5		13	0
7	{1,4}	19		12	0.5		2	1
8	{2,3}	10		13	0.5		4	1
9	{2,4}	12		14	0.5		7	1
10	{3,4}	12		6	1		12	1
11	{1,2,3}	10		7	1		14	1
12	{1,2,4}	19		9	1		9	2
13	{1,3,4}	20		8	8		3	8
14	{2,3,4}	13		3	8.5		8	8
15	{1,2,3,4}	20						

The allocation $\bar{x}$ is a solution to the first LP in the series but not to the second one. In the solution that we obtain for the first LP, there are seven coalitions whose excess is $\varepsilon = 0$, but only two of the corresponding optimal dual values are positive (y_5 and y_{10}). By defining $F_1 = \{5, 10\}$ and running the second LP, we obtain as solution the allocation x, which defines a single allocation for all players and is the correct nucleolus for this game. Note the excess vector at x is lexicographically greater than the excess vector at $\bar{x}$, since the first two components of the vectors are equal but the third component of x is greater than the third component of $\bar{x}$.

3.6 Manufacturing

Oh and Shin (2012) address a cost sharing problem on joint network-centric manufacturing. They compute the nucleolus for an example using the characteristic function given in Table 6. Their solution $\bar{x}$ solves the first LP in the sequence (we neglect the last digit of the excess), but fails to solve the second one. In the solution we obtain for the first LP, y_3 and y_4 are the only dual variables with positive values. By defining $F_1 = \{3, 4\}$ and solving the second LP, the optimal solution x is obtained, which is the correct nucleolus for this game.

Table 6: Data, correct and wrong results for the game on manufacturing by Oh and Shin (2012)

			Correct			Wrong		
c	S	$v(S)$	x	c	ε	$\bar{x}$	$\bar{c}$	$\bar{\varepsilon}$
1	{1}	375 144	331 695.3	3	3 485.5	339 171.0	3	3 485.0
2	{2}	245 280	233 051.3	4	3 485.5	225 575.0	4	3 486.0
3	{3}	211 239	207 753.5	5	11 606.3	207 754.0	5	4 130.0
4	{1,2}	568 232		6	11 606.3		6	19 082.0
5	{1,3}	551 055		2	12 228.8		2	19 705.0
6	{2,3}	452 411		1	43 448.8		1	35 973.0
7	{1,2,3}	772 500						

3.7 Computational aspects

We would like to conclude this section with some remarks on the computational implementation of the sequence of LPs. First of all, the explicit implementation of the dual model is not needed as most optimization software can provide information on the dual solution after the primal model has been solved. For example, in our computations we used AMPL/CPLEX 12.6, which include a command called *.dual* for this purpose.

Second, some numerical issues may arise in the solution process. For example, when fixing the excesses ε_k according to the equality constraints (7), the precision settings of the solver may affect the solution. Also, when identifying the dual variables with positive values, one may include certain tolerance to numbers that differ from zero by an insignificant amount (e.g. 10^{-9}), in order to avoid a wrong definition of the sets F_i.

Third, as well as the primal, the dual model may have multiple solutions. This might affect the sequence through one gets to the nucleolus, but not the nucleolus itself (recall this is unique).

Recently, Puerto and Perea (2013) develop an approach to compute the nucleolus by solving only one LP, though of much larger dimension than the LPs in the sequential approach. They also point out the computation may be affected by numerical precision issues and propose a procedure to avoid them. Comparing the two approaches define a possible avenue for future research.

4 Final remarks

We as readers and researchers can certainly tolerate the existence of errors in past literature. However, when we realized that several authors in several different contexts have incurred in similar errors and obtained conclusions based on wrong results, we found worthy to write this note in order to facilitate correctness in future works. Especially, because of the growing interest that cooperative game solution concepts have captivated in our field recently. It is fair to mention that when elaborating this note, we also verified that several other authors have calculated the nucleolus correctly (e.g. Frisk et al. (2010); Lozano (2012); Lozano et al. (2013)).

Granot and Granot (1992) and Skorin-Kapov and Skorin-Kapov (2005) assert that a method for computing the nucleolus by solving a sequence of LPs was "implicitly suggested" by Schmeidler (1969). Whether an explicit algorithm in Schmeidler's seminal work would have prevented mistakes will remain unknown. We attempted to trace where in the literature the errors originated. Although we do not have an answer for such question, when preparing this note we found a number of fuzzy statements which might lead to confusion. For example, Faigle et al. (2001) state that when solving the second LP in the sequence one must fix the first lowest excess for "all coalitions that become tight at $\varepsilon = \varepsilon_1$". Would this be interpreted as all coalitions for which constraint (2) is satisfied with equality at an optimal solution of P, it may lead to a *wrong nucleolus*. Another example is the highly cited manuscript by Lemaire (1984) that we mentioned in Section 3.1. He states that in order to compute the nucleolus "one has to solve the linear program", followed by a formulation which is equivalent to P. A mention to the possibility of this LP having multiple solutions is omitted. Unfortunately, the LP in his *Example 3* has multiple solutions and the author reports a wrong nucleolus. This manuscript is curiously one of the first hits in Google Scholar when searching for "game theory" and "cost allocation". In his later article, Lemaire (1991) briefly mentions that the nucleolus is computed by solving a finite sequence of linear programs, without any additional specification on this sequence. However, he includes the same wrong example from the previous manuscript. Lemaire cite an even earlier article by Hamlen et al. (1977), which presents a single LP model (equivalent to P) to compute the nucleolus for a 3-player game. Despite the years gone by, these ambiguities appear to still induce errors in recent literature, as SatyaRamesh and Radhakrishna (2009) and Hasan et al. (2011) cite Lemaire (1984) as the basis for the allocation methods they use.

Finally, although our attention focused on the the nucleolus, several other similar *nucleoli* variations have been given in the literature. For example, the same article by Kruś and Bronisz (2000) include the *weak nucleolus*, the *concession nucleolus*, the *proportional nucleolus* and the *disruption nucleolus*. These last two are also used by Lemaire (1984), who unfortunately also reports a wrong solution for the proportional nucleolus. A similar algorithm as for the nucleolus is generally used in order to compute these variations, being the main difference the way in which the excess is defined. Therefore, we hope our note will help to prevent errors not only in the computation of the nucleolus, but also in the computation of all types of nucleoli.

Acknowledgements
We would like to thank Kaja Lorentzen, Tone Lund and Evangelos Toumasatos, for fruitful discussion which helped us to elaborate this note.

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