NHH

INSTITUTT FOR SAMFUNNSØKONOMI

DEPARTMENT OF ECONOMICS

SAM 17 2018

ISSN: 0804-6824

August 2018

Discussion paper

An analysis of the two-bidder all-pay auction with common values

BY

Chang Koo Chi

Norges Handelshøyskole

NORWEGIAN SCHOOL OF ECONOMICS

An Analysis of the Two-bidder All-pay Auction with Common Values

Chang Koo Chi*[1]

[1]Department of Economics, Norwegian School of Economics

August 28, 2018

Abstract

This paper studies a symmetric two-bidder all-pay auction where the bidders compete for a prize whose unknown common value is either high or low. The bidders' private signals (or types) are discrete and affiliated through the value. Even with affiliated signals, monotonicity of equilibria can fail in the sense that the bidder with a higher signal does not always win the auction. I show that when monotonicity fails, there exist multiple symmetric equilibria but the bidder's type-dependent payoff is invariant across the equilibria. The paper provides a closed-form formula for the equilibrium payoffs and a condition for rent dissipation.

JEL Classification: D44, D72, D88

Keywords: All-pay auctions, common values, correlated signals, non-monotone equilibria, rent dissipation

1. Introduction

Suppose two lobbyists decide to make contributions for a candidate's campaigns in anticipation of a political prize following an election. The value of the prize to the lobbyists

*The main idea of this paper has emerged from a joint project with Pauli Murto and Juuso Välimäki. I am grateful to their insightful comments. Part of this paper has been written when the author visited the Hanyang unversity in Seoul, South Korea. I would like to thank Youngwoo Koh and the Economics Department at Hanyang university for their kind hospitality and support. All remaining errors are my own. Chang Koo Chi: chang-koo.chi@nhh.no

is common and dependent upon the electoral outcome. The lobbyists have private estimates about the candidate's chance of getting elected, but their estimates are positively correlated. The prize is awarded to one of them who made the most contributions. Between the two lobbyists, who does expend more for the campaigns and take up the prize? The more optimistic or pessimistic one about the election chance? How does each lobbyist's expected payoff depend on his private information? Do the lobbyists benefit from a more precise estimate?

I address these questions in a symmetric two-player all-pay auction setting. An all-pay auction describes a game in which a fixed set of players compete for a fixed prize by simultaneously submitting bids, under the rule that the player with the highest bid achieves the prize but all submitted bids are forfeited. Even though auctions with such all-pay rules are seldom conducted in the real world, this format has been extensively studied because of its theoretical connection to winner-takes-all contests. In my model, there are two bidders vying for a *common-value* prize and its true value depends on a *binary* state of the world. Prior to bidding, each bidder is partially informed about the state by observing a private *discrete* signal. The bidder's signal or type is independently drawn from a common distribution conditional on the state. Hence the types are *affiliated* through the binary state. Formulating this auction environment as a static game with incomplete information, I analyze its symmetric equilibria.[1]

Despite its simple structure, it is a difficult problem to characterize equilibria of this game in full. The main challenge comes from the fact that affiliation between the bidders' types may preclude a monotone bidding behavior in auctions with an all-pay feature, as is well documented in literature (e.g, Milgrom and Weber (1982), Krishna and Morgan (1997), Landsberger (2007), Rentschler and Turocy (2016)). In particular, if the types are strongly correlated, then a monotone-strategy equilibrium may not exist. The idea is straightforward. Compared to a low-type one, a high-type bidder expects his opponent to be more optimistic about the state like himself and thus expects the level of competition in the auction to be higher. When bidding is costly, therefore, the high type may optimally respond by bidding cautiously. For this reason, the event of observing a high signal is not necessarily good news to bidders in the all-pay auction.

When monotonicity fails to hold, the standard techniques cannot be applied to find an equilibrium. Within the two-bidder setting, a recent paper by Rentschler and Turocy (2016) develops a method of constructing non-monotone equilibria for a general interdependent values model, and their model subsumes the model of this paper. However, such algorithmic characterizations make further equilibrium analysis difficult. To gain

[1]Proposition 2 in Rentschler and Turocy (2016) establishes the existence of symmetric equilibria in the symmetric two-bidder setting with interdependent valuations.

more insights about non-monotone equilibria, I assume in this paper that the common value of the prize is either high or low. Although this assumption is restrictive, there are a few number of contest environments in which the key factor in valuations for the prize is a binary outcome. In addition to the lobbying example above, undertaking R&D to obtain a patent, vying for a monopoly position in an industry with potential entrants, and competing in an elimination tournament for proceeding to the next round without knowing the next competitor fall into this category.

The main result of this paper is Theorem 1 in Section 3 that provides a full characterization of the expected payoffs in symmetric equilibria. The bidder's type-dependent payoff is characterized by the power of his type relative to a *threshold*. To be specific, I show that the profile of possible types can be categorized into two groups with the threshold. The group of types lower than or equal to this threshold does not obtain positive payoffs, put differently, the information rents for this group are fully dissipated. In contrast, the group of types above the threshold enjoys positive rents, and their equilibrium bidding strategies are monotone: Every type above the threshold outbids the lower types with probability one.

The threshold is uniquely determined by the bidder's valuations for the prize and the distribution of types, but it is independent of the bidder's prior beliefs about the states. In addition to the bidder's payoffs, the threshold also determines the structure of equilibria. Therefore, the payoff result shows how the equilibrium structure changes depending on the configurations of parameter values. If the threshold is determined at the lowest type, a monotone strategy equilibrium exists and this is a unique symmetric equilibrium.[2] In such an equilibrium, every type except the lowest one receives a positive payoff. The other extreme case arises when the threshold is determined at the highest type. In this case, every symmetric equilibrium involves full rent dissipation, and even the highest type can be beaten by a low type.

Theorem 1 does not rely on equilibrium uniqueness. In Section 3, I show through an example that when the threshold lies above the second lowest type, the all-pay auction may have multiple symmetric (non-monotone) equilibria. The probability of one type winning against another is not constant and thus the allocation is different across this set of equilibria. Nonetheless, the expected payoff of each type is invariant and uniquely determined. In other words, the set of symmetric equilibria is not outcome-equivalent but payoff-equivalent.

Lastly, the payoff characterization result can be applied to the information design problem. I investigate how a change in the information structure affects the bidder's

[2]For monotonicity, the condition is also necessary. That is, if the threshold is not the lowest type, a high type can be defeated by a low type with positive probability in equilibrium.

information rents or the expected revenue accruing to the auctioneer.[3] An improvement in the signal's quality has nontrivial effects on the rents. As the bidder's type becomes more informative about the state, his information rents increase due to the higher value of information. On the other hand, there is a countervailing effect on the structure of equilibria: a more informative signal results in stronger affiliation between the bidders' types. Consequently, the threshold is determined at a higher level, thereby undermining the power of each type.

This paper contributes to the auction and contest theory literature in that all-pay auctions lie in their intersection.[4] The early literature of all-pay auctions has generally focused on environments where bidders have complete information about each player's value of the object and cost of bidding (e.g., Hillman and Riley (1989), Baye, Kovenock, and de Vries (1993, 1996), Che and Gale (1998)). Siegel (2009) provides a definitive treatment of this model, by allowing heterogeneity on the player's characteristics, and characterizes the expected payoff in equilibrium in terms of the player's power. Theorem 1 in this paper gives a similar flavor to his payoff result, but their key difference lies in how to define the power. In a model with complete information and asymmetries among bidders, the power is determined by a player's *reach*, which indicates the player's maximum willingness to pay if he wins a prize for certain. On the other hand, in a model with incomplete information, the power is determined by a player's type and the threshold.

The literature of all-pay auctions with incomplete information has developed by relaxing the assumptions of the revenue equivalence theorem. Amann and Leininger (1996) study the two-bidder case with independent but asymmetrically distributed private values, and characterize a unique monotone pure-strategy equilibrium (MPSE).[5] Siegel (2014) studies the two-bidder model with (asymmetric) interdependent values and discrete types, and provides an algorithmic way to construct a monotone mixed-strategy equilibrium. Krishna and Morgan (1997) derive a symmetric MPSE from the model with symmetric interdependent values and affiliated signals á la Milgrom and Weber (1982).[6]

However, when bidders' signals are affiliated, such a monotone equilibrium exists

[3] In a common values model, there is no surplus loss from misallocation. Hence the revenue can be easily computed by subtracting away the aggregate rents given away to the bidders from the entire surplus.

[4] An all-pay auction corresponds to a contest with the success function where the prize is awarded to the contestant who put forth the greatest effort. The recent survey paper by Kaplan and Zamir (2015) gives a comprehensive picture of recent developments in the auction and contest theory.

[5] Parreiras and Rubinchik (2010) find that in the same environment with more than two bidders (possibly risk-averse), the monotone strategy equilibrium exhibits an "all-or-nothing" feature. In particular, a bidder may optimally drop out by bidding zero even though he has a chance of winning.

[6] McAdams (2007) finds that the symmetric equilibrium of Krishna and Morgan (1997) is unique among the set of monotone pure-strategy equilibria. More precisely, there is no asymmetric MPSE in symmetric all-pay auctions. Note that his uniqueness result does not apply to the current paper, because all-pay auctions with discrete types give rise to equilibrium in mixed strategies.

under a restrictive condition as it has been pointed out earlier. If the condition for monotonicity is violated, the equilibrium is non-monotonic and cannot be derived with standard methods. A recent paper by Chi, Murto, and Välimäki (2017) provides a full characterization in a symmetric environment with an arbitrary number of bidders, at the price of restricting to binary signals.[7] Within the two-bidder setting, Rentschler and Turocy (2016) provide algorithmic characterizations for a symmetric interdependent values model, and Appendix B of the paper by Lu and Parreiras (2017) contains an example of non-monotone equilibrium under a specific functional form. The common values model of this paper is a special case of theirs, but this simple model allows for a full characterization of the equilibrium payoffs. Furthermore, through the payoff characterization, the paper establishes a systematic link between the structure of equilibria and the payoff.

The rest of the paper is organized as follows. Section 2 lays out the model and its primitives. Section 3 presents the main result on the bidder's equilibrium payoffs and provides an example that shows the possibility of multiple equilibria in the environment. Section 4 concludes and suggests avenues for future research. The proof of the main theorem can be found in Appendix A. The other omitted proofs are relegated to Appendix B.

2. The Model

There are two risk-neutral players who compete for a single prize. The players' valuations for the prize depend solely on a binary random variable $\theta \in \{\theta_L, \theta_H\}$, which I call the *state* hereafter. Specifically, the value of the prize is common to both players and given by $v(\theta)$. I assume $v(\theta_H) > v(\theta_L) > 0$ and denote by $q \equiv \Pr(\theta = \theta_H) \in (0,1)$ the common prior. The realization of θ is not observable to the players until the prize is awarded to one of them. Instead, each player $i = 1,2$ is privately informed about θ by observing a private signal (or type) $t_i \in \mathcal{T} = \{t^1, \cdots, t^M\}$. I assume that the player's type is independently drawn from an identical probability mass function $\pi_\theta(k) \equiv \Pr(t_i = t^k|\theta)$ conditional on θ. Throughout the paper, the typical element of $\mathcal{T}$ is denoted t^k and called "type k". I assume that the information structure has full support: $\pi_\theta(k) > 0$ for all possible realizations of θ and t^k. In addition, I assume that the players' types are *strictly* affiliated with θ. Within the current framework, strict affiliation implies that $\pi_\theta(k)$ is log-supermodular, or

$$\frac{\pi_H(k)}{\pi_L(k)} \text{ is strictly increasing in } k.$$

[7] Their model accommodates both common values and affiliated private values models. The paper proves the uniqueness of symmetric equilibrium and determines the revenue ranking between all-pay auctions and standard auctions.

After observing a type, each player updates his beliefs about θ and simultaneously expends efforts or money in order to influence his winning probability. The prize is awarded to the one exerting the highest effort. Interpreting the effort as a non-refundable bid, this winner-takes-all contest corresponds to an all-pay auction where the two players simultaneously submit a bid of $b_i \geq 0$, the high bid wins the prize, but both have to pay their bids regardless of the outcomes. I therefore adopt the language of auctions and refer to players as bidders.

With a profile of bids $\mathbf{b} = (b_1, b_2)$ and an arbitrary tie-breaking rule $\sigma_i(\mathbf{b}) = \Pr(\text{i wins}|b_1 = b_2) \in (0,1)$, the (ex post) payoff to bidder i can be written as

$$\widetilde{u}_i(\mathbf{b};\theta) \ = \ -b_i + v(\theta)\left[\mathbb{1}_{\{b_i > b_j\}} + \sigma_i(\mathbf{b})\mathbb{1}_{\{b_i = b_j\}}\right].$$

At the moment of bidding, the value of the prize is uncertain and each bidder receives information on the value only through his type. Hence each bidder i's strategy must be a function of t_i only. Moreover, there is no equilibrium in pure strategies in this setting with a finite set of types. Let $F_i^k(b) \equiv \Pr(b_i \leq b | t_i = t^k)$ denote the bid distribution function from which bidder i who observed $t_i = t^k$ randomly draws a bid. With this notation, bidder i's mixed strategy can be represented by a vector of M distribution functions, $\mathbf{F}_i = (F_i^1, \cdots, F_i^M)$. For each F_i^k, let $\text{supp}[F_i^k]$ denote its support, i.e., the smallest closed set satisfying $\Pr(X \in \text{supp}[F_i^k]) = 1$ when the random bid X follows distribution F_i^k.

When choosing a bid in a common value auction, in order to avoid the winner's curse, rational bidders consider the expected value of the prize conditional on their own types plus conditional on the event of winning. The latter conditioning naturally depends on the bidding strategy chosen by the opponent. Suppose that bidder i observes type k and his opponent adopts a strategy $\mathbf{F}_j$. The expected payoff from submitting a bid of b to bidder i can be written as:

$$u_i\left(b, k | \mathbf{F}_j\right) \ = \ \mathbb{E}\left[\widetilde{u}_i(\mathbf{b};\theta) \mid b_i = b,\ t_i = t^k,\ b_j \sim \mathbf{F}_j\right]. \tag{1}$$

With this payoff expression, an equilibrium of the game is defined as follows:

Definition 1. *An equilibrium of the all-pay auction is a profile of strategies $(\mathbf{F}_1, \mathbf{F}_2)$ such that for each bidder $i = 1, 2$, for all $t^k \in \mathcal{T}$, and for all elements b_i of the interior of* $\text{supp}[F_i^k]$,

$$b_i \ \in \ \underset{b}{\text{argmax}} \ u_i\left(b, k | \mathbf{F}_j\right).$$

An equilibrium $(\mathbf{F}_1, \mathbf{F}_2)$ is symmetric if both bidders employ the same bidding strategy: $\mathbf{F}_i = \mathbf{F}_* = (F_*^1, \cdots, F_*^M)$ for $i = 1, 2$. As the model describes a symmetric environment, the rest of the paper focuses on symmetric equilibria. In such an equilibrium $\mathbf{F}_*$, the bidder's expected payoff depends on his type. To each type k, I denote by $U^{\mathbf{F}_*}(k)$ the corresponding payoff and refer to it as the information rent. In order for type k to randomize over $\text{supp}[F_*^k]$ in equilibrium, all bids in the interior of the bid support must yield the same payoff. Using this indifference condition, the rent can be easily computed by choosing an arbitrary bid from the support:

$$U^{\mathbf{F}_*}(k) \;=\; u(b,k|\mathbf{F}_*) \quad \text{for every } \; b \in \text{int}\left(\text{supp}[F_*^k]\right).$$

Lastly, I say that a symmetric equilibrium $\mathbf{F}_*$ is in monotone strategies if for all pairs of types k and m with $k > m$, $b_k \in \text{supp}[F_*^k]$ and $b_m \in \text{supp}[F_*^m]$ imply $b_k \geq b_m$. In a monotone equilibrium, therefore, bidder i wins against bidder j with probability one if $t_i > t_j$.

3. Equilibrium Payoff Characterization

In this section, I analyze the symmetric equilibria of the all-pay auction and characterize the bidder's expected equilibrium payoffs.

Before turning to the analysis, I make two preliminary observations that provide necessary conditions for any symmetric equilibria. First, there cannot be atoms in the equilibrium bid distribution for every type. To see this, suppose that distribution F_*^k has an atom at some $x \geq 0$ and bidder 2 employs this equilibrium strategy. Then bidder 1's expected payoff function $u(b, t_1|\mathbf{F}_*)$ has an upward jump at $b = x$, and thus if bidder 1 observed type k, he would be strictly better off by bidding slightly higher than x. Hence every equilibrium bid distribution function admits no atoms. Second, the union of the whole bid supports, $\cup_{k=1}^{M} \text{supp}[F_*^k]$, must form an interval starting at zero with no internal gaps. Since the bidders must pay their own bids regardless of the outcome, if no bidders are active over an interval (b_1, b_2), the bidder who planned to submit b_2 could extenuate his payment by deviating downwards, without any changes in the winning probability.[8]

Lemma 1. *In every symmetric equilibrium $\mathbf{F}_* = (F_*^1, \cdots, F_*^M)$ of the all-pay auction, the following properties must hold:*

1. *For each type k, F_*^k is continuous, i.e., no distribution has mass points.*
2. *The union of all bid supports, $\cup_{k=1}^{M} \text{supp}[F_*^k]$, is a connected interval that includes zero.*

One important implication of Lemma 1 is that a tie does not occur in equilibrium and hence the expected payoff function in (1) becomes continuous at every $b \geq 0$. By virtue

[8] The two properties in Lemma 1 also hold in a general interdependent values model with an arbitrary number of bidders (Chi et al. (2017)). Refer to Siegel (2014) for other properties of (asymmetric) equilibria.

of this property, I can simplify the payoff function into

$$\begin{aligned} u(b,k|\mathbf{F}_*) &= -b + \mathbb{E}\left[v(\theta)\,\mathbb{1}_{\{b_j<b\}}\,\middle|\, t_i = t^k,\, b_j \sim \mathbf{F}_*\right] \\ &= -b + \sum_{m=1}^{M} V(k,m)p_k(m)F_*^m(b). \end{aligned} \tag{2}$$

In (2), $V(k,m) \equiv \mathbb{E}[v(\theta) \mid t_i = t^k,\, t_j = t^m]$ denotes the expected value of the prize to bidder i conditional on the types, and $p_k(m) \equiv \Pr\left(t_j = t^m | \, t_i = t^k\right)$ represents bidder i's beliefs about the opponent's type after learning his own. As the bidders' types are affiliated through the state, it follows from Milgrom and Weber (1982) that $V(k,m)$ is increasing in each argument and $p_k(m)$ is log-supermodular. For an illustration of the reduced payoff expression (2), observe that when the opponent of $t_j = t^m$ employs a bidding strategy $\mathbf{F}_*$, bidder i wins with probability $F_*^m(b)$ and thereby gains $V(k,m)$. The expected payoff is therefore the average gain from winning weighted by bidder i's beliefs about t_j less his unconditional bid.

Let $\phi(k,m) \equiv V(k,m)p_k(m)$ and slightly abusing notation, for each type k, let $p_k \equiv \Pr\left(t_i = t^k\right) = q\pi_H(k) + (1-q)\pi_L(k)$ denote the marginal distribution of the bidder's type. Interpreting $\phi(k,m)$ as the expected gain to bidder i with $t_i = t^k$ from winning against his opponent with $t_j = t^m$, consider the difference $\phi(k',m) - \phi(k,m)$ for $k' > k$. Using Bayes' rule and doing simple algebra, it can be shown that the incremental gains to bidder i when his type rises to k' take a form of

$$\phi(k',m) - \phi(k,m) = C(k',k)\Big[v(\theta_H)\pi_H(m) - v(\theta_L)\pi_L(m)\Big], \tag{3}$$

where

$$C(k',k) \equiv q(1-q)\frac{\pi_H(k')\pi_L(k) - \pi_H(k)\pi_L(k')}{p_k p_{k'}}.$$

Note that the defined expression C is independent of the opponent's type m and is always positive. Its positive sign is due to affiliation between t_i and θ: the expression on the top of the quotient above, which is the unique factor that determines the sign of C, is positive for every $k' > k$. As a result, the sign of the incremental gains is determined by the sign of the bracketed expression in (3), which changes the sign at most once from negative to positive as m increases from 1 to M. To put it another way, the function $\phi(k,m)$ satisfies the single-crossing property in $(k;m)$, in the sense that for every $k' > k$,

$$\phi(k',m) - \phi(k,m) \begin{cases} \leq 0 & \text{for all } m < \tau \\ > 0 & \text{for all } m \geq \tau. \end{cases}$$

The Greek letter τ indicates the tipping point at which the incremental gains, or equivalently the expression $v(\theta_H)\pi_H(m) - v(\theta_L)\pi_L(m)$ in (3), change the sign from negative to positive. Since $\pi_H(M) > \pi_L(M)$ always holds by affiliation, the expression must take on a positive value at $m = M$ at least. Hence the sign-changing point $\tau \leq M$ is well defined.[9]

It is worthwhile to note that the point τ does not depend on the bidder i's types $k' > k$, nor on the prior beliefs about the state. This property of τ is a salient feature of the binary states. The assumption of θ being a binary random variable is tantamount to having the fixed sign-changing point of $\phi(k', m) - \phi(k, m)$ regardless of $k' > k$. The property plays an essential role in the subsequent equilibrium analysis.[10] Its key implication is the next:

Lemma 2. *Given a symmetric equilibrium $\mathbf{F}_*$, choose a bid of $b > 0$ from $\cup_{k=1}^{M} \operatorname{supp}[F_*^k]$.*

(a) Suppose that the bid yields a nonnegative payoff to type k but a nonpositive payoff to type $m < k$ and type k' in the equilibrium. Then the expected payoff from bidding b to all types below k' is at most zero.

(b) Suppose that the bid yields a payoff of zero to type m but a nonpositive payoff to type $k > m$ in the equilibrium. Then the expected payoff from bidding b to all types above m is at most zero.

PROOF OF LEMMA 2: See Appendix B.1. □

Utilizing Lemma 2, I derive an intuitive characterization for the payoffs in equilibrium of the all-pay auction. To be specific, I show that given the primitives of the model, the type space $\mathcal{T} = \{t^1, \cdots, t^M\}$ can be partitioned into two groups with a *threshold* as follows. In every symmetric equilibrium, the bidder who observed a type lower than or equal to this threshold τ^* earns an expected payoff of zero. On the other hand, if the bidder's type lies above τ^*, then he obtains a positive rent. Depending on the information structure $\langle \pi_L(k), \pi_H(k) \rangle_{k=1}^{M}$ of the game and the bidder's valuations for the prize, τ^* can be determined at the lowest or the highest type so that one of these groups is to be empty. For example, if $\tau^* = 1$, then every type except the lowest type earns a positive expected payoff in equilibrium. In contrast, if $\tau^* = M$, no bidders earn positive payoffs in any symmetric equilibrium. In other words, equilibrium must involve full rent dissipation.

To illustrate how to determine the threshold τ^*, consider a partial sum of the difference $\phi(k', t) - \phi(k, t)$ with $k' > k$ from $t = 1$ to $t = m$. Using the formula (3), this partial sum

[9] In fact, it can be shown that the function $\phi(k, m)$ is log-supermodular (so it satisfies the single-crossing property). Since log-supermodularity is preserved under integration, the expected value $V(k, m)$ is log-supermodular (See Karlin and Rinott (1980)) in this common value setting. Therefore, $\phi(k, m)$ is the product of the two log-supermodular functions and thus is log-supermodular (See Lemma 2 in Chi et al. (2017)).

[10] The main difficulty for going beyond the binary state comes from the fact that the invariance property does not hold in general when the number of possible states is more than two.

can be written as

$$\sum_{t=1}^{m}\Big(\phi(k',t)-\phi(k,t)\Big) \;=\; C(k',k)\sum_{t=1}^{m}\Big(v(\theta_H)\pi_H(t)-v(\theta_L)\pi_L(t)\Big)$$
$$\;=\; C(k',k)\Big(v(\theta_H)\Pi_H(m)-v(\theta_L)\Pi_L(m)\Big),$$

where $\Pi_\theta(m) = \sum_{t=1}^{m}\pi_\theta(t)$ on the bottom line indicates the cumulative distribution of types conditional on θ. To see how the sign of the partial sum changes as m varies, recall that $C(k',k) > 0$ for every $k' > k$ and affiliation implies the monotone probability ratio property, i.e., the increasing ratio $\Pi_H(m)/\Pi_L(m)$ over m. Consequently, like the incremental gains in (3), the partial sum changes its sign at most once as m increases.[11]

With this single-crossing property in hand, define the threshold τ^* as the tipping point at which the m-th partial sum changes the sign. Like the previously defined point τ, the threshold τ^* is well-defined because $v(\theta_H)\Pi_H(M) - v(\theta_L)\Pi_L(M) = v(\theta_H) - v(\theta_L) > 0$. Also, it is uniquely determined and independent of the bidder i's types $k' > k$. To compare these two points, note that a positive m-th partial sum calls for $\phi(k',m) - \phi(k,m) > 0$. Hence τ must be below τ^*.

I now state the main result of the paper:

Theorem 1. *In every symmetric equilibrium* $\mathbf{F}_*$ *of the all-pay auction, the expected payoff of the type-*k *bidder equals*

$$U^{\mathbf{F}_*}(k) \;=\; \max\Big\{0,\, w_{\tau^*}(k)\Big\},$$

where the function $w_{\tau^*}(k)$ *represents the power of type* k *relative to the threshold* τ^**:*

$$w_{\tau^*}(k) \;\equiv\; \sum_{m=1}^{k}\phi(k,m) - \left[\sum_{m=1}^{\tau^*}\phi(\tau^*,m) + \mathbb{1}_{\{k\geq\tau^*+1\}}\sum_{m=\tau^*+1}^{k}\phi(m,m)\right].$$

PROOF OF THEOREM 1: See Appendix A. □

The main implication of Theorem 1 comes from the defined power function $w_{\tau^*}(k)$. In order to explain how the function is related to the power of type k, I demonstrate that $w_{\tau^*}(k)$ takes on a nonpositive value for $k \leq \tau^*$ and a positive value for $k > \tau^*$. When type k lies below the threshold,

$$w_{\tau^*}(k) \;=\; \sum_{m=1}^{k}\phi(k,m) - \sum_{m=1}^{\tau^*}\phi(\tau^*,m) \;\leq\; -\sum_{m=1}^{\tau^*}\left[\phi(\tau^*,m)-\phi(k,m)\right] \;\leq\; 0,$$

[11]More generally, if a real-valued function $f(x)$ is single-crossing, then so is $F(x) \equiv \int \mathbb{1}_{\{t\leq x\}} f(t)dt$.

where the last inequality follows by definition of τ^* and becomes strict when $k < \tau^*$. Accordingly, Theorem 1 implies that if a bidder observes a type in subspace $\mathcal{T}_L^* \equiv \{t^1, \cdots, t^{\tau^*}\}$, then there is no symmetric equilibrium where the bidder obtains a positive expected payoff. On the other hand, when type k lies above the threshold, the power function takes a value of

$$\begin{aligned} w_{\tau^*}(k) &= \sum_{t=1}^{k} \phi(k,t) - \left[\sum_{t=1}^{\tau^*} \phi(\tau^*, t) + \sum_{t=\tau^*+1}^{k} \phi(t,t) \right] \\ &= \sum_{t=1}^{\tau^*} \Big(\phi(k,t) - \phi(\tau^*, t) \Big) + \sum_{t=\tau^*+1}^{k} \Big(\phi(k,t) - \phi(t,t) \Big). \end{aligned}$$

Observe that the last derived expression is strictly positive. This is because its first τ^*-th partial sum is positive for every type $k > \tau^*$ and the difference $\phi(k,t) - \phi(t,t)$ in the second sum is positive for every type t between $\tau^* + 1$ and k. Therefore, it results from Theorem 1 that a bidder who observes a type in subspace $\mathcal{T}_H^* = \{t^{\tau^*+1}, \cdots, t^M\}$ earns a positive rent of $w_{\tau^*}(k)$ in equilibrium. Furthermore, it can be shown that $w_{\tau^*}(k+1) > w_{\tau^*}(k)$ for every type k in $\mathcal{T}_H^*$, meaning that the bidder's equilibrium payoff increases with his type.

Therefore, the theorem shows that the equilibrium payoff of each type is determined by its relative power to the threshold. A configuration of the information structure and the prize values uniquely identifies the threshold, which in turn partitions the type space into $\mathcal{T}_L^*$ and $\mathcal{T}_H^*$. More importantly, this type-dependent rent does not depend on equilibrium uniqueness. If the type space contains more than 3 elements and $\tau^* \geq 3$, then the all-pay auction may possess multiple symmetric equilibria in non-monotone strategies and characterizing such equilibria in full is a challenging task. Furthermore, the probability of type k winning against another type m is not constant across equilibria. See the example at the end of the current section for a case with a continuum of equilibria.

The following two results are by-products of Theorem 1 which concern the structure of symmetric equilibria. Corollary 2 provides a sufficient and necessary condition for the existence of monotone strategy equilibrium in the all-pay auction. The condition is also sufficient for the uniqueness of symmetric equilibrium: if the condition is met, a unique symmetric equilibrium exists and this equilibrium is in monotone strategies. In contrast, Corollary 3 provides a condition under which the bidder's rent is fully dissipated regardless of his type.

Corollary 2. *For types k in $\mathcal{T}_H^*$, the equilibrium bid distributions are uniquely determined. Specifically, a type-$k > \tau^*$ bidder randomizes over the bid support* $\text{supp}[F_*^k] = [\underline{B}_k, \overline{B}_k]$ *according to* *the uniform distribution, where*

$$\underline{B}_k \equiv \sum_{t=1}^{\tau^*} \phi(\tau^*, t) + \sum_{t=\tau^*+1}^{k-1} \phi(t,t) \quad \textit{and} \quad \overline{B}_k \equiv \sum_{t=1}^{\tau^*} \phi(\tau^*, t) + \sum_{t=\tau^*+1}^{k} \phi(t,t).$$

Consequently, the all-pay auction has a unique symmetric equilibrium that is in monotone strategies if and only if $\tau^* = 1$, or equivalently

$$v(\theta_H)\pi_H(1) > v(\theta_L)\pi_L(1). \tag{MC}$$

PROOF OF COROLLARY 2: See Appendix B.2. □

Corollary 2 provides a simple but tight condition under which a high signal becomes unambiguously good news in the all-pay auction, so that there exists a symmetric equilibrium where a high-type bidder always outbids a low-type one. To understand the intuition behind (MC), observe that if the condition holds, then $v(\theta_H)\pi_H(t) > v(\theta_L)\pi_L(t)$ for all types $t = 2, \cdots, M$. This in turn implies that the expected gain $\phi(k,t)$ to bidder i is increasing when his type rises, regardless of the opponent's type. As a result, the bidder is willing to bid more aggressively as he observes a higher type.

It is worth noting that (MC) is in line with the existing monotonicity conditions in the literature. For instance, Krishna and Morgan (1997) and Siegel (2014) have shown that if $\phi(k,t)$ increases in k for every t, then the all-pay auction with general interdependent valuations possesses a monotone strategy equilibrium.[12] In this binary common values model, however, the condition is also necessary for monotonicity. Indeed, if the expected gain $\phi(k,t)$ is not monotone for some type t, it turns out that the type-t bidder is defeated by a lower type with positive probability in any symmetric equilibrium.

The next result describes the opposite case to Corollay 2 and provides a condition for rent dissipation.

Corollary 3. *Every symmetric equilibrium of the all-pay auction involves full rent dissipation if and only if $\tau^* = M$, equivalently,*

$$v(\theta_H)\left(1 - \pi_H(M)\right) < v(\theta_L)\left(1 - \pi_L(M)\right). \tag{FD}$$

Since $\pi_H(M) > \pi_L(M)$ by affiliation, the all-pay auction extracts full rents from the bidders when $v(\theta_H)$ is not sufficiently higher than $v(\theta_L)$. To gain some insights on the condition, observe that the expression on each side of (FD) represents the gain from winning

[12] In a (asymmetric) two-bidder case with continuous signal distributions, Lu and Parreiras (2017) identify a sufficient and necessary condition for the existence of a pure-strategy monotone equilibrium.

to the highest type when the state is high and low, respectively. Hence if the condition is met, then the event of winning in the high state becomes bad news to the bidder who is the most optimistic about the chance of θ_H. This results in a cautious bidding in the auction, thereby extracting the entire surplus from the bidders.

I conclude this section with an example that illustrates how Theorem 1 is applied to the case with three possible types. The example also shows the possibility of multiple symmetric equilibria and investigates the impact of a change in the information structure on the payoffs.

Example 1. *Consider the following information structure with type space $\mathcal{T} = \{t^1, t^2, t^3\}$:*

		t^1	t^2	t^3
$\pi_\theta(k)$	$\theta = \theta_L$	$\alpha(1-\lambda)$	λ	$(1-\alpha)(1-\lambda)$
	$\theta = \theta_H$	$(1-\alpha)(1-\kappa\lambda)$	$\kappa\lambda$	$\alpha(1-\kappa\lambda)$

The parameter $\lambda \in (0,1)$ controls the chance of the second type relative to the other types conditional on the state, and another parameter $\kappa = \frac{\pi_H(2)}{\pi_L(2)} > 0$ indicates its likelihood ratio. Define $\hat{\alpha} \equiv \frac{\alpha}{1-\alpha}$ and assume $\hat{\alpha} > 1$. The parameter α or $\hat{\alpha}$ measures the informativeness of t^1 and t^3. As α increases, it is more likely that each bidder observes t^1 in the low state and t^3 in the high state. The given information structure satisfies the strict monotone likelihood ratio property (sMLRP) if

$$\frac{1}{\hat{\alpha}(1-\lambda)+\lambda} < \kappa < \frac{\hat{\alpha}}{\hat{\alpha}\lambda + (1-\lambda)}.$$

Suppose that the threshold is determined at the lowest type, i.e. $\tau^* = 1$. This is the case when

$$\frac{v(\theta_H)}{v(\theta_L)} > \hat{\alpha} \cdot \frac{1-\lambda}{1-\kappa\lambda},$$

that is, the true value of the prize in the high state is sufficiently higher than in the low state. In this case, there exists a unique symmetric equilibrium, and the equilibrium is in monotone strategies as displayed in Figure 1. Each type k randomizes over its own support with uniform density $\frac{1}{\phi(k,k)}$. The equilibrium payoff is

$$\begin{aligned} U^{\mathbf{F}_*}(1) &= w_1(1) = 0 \\ U^{\mathbf{F}_*}(2) &= w_1(2) = \phi(2,1) - \phi(1,1) > 0 \\ U^{\mathbf{F}_*}(3) &= w_1(3) = \phi(3,1) + \phi(3,2) - \phi(1,1) - \phi(2,2) > 0, \end{aligned}$$

respectively.

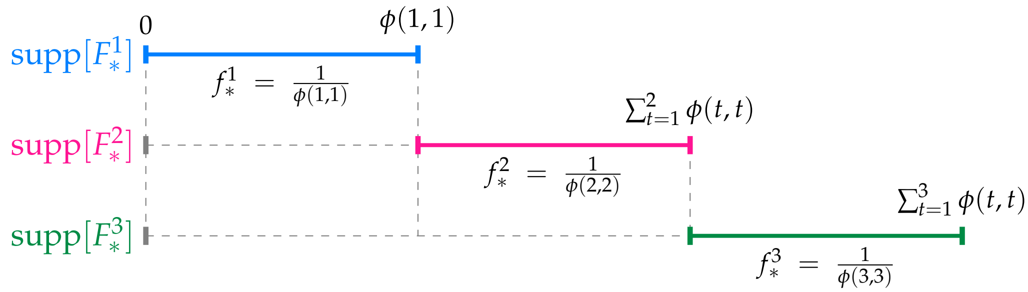

Figure 1: The unique symmetric equilibrium is monotonic when $\tau^* = 1$. Horizontal bars indicate the equilibrium bid support of the corresponding type, and f_*^k represents the uniform bid density for type k.

The threshold is determined at $\tau^* = 2$ if

$$\hat{\alpha} \cdot \frac{1-\lambda}{1-\kappa\lambda} \geq \frac{v(\theta_H)}{v(\theta_L)} > \frac{\hat{\alpha}+\lambda}{1+\hat{\alpha}\kappa\lambda}.$$

It can be shown that there exists a unique symmetric equilibrium like in the previous case, but the equilibrium is not in monotone strategies. Specifically, in every symmetric equilibrium, the bid supports of type 1 and 2 are connected intervals which begin at zero.[13] As a result, both type-1 and type-2 bidders obtain a payoff of zero. In the overlapping support, the equilibrium bid distribution is uniquely determined by the no-rent condition:[14]

$$\begin{pmatrix} \phi(1,1) & \phi(1,2) \\ \phi(2,1) & \phi(2,2) \end{pmatrix} \begin{pmatrix} F_*^1(b) \\ F_*^2(b) \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix}.$$

As opposed to the two low types, the highest type obtains a positive payoff of

$$U^{\mathbf{F}_*}(3) = w_2(3) = \phi(3,1) + \phi(3,2) - \phi(2,1) - \phi(2,2).$$

Figure 2 provides a complete characterization of this non-monotone equilibrium.

Lastly, if

$$\frac{\hat{\alpha}+\lambda}{1+\hat{\alpha}\kappa\lambda} \geq \frac{v(\theta_H)}{v(\theta_L)},$$

then the threshold is determined at the highest type, $\tau^* = 3$. In this case, the all-pay auction does not give away any rents to all types and extracts the entire surplus. In

[13] The uniqueness result does not rely on the 3-tuple type space in this example, but it holds for any finite set of types whenever $\tau^* = 2$. The way to prove the uniqueness of equilibrium is similar to the proof of Proposition 3 in Chi et al. (2017).

[14] Recall that the expected gain $\phi(k,t)$ is log-supermodular in my framework (See Footnote 9). Hence the 2-by-2 matrix in the no-rent condition has a positive determinant, thereby giving rise to positive bid densities in the overlapping region.

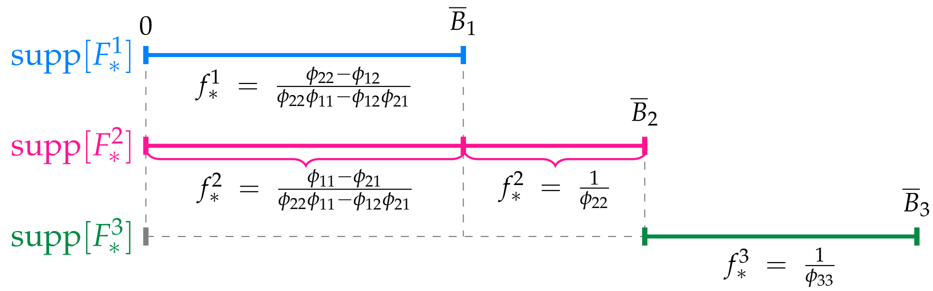

Figure 2: Characterization of the unique symmetric equilibrium that arises when $\tau^* = 2$. When $\tau^* \neq 1$, the equilibrium is non-monotonic so that a type-2 bidder can be defeated by type 1 with positive probability. In this unique equilibrium, the bid support of type 1 is included in that of type 2. In contrast, the type-3 bidder outbids the lower types with probability one. The upper support for each type is $\overline{B}_1 = (f_*^1)^{-1}$, $\overline{B}_2 = \sum_{t=1}^{2} \phi(2,t)$, and $\overline{B}_3 = \overline{B}_2 + \phi(3,3)$. The function $\phi(k,t)$ is abbreviated to ϕ_{kt} in this figure.

addition, the symmetric equilibrium is not unique. In fact, there exists a continuum of symmetric equilibria in this game, depending on the configurations of parameter values.

Figure 3 describes a continuum of symmetric equilibria that arises when $\tau^* = \tau = 3$ and features supp$[F_*^2]$ having an internal gap.[15] In this set, type 1 and type 2 are active until the bid reaches $\underline{B}_3$, the lower bound of supp$[F_*^3]$. Here the choice of $\underline{B}_3$ is arbitrary. It can vary from zero to the point where the type-2 bidder exhausts the bid density:

$$\underline{B}_3 \in \left[0, \frac{\phi(2,2)\phi(1,1) - \phi(1,2)\phi(2,1)}{\phi(1,1) - \phi(2,1)}\right].$$

Each choice of $\underline{B}_3$ generates a symmetric equilibrium, ranging from the equilibrium without I_1 (corresponding to $\underline{B}_3 = 0$) to the equilibrium without I_3.

To see that this is an equilibrium, observe that the proposed equilibrium is constructed in a way that two types are active on each piece of the interval $[0, \overline{B}_2]$. For example, t^1 and t^2 are active on I_1. Together with Theorem 1, this means that any bids in I_1 yield a payoff of zero to those types:

$$\begin{aligned} v(\theta_H)q^1 W(b|\mathbf{F}_*, \theta_H) \; + \; v(\theta_L)(1-q^1)W(b|\mathbf{F}_*, \theta_L) \; &= \; b \\ v(\theta_H)q^2 W(b|\mathbf{F}_*, \theta_H) \; + \; v(\theta_L)(1-q^2)W(b|\mathbf{F}_*, \theta_L) \; &= \; b, \end{aligned}$$

where I reformulate the no rents condition in terms of the probability of winning at b given state θ:

$$W(b|\mathbf{F}_*, \theta) \; \equiv \; \sum_{t=1}^{3} \pi_\theta(t) F_*^t(b) \quad \text{for each } \; \theta = \theta_L, \, \theta_H.$$

[15]The condition $\tau = 3$ allows for an overlap between supp$[F_*^2]$ and supp$[F_*^3]$.

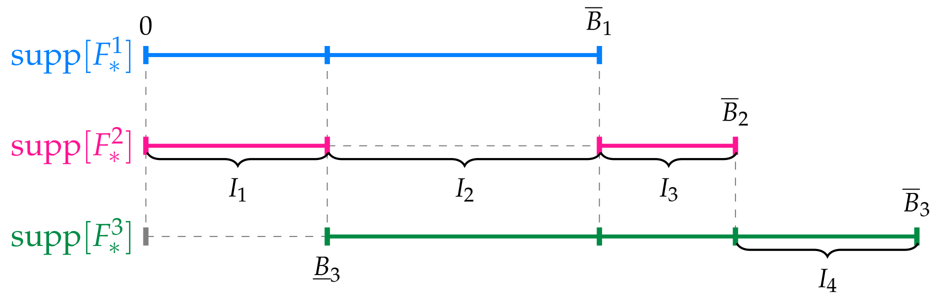

Figure 3: A set of non-monotone symmetric equilibria that arises when $\tau^* = \tau = 3$.

Since the posterior beliefs satisfy $q^2 \equiv \Pr(\theta_H|t^2) > q^1 \equiv \Pr(\theta_H|t^1)$ by affiliation, solving the system of equations above leads to $v(\theta_H)W(b|\mathbf{F}_*, \theta_H) = v(\theta_L)W(b|\mathbf{F}_*, \theta_L)$. This implies that the bidders are indifferent between winning at b in the high state and winning at b in the low state, irrespective of types. Therefore, the inactive type t^3 cannot obtain a positive payoff by deviating and choosing a bid in I_1.

The same argument can be applied to I_2 and I_3, establishing that the inactive type has no incentives to deviate over the support $[0, \overline{B}_2]$. If type $m = 1, 2$ deviates from $\overline{B}_m$ by making a bid of $\overline{B}_m + \Delta$ in I_4, his net payoff is negative as $\phi(m, 3) \cdot \frac{\Delta}{\phi(3,3)} - \Delta < 0$. This precludes any global deviations to I_4 for the two low types, and hence the bid support described in Figure 3 is indeed an equilibrium.

The discussion shows how the structure of symmetric equilibrium changes depending on the configurations of parameter values. Intuitively, a higher ratio of the prize values would enhance the informational advantage of high types and thus lead to their aggressive bidding behavior, resulting in a monotone equilibrium.

The example also can be used to investigate the effect of a change in the information structure on the equilibrium outcomes. To this end, suppose that the auctioneer is able to control the parameter α by supplying her own information about the state.[16] Observe that as α increases, the information structure becomes more statistically precise about the state. This has two effects on the bidder's information rents. On the one hand, a higher α renders the bidder's private information more valuable and thus increases the information rent. On the other hand, a higher α implies a higher degree of affiliation between the bidders' types, put differently, a higher degree of competition between the bidders. Hence the bidders optimally respond by bidding cautiously, and thus the threshold tends to be higher, thereby undermining the power of each type.

It depends on the parameter values which effect overwhelms the other. Figure 4 dis-

[16]This is related to the information design problem in auction literature. Refer to Ganuza and Penalva (2010).

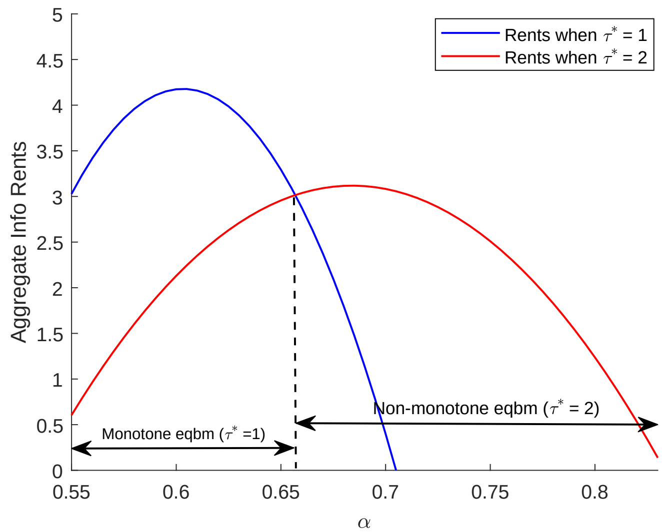

Figure 4: The impact of an increase α on aggregate information rents. The specified information structure satisfies the sMLRP if $\alpha > \frac{23}{43} \approx 0.535$. The threshold is determined at $\tau^* = 3$ if $\alpha \geq \frac{5}{6}$, and thus the aggregate rents boil down to zero.

plays a numerical result when the parameters are given by $v(\theta_H) = 200$, $v(\theta_L) = 100$, $\lambda = 0.3$, $\kappa = 1.1$, and $q = 0.5$. Under these specifications, the all-pay auction has a monotone symmetric equilibrium iff $\alpha > \frac{67}{102} \approx 0.657$, and the auction extracts a full surplus from the bidders iff $\alpha \geq \frac{5}{6}$. Formulating the aggregate information rents as a function of α, Figure 4 shows how the rents and the equilibrium structure change as α varies. Each function takes on a hump shape, suggesting that the effect of precision dominates the effect of affiliation for a low value of α. For $\alpha \geq 0.685$, however, the ranking is reversed so that the rents given away to the bidders are monotone decreasing in α.

4. Conclusion

This paper studies a symmetric two-bidder all-pay auction with common values. When the bidders' private signals are highly affiliated, or when their valuations for the prize are not much sensitive to a change in the underlying state, the monotonicity of equilibria fails to hold and non-monotone ones inevitably arise. In the model with binary common values, this paper proposes a novel way to analyze such equilibria and provides a closed-form formula for the expected payoffs in both types of equilibria. The threshold is key in the main result. The bidder's payoff is characterized by the power of his type relative to the threshold. The result presents simple conditions for existence of monotone equilibrium and rent dissipation, and enables further interesting equilibrium analysis.

Furthermore, the techniques and insights developed in this paper can be used as a building block for future research on non-monotone equilibrium analysis in contests or in auctions with costly bidding. One avenue for future research is to extend the payoff result into more general information settings that accommodate more-than-two possible states and asymmetric signal distributions. These settings naturally give rise to type-dependent thresholds, and the main hurdle is to sort out which one is relevant to the equilibrium payoffs. Another promising avenue is to extend the analysis into contests with more general stochastic success functions, including the two specifications—logit and probit—commonly used in the literature. Further analysis of these models is left for future work.

A. Proof of Theorem 1

I prove Theorem 1 through a series of lemmas. The proof is comprised of 3 parts.

Part 1

In the first part of the proof, I use the sign-changing point τ of the incremental gains to break the type space $\mathcal{T}$ into two separate groups, $\mathcal{T}_L = \{t^1, \cdots, t^{\tau-1}\}$ and $\mathcal{T}_H = \{t^{\tau}, \cdots, t^M\}$. The main result of Part 1 is Proposition A.1, which proves that in any symmetric equilibrium the bidder in $\mathcal{T}_L$ obtains no information rents. In what follows, the supremum and the infimum of a set A are denoted by $\sup A$ and $\inf A$, respectively.

Lemma A.1. *In every symmetric equilibrium* $\mathbf{F}_*$ *of the all-pay auction,* $\inf \bigcup_{t \in \mathcal{T}_L} \text{supp}[F^t_*] = \inf \bigcup_{t \in \mathcal{T}_H} \text{supp}[F^t_*] = 0$.

Proof of Lemma A.1: Define $B \equiv \inf \bigcup_{t \in \mathcal{T}_L} \text{supp}[F^t_*]$ and $B' \equiv \inf \bigcup_{t \in \mathcal{T}_H} \text{supp}[F^t_*]$. Suppose to the contrary that $B > 0$. Then $B' = 0$ follows from Lemma 1. However, $B' = 0$ implies that there exists a type k in $\mathcal{T}_H$ whose bid distribution function satisfies $F^k_*(B) > 0$ and the expected payoff from submitting the bid B is at most zero. In parallel, as the bid B is an element of $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F^t_*]$, there must exist a type m in $\mathcal{T}_L$ who earns a nonnegative expected payoff from submitting B. These two derived conditions can be put into

$$-B + \sum_{t \in \mathcal{T}_H} \phi(m,t) F^t_*(B) \geq 0 \geq -B + \sum_{t \in \mathcal{T}_H} \phi(k,t) F^t_*(B),$$

where I used $F^t_*(B) = 0$ for all $t \in \mathcal{T}_L$. The two inequalities above can be reduced further into

$$\sum_{t \in \mathcal{T}_H} \left[\phi(k,t) - \phi(m,t)\right] F^t_*(B) \leq 0.$$

But this leads to a contradiction, since $\phi(k,t) - \phi(m,t) > 0$ for all $t \in \mathcal{T}_H$ and $F^k_*(B) > 0$ at least for the type k in $\mathcal{T}_H$. $B' = 0$ can be established in an analogous way. $\square$

Lemma A.2. *In every symmetric equilibrium* $\mathbf{F}_*$*, both* $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F^t_*]$ *and* $\bigcup_{t \in \mathcal{T}_H} \text{supp}[F^t_*]$ *are connected intervals.*

Proof of Lemma A.2: Suppose to the contrary that $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F^t_*]$ is not a connected interval. To be precise, there exists a closed interval $[b_1, b_2]$ such that $[b_1, b_2] \cap \left(\bigcup_{t \in \mathcal{T}_L} \text{supp}[F^t_*]\right) = \{b_1, b_2\}$. Observe that in this case, the interval $[b_1, b_2]$ must be a subset of $\bigcup_{t \in \mathcal{T}_H} \text{supp}[F^t_*]$ due to Lemma 1. Then I can choose a type k from $\mathcal{T}_H$ and a type m from $\mathcal{T}_L$ such that the incremental returns from bidding b_2 rather than b_1

are nonpositive for type k but nonnegative for type m. Using the fact that $F_*^t(b_2) = F_*^t(b_1)$ for all $t \in \mathcal{T}_L$, the last statement can be translated into

$$\sum_{t \in \mathcal{T}_H} [\phi(k,t) - \phi(m,t)] \left[F_*^t(b_2) - F_*^t(b_1)\right] \leq 0. \tag{4}$$

Since $\phi(k,t) - \phi(m,t) > 0$ and $F_*^t(b_2) - F_*^t(b_1) \geq 0$ for all $t \in \mathcal{T}_H$, and since the latter inequality becomes strict for at least one type $t \in \mathcal{T}_H$, the inequality (4) gives a contradiction. It can be shown in a similar way that $\bigcup_{t \in \mathcal{T}_H} \text{supp}[F_*^t]$ is a connected interval. □

The previous lemmas are used to prove the next result:

Lemma A.3. *In every symmetric equilibrium* $\mathbf{F}_*$, $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t] \subset \bigcup_{t \in \mathcal{T}_H} \text{supp}[F_*^t] = [0, B]$.

PROOF OF LEMMA A.3: In light of the previous lemmas, for the proof, it suffices to show that the supremum of $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$ is no greater than the supremum of $\bigcup_{t \in \mathcal{T}_H} \text{supp}[F_*^t]$. Suppose there is an element b in the set $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$ which is greater than $B \equiv \sup \bigcup_{t \in \mathcal{T}_H} \text{supp}[F_*^t]$. Then it follows from Lemma A.1 that making the bid b yields a nonnegative expected payoff for some type m in $\mathcal{T}_L$ but a nonpositive payoff for some type k in $\mathcal{T}_H$. Since $b > B$ is assumed, $F_*^t(b) = 1$ for all $t \in \mathcal{T}_H$. Hence I have

$$u(b, k|\mathbf{F}_*) - u(b, m|\mathbf{F}_*) = \sum_{t \in \mathcal{T}_L} [\phi(k,t) - \phi(m,t)] F_*^t(b) + \sum_{t \in \mathcal{T}_H} [\phi(k,t) - \phi(m,t)] \leq 0. \tag{5}$$

However, since the total sum $\sum_{t \in \mathcal{T}} [\phi(k,t) - \phi(m,t)]$ is strictly positive and $\phi(k,t) - \phi(m,t) \leq 0$ for all $t \in \mathcal{T}_L$, the payoff difference in (5) must be strictly positive, thereby leading to a contradiction. □

Proposition A.1. *There is no information rent to a bidder of type $t \in \mathcal{T}_L$ in any symmetric equilibrium.*

PROOF OF PROPOSITION A.1: Choose an arbitrary bid b from $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$. Below I demonstrate that the expected payoff from bidding b is at most zero for all $t \in \mathcal{T}_L$, implying that the rents accruing to type $t \in \mathcal{T}_L$ are completely dissipated in equilibrium.

In light of Lemma A.3, the chosen bid b must be an element of $\bigcup_{t \in \mathcal{T}_H} \text{supp}[F_*^t]$. Hence $b \in \text{supp}[F_*^k]$ for some type k in $\mathcal{T}_H$, meaning that the bid must yield a nonnegative payoff to the type. On the other hand, in light of Lemma A.1, there is at least one type m in $\mathcal{T}_L$ who earns at most a payoff of zero by bidding b. Note $k > m$. In addition, Lemma A.1 suggests that such a type, say type k', also exists in the group $\mathcal{T}_H$, for whom the payoff from bidding b is nonpositive.

The previous discussion confirms that all of the given conditions in Lemma 2-(a) are satisfied for the bid b. Therefore, it follows that $u(b, s|\mathbf{F}_*) \leq 0$ for all $s \leq k'$. Since type k'

was selected from $\mathcal{T}_H$ and the bid b was arbitrarily chosen from the bid support for $\mathcal{T}_L$, I conclude that there are no bids that ensure a positive payoff for $\mathcal{T}_L$. □

Part 2

In the second part of the proof, I partition the type space $\mathcal{T}$ again but this time with the threshold τ^*: $\mathcal{T}_L' = \{t^1, \cdots, t^{\tau^*-1}\}$ and $\mathcal{T}_H' = \{t^{\tau^*}, \cdots, t^M\}$.[17] The main goal of this part is (i) to extend the rent dissipation result in Proposition A.1 to the group $\mathcal{T}_L'$, as is stated in Lemma A.5, and (ii) to characterize the equilibrium bid supports for each group, as is displayed in Figure 5.

The next lemma is parallel with Lemma A.1 in the first part. It demonstrates that at least one type in the group $\mathcal{T}_H'$ obtains an expected payoff of zero in any symmetric equilibrium. The lemma plays an essential role in establishing the ensuing results. In the following, I use $A \backslash B$ to denote the complement of set B in set A.

Lemma A.4. *There exists one type in $\mathcal{T}_H'$ who earns no rents in any symmetric equilibrium.*

PROOF OF LEMMA A.4: Let $B \equiv \inf \bigcup_{t \in \mathcal{T}_H'} \text{supp}[F_*^t]$ denote the lower support for $\mathcal{T}_H'$. I assume $B > 0$, for otherwise there is nothing to prove. In addition, I assume $\tau < \tau^*$, or equivalently $\mathcal{T}_H'$ is a strict subset of $\mathcal{T}_H$, for otherwise the result is immediate from Lemma A.1. I first prove by contradiction that $B \leq B' \equiv \sup \bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$, where B' represents the upper support for $\mathcal{T}_L$.

For this purpose, suppose $B > B'$ and consider a bid $b \in (B', B)$. By definition of B', $b \notin \cup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$, and thus it results from Proposition A.1 that $u(b, m|\mathbf{F}_*) \leq 0$ for all types $m \in \mathcal{T}_L$. In addition, since $b > B'$ and $b < B$, I have $F_*^t(b) = 1$ for all $t \in \mathcal{T}_L$ and $F_*^t(b) = 0$ for $t \in \mathcal{T}_H'$. Accordingly, I can write the expected payoff from bidding b to type m in $\mathcal{T}_L$ as

$$u(b, m|\mathbf{F}_*) \; = \; -b \; + \sum_{t \in \mathcal{T}_L'} \phi(m, t) F_*^t(b) \; \leq \; 0. \tag{6}$$

Observe that the bid b must be an element of $\bigcup_{t \in \mathcal{T}_L' \backslash \mathcal{T}_L} \text{supp}[F_*^t]$. Furthermore, $F_*^t(b) < 1$ for at least one type $t \in \mathcal{T}_L' \backslash \mathcal{T}_L = \{t^\tau, \cdots, t^{\tau^*-1}\}$ due to $b < B$ and Lemma 1. Consequently, the expected payoff from bidding b to type k in $\mathcal{T}_L' \backslash \mathcal{T}_L$ is

$$\begin{aligned} u(b, k|\mathbf{F}_*) \; &= \; -b \; + \sum_{t \in \mathcal{T}_L'} \phi(k, t) F_*^t(b) \\ &\leq \; \sum_{t=1}^{\tau-1} [\phi(k, t) - \phi(m, t)] \; + \; \sum_{t=\tau}^{\tau^*-1} [\phi(k, t) - \phi(m, t)] \, F_*^t(b) \end{aligned}$$

[17] Recall that $\tau \leq \tau^*$ by definition. Hence $\mathcal{T}_L \subset \mathcal{T}_L'$ and $\mathcal{T}_H' \subset \mathcal{T}_H$.

$$< \sum_{t=1}^{\tau^*-1} [\phi(k,t) - \phi(m,t)]$$
$$\leq 0,$$

where the first inequality follows from (6) and the fact that $F_*^t(b) = 1$ for all $t \in \mathcal{T}_L$, the second strict inequality from the fact that $F_*^t(b) < 1$ for at least one type $t \in \mathcal{T}_L' \backslash \mathcal{T}_L$ and $\phi(k,t) - \phi(m,t) > 0$ for all types $t \in \mathcal{T}_L' \backslash \mathcal{T}_L$, and the last inequality from the definition of the threshold τ^*. Hence the expected payoff is strictly negative for all types within $\mathcal{T}_L' \backslash \mathcal{T}_L$, which contradicts with the fact that the bid b is an element of the support for $\mathcal{T}_L' \backslash \mathcal{T}_L$. This establishes $B \leq B'$, which in turn together with Lemma A.2 implies that the lower support for $\mathcal{T}_H'$ must be an element of the bid support for $\mathcal{T}_L$: $B \in \bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$.

I next claim that the payoff from bidding B is at most zero for every type $t \in \mathcal{T}_H'$. Observe that this will complete the proof of the lemma, since $B \in \bigcup_{t \in \mathcal{T}_H'} \text{supp}[F_*^t]$ implies that at least one type within that group has no information rents in equilibrium. For the proof of the claim, I make use of Lemma 2-(b). Proposition A.1 and $B \in \bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$ suggest that there exists a type m in $\mathcal{T}_L$ who obtains an expected payoff of zero from bidding B. On the other hand, it follows from Lemma A.1 that there exists a type k in $\mathcal{T}_H$ (so $k > m$) who weakly prefers to bid zero rather than B. Therefore, for every $s \geq m$, $u(B, s|\mathbf{F}_*) \leq 0$ is immediate from Lemma 2-(b). Since the type m is an element of $\mathcal{T}_L$, the obtained result ensures a nonpositive payoff from B for every type in $\mathcal{T}_H'$. □

Lemma A.4 is used to establish rent dissipation for all types $m \leq \tau^*$.

Lemma A.5. *There is no information rent to a bidder of type $t \in \mathcal{T}_L^* \equiv \{t^1, \cdots, t^{\tau^*}\}$ in any symmetric equilibrium.*

PROOF OF LEMMA A.5: Let type k' in $\mathcal{T}_H'$ denote the one who receives no information rent in the proof of Lemma A.4. I use Lemma 2 to demonstrate that there is no bid in the whole support yielding a strictly positive payoff to any types $s \leq k'$. Since type k' is equal to or higher than the threshold τ^*, this proves the current lemma.

To proceed, choose an arbitrary bid b from $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$. Since this bid belongs to some type $m \leq \tau - 1$, it follows from Proposition A.1 that the bid b yields a payoff of zero to the type m. In contrast, the bid yields a nonpositive payoff to the type $k' > m$. Then the part (b) of Lemma 2 can be used to establish that $u(b, s|\mathbf{F}_*) \leq 0$ for all types $s \geq m$. Together with Proposition A.1, this result guarantees no rents for $\mathcal{T}_L^*$ from any bids in the bid support for $\mathcal{T}_L$.

To complete the proof, consider now a bid B in the complement of $\bigcup_{t \in \mathcal{T}_L} \text{supp}[F_*^t]$ in $\bigcup_{t \in \mathcal{T}_H} \text{supp}[F_*^t]$ (Recall Lemma A.3). Since this bid must be an element of the bid support for some type k in $\mathcal{T}_H$, $u(B, k|\mathbf{F}_*) \geq 0$ follows. However, making that bid is not profitable for every type m in $\mathcal{T}_L$ (so $m < k$) and the type k' in $\mathcal{T}'_H$. Then the part (a) of Lemma 2 can be used to prove $u(B, s|\mathbf{F}_*) \leq 0$ for every type $s \leq k'$. As no bids in the whole support can earn a positive payoff to types $s \leq k'$, the desired result follows. □

The next lemma characterizes the equilibrium bid support of the no-rent group, $\mathcal{T}_L^*$.

Lemma A.6. *In every symmetric equilibrium* $\mathbf{F}_*$, $\bigcup_{k=1}^{\tau^*} \text{supp}[F_*^k] = [0, \overline{B}_{\tau^*}]$.

PROOF OF LEMMA A.6: Let $B \equiv \sup \bigcup_{t \in \mathcal{T}'_L} \text{supp}[F_*^t]$. I first demonstrate that $B < \overline{B}_{\tau^*} \equiv \sup \text{supp}[F_*^{\tau^*}]$ and thus $\overline{B}_{\tau^*}$ constitutes the upper support for $\mathcal{T}_L^*$. To this end, I begin by finding an alternative expression of B. Since there is a type m in $\mathcal{T}'_L$ whose bid support includes the bid B, and since the expected payoff to the type m must be zero by Lemma A.5, I can write $B = \sum_{t \in \mathcal{T}} \phi(m, t) F_*^t(B)$. Using this expression, the expected payoff from bidding B to the threshold τ^* can be decomposed into the following three terms:

$$u(B, \tau^*|\mathbf{F}_*) = \sum_{k=1}^{\tau^*-1} [\phi(\tau^*, k) - \phi(m, k)] + [\phi(\tau^*, \tau^*) - \phi(m, \tau^*)] F_*^{\tau^*}(B) + \sum_{k=\tau^*+1}^{K} [\phi(\tau^*, k) - \phi(m, k)] F_*^k(B).$$

In this decomposition, I used $F_*^t(B) = 1$ for all $t \in \mathcal{T}'_L$. Recall that the expected payoff to the threshold must be nonpositive due to Lemma A.5. In order to have $u(B, \tau^*|\mathbf{F}_*) \leq 0$, however, $F_*^{\tau^*}(B)$ in the second term must be strictly smaller than 1, since otherwise the sum of the first two terms becomes strictly positive (by the definition of τ^*) and the last term is always nonnegative, thereby leading to a contradiction. Hence $B < \overline{B}_{\tau^*}$ must be the case.

The fact that $\bigcup_{k=1}^{\tau^*} \text{supp}[F_*^k]$ forms a connected interval without any internal gaps follows by a similar argument used in the proof of Lemma A.2. The proof of the minimum element being zero is already done in Lemma A.1. The proof is now complete. □

The next proposition characterizes the equilibrium bid support for the other group $\mathcal{T}_H^* = \{t^{\tau^*+1}, \cdots, t^M\}$. It shows that the upper support for the threshold τ^* separates the bid support for $\mathcal{T}_L^*$ and the support for $\mathcal{T}_H^*$, as is described in Figure 5.

Proposition A.2. *In every symmetric equilibrium* $\mathbf{F}_*$, $\cup_{k=\tau^*+1}^{M} \text{supp}[F_*^k] = [\overline{B}_{\tau^*}, \overline{B}]$, *where the maximum element of* $\text{supp}[F_*^{\tau^*}]$ *is given by*

$$\overline{B}_{\tau^*} = \sum_{t=1}^{\tau^*} \phi(\tau^*, t).$$

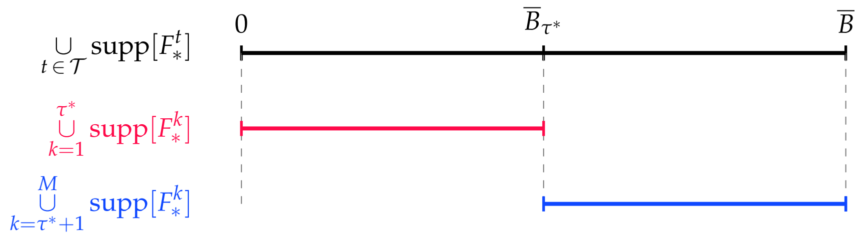

Figure 5: Proof of Part 2 - In every symmetric equilibrium $\mathbf{F}_*$ of the all-pay auction, the full bid support is partitioned at $\overline{B}_{\tau^*}$ into two parts, the union of supports for $\mathcal{T}_L^*$ and the union of supports for $\mathcal{T}_H^*$.

PROOF OF PROPOSITION A.2: For the proof of the bid support, I show that the defined $\overline{B}_{\tau^*}$ constitutes the lower support for $\mathcal{T}_H^*$, which is denoted by $B' \equiv \inf \cup_{k=\tau^*+1}^{M} \text{supp}[F_*^k]$. To this end, it suffices to show that $\overline{B}_{\tau^*} \leq B'$ because $\overline{B}_{\tau^*} \geq B'$ follows from Lemma A.6.

Suppose to the contrary that $\overline{B}_{\tau^*} > B'$. Below I demonstrate that in this case, submitting the bid $\overline{B}_{\tau^*}$ rather than B' is a profitable deviation for every bidder with type $k \geq \tau^* + 1$, i.e., $u(\overline{B}_{\tau^*}, k|\mathbf{F}_*) > u(B', k|\mathbf{F}_*)$ for every $t^k \in \mathcal{T}_H^*$. This contradicts with the fact that the bid B' is an element of $\cup_{k=\tau^*+1}^{M} \text{supp}[F_*^k]$.

I proceed by showing that if $\overline{B}_{\tau^*} > B'$, then $u(B', k|\mathbf{F}_*)$ is at most zero for every type $k \geq \tau^* + 1$. For this, note that due to Lemma A.5 and A.6, $\overline{B}_{\tau^*} > B'$ implies either (i) $B' = \sum_{t\in\mathcal{T}} \phi(m,t) F_*^t(B')$ for some type $m \leq \tau^* - 1$ or (ii) $B' = \sum_{t\in\mathcal{T}} \phi(\tau^*, t) F_*^t(B')$. In words, according to whether B' belongs to $\cup_{t=1}^{\tau^*-1} \text{supp}[F_*^t]$ or $\text{supp}[F_*^{\tau^*}]$, the bid B' can be rewritten as the expected gain conditional on winning. Then a similar argument to the proof of Lemma A.4 establishes that the bidder with type $k \geq \tau^* + 1$ earns at most an expected payoff of zero from bidding B'.

I now compute the expected payoff from submitting the bid of $\overline{B}_{\tau^*}$ to type $k \geq \tau^* + 1$. Since I have $u(\overline{B}_{\tau^*}, \tau^*|\mathbf{F}_*) = 0$ from Lemma A.5, and since I have $F_*^m(\overline{B}_{\tau^*}) = 1$ for every type $m \leq \tau^*$ from Lemma A.6, I can express the bid $\overline{B}_{\tau^*}$ as

$$\overline{B}_{\tau^*} = \sum_{t=1}^{\tau^*} \phi(\tau^*, t) + \sum_{t=\tau^*+1}^{M} \phi(\tau^*, t) F_*^t(\overline{B}_{\tau^*}). \tag{7}$$

Using (7), I can write the expected payoff to type $k \geq \tau^* + 1$ as

$$u(\overline{B}_{\tau^*}, k|\mathbf{F}_*) = \sum_{t=1}^{\tau^*} [\phi(k,t) - \phi(\tau^*, t)] + \sum_{t=\tau^*+1}^{M} [\phi(k,t) - \phi(\tau^*, t)] F_*^t(\overline{B}_{\tau^*}).$$

Observe that both summations above take a strictly positive value. Hence the expected payoff must be strictly positive. This implies that in contrast with B', the bid $\overline{B}_{\tau^*}$ yields a strictly positive expected payoff, and thus the deviation to $\overline{B}_{\tau^*}$ is indeed profitable for all types $k \geq \tau^* + 1$. Therefore, if $\overline{B}_{\tau^*} < B'$, then B' cannot be an element of the bid support for $\mathcal{T}_H^*$, establishing the desired result, $\overline{B}_{\tau^*} = B'$. The bid $\overline{B}_{\tau^*}$ identifies the lowest bid of $\cup_{k=\tau^*+1}^M \mathrm{supp}[F_*^k]$ and thus it must be the case that $F_*^k(\overline{B}_{\tau^*}) = 0$ for all types $k \geq \tau^* + 1$ in every symmetric equilibrium $\mathbf{F}_*$. The closed-form expression of $\overline{B}_{\tau^*}$ is then immediate from (7). □

Part 3

The third part completes the proof of Theorem 1 by characterizing the equilibrium payoff accruing to the types in $\mathcal{T}_H^* = \{t^{\tau^*+1}, \cdots, t^M\}$. To this end, I first establish the following lemma. It shows that the single-crossing condition developed by Athey (2001) is satisfied for the types in $\mathcal{T}_H^*$.

Lemma A.7. *For every pair of types $k' > k$ in $\mathcal{T}_H^*$, $\mathrm{supp}[F_*^{k'}]$ is larger than $\mathrm{supp}[F_*^k]$ in the strong set order.*

PROOF OF LEMMA A.7: Choose two arbitrary bids, b_1 and b_2 with $b_2 > b_1$, from $\cup_{k=\tau^*+1}^M \mathrm{supp}[F_*^t] = [\overline{B}_{\tau^*}, \overline{B}]$. Using the fact that $F_*^m(b_2) = F_*^m(b_1) = 1$ for every type m in $\mathcal{T}_L^*$, the incremental return from making the bid b_2 rather than b_1 to type k in $\mathcal{T}_H^*$ can be written as

$$u(b_2, k|\mathbf{F}_*) - u(b_1, k|\mathbf{F}_*) \;=\; -(b_2 - b_1) + \sum_{t=\tau^*+1}^{M} \phi(k,t)\left[F_*^t(b_2) - F_*^t(b_1)\right].$$

Similarly, the corresponding return to type $k' > k$ can be written as

$$u(b_2, k'|\mathbf{F}_*) - u(b_1, k'|\mathbf{F}_*) \;=\; -(b_2 - b_1) + \sum_{t=\tau^*+1}^{M} \phi(k',t)\left[F_*^t(b_2) - F_*^t(b_1)\right].$$

Note that $\phi(k',t) > \phi(k,t)$ for all types $t \geq \tau^* + 1$ and that $F_*^t(b_2) > F_*^t(b_1)$ for at least one type $t \geq \tau^* + 1$. Hence the incremental return must be higher for type k'. This implies that the expected payoff function, when the opponent adopts an equilibrium strategy, satisfies (strict) increasing differences in (b, k) for $k \geq \tau^* + 1$. The result then follows from the Monotonicity Theorem in Milgrom and Shannon (1994). □

In the next lemma, I enumerate implications of Lemma A.7 on the equilibrium bid support for types in $\mathcal{T}_H^*$.

Lemma A.8. *In every symmetric equilibrium $\mathbf{F}_*$ of the all-pay auction, the following properties hold:*

(i) *For each type* $k \geq \tau^* + 1$, $\text{supp}[F_*^k]$ *is a connected interval.*

(ii) $\inf \cup_{k=\tau^*+1}^{M} \text{supp}[F_*^k] = \underline{B}_{\tau^*+1}$.

(iii) *For the group* $\mathcal{T}_H^*$, *the bid supports are fully separate, that is,* $\overline{B}_k = \underline{B}_{k+1}$ *for each type* $k = \{\tau^* + 1, \cdots, M-1\}$.

PROOF OF LEMMA A.8: The first two statements are straightforward. To prove (iii), note that Lemma A.7 implies $\overline{B}_k \geq \underline{B}_{k+1}$. Hence it is enough to show that $\overline{B}_k > \underline{B}_{k+1}$ cannot occur in any symmetric equilibrium. Suppose to the contrary that $\overline{B}_k > \underline{B}_{k+1}$ is the case. Then it follows from part (i) of the current lemma that the two distinct bids must be indifferent to type k, i.e., $u(\overline{B}_k, k|\mathbf{F}_*) = u(\underline{B}_{k+1}, k|\mathbf{F}_*)$, and $F_*^{k+1}(\overline{B}_k) - F_*^{k+1}(\underline{B}_{k+1}) > 0$. However, this implies

$$u(\overline{B}_k, k+1|\mathbf{F}_*) \; > \; u(\underline{B}_{k+1}, k+1|\mathbf{F}_*).$$

That is, type $k+1$ strictly prefers to submit the bid $\overline{B}_k$ rather than $\overline{B}_{k+1}$, contradicting with $\overline{B}_{k+1} \in \text{supp}[F_*^{k+1}]$. □

The previous lemma pins down the bid support for types $k \geq \tau^* + 1$ in the all-pay auction, and thus the equilibrium bidding strategy for those types is uniquely determined by the standard indifference conditions.

Proposition A.3. *In every symmetric equilibrium* $\mathbf{F}_*$ *of the all-pay auction, a type-*$k \geq \tau^* + 1$ *bidder randomly draws a bid from the uniform distribution over* $[\underline{B}_k, \overline{B}_k]$. *The constant bid density is*

$$f_*^k \; = \; \frac{1}{\phi(k,k)} \quad \textit{over the support } [\underline{B}_k, \overline{B}_k],$$

where

$$\underline{B}_k \; = \; \overline{B}_{k-1} \quad \textit{and} \quad \overline{B}_k \; = \; \overline{B}_{\tau^*} + \sum_{t=\tau^*+1}^{k} \phi(t,t).$$

PROOF OF PROPOSITION A.3: Lemma A.8 implies that each $\text{supp}[F_*^k]$ is disjoint with other supports and is a connected interval. Hence the indifference condition between b_1 and b_2 chosen from $\text{supp}[F_*^k]$ leads to

$$u(b_2, k|\mathbf{F}_*) - u(b_1, k|\mathbf{F}_*) \; = \; -(b_2 - b_1) + \phi(k,k) \left[F_*^k(b_2) - F_*^k(b_1) \right] \; = \; 0.$$

Thus, the equilibrium bid distribution F_*^k follows a uniform distribution with density $\frac{1}{\phi(k,k)}$ over the support $[\underline{B}_k, \overline{B}_k]$. The lower and upper bound of each support can be obtained by mathematical induction. □

The next proposition completes the proof of Theorem 1.

Proposition A.4. *For each type $k \in \{\tau^* + 1, \cdots, M\}$, the equilibrium expected payoff is*

$$U^{\mathbf{F}_*}(k) \;=\; \sum_{t=1}^{k} \phi(k,t) - \left[\sum_{t=1}^{\tau^*} \phi(\tau^*, t) + \sum_{t=\tau^*+1}^{k} \phi(t,t) \right].$$

PROOF OF PROPOSITION A.4: Note that $u(b, k|\mathbf{F}_*) = u(\overline{B}_k, k|\mathbf{F}_*)$ for all $b \in [\underline{B}_k, \overline{B}_k]$ due to the indifference condition, and that by submitting the bid $\overline{B}_k$, the bidder of type $k \geq \tau^* + 1$ can outbid the opponent with type $t \leq k$ for certain. Therefore, the information rents for type k in $\mathcal{T}_H^*$ are

$$\begin{aligned} U^{\mathbf{F}_*}(k) \;&=\; \int_{\underline{B}_k}^{\overline{B}_k} u(b, k|\mathbf{F}_*) dF_*^k(b) \;=\; u(\overline{B}_k, k|\mathbf{F}_*) \\ &=\; -\overline{B}_k + \sum_{t=1}^{k} \phi(k,t) \\ &=\; -\left(\overline{B}_{\tau^*} + \sum_{t=\tau^*+1}^{k} \phi(t,t) \right) + \sum_{t=1}^{k} \phi(k,t), \end{aligned}$$

where the last expression results from Proposition A.3. □

B. Other Omitted Proofs

B.1. Proof of Lemma 2

Using the payoff expression (2), I first rewrite the given conditions in part (a) as $b \leq \sum_{t=1}^{M} \phi(k,t) F_*^t(b)$, $b \geq \sum_{t=1}^{M} \phi(m,t) F_*^t(b)$ and $b \geq \sum_{t=1}^{M} \phi(k',t) F_*^t(b)$, respectively. Combining the first two inequalities, I obtain

$$\sum_{t=1}^{M} \Big(\phi(k,t) - \phi(m,t) \Big) F_*^t(b) \;\geq\; 0. \tag{8}$$

The equilibrium payoff to type $s \leq k'$ from submitting b is then

$$u(b, s|\mathbf{F}_*) \;=\; -b + \sum_{t=1}^{M} \phi(s,t) F_*^t(b) \;\leq\; -\sum_{t=1}^{M} \Big(\phi(k',t) - \phi(s,t) \Big) F_*^t(b)$$

$$= -\frac{C(k',s)}{C(k,m)} \sum_{t=1}^{M} \Big(\phi(k,t) - \phi(m,t)\Big) F_*^t(b) \;\leq\; 0,$$

where the first inequality follows from $u(b,k'|\mathbf{F}_*) \leq 0$, and the last inequality follows from (8) and $C(k',s)/C(k,m) \geq 0$ for all $s \leq k'$. This proves part (a) of the lemma.

For part (b), I first obtain an alternative expression of b from the given condition:

$$u(b,m|\mathbf{F}_*) \;=\; 0 \quad \Rightarrow \quad b \;=\; \sum_{t=1}^{M} \phi(m,t) F_*^t(b).$$

With this expression, the condition $u(b,k|\mathbf{F}_*) \leq 0$ can be translated into

$$\sum_{t=1}^{M} \Big(\phi(k,t) - \phi(m,t)\Big) F_*^t(b) \;\leq\; 0. \tag{9}$$

The equilibrium payoff to type $s \geq m$ is then

$$u(b,s|\mathbf{F}_*) \;=\; \sum_{t=1}^{M} \Big(\phi(s,t) - \phi(m,t)\Big) F_*^t(b) \;=\; \frac{C(s,m)}{C(k,m)} \sum_{t=1}^{M} \Big(\phi(k,t) - \phi(m,t)\Big) F_*^t(b) \;\leq\; 0,$$

where the last inequality results from (9). The proof is now complete.

B.2. Proof of Corollary 2

The equilibrium bid distributions along with their supports are derived in the proof of Proposition A.3 in Appendix A. The sufficiency of $\tau^* = 1$ for the uniqueness of $\mathbf{F}_*$ and monotonicity is then immediate from $\underline{B}_k = \overline{B}_{k-1}$. To prove its necessity, it is enough to show that $\tau^* > 1$ leads to

$$\text{supp}[F_*^{\tau^*}] \cap \Big(\cup_{t=1}^{\tau^*-1} \text{supp}[F_*^t]\Big) \;\neq\; \emptyset. \tag{10}$$

If (10) holds, then a bidder with type $m \leq \tau^* - 1$ outbids a type-τ^* opponent with positive probability in any symmetric equilibrium $\mathbf{F}_*$, thereby breaking its monotonicity.

I prove (10) by contradiction. Suppose to the contrary that there exists a bid of b on the boundary between the two supports. Then it must be the case that (i) $F_*^k(b) = 0$ for all $k \geq \tau^*$, (ii) $F_*^m(b) = 1$ for all types $m \leq \tau^* - 1$, and the bid belongs to both supports, that is, (iii) $b \in \text{supp}[F_*^{\tau^*}]$ and (iv) $b \in \text{supp}[F_*^m]$ for some type $m \leq \tau^* - 1$. By Theorem 1, (iv) implies $u(b, m|\mathbf{F}_*) = 0$, and using (i) and (ii), the bid b can be written as

$$b \; = \; \sum_{t=1}^{\tau^*-1} \phi(m, t).$$

But this implies that a type-τ^* bidder obtains a negative payoff from bidding b, because

$$u(b, \tau^*|\mathbf{F}_*) \; = \; -b + \sum_{t=1}^{\tau^*-1} \phi(\tau^*, t) \; = \; \sum_{t=1}^{\tau^*-1} \Big(\phi(\tau^*, t) - \phi(m, t)\Big) \; < \; 0.$$

The negative expected payoff contradicts with (iii) that the bid b is an element of $\text{supp}[F_*^{\tau^*}]$.

References

AMANN, E. AND W. LEININGER (1996): "Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case," *Games and Economic Behavior*, 14, 1–18.

ATHEY, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, 861–89.

——— (2002): "Monotone Comparative Statics under Uncertainty," *The Quarterly Journal of Economics*, Vol. 117, pp. 187 223.

BAYE, M. R., D. KOVENOCK, AND C. G. DE VRIES (1993): "Rigging the Lobbying Process: An Application of the All-Pay Auction," *American Economic Review*, 83, 289–94.

——— (1996): "The all-pay auction with complete information (*)," *Economic Theory*, 8, 291–305.

CHE, Y.-K. AND I. L. GALE (1998): "Caps on Political Lobbying," *American Economic Review*, 88, 643–651.

CHI, C. K., P. MURTO, AND J. VÄLIMÄKI (2017): "All-pay auctions with affiliated values," Discussion Paper Series in Economics 13/2017, Norwegian School of Economics, Department of Economics.

GANUZA, J.-J. AND J. S. PENALVA (2010): "Signal Orderings Based on Dispersion and the Supply of Private Information in Auctions," *Econometrica*, 78, 1007–1030.

HILLMAN, A. L. AND J. G. RILEY (1989): "Politically Contestable Rents And Transfers," *Economics and Politics*, 1, 17–39.

KAJII, A. AND S. MORRIS (1997): "The Robustness of Equilibria to Incomplete Information," *Econometrica*, 65, 1283–1310.

KAPLAN, T. AND S. ZAMIR (2015): "Advances in Auctions," in *Handbook of Game Theory with Economic Applications*, Elsevier, vol. 4, chap. 7, 381–453.

KARLIN, S. AND Y. RINOTT (1980): "Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions," *Journal of Multivariate Analysis*, 10, 467–498.

KRISHNA, V. AND J. MORGAN (1997): "An Analysis of the War of Attrition and the All-Pay Auction," *Journal of Economic Theory*, 72, 343–362.

LANDSBERGER, M. (2007): "Non-existence of monotone equilibria in games with correlated signals," *Journal of Economic Theory*, 132, 119–136.

LU, J. AND S. O. PARREIRAS (2017): "Monotone equilibrium of two-bidder all-pay auctions Redux," *Games and Economic Behavior*, 104, 78–91.

MCADAMS, D. (2007): "Uniqueness in symmetric first-price auctions with affiliation," *Journal of Economic Theory*, 136, 144–166.

MILGROM, P. AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–80.

MILGROM, P. R. AND R. J. WEBER (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.

PARREIRAS, S. O. AND A. RUBINCHIK (2010): "Contests with three or more heterogeneous agents," *Games and Economic Behavior*, 68, 703–715.

RENTSCHLER, L. AND T. L. TUROCY (2016): "Two-bidder all-pay auctions with interdependent valuations, including the highly competitive case," *Journal of Economic Theory*, 163, 435–466.

SIEGEL, R. (2009): "All-Pay Contests," *Econometrica*, 77, 71–92.

——— (2014): "Asymmetric all-pay auctions with interdependent valuations," *Journal of Economic Theory*, 153, 684–702.

Issued in the series Discussion Papers 2017

2017

01/17 January, **Agnar Sandmo**, "Should the marginal tax rate be negative? Ragnar Frisch on the socially optimal amount of work"

02/17 February, **Luca Picariello**, "Organizational Design with Portable Skills"

03/17 March, **Kurt R. Brekke**, Tor Helge Holmås, Karin Monstad og Odd Rune Straume, "Competition and physician behaviour: Does the competitive environment affect the propensity to issue sickness certificates?"

04/17 March, **Mathias Ekström**, "Seasonal Social Preferences".

05/17 April, Orazio Attanasio, Agnes Kovacs, and **Krisztina Molnar**: "Euler Equations, Subjective Expectations and Income Shocks"

06/17 April, **Alexander W. Cappelen**, Karl Ove Moene, Siv-Elisabeth Skjelbred, and **Bertil Tungodden**, "The merit primacy effect"

07/17 May, **Jan Tore Klovland**, "Navigating through torpedo attacks and enemy raiders: Merchant shipping and freight rates during World War I"

08/17 May, **Alexander W. Cappelen**, Gary Charness, **Mathias Ekström**, Uri Gneezy, and **Bertil Tungodden: "**Exercise Improves Academic Performance"

09/17 June, **Astrid Kunze**, "The gender wage gap in developed countries"

10/17 June, **Kristina M. Bott, Alexander W. Cappelen, Erik Ø. Sørensen** and **Bertil Tungodden**, "You've got mail: A randomized field experiment on tax evasion"

11/17 August, Marco Pagano and **Luca Picariello**, "Talent Discovery, Layoff Risk and Unemployment Insurance"

12/17 August, Ingrid Kristine Folgerø, **Torfinn Harding** and Benjamin S. Westby, «Going fast or going green? Evidence from environmental speed limits in Norway"

13/17 August, **Chang-Koo Chi**, Pauli Murto, and Juuso Välimäki, "All-pay auctions with affiliated values"

14/17 August, **Helge Sandvig Thorsen**, "The effect of school consolidation on student achievement".

15/17 September, Arild Sæther, "Samuel Pufendorf and Ludvig Holberg on Political Economy".

16/17 September**, Chang-Koo Chi,** Pauli Murto, and Juuso Välimäki**, "**War of attrition with affiliated values".

17/17 September, **Aline Bütikofer** and Giovanni Peri, "The Effects of Cognitive and Noncognitive Skills on Migration Decisions"

18/17 October, **Øivind Schøyen**, "What limits the powerful in imposing the morality of their authority?

19/17 October, **Charlotte Ringdal** and **Ingrid Hoem Sjursen**, "Household bargaining and spending on children: Experimental evidence from Tanzania"

20/17 December, **Fred Schroyen** and Karl Ove Aarbu, "Attitudes towards large income risk in welfare states: an international comparison"

21/17 December, **Alexander W. Cappelen, Ranveig Falch**, and **Bertil Tungodden**, "The Boy Crisis: Experimental Evidence on the Acceptance of Males Falling Behind

2018

01/18 January, Øystein Foros, Mai Nguyen-Ones, and **Frode Steen**, "Evidence on consumer behavior and firm performance in gasoline retailing"

02/18 January, **Agnar Sandmo**, "A fundamental externality in the Labour Market? Ragnar Frisch on the socially optimal amount of work"

03/18 February, Pierre Dubois and **Morten Sæthre, "**On the Effect of Parallel Trade on Manufacturers' and Retailers' Profits in the Pharmaceutical Sector"

04/18 March, **Aline Bütikofer**, Julie Riise, and Meghan Skira, "The Impact of Paid Maternity Leave on Maternal Health"

05/18 March, **Kjetil Bjorvatn** and **Bertil Tungodden**, "Empowering the disabled through savings groups: Experimental evidence from Uganda"

06/18 April, Mai Nguyen-Ones and **Frode Steen,** "Measuring Market Power in Gasoline Retailing: A Market- or Station Phenomenon?

07/18 April, **Chang Koo Chi** and Trond Olsen, "Relational incentive contracts and performance"

08/18 April, **Björn Bartling**, **Alexander W. Cappelen, Mathias Ekström, Erik Ø. Sørensen,** and **Bertil Tungodden**, «Fairness in Winner-Take-All Markets»

09/18 April, **Aline Bütikofer, Sissel Jensen**, and **Kjell G. Salvanes**, «The Role of Parenthood on the Gender Gap among Top Earners»

10/18 May, **Mathias Ekström**, "The (un)compromise effect"

11/18 May, Yilong Xu**, Xiaogeng Xu**, and Steven Tucker**,** «Ambiguity Attitudes in the Loss Domain: Decisions for Self versus Others»

12/18 June, **Øivind A. Nilsen**, Per Marius Pettersen, and Joakim Bratlie, "Time-Dependency in producers' price adjustments: Evidence from micro panel data"

13/18 June, **Øivind A. Nilsen**, Arvid Raknerud, and Diana-Cristina Iancu, "Public R&D support and firms' performance. A Panel Data Study"

14/18 June, Manudeep Bhuller, Gordon B. Dahl, **Katrine V. Løken**, and Magne Mogstad: «Incarceration, Recidivism, and Employment"

15/18 August, Manudeep Bhuller, Gordon B. Dahl, **Katrine V. Løken**, and Magne Mogstad: «Incarceration Spillovers in Criminal and Family Networks"

16/18 August, Pedro Carneiro, Kai Liu, and **Kjell G. Salvanes**: "The Supply of Skill and Endogenous Technical Change: Evidence From a College Expansion Reform"

17/18 August, **Chang Koo Chi**, "An analysis of the two-bidder all-pay auction with common values"

NHH

Norges Handelshøyskole

Norwegian School of Economics

NHH
Helleveien 30
NO-5045 Bergen
Norway

Tlf/Tel: +47 55 95 90 00
Faks/Fax: +47 55 95 91 00
nhh.postmottak@nhh.no
www.nhh.no