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Discussion paper

What is wrong with IRV?

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Abstract Struggles over the single-seat preferential election method IRV, Instant Runoff Voting, (a.k.a. AV, Alternative Vote or RCV, Ranked-Choice Voting) go on in many arenas: legislatures, courts, websites, and scholarly journals.

Monotonicity failures, i.e. elections (preference distributions) that may allow the startling tactical voting of *Pushover* or its reverse, has come to the forefront. An analysis of 3-candidate elections concludes that monotonicity failures, while not rare, are hard to predict and risky to exploit; it also explains the scarcity of evidence for effects on election results.

A more unfortunate possibility is the *No-Show accident*; the number of ballots with preference order XYZ grows beyond a critical size and cause Z to win instead of Y. An analysis concludes that this must happen often enough to justify a modification of the rules.

Pictograms and **constellation diagrams** are visualization tools that organize the set of possible elections efficiently for the analysis, which obtains explicit classification of elections where Pushover or a No-Show accident may occur or may already have occurred, and of bounds for the number of voters that must be involved.

The analysis takes place in close contact with two frameworks for preferential election methods, one mathematical and one legal/political; these frameworks are themes for two survey sections.

AMS subject classification 91B12

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Introduction: Non-monotonicity and constellations in 3-candidate IRV

IRV (Instant Runoff Voting), also known as Alternative Vote and Ranked-Choice Voting, is one of many single seat preferential election methods. Every ballot contains a ranking of the candidates. The IRV tally has several rounds. In each round, the candidate with the smallest number of top-ranks is eliminated from all ballots; here some tiebreak rule is assumed. The ballot counts for the top-ranked among its remaining candidates. A candidate who reaches $> 1/2$ of the top-ranks becomes IRV-winner. A candidate with exactly $1/2$ will at least qualify for tiebreaks. We focus first on the round with three candidates and assume each candidate has $< 1/2$ of the top ranks.

Notation Consider an IRV tally with N voters, where 3 candidates, X , Y , and Z remain. Each ballot contains one of six orderings: XYZ , XZY , ZXY , ZYX , YZX , or YXZ . Let $|X|$ voters rank X on top; let $|XYZ|$ of them have ranking XYZ . Thus,

$$(0.1) \quad N = |X| + |Y| + |Z|, \quad |X| = |XYZ| + |XZY|, \quad \text{etc.}$$

(0.2) **Deficiencies** $\delta_x, \delta_y, \delta_z$ tell how many top-ranks X, Y, Z are away from 50%:

$$|X| + \delta_x = |Y| + \delta_y = |Z| + \delta_z = N/2. \quad \text{Thus,}$$
$$\delta_x + \delta_y + \delta_z = N/2, \quad |X| = \delta_y + \delta_z, \quad |Y| = \delta_z + \delta_x, \quad |Z| = \delta_x + \delta_y.$$

The supporters of X decide the *pairwise comparison* in $\{Y, Z\}$; equality occurs when

$$|Y| + |XYZ| = |Z| + |XZY|, \quad \text{i.e.} \quad |XYZ| = \delta_y, \quad |XZY| = \delta_z, \quad \text{etc.}$$

When three candidates remain, the IRV tally uses two social preference relations. One relation orders the candidates by number of top-ranks. The other is the *Condorcet relation* (the relation of pairwise comparisons). Both may be any of the six orderings, but the Condorcet relation may also be one of two cycles, for short denoted $XYZX$ and $XZYX$. A main theme in this paper is effects on the outcome, more or less unwanted, from voter actions that either change the size $|XYZ|$ of a single voter category or moves voters from one category to another, e.g. in strategic voting.

(0.3) **Strategic voting** In single-seat preferential elections, three kinds of *strategic* (also called *tactical*) *voting* get most attention. With three candidates, X, Y , and Z , they are as follows:

- 1) **“Compromise”**: original ballot ranking XYZ lets Z win; new ranking YXZ lets Y win.
- 2) **“Burying”**: original ballot ranking XYZ lets Y win; new ranking XZY lets X win.
- 3) **“Pushover”**: original ballot ranking XYZ lets Z win; new ranking YXZ lets X win.

Standard labelling When a 3-candidate preferential election is a reference throughout a discussion, it is convenient to label the candidates according to how they fare in an IRV tally of the reference election. With two rounds left in IRV, three candidates, A, B, and C, remain:

- (0.4) C is *eliminated* because $|C| < |A|$ and $|C| < |B|$;
 B is *runner-up* because $|B| + |CBA| < |A| + |CAB|$;
 A is *IRV-winner*.

With $(x, y) = (|ACB|, |BCA|)$, information ignored in the IRV tally, the *vote vector* is:

(0.5) $(|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|)$
 $= (|A|-x, x, |CAB|, |CBA|, y, |B|-y)$.

Monotonicity failure and Frome 2009

Our main reference is an IRV election in Frome, South Australia, analyzed in section 1. The electoral board published more data, in effect revealing $x = 3801$ in the vote vector (0.5). An estimate, $y = 2748$, then gives the vote vector in row 1 of the table in (0.6), visualized in the middle “pictogram” of Figure (1.3). In both, the labelling is as in (0.4). Thus:

After elimination of C, A defeated B.

Two different changes of row 1 are ingredients in the Pushover strategy of (0.3):

(*) 100 new voters join the election of row 1 in category CBA. C passes A in top-ranks:

After elimination of A, B defeats C.

(**) 100 voters leave the election of row 1 from category BCA. Alone, (**) keeps A as winner:

After elimination of C, A defeats B.

If (**) happens alone, it gives row 2 in the table (0.6); the original tally is repeated with adjusted numbers, and A wins. Constant electorate size is obtained by balancing (*) with (**). Together, (**) and (*) give row 3. With adjusted numbers, the tally goes as if (*) happens alone, and B wins.

(0.6)

	ABC	ACB	CAB	CBA	BCA	BAC	A	C	B
	1761	3801	4425	1107	2748	5467	5562	5532	8215
(**)	1761	3801	4425	1107	<u>2648</u>	5467	5562	5532	<u>8115</u>
(**) & (*)	1761	3801	4425	<u>1207</u>	<u>2648</u>	5467	5562	<u>5632</u>	<u>8115</u>

To construct the Pushover mechanism, glue together (**) and (*): Identify those who leave BCA in (**) with those who enter CBA in (*). Now, concerning rows 1 and 3, the narrative is:

Down-ranking of B in 100 ballots is the only change, and B wins instead of A;
 in reverse: *Up-ranking* of B in 100 ballots is the only change, and A wins instead of B.

A single winner preferential election method is *monotonic* if up-ranking winner W cannot make W a non-winner [down-ranking a non-winner L cannot make L a winner], while nothing else is altered on any ballot. Thus, IRV is a *non-monotonic method*, and this term is a misnomer. Focus on (**), i.e. less top-ranks to B , is misleading. It has nothing to do with B 's victory. A competes with C to challenge B in the final tally round: The explanation is (*), i.e. more top-ranks to C .

The expression “down-ranking of B ” deflects readers’ attention away from the decisive (*) by hiding it as a chosen but not mentioned concomitant to (**); see Example (0.1). By construction, Pushover or its reverse are *possible* in some preference distributions called “*monotonicity failures*”, and receive an attention it is hard to ignore, e.g. Gierzinsky (2009, 2011), Ornstein and Norman (2014), Miller (2017), and Supreme Court treatment: Minnesota (2009), Maine (2017).

If some voters with a BCA-ranking see Pushover as a realistic way to win, they have an incentive to perform it, but how realistic is it? Are there reasons for practical concern? Points in case are

- how frequently opportunities to win a 3-candidate IRV-election by Pushover occur;
- which structural features in a preference distribution that allow Pushover;
- how easy it is to detect an opportunity before election and perform Pushover;
- how strong the incentives to join a Pushover action really are.

Empirical evidence for Pushover actions in IRV is hard to find, but in this paper, the tally mechanism itself illuminates possibilities and incentives. The conclusions are, in short version:

- quite frequent; • $N/4 < |A| < N/3$ and $|A| + |ACB| < N/2 - 1$; • difficult and risky; • very weak.

EXAMPLE (0.1) A website for election science claims about Frome 2009: “*That is, the Liberal Party [B] lost because some voters ranked him too high*”. This is misleading. In IRV, a ballot never harms its top-ranked candidate X : An extra ballot with top-rank strengthens X in all tally rounds. In row 1 of (0.6), each BCA-ballot has changed exactly one account, i.e. $|BCA|$. The context makes it clear that the writer had in mind the reverse of Pushover.¹ *Theoretically*, row 1 of (0.6) may include 100 (say) BCA voters who had consciously moved from a planned CBA to BCA in their ballots and changed an imagined row 3 to row 1. If so, they may regret that they thereby caused A to replace C in the final tally round and to go on and win. Without evidence, a claim that this *really* happened, and that $|CBA|$ *really* had been significantly larger, is creative accounting.

¹ Out of context, the formulation is a false claim of a “No-Show accident”, actually of its “strong version”, which by Theorem (0.2) cannot even occur in IRV. Neither did the “weak version”, which occasionally is a nuisance in IRV, occur in Frome 2009. The reality was that the B -supporters missed an opportunity to win by Pushover.

No-Show Arguably, a more unfortunate possibility than Pushover and its reverse is the *No-Show Paradox*, which also may hit row 1 of (0.6): Let 100 new voters enter the election and vote CAB. Then $|C|$ increases to 5632, elimination of A follows and B defeats C. The new voters would have gotten a better winner according to their own ballots, i.e. A instead of B, by not showing up at the poll-site to participate. This “*No-Show accident*” changes only one component of the vote vector: There is no creative accounting. Evidence is in the new vote vector: The new vote vector then contains the evidence: Reduction of the $|CAB|$ -account by 100 leads back to row 1 of (0.6), and restores A as winner instead of B.

Standard tally In 3-candidate IRV with labelling (0.4), also the tally will be called *standard* if (0.7) $(|A|, |B|, |CAB|, |CBA|)$ is the information revealed.

However, the unknown (x, y) in (0.5) determines how action by a voter group may cause a change of winner, e.g. by means of Pushover or a No-Show accident. The interplay between two social relations, ranking by top-ranks and Condorcet’s pairwise comparison, is central to such changes. A *constellation diagram* is a tool for visualization of the interplay and for natural reasoning.

Constellations

(0.8) **Definition** There are eight *constellations* shown in the *diagrams* (3x3-tables) in Figure (0.1), and labelled *i*, *ii*, *iii*, *iv*, *v*, *vi*, $i_{(cyclic)}$ and $iii_{(cyclic)}$. There is one candidate in each column and one in each row. Number 1 (Plurality winner), 2, and 3 in top-ranks are in column 1, 2, and 3, respectively. In pairwise comparison, the candidate in row 1 [2] beats the one in row 2 [3]. In the cyclic cases i.e. $i_{(cyclic)}$ and $iii_{(cyclic)}$, the candidate in row 3 beats the one in row 1.

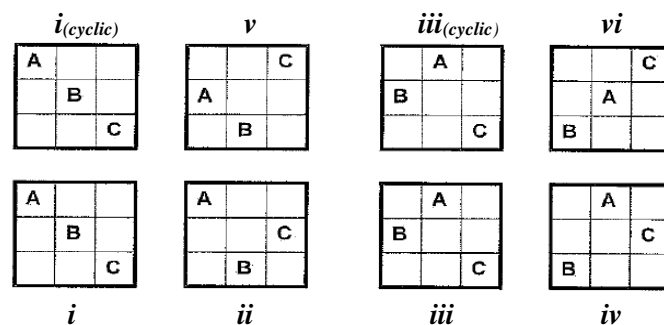


FIGURE (0.1) Information not revealed in the standard IRV tally

All eight diagrams show that C is eliminated (C in the right hand column) and that A wins over B in the final round (A in the higher row). The standard IRV-tally tells nothing about how C does in pairwise comparisons. By increasing y in (0.5), B-supporters may change *i* to $i_{(cyclic)}$ and *iii* to $iii_{(cyclic)}$. By decreasing x , A-supporters may change *v* to $i_{(cyclic)}$ and *vi* to $iii_{(cyclic)}$. Only *vi* is non-cyclic and gives an IRV-winner who is neither Condorcet- nor Plurality winner.

In Figure (0.1), the notation C, B, and A for candidates is according to the IRV tally as in (0.4). Similar cyclic versions of ii , iv , v , and vi exist, but cyclic permutations of the rows show that

$$i_{(cyclic)} = iv_{(cyclic)} = v_{(cyclic)} \text{ and } iii_{(cyclic)} = ii_{(cyclic)} = vi_{(cyclic)}.$$

With tiebreak rules in cases of equality, every 3-candidate vote vector (0.5) belongs to one of the eight constellations.

(0.9) Definition The *constellation family* \mathcal{A} consists of i , ii , v , and $i_{(cyclic)}$, where the IRV-winner A is also Plurality winner. *Family* \mathcal{B} consists of iii , iv , vi , and $iii_{(cyclic)}$, where the runner-up B is Plurality winner. See (0.8) and Figure (0.1). The standard 3-candidate IRV tally (0.7) reveals what family the election belongs to, but all constellations in the family are compatible with the tally.

Two facts, known in other formulations, are that only constellations iii and $iii_{(cyclic)}$

- may allow some supporters of another candidate to let their favorite snatch victory from A by applying the strategic (tactical) voting of *Pushover*;
- may let additional voters in one voting category cause a worse result according to their own ballot ranking through a *No-Show accident*.

There are different ways to establish these facts. However, constellation diagrams allow a hands-on reasoning, in close touch with the tally process. They also help to explain why the only voter actions that may cause these effects are the Pushover strategy and the No-Show accident.

THEOREM (0.1) The preference distributions that allow supporters of B or C to make their favorite become IRV-winner with any kind of strategic voting, form a subset of all preference distributions in constellations iii and $iii_{(cyclic)}$. The only possibility is then that suitably many supporters of B yield their top-rank to C, as in the Pushover strategy.

Proof: The voters who rank C on top cannot change their ballots in a way that prevents elimination of C. We must consider what may be possible for the supporters of B.

The voters who rank B on top cannot make B an IRV-winner in constellations ii , iv , v or vi , because no change in their ballots can change the fact that B is Condorcet loser and, if promoted to the final round, will lose whether the opponent is C or A.

The voters who rank B on top cannot make B an IRV-winner in constellations i or $i_{(cyclic)}$ either: No change in their ballots can prevent that A, as Plurality winner, thus with more than 1/3 of the top-

ranks, qualifies for the final round. In order to win, they must ensure that B still qualifies for the final round, but no change in their ballots can prevent that A wins over B in pairwise comparison. Only constellations *iii* and *iii_(cyclic)* remain. The supporters of B cannot change the fact that a majority prefers A to B. The only possibility is to get rid of A; a suitable number of B-supporters yield top-rank to C, promote C to the final round and get A eliminated. \square

If $|A| < N/4$, then $|A| + |C| < N/2 < |B|$ and B wins. If $N/3 < |A|$, then A cannot be eliminated:

(0.10) With regard to Pushover and its reverse, the scope of this paper is $N/4 < |A| < N/3$.

REMARK (0.1) By Theorem (0.1), a move from BAC or BCA into categories CBA or CAB is the only way for supporters of another candidate X to make X defeat A through strategic voting. Moves into CBA and CAB have the same effect after elimination of A, so it is enough to consider moves into CBA. Moves from BAC and BCA also have the same effect. A contribution from category BAC is required only if $|BCA|$ is too small. It may be decomposed: $BAC \rightarrow BCA \rightarrow CBA$.

Thus, it is enough to concentrate on categories BCA and CBA, i.e. voters who rank IRV-winner A last; then the action is a case of Pushover (0.3). A Pushover attempt to help B may miss, but cause C to win, e.g. if 400 voters up-rank C from BCA to CBA in row 1 of (0.6). This is an improvement for both voter groups involved, BCA and CBA. C is then a *fallback security* for the actionists who start from BCA. In effect, the action becomes a case of Compromise strategy (0.3), common in Plurality elections. However, voters who start from BAC, run a risk to turn their bottom-ranked candidate C into a winner; this risk is an argument against joining a Pushover attempt.

THEOREM (0.2) The preference distributions that allow new voters to be added to one of the six voter categories and cause a candidate whom they rank after the IRV-winner A to become new IRV-winner, form a subset of all preference distributions in constellations *iii* and *iii_(cyclic)*. The only possibility is then that the new voters have preference CAB.

Proof: We first establish that the new voters must have preference CAB. The new voters cannot give top-rank to IRV-winner A, because with higher margin than before, A would qualify for the final round, and there win against B with higher margin than before.

They cannot rank A last either, because then there cannot be a new winner whom they rank *after* A. Thus the extra voters must give A second rank and vote either BAC or CAB.

If they vote BAC, then C still is last in top-ranks, and the new winner cannot be the one they rank after A. Therefore, the only possibility is that they vote CAB and make B new IRV-winner. However, A will still beat B in a final, so they must eliminate A.

In what constellations from Figure (0.1) may additional CAB-ballots cause elimination of the IRV-winner A? In constellations i , $i_{(cyclic)}$, ii , and v , A is ahead of B in top ranks, and cannot possibly be eliminated.

In constellations iv and vi B is already Condorcet loser and cannot possibly win the final tally round. Thus only iii and $iii_{(cyclic)}$ remain. \square

REMARK (0.2) *If Pushover is possible, then a No-Show accident is possible too.*

To see this, suppose that h voters switching from BCA to CBA will succeed in helping B to win with Pushover. With some good luck, B may become IRV-winner without Pushover: Suppose instead that extra top-ranks for C come from h new voters who vote CBA or CAB. A is eliminated, and B wins over C with higher margin than if the new top-ranks for C came through Pushover.

If the new voters vote CBA, they have B as a fallback security, and may feel some happiness if they cause B to win instead of A. If they change mind and switch from CBA to CAB, they still eliminate A and still get B as IRV-winner. Causing B to win, they are “victims” of a No-Show accident, i.e. they would have been better off if they had not participated. Thus, IRV fails the *Participation criterion*, i.e. that a ballot never causes a worse result according to the ballot’s own ranking.

The possibility of a No-Show accident is a frequently used argument against the use of IRV. However, a preferential election method cannot be blamed if h switches from CBA to CAB are insufficient to let A win instead of B. The h new CAB-voters simply let C spoil the election for A.

Non-participation in order to avoid B would be a very artificial remedy. Participation with h new ACB-ballots instead would just have made A’s win more secure. If some of the unfortunate new voters had switched from CAB to ACB, they could have helped A by Compromise (0.3). One cannot blame CAB-voters for not seeing that their ballots are counter-productive, but adjustments in the IRV tally rules considered in section 4 will eliminate No-Show events in 3-candidate IRV.

(0.10) Equal preference Like in most Australian IRV-elections, the voters in our main example (Frome 2009) were obliged to express a complete and strict ranking of the candidates. If voters

are allowed to declare equal preference, e.g. that j candidates share the ranks $k+1, k+2, \dots, k+j$, the tally can be done through *symmetrizing*: It is done as if such a ballot is replaced by $j!$ “mini-ballots” of weight $1/(j!)$, one for each permutation of the j candidates. In many IRV elections, like the one in Example (3.2) (Burlington 2009), voters are allowed to rank strictly k of n candidates from top, $1, 2, \dots, k$, and leave out the remaining $n-k$, and the result is the same as if equal preference for the remaining $n-k$ is treated with symmetrizing.

Structure of the paper

Section 1: In Figure (1.1) of the xy -plane, the sub-rectangles show the changes of constellation. The curves illustrate a problem too often ignored: what preferential distributions are normal? A cycle (Condorcet Paradox) is visibly not normal. With a survey figure like Figure (1.1) for every standard IRV tally (0.7), the set of all 3-candidate IRV elections is organized in a convenient way for a study of monotonicity failures and No-Show events, because the criteria derived in sections 2 and 3 are intervals that x or y must belong to, defined by the standard tally. There are just two types of figures, like Figure (1.1) for family \mathfrak{B} and a similar one for family \mathfrak{Q} .

Section 2: The topic is monotonicity failure in 3-candidate IRV. When does (x, y) define an election where a non-monotonic event (“trick” or “trap”) may happen or may have happened? For each standard tally (0.7), the unknown (x, y) defines a trick position [a trap position] if and only if x [y] is in a specified interval. The numerical example in Figure (2.5) shows monotonicity failures of both types. Central to the paper is the description of the “action space” for Pushover trick in Figure (2.1) and Figure (2.2). Theorem (0.1) is the background.

Section 3: The topic is the anomaly of No-Show in 3-candidate IRV. When does (x, y) define an election where *new voters* in one component of the vote vector may cause a *worse* result according to their ballots? The dual question is also studied: when may a suitable number of voters leave a component (*abstain* from participation, or theoretically, *cancel* their ballots) and cause an improved result? For each standard tally (0.7), (x, y) allows a No-Show accident [abstention strategy] if and only if x [y] is in a specified interval. The numerical example in Figure (3.3) shows both a No-Show accident and a win by abstention. Theorem (0.2) is the background.

Section 4: The findings in previous sections are included in a discussion of the IRV-rules in a context of some basic methods of single seat preferential election. Arguably, IRV will be more

widely accepted as a fair election method if pairwise comparisons come already in the penultimate round, i.e. one round earlier than now.

Section 5: Results of Arrow (1950) and Black (1948) form a mathematical framework for the theory of preferential elections. Actually, it is Wilson's improved version (1972) of Arrow's "impossibility theorem" that is useful in this context. The purpose of Black's "Single Peak condition" was to describe a structural condition on the electorate's preference distribution that avoided "Condorcet cycles". The concept of "Perfect Pie-sharing" appears as a natural extension of Black's condition.

Section 6: Two kinds of elections to legislatures are common: In some, each elected candidate is the representative of one local community, e.g. in UK and US. Others give several parties that pass a certain threshold criterion, a number of representatives roughly proportional to their number of votes. The MMP (*Mixed Member Proportional*) method combines these two ideas, but with present rules, the price is that the total number of representatives may vary a lot. IRV for local representatives seems to improve the situation.

The supreme courts of Minnesota 2009 and Maine 2017 treated the question whether IRV is unconstitutional. The first concerned the modern theme of monotonicity failures, but a claimed precedent from 1915 gives a time perspective back to the "Progressive Era". An obstacle to IRV in Maine's constitution has a special historic root from events in 1879-80.

1 The standard IRV tally and Frome 2009

A by-election in Frome, South Australia in 2009, for a single seat in the state assembly, had six candidates. As usual in Australian IRV elections, ballot were required to specify one of the $6! = 720$ complete, strict, and transitive preferences. Three candidates remained for tally round 4:

(1.1) (A, B, C) = (Brock – independent, Boylan – liberal, Rohde – labor)

(1.2) Round 4 established $|A| = 5562$, $|B| = 8215$, and $|C| = 5532$.

Round 5 established $|CAB| = 4425$ and $|CBA| = 1107$. Thus,

A became IRV-winner with $5562 + 4425 = 9987$, while B got $8215 + 1107 = 9322$

After the standard tally in (1.2), the *vote vector* (0.5) still keeps the number of subsidiary ranks given to C by the supporters of A and B as unknowns x and y :

(1.3) $(|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|)$
 $= (5562-x, x, 4425, 1107, y, 8215-y)$.

The unknowns are $x = |ACB| \in [0, 5562]$, $y = |BCA| \in [0, 8215]$.

The standard IRV-tally is the same for all (x, y) in Figure (1.1). However, properties of IRV that many voters find important depend on (x, y) . How often will (x, y) define a vote vector (0.5) which allows or may be caused by either Pushover or a No-Show accident? Figure (1.1) illustrates a discussion that uses both constellation diagrams and *pictograms*.

Pictograms A unique *pictogram* represents the 3-candidate vote vector (0.5) faithfully (Stensholt 1996). A pictogram for (0.5) with specified (x, y) consists of a unit circle and three chords that meet pairwise inside or on the circle. Distinct chords form an “empty” triangle T , i.e. not corresponding to any voter group. In real elections, T is usually very small. The pictograms of Figure (1.2) illustrate three realistic choices of (x, y) in (1.3).

“*Ideal points*” A , B , and C , are the corners of a “*candidate triangle*” ΔABC , inserted so that its perpendicular bisectors almost coincide with the chords of the pictogram. Exact coincidence occurs in “*Perfect Pie-sharing*”, and does not “waste” any area with T ; then the chords are not distinct or T has zero area. The other areas are proportional to the components of the vote vector. Perfect Pie-sharing is a 2D model of spatial voting inspired by the familiar 1D “*Single Peak*”-model (Black 1948): The voters are distributed uniformly in the unit circle, and each voter ranks the candidates according to the Euclidean distance from the voter to the ideal points. ²

² The candidate triangle ΔABC is unique in shape, but not in size. One may see from Figure (1.2) that the size changes with homothetic transformations centered on the intersection point of the perpendicular bisectors.

Main features of Figure (1.1) The 5563x8216 possible points (x, y) with integer coordinates form a grid in the big rectangle of Figure (1.1). One main feature is the partitioning into four sub-rectangles labelled SW, SE, NE, and NW, defined by two lines where A or B, in the rôle of X in (0.2), gives $N/2 - |C|$ second ranks to C, $N = |A| + |B| + |C|$. In Condorcet's pairwise comparison,

$$(1.4) \quad A \text{ makes C tie with B at } x = N/2 - |C| = 4122.5 = \delta_c$$

$$(1.5) \quad B \text{ makes C tie with A at } y = N/2 - |C| = 4122.5 = \delta_c$$

In each sub-rectangle is a constellation diagram from Figure (0.1) that changes when a line, (1.4) or (1.5), is crossed. Only pairwise comparisons change; A, B, and C stay in columns 2, 1, and 3, respectively. When $x < \delta_c$ and $y < \delta_c$, Plurality loser C is also Condorcet loser. Thus, in SW, C occupies the lower right corner in the constellation diagram, and the constellation in SW is either *i* or *iii*; see Figure (0.1). According to the standard tally in (1.2), B is Plurality winner in Frome 2009; thus, the constellation is *iii*.

Another main feature is a set of three curves. The middle curve connects two corners. In the corner $(x, y) = (0, 0)$, no voter ranks C as number 2. The pictogram degenerates, i.e. two chords coincide, and there is Perfect Pie-sharing. In the NE corner, $(x, y) = (|A|, |B|)$, no voter ranks C last; the preference distribution is single peaked, and there is again Perfect Pie-sharing.

Along the middle curve, (x, y) maintains Perfect Pie-sharing. Three choices of grid points close to the curve in SW, SE, and NE, give the pictograms in Figure (1.2). They illustrate that *the Condorcet relation is transitive along the middle curve*. The reason is that, with Perfect Pie-sharing, the Condorcet relation ranks the candidates according to their distance from the circle center.

The transitivity implies that the middle curve cannot pass through the NW sub-rectangle where all points (x, y) define a cyclic vote vector. The other curves have endpoints on the edges of the rectangle and give pictograms where triangle T covers a fraction 0.001 of the circle area.

Why Condorcet cycles are rare In real 3-candidate elections with this many voters, mostly with reasonably similar perceptions of the political landscape, a pictogram usually fits the Perfect Pie-sharing model visually well; (x, y) is often very close to the middle curve and is very unlikely to be outside the "0.001-zone" between the upper and the lower curve. Only a tiny part of the 0.001-zone may be in the NW. Figure (1.1) shows that, in Frome 2009, the 0.001-zone does contain a small set of cycles.

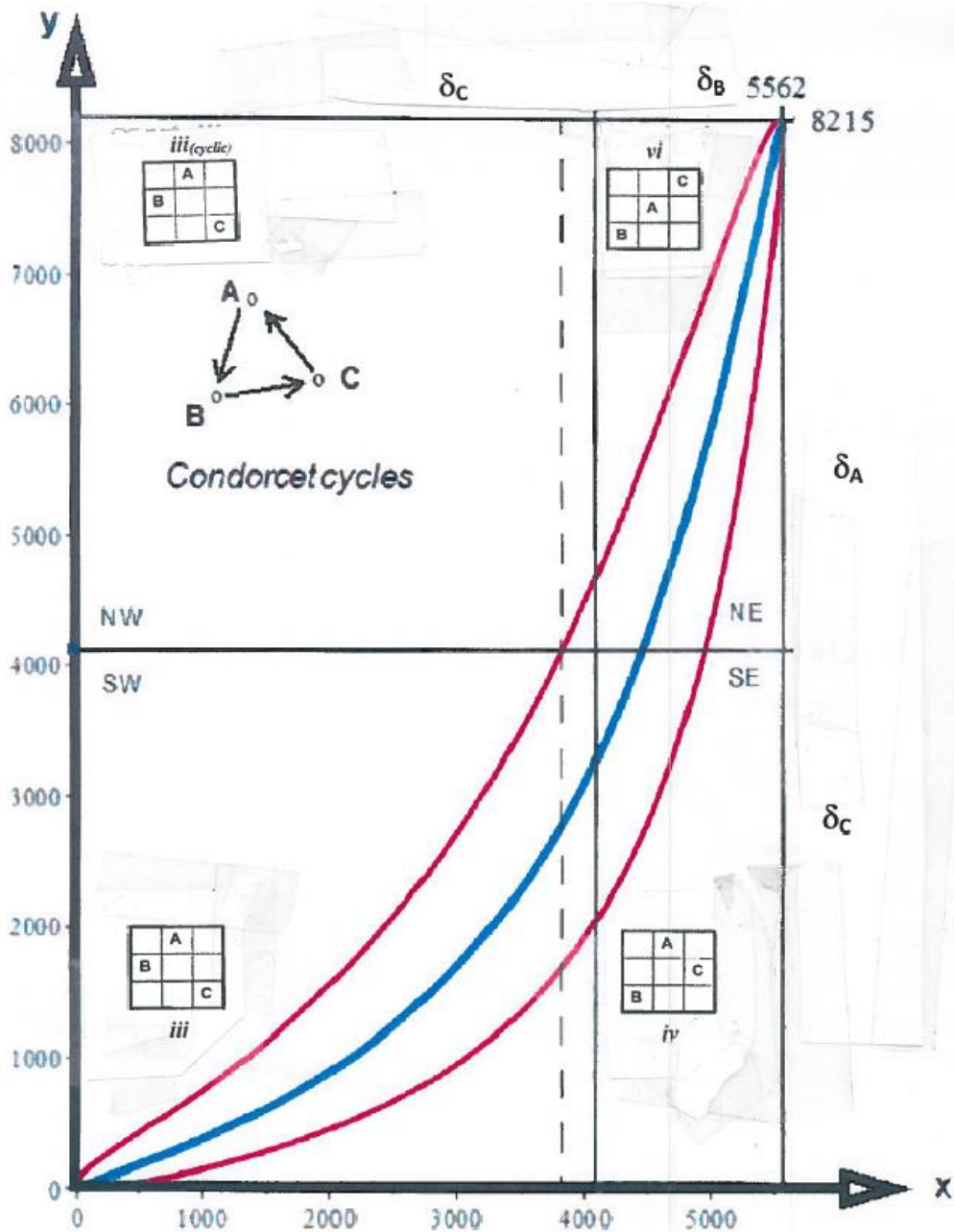


FIGURE (1.1) *The last two rounds of the standard IRV tally in Frome 2009*

The quantitative information $(|A|, |B|, |CAB|, |CBA|) = (5562, 8215, 4425, 1107)$ is all that is found in the standard tally. Both $x = |ACB|$ and $y = |BCA|$ are unknown.

The constellation changes within family \mathfrak{B} when x or y passes $\delta_c = 4122.5$. *Additional information* from the electoral board revealed that $x = 3801$. The real election corresponds to an unknown point on the stapled line. Pictograms for selected points on the line are in Figure (1.3), Figure (2.5), and Figure (3.3). Except for the stapled line, the figure builds on information from the standard tally (1.2).

Along the middle curve are the (x, y) that represent a vote vector (1.3) with Perfect Pie-sharing; pictograms for selected (x, y) are in Figure (1.2). Along the other curves, T covers 0.001 of the pictogram area.

Constellation *vi* The NE sub-rectangle in Figure (1.1) has a diagram of constellation *vi*. It is the only noncyclic constellation where the IRV-winner is neither Plurality winner nor Condorcet winner. B is Plurality winner but Condorcet loser; for C it is the other way around. The diagram visualizes a double incentive IRV gives to candidates and their parties: Work for primary support from enthusiastic followers (A beats C in top-ranks), but also for subsidiary support from political neighbors (A beats B in pairwise comparison). That “two silver” are better than “one gold and one bronze” has, sometimes, a didactic value since an idea behind IRV is to achieve a balance between high rankings and broad acceptance. However, in IRV, “two silver”, are *always* the best, and there are cases that most people probably would find unreasonable, e.g the one illustrated with the third pictogram in Figure (1.2).

The fate of the Condorcet winner The double incentive seen in constellation *vi* is a common argument in favor of IRV, but this does not imply that it is wise to keep a rule that always declares A as winner in constellation *vi*. Consider $(x, y) = (5300, 7062)$, close to the middle curve in Figure (1.1), which gives the third pictogram in Figure (1.2). C defeats both A and B pairwise, with wide margins: $(12794 - 6715)$ and $(10832 - 8477)$. With A having just a tiny advantage over C in top-ranks, what will the public reaction be if this is a real election and A wins ahead of C?

All three pictograms in Figure (1.2) correspond to a grid-point (x, y) close to the middle curve:

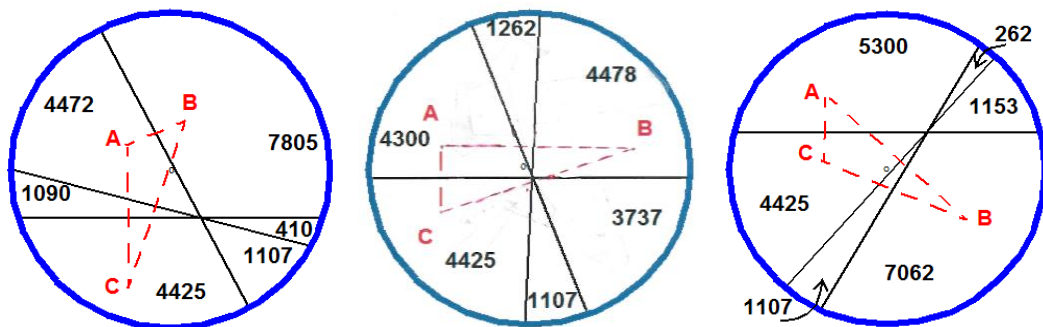


FIGURE (1.2) Perfect Pie-sharing compatible with the standard tally

Imagine that (x, y) moves along the middle curve in Figure (1.1), with snapshots taken at $(x, y) = (1090, 410)$ in SW; $(4300, 3737)$ in SE, and $(5300, 7062)$ in NE.

Moving closer to the corner in SW, all subsidiary support for C disappears; in the limit, two chords coincide, and the pictogram degenerates.

In NE, the constellation is *vi*, and the eliminated candidate C is Condorcet-winner. C has the smallest number of bottom-ranks: $262+1153$. Moving closer to the corner, all bottom-ranks for C disappear; in the limit, the preference distribution is single-peaked.

The constellation in the NE is either v (family \mathcal{A}) or vi (family \mathcal{B}); in both cases, C is Condorcet winner, but last in top-ranks.

Cycles are too rare to influence a discussion. When is it justified to eliminate a Condorcet winner? In Frome 2009, both A and C had between 28% and 29 % of the top-ranks. It seems reasonable to declare both as eligible, and C as winner in constellation vi . One idea is to stop the elimination based on top-ranks at some stage, and then use a Condorcet method; see section 4.

Additional information in Frome 2009 Fortunately, the reality in Frome 2009 was far from the scenario of the third pictogram in Figure (1.2). Since the winner A was an independent candidate, the electoral board published also the pairwise comparison between the candidates of the major parties. There,

(1.6) in the pair (B, C), B won with 9976 versus C with 9333.

This reveals that in (1.3), $x = |ACB| = 3801$. With only one unknown, the vote vector was

$$(1.7) \quad (|ABC|, |ACB|, |CAB|, |CBA|, |BCA|, |BAC|) \\ = (1761, 3801, 4425, 1107, y, 8215-y),$$

It is likely that the real (x, y) was in the 0.001-zone with $1662 < y < 4067$. In the *pictograms* below, $y = |BCA| = 1663; 2748; 4066$. A natural estimate for the unknown y is 2748.

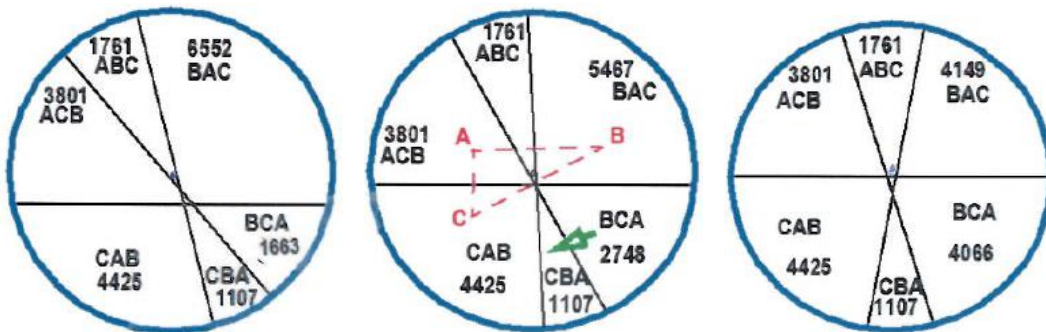


FIGURE (1.3) Pictograms on the stapled line in Figure (1.1)

The grid points $(x, y) = (3801, 1663); (3801, 2748); (3801, 4066)$ are close to the curves in Figure (1.1). For $y = |BCA| = 2748$, the triangle T defined by the chords covers a fraction $4 \cdot 10^{-11}$ of the circle area, a good approximation to Perfect Pie-sharing.

T changes continuously with y . For $1663 \leq y \leq 4066$, T covers < 0.001 of the circle area. This illustrates the robustness of the Perfect Pie-sharing model. The arrow shows how, in all three cases, h voters perform Pushover by switching from BCA to CBA, $31 \leq h \leq 321$.

Figures (2.5) and (3.3) show pictograms of cyclic elections in NW, on the stapled line in Figure (1.1).

2 How Pushover works

Only in some elections in constellations *iii* or *iii*_(cyclic) from family \mathfrak{B} , is it possible to make one's top-ranked candidate defeat an IRV-winner by means of strategic voting, see Theorem (0.1). A suitable number (h) of supporters of Plurality winner B in Figure (1.1) may, for some values of $x = |ACB|$, apply Pushover (0.3) to snatch victory from the IRV-winner A in Figure (1.1). The target A is also Condorcet winner in constellation *iii*.

It is convenient to focus on the anti-A voters, categories BCA and CBA; see Remark (0.1). They turn A into Plurality loser, and B beats C in the final round. Figures (2.1) and (2.2) show the possibilities the anti-A voters have to avoid election of A.

Action space with pitfalls

Let h Pushover actionists switch ballot ranking from BCA to CBA, starting at $h = 0$ in constellation *iii* or *iii*_(cyclic), in the SW or NW of Figure (1.1). Increasing h may cause B to snatch the IRV victory from A by creating a new constellation where B is winner. The anti-A group has $|CBA| + |BCA|$ members, so the full *action space* is given by $-|CBA| \leq h \leq |BCA|$. By Remark (0.1), all B-supporters may join a Pushover action by first increasing $|BCA|$. In notation from (0.2), the ranking by top-rank changes when h passes α , β , and γ , where

$$(2.1) \quad \begin{aligned} \alpha &= |A| - |C| = \delta_C - \delta_A; \\ \gamma &= |B| - |A| = \delta_A - \delta_B; \\ \beta &= (|B| - |C|)/2 = (\alpha + \gamma)/2 = (\delta_C - \delta_B)/2 \end{aligned}$$

$|A| < N/3$, see condition (0.10), is necessary for elimination of A without tiebreak. Then

$$(2.2) \quad \alpha < \beta < \gamma$$

When an increasing h passes α ; β ; γ , then, respectively, in terms of top-ranks,

$$C \text{ passes } A; \quad C \text{ passes } B; \quad B \text{ passes } A.$$

In the constellation diagram, the *columns* are switched for A and C; B and C; A and B.

The *pairwise comparison* changes once with h . This happens in $\{B, C\}$, but obviously not in $\{A, B\}$ or $\{A, C\}$. After h ballot switches from BCA to CBA, the vote vector (0.5) is

$$(2.3) \quad \begin{aligned} &(|ABC|, |ACB|, |CAB|, |CBA| + h, |BCA| - h, |BAC|) \\ &= (|A| - x, x, |CAB|, |CBA| + h, y - h, |B| - y) \end{aligned}$$

B and C are equal in pairwise comparison when

$$(2.4) \quad \begin{aligned} (y - h) + (|B| - y) + (|A| - x) &= x + |CAB| + (|CBA| + h), \\ 2\delta_C = N - 2|C| &= |A| + |B| - |C| = 2(x + h) \end{aligned}$$

Thus, the rows of the constellation diagram are permuted once, i.e. at $h = \xi$, where

$$(2.5) \quad \xi + x = N/2 - |C| = \delta_c$$

By (1.4) and Figure (1.1), $0 \leq x \leq \delta_c$ in constellation *iii* and *iii_(cyclic)*; thus, $0 \leq \xi \leq \delta_c$ and

$$(2.6) \quad \xi = \alpha \text{ for } x = \delta_A ; \quad \xi = \beta \text{ for } x = (\delta_B + \delta_C)/2 ; \quad \xi = \gamma \text{ for } x = \delta_C - \delta_A + \delta_B$$

When α, β, γ , and ξ are distinct, $x = |ACB|$ defines one of four different sequences:

- (2.7)
- sequence 1: $\xi < \alpha < \beta < \gamma$ for $\delta_A < x \leq \delta_C$
 - sequence 2: $\alpha < \xi < \beta < \gamma$ for $(\delta_B + \delta_C)/2 < x < \delta_A$
 - sequence 3: $\alpha < \beta < \xi < \gamma$ for $\delta_C - \delta_A + \delta_B < x < (\delta_B + \delta_C)/2$
 - sequence 4: $\alpha < \beta < \gamma < \xi$ for $0 \leq x < \delta_C - \delta_A + \delta_B$

Cycles are very rare, so in the vast majority of cases, a Pushover action must start at $h = 0$ in constellation *iii*. All four sequences end up in constellation *iii*, with new rôles for B and C. In Figure (2.1), each sequence follows a unique path of arrows from the upper left to the lower right constellation diagram. Figure (2.2) shows the sequences after a start in *iii_(cyclic)*.

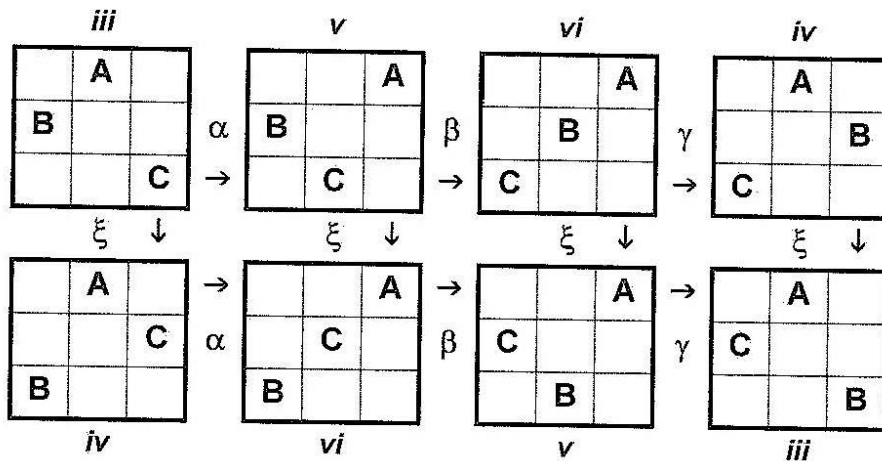


FIGURE (2.1) *Action space: Pushover from constellation iii, $N/4 < |A| < N/3$*

Starting at $h = 0$ in the upper left constellation *iii*, sequences 2, 3, and 4 lead to *v* or *vi*, and B wins by Pushover if the ballot changes stop in time, before the pitfalls at γ and ξ : Passing γ makes B Plurality loser, and passing ξ makes B Condorcet loser. Passing α and ξ but not γ makes C win by Compromise (0.3). At the other end of the action space, by symmetry, C-supporters may follow sequences 3, 2, and 1 in reverse, reduce h , move from constellation *iii* to *v* or *vi*, and win if they stop in time, before the pitfalls at $h = \alpha$ and $h = \xi$.

Frome 2009 had sequence 2, $x = 3801$ and $(\alpha, \xi, \beta, \gamma) = (32, 321.5, 1342.5, 2653)$.

The symmetry of Figure (2.1) shows that in the end constellation *iii*, C and B have changed rôles: C has become Plurality winner, and reversing all arrows illustrate the possibilities for C to win over A by Pushover, moving voters from CBA to BCA and reducing h .

Since Condorcet-cycles are rare, a start in constellation *iii* has more practical interest than a start in *iii_(cyclic)*. The transition points α , β , γ , and ξ in Figure (2.2), with start in *iii_(cyclic)*, are as in Figure (2.1), with start in *iii*. However, with start in *iii_(cyclic)*, all four sequences in (2.4) lead to constellation *i*, and there is no complete reversal symmetry like the one in Figure (2.1).

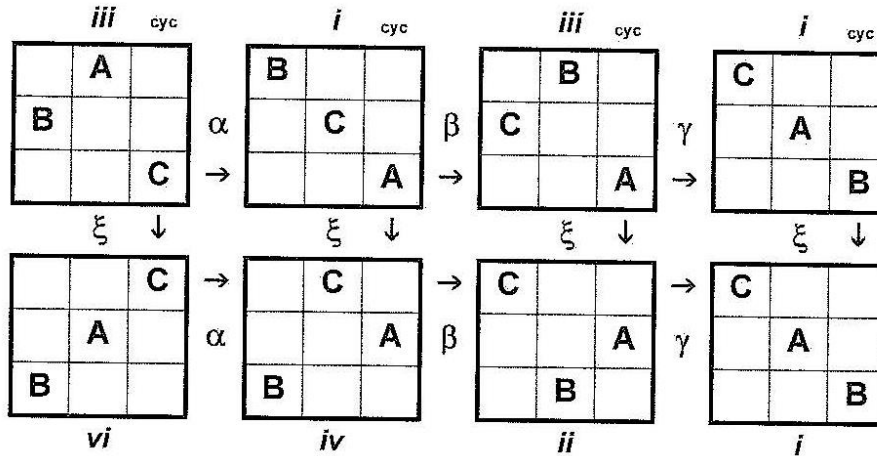


FIGURE (2.2) Action space: Pushover from constellation *iii_(cyclic)*, $N/4 < |A| < N/3$

The figure shows all possibilities for supporters of B to win by Pushover starting from a Condorcet cycle. The points of change in (2.1) and (2.5) depend on $|A|$, $|B|$, $|C|$, and $x = |ACB|$; they are the same for a start in *iii*, see Figure (2.1), and a start in *iii_(cyclic)*.

The IRV-winner A and Pushover attempts from B-supporters

The supporters of A determine x and therefore have the power to prevent Pushover: By making $\xi \leq \alpha$, they avoid sequences 2, 3, 4 in (2.1), and Figures (2.1) and (2.2) show that increasing h -values then do not lead to any constellation won by B. By (2.5–7), $\xi \leq \alpha$, means that

$$(2.8) \quad N/2 - |A| = \delta_A \leq x = |ACB|$$

Obviously $|ACB| \leq |A|$, so this is possible since $N/4 \leq |A|$.

With $|A| < N/3$, (2.8) shows that a necessary condition for sequence 1 is

$$(2.9) \quad N/6 < x$$

If $N/3 < |A|$, then A is not vulnerable to Pushover. Otherwise, Figures (2.1) and (2.2) show that by (2.5), A-supporters may choose x and ξ so that participants in a Pushover attempt for B [C] drops into a pitfall, by making $\xi \leq \alpha$ [$\gamma \leq \xi$]. This is a “prophylactic remedy” against a Pushover attempt from the B-supporters. Figures (2.1) and (2.2) show that if, say, BCA-voters then attempt Pushover and let h pass α , h must pass ξ too, and C wins.

However, it is not a practical remedy. Raising x and lowering ξ requires A-supporters moving from ABC to ACB in order to move the pitfall location, and thus prophylactically prevent Pushover. However, this prophylactic remedy is not good for ABC-voters. The reason is that if supporters of B really make a Pushover attempt, then the ABC-voters are better off with $\alpha < \xi$, because $\alpha < h$ does not keep A as winner anyway, and $\alpha < h < \xi$ lets B win instead of C.

Problems for Pushover actionists: Risk and motivation

The structure of the action space, shown in Figures (2.1) and (2.2), enables us to follow up the introductory discussion of Pushover based on the table in (0.6). Pushover with h actionists switching from BCA to CBA, works if and only if h passes α but none of the pitfalls at γ or ξ :

$$(2.10) \quad h \in (\alpha, \min(\gamma, \xi)) = (|A| - |C|, \min(|B| - |A|, N/2 - |C| - x))$$

An h outside the interval in (2.10) means a failed Pushover attempt. For several reasons it will be problematic for the B-supporters in constellation *iii* to plan and conduct a Pushover action:

- Predictions of $|A|$, $|B|$, $|C|$, and x will often not be reliable enough to give a useful estimate of the interval in (2.10) that h must belong to.
- It is unlikely that organizers of a Pushover action can reliably conduct BCA-voters' switches to CBA such that h hits the stochastic interval of (2.10).

With $\alpha < h < \gamma$, either B wins by Pushover, or C wins by Compromise. The voters in BCA and CBA are all better off if they avoid election of A. However, usually A is the second choice for a majority of B-supporters in constellation *iii*; see Figure (1.1). A Pushover action, with switches from BCA to CBA will hardly be acceptable for BAC-voters, and may damage the unity in B's party. Some BCA-voters, with a strong anti-A attitude may want to join a combined Pushover/Compromise action with a CBA-ballot, but most BCA-voters are likely to share many opinions with the BAC-voters, and their subsidiary preference for C to A is likely to be weak.

- Therefore, the motivation for joining a Pushover action is also likely to be weak.
- BCA-voters may see the tally method as fair when it picks Condorcet-winner A.

Tricks and traps

There are two kinds of monotonicity failure. They allow Pushover or its reverse, and the terms "*trick*" and "*trap*" emphasize that they are closely related. Figures (2.1) and (2.2) show how Pushover actionists perform a "*non-monotonicity trick*": Voters move from BCA to CBA, down-

rank B, up-rank C and eliminate A in sequences 2, 3, 4. They cause B to be elected instead of A, provided they stop before the pitfall at $h = \min(\gamma, \xi)$.

Arrows in Figure (2.1) show that Pushover, with start in constellation iii , leads to constellation v (sequences 2, 3, and 4) or to vi by passing through v (sequences 3 and 4). Figure (2.2) shows that Pushover with start in $iii_{(cyclic)}$ leads to $i_{(cyclic)}$ or $iii_{(cyclic)}$, but the possibility is rare. The point of the Pushover trick is that the anti-A voters distribute their top-ranks on B and C in such a balanced way that IRV-winner A (before the trick) loses on top-ranks and becomes eliminated.

A reversal of these moves leads into a “*non-monotonicity trap*”: Voters who originally rank CBA, but change their mind and sincerely switch to BCA, quite likely intending to strengthen IRV-winner B in order to avoid A, are trapped. They move the election from constellation v or vi [$i_{(cyclic)}$ or $iii_{(cyclic)}$] and back to iii [$iii_{(cyclic)}$], and thereby cause A to snatch the IRV-victory from B. This is bad for all anti-A voters, both the BCA-group and the CBA-group.

Thus, tricks and traps belong together in the same action space, which is on a straight line crossing the xy-plane in Figure (1.1) when $h = 0$. By Theorem (0.1), tricks are only possible from certain “*trick positions*” in constellations iii and $iii_{(cyclic)}$. The *trick effect* leads to a trap position; the *trap effect* leads to a trick position. Trap positions exist only in constellations that may be reached by a trick: v or vi in Figure (2.1) and $i_{(cyclic)}$ or $iii_{(cyclic)}$ in Figure (2.2); v and $i_{(cyclic)}$ belong to family \mathcal{Q} .

REMARK (2.1) An obvious complication for Pushover comes with a larger number of candidates. Will B-supporters in a 6-candidate election anticipate eliminations of L, M, and N and up-rank C from BLMNCA to CBLMNA in order to join a Pushover action against A? Will they become aware of an opportunity? In Frome 2009, B, C, A, L, M, N, got respectively, 7576, 5041, 4557, 1267, 734, 134 top-ranks in the first tally round. Even a rough pre-election opinion poll should detect the big gap from A to L and establish that up-ranking C would not harm L, M, or N. However, some sincere voters intend a ballot ranking BLMNCA as expression of their true opinion. In the 6-candidate election, B-supporters have reason to be even more unwilling than in a 3-candidate election to join a Pushover action.

Moreover, that the independent A besides getting so many top-ranks, also should receive as much as $5562 - 4557 = 1005$ of the 2135 votes transferred from $\{L, M, N\}$, and thereby become a serious challenger to the major parties, was clearly a surprise.

Natural sortition

One source of randomness in α , β , γ , and ξ is the unpredictability of changes in voter rankings, perhaps influenced by the last days of campaigning. Another source is “*natural sortition*”.

Ordinary sortition is a planned lottery in an appointment process. There are good reasons to use lottery in some situations, e.g. selection of jury members in many countries. The idea is old, but Socrates criticized the practice in Athens to appoint Archons entirely by lottery.³ Could he have accepted a sortition that only influences the composition of an electoral college? Venice, a city republic until 1797, had an elaborate sortition process to establish an electoral college that finally elected the Doge (Finlay 1980).

In most countries, the fraction of the electorate that will participate in an election is quite unpredictable. Everyday random events have effects that cause non-participation. Even bad weather in parts of a country may reduce participation and change the relative strengths of the parties. The final campaigning efforts have an effect, but many other last-day effects are a “*natural sortition*” of the electorate that brings even more randomness to the vote vector.

In Australia voting is compulsory, and non-participation leads to a fine, but voters who are not motivated to vote according to their own individual assessments, values and interests may produce ballot rankings with some other kind of randomness.

Occasionally, a 3-candidate vote vector lands in “A-territory”, close to the territory of B or C, or even both. Natural sortition and other random last day events have consequences for all six components. Together they may result in a border crossing, in either direction. This can hardly be detected and, even more unlikely, be identified as cause of a non-monotonicity event.

EXAMPLE (2.1) Frome 2009 was a trick situation, and could theoretically have been caused by a trap effect where a group, of at least 31 voters, had moved from CBA to BCA; see Example (0.1). Moves from $|CAB|$ to $|ACB|$, changing $|A| < |C|$ to $|A| > |C|$, are more likely than a non-monotonicity event. Since natural sortition is unavoidable and perturbs all six voting categories, it remains unknown if such moves took place and had the effect they theoretically may have.

³ “But, said his accuser, he taught his companions to despise the established laws by insisting on the folly of appointing public officials by lot, when none would choose a pilot or builder or flautist by lot, nor any other craftsman for work in which mistakes are far less disastrous than mistakes in statecraft. Such sayings, he argued, led the young to despise the established constitution and made them violent.” (Xenophon ca 370 BC).

Non-monotonicity events; conditions and requirements

The non-monotonicity trick (Pushover) The vote vector in (2.3) shows when the move of h voters from BCA to CBA causes Pushover. First, C must pass A in top-ranks:

$$\begin{aligned} (|A| - x) + x &< |CAB| + (|CBA| + h) \\ |A| - |C| &< h \end{aligned}$$

Next, B must stay ahead of C in pairwise comparison:

$$\begin{aligned} x + |CAB| + (|CBA| + h) &< (y - h) + (|B| - y) + (|A| - x) \\ h &< N/2 - |C| - x \end{aligned}$$

Thus, since h and x are integers, (2.10) requires h actionists, where

$$(2.11) \quad |A| - |C| + 1 \leq h < \min(|B| - |A|, N/2 - |C| - x)$$

$$(2.12) \quad 0 \leq x < N/2 - |A| - 1$$

Figures (2.1) and (2.2) show in detail how h voters moving from BCA to CBA make B snatch the IRV-victory from A with Pushover. Pushover changes the rôles of the candidates. Row II of Figure (2.3) shows an unspecified election with candidates A, B, and C in their standard rôles of (0.4). Row I shows their new rôles after a Pushover action. They are the same whether the Pushover trick leads to constellation v , vi , $i_{(cyclic)}$ or $iii_{(cyclic)}$.

	C	B	A	
I	runner-up	IRV-winner	eliminated	
	↑	↑	↑	h actionists switch from BCA to CBA
II	eliminated	runner-up	IRV-winner	

FIGURE (2.3) *Candidate rôles changed by the non-monotonicity trick*

At the start of a non-monotonicity trick, i.e. a Pushover action, candidates C, B, A have their standard rôles in row II. By Remark (0.1), the only strategy that can make B, the runner-up, snatch IRV-victory from A includes Pushover: a suitable number h of actionists switch from BCA to CBA and give the candidates the new rôles shown in row I.

The non-monotonicity trap Where are the points (x, y) in Figure (1.1) that make a trap? First, consider how a trap changes the rôles. Row II of Figure (2.4) shows an unspecified election with candidates A, B, C in their standard rôles of (0.4). Row III shows their new rôles after a trap effect. Figure (2.3) forces row III because the reverse action, from row III to row II, is a trick effect (Pushover). Thus, the rôle changes are the same as those in Figure (2.3). Equivalently, Figure (2.4) states that a non-monotonic trap works against the arrows and in effect changes the rôles from those of row II to those of row III. Changing IRV-winner from A to C, the figure shows how the trap requires a suitable number g of voters to switch their ranking from BAC to ABC:

II	C eliminated	B runner-up	A IRV-winner	
	↑	↑	↑	g voters switch (upwards) from ABC to BAC
III	IRV-winner	eliminated	runner-up	

FIGURE (2.4) Candidate rôles changed by the non-monotonicity trap

At the end of a Pushover action, candidates C, B, A have their standard rôles in row II. The only way that the rôles of row II can be due to Pushover performed by g voters is that the Pushover action started with the non-standard rôles in row III. The rôle changes are the same as those in Figure (2.3). The trap requires g voters to move against the arrows.

According to Figure (2.4), a non-monotonicity trap effect changes the vote vector (2.3) to

$$\begin{aligned} & (|ABC| + g, |ACB|, |CAB|, |CBA|, |BCA|, |BAC| - g) \\ = & (|A| - x + g, x, |CAB|, |CBA|, y, |B| - y - g) \end{aligned}$$

Clearly $g \leq |B| - y$. The trap requires that B is eliminated, so that C meets A in the final. This is obtained without tiebreak if C gets more top-ranks than B:

$$\begin{aligned} & y + (|B| - y - g) < |CAB| + |CBA|, \text{ i.e. } |B| - |C| < g, \text{ and so} \\ (2.13) \quad & 1 + |B| - |C| \leq g \leq |B| - y \end{aligned}$$

The trap also requires C to win the final pairwise comparison with A. Without tiebreaks this means

$$\begin{aligned} & (|B| - y - g) + (|A| - x + g) + x < |CAB| + |CBA| + y; \text{ i.e. } N/2 - |C| < y \\ \text{From (2.13) follows } & 1 - |C| \leq -y. \text{ Thus, a non-monotonicity trap requires} \end{aligned}$$

$$(2.14) \quad N/2 - |C| + 1 \leq y \leq |C| - 1.$$

This implies that

$$(2.15) \quad \begin{aligned} & \text{a necessary condition for a non-monotonicity trap} \\ & \text{in a 3-candidate IRV election is that } N/4 < |C| \end{aligned}$$

In Frome 2009, the condition (2.12) becomes $x \leq 4191$ and is satisfied; (2.14) and (2.13) become

$$(2.16) \quad 4123 \leq y \leq 5531$$

$$(2.17) \quad 2684 \leq g \leq 8215 - y$$

Trick positions that also are trap positions Each (x, y) in Figure (1.1) defines an election where C is eliminated and A wins. Inequalities (2.12) and (2.14) together define a sub-rectangle in the NW where (x, y) defines a cyclic election that allows both a trick and a trap:

EXAMPLE (2.2) The point $(x, y) = (3801, 4200)$ is on the stapled line in Figure (1.1), in the NW, quite close to the 0.001-zone, see Figure (1.1). The pictogram is in Figure (2.5).

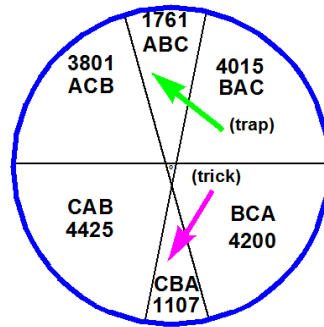


FIGURE (2.5) A trick (Pushover) and a trap are both possible ⁴:

With $(x, y) = (3801, 4200)$, on the stapled line in Figure (1.1), the election is cyclic; thus, triangle T covers the circle center. T also covers a fraction 0.001213 of the circle area.

The election is compatible with the standard tally of Frome 2009: C gets eliminated and A becomes IRV-winner.

The constellation is $iii_{(cyclic)}$; the vote vector (1.3) is (1761, 3801, 4425, 1107, 4200, 4015).

Conditions (2.12) for a trick and (2.14) for a trap are satisfied.

If h voters switch from BCA to CBA, $31 \leq h \leq 321$, A is eliminated, and B wins.

If g voters switch from BAC to ABC, $2684 \leq g \leq 4015$, B is eliminated, and C wins.

With a trap position in family \mathcal{B} , the trap needs usually, like here, a relatively large number g of “victims”. This is because the line that connects the trap position with a trick position must pass through a constellation in family \mathcal{A} ; see Figures (2.1) and (2.2).

(2.18) Monotonicity failures – a summary As theoretic possibilities, the two kinds of failure occur quite often according to (2.12) and (2.14), i.e. tricks and traps. The trick (Pushover) is a voting strategy that an action group, conceivably, might attempt to use. The structure of the action space, Figure (2.1) and Figure (2.2), shows how a lack of reliable predictions of the standard tally (0.7) and of the unknown $x = |ACB|$ makes a Pushover action a risky enterprise. It requires a number h of actionists that satisfies (2.10). If reliable estimates makes the interval for h so large that a conducted action seems worthwhile, there is still a question of motivation. Actionists reduce the BCA-category and increase the CBA-category, targeting Condorcet winner A in the non-cyclic cases, but must state their ranking in $\{B, C\}$ insincerely. *A is a central candidate and widely accepted as winner. Voters with sincere BCA-ranking will not be enthusiastic actionists.*

Last day random events always influence voter participation and change the vote vector. They decide when the vote vector lands in “A-territory”, close to the territory of B or of C, or even both. Moves between BCA and CBA are included, but so are, say, moves between ACB and CAB, with a more obvious impact on C v A. When tally data reveal such closeness, a claim that a

⁴ As indicated in Figure (1.1), cycles are very rare in political elections, but are useful to illustrate, as in Figure (2.5), both non-monotonic events, tricks and traps, with the same choice of (x, y) , i.e. in the same pictogram.

monotonicity failure really caused such moves to help A to win, may be true, but not a likely one in the case of Frome 2009. As usual in elections won with small margins, last day ranking changes may have changed the outcome. The very possibility that a trap effect occurred is upsetting, but it should not derail public attention away from other arguments, both for and against IRV.

However, one must expect, in very rare cases, identification of some “victims” of a trap effect:

(2.19) Caught in the trap – reaction of the “victims” When it was reported that the election in Frome 2009 landed on the trick side close to a border, Example (2.1), many saw it as revealing a serious flaw of IRV. Imagine that h voters, $31 \leq h \leq 321$, had moved from CBA to BCA because the last days of campaigning made them *agree* that B was the best candidate. If such agreement is a proven fact, the trap effect *appears* as real, not just as a case of creative accounting.

The trap is just the Pushover trick in reverse, but often preferred as argument in attacks on IRV. It appears as a more serious flaw than the trick, because voters who seriously up-rank B, not only destroy B’s victory, but even cause their bottom-ranked A to become the new IRV-winner instead of B. This trap narrative has impact on public opinion, but courts want evidence. The Minnesota Supreme Court 2009 stated that the appellants who attacked IRV, gave “*no evidence, much less proof, of the extent to which it [the trap effect] might occur*”. See Section 6.

Trap “victims” will perhaps not be amenable to explanations based on natural sortition or to consolation by the fact that it was a Condorcet winner (A) they happened to save from elimination. However, in other words, they just contributed to create a ballot distribution where BCA-voters missed an opportunity to win with Pushover.

When A is Condorcet winner and B Plurality winner, $|CAB| > |CBA|$; in Frome 2009 almost 80% of the C-supporters voted CAB. In most of constellation *iii*, a large majority of B-supporters vote BAC. Voters who move from CBA to BCA, into the trap, do not fit a “victim rôle” more than other BCA-voters do; see also Figure (1.3). The dual emphasis on primary support and general acceptance is the basic property of IRV. In non-cyclic 3-candidate cases, non-monotonicity events occur on the border of constellations *iii* (Condorcet-winner wins) and *v* (Plurality winner wins).

In the elimination process of ITV only primary support counts. In the context of monotonicity failure, elimination of the Condorcet-winner in constellations *v* or *vi* should be seen as, at most, a minor nuisance. However, the third pictogram of Figure (1.2) shows the result of a No-Show accident, and is a reason to reconsider the IRV rules.

3 The No-Show accident

A No-Show accident occurs when a group of new voters with the same ranking causes a new result that is worse than the original result according to their own ranking. In a 3-candidate IRV election, this is only possible in constellations *iii* and *iii_(cyclic)* and requires a sufficient number k of new CAB voters; see Theorem (0.2). The first known example (constructed) of a No-Show accident, and equivalently, its reverse, is in Fishburn and Brams (1983).

The accident, when and how? New voters change the vote vector from (1.3) to:

$$\begin{aligned} & (|ABC|, |ACB|, |CAB| + k, |CBA|, |BCA|, |BAC|) \\ & = (|A| - x, x, |CAB| + k, |CBA|, y, |B| - y). \end{aligned}$$

One condition is that C passes A in top-ranks. This happens without tiebreak if

$$\begin{aligned} & (|A| - x) + x < (|CAB| + k) + |CBA| \\ (3.1) \quad & |A| - |C| < k \end{aligned}$$

Another condition is that B still wins over C in pairwise comparison. This happens without tiebreak if

$$\begin{aligned} & x + (|CAB| + k) + |CBA| < y + (|B| - y) + (|A| - x) \\ (3.2) \quad & k < N - 2|C| - 2x \end{aligned}$$

with $N = |A| + |B| + |C|$. Since the numbers are integers, (3.1) and (3.2) are equivalent to

$$(3.3) \quad |A| - |C| + 1 \leq k \leq N - 2|C| - 2x - 1$$

The maximal value for x follows from (3.3):

$$(3.4) \quad x \leq |B|/2 - 1$$

No-Show and change of rôles In a 3-candidate IRV-election with N voters, a No-Show accident may occur if a suitable number k of new voters join one of the six voting categories. Row II in Figure (3.1) describes the start, with the candidates in the rôles from (0.4). According to Theorem (0.2), the start in row II must be in constellation *iii* or *iii_(cyclic)* and the No-Show accident occurs when suitably many join the election and vote CAB, i.e.

$$(3.5) \quad \text{1) eliminated; 2) IRV-winner; 3) runner-up}$$

Row I in Figure (3.1) shows how the rôles of the candidates change for C, B, and A, i.e. from

$$(3.6) \quad \text{eliminated to runner-up; runner-up to IRV-winner; IRV-winner to eliminated.}$$

	Voters	C	B	A
I	N + k	runner-up	IRV-winner	eliminated
		↑	↑	↑
II	N	eliminated	runner-up	IRV-winner

FIGURE (3.1) Candidate rôles changed by a No-Show accident

The candidates have labels C, B, and A, according to their rôles in row II. The k new CAB-voters cause the No-Show accident and the candidates get the new rôles in row I.

No-Show accident reversed The dual question is: When may an election result itself be explained as the result of a No-Show accident caused by q new voters? In Figure (3.2), row II is now an unspecified preference distribution with candidates labelled according to (0.4). With start in row III, the rôles change according to (3.6). This forces row III of Figure (3.2). The q new voters must have voted as in (3.5). According to row III, this means BCA.

	Voters	C	B	A
II	N	eliminated	runner-up	IRV-winner
		↑	↑	↑
III	N - q	IRV-winner	eliminated	runner-up

FIGURE (3.2) Candidate rôles changed by reversal of a No-Show accident

The candidates have labels C, B, and A, according to their rôles in row II. Row III describes what rôles A, B, C must have had at the start. They must lead to the same changes as described in Figure (3.1). The reversed accident reverses the arrows and reduces |BCA|.

The original vote vector, before the q BCA voters entered in row III, was

$$\begin{aligned} & (|ABC|, |ACB|, |CAB|, |CBA|, |BCA| - q, |BAC|) \\ & = (|A| - x, x, |CAB|, |CBA|, y - q, |B| - y). \end{aligned}$$

In row III, B loses to C in top-ranks. This happens without tiebreaks if

$$(3.7) \quad \begin{aligned} (y - q) + (|B| - y) &< |CAB| + |CBA| \\ |B| - |C| &< q \end{aligned}$$

A must lose to C in pairwise comparison. This happens without tiebreak if

$$(3.8) \quad \begin{aligned} (|B| - y) + (|A| - x) + x &< |CAB| + |CBA| + (y - q) \\ q &< 2|C| - N + 2y \end{aligned}$$

Since the numbers are integers, (3.7) and (3.8) imply

$$(3.9) \quad |B| - |C| + 1 \leq q \leq 2|C| - N + 2y - 1$$

Clearly $y \leq |B|$; another condition follows from (3.9):

$$(3.10) \quad |B| - |C| + |A|/2 + 1 \leq y \leq |B|$$

From (3.10) follows a necessary condition for an election to be the result of a No-Show accident:

$$(3.11) \quad |A|/2 \leq |C| - 1$$

In case (3.11) is satisfied, choose y to satisfy (3.10); if q then satisfies (3.9) and q BCA-votes *are cancelled*, the election drops from row II to row III in Figure (3.2). C replaces A as IRV-winner.

EXAMPLE (3.1) Assuming that $y = 5600$ in Frome 2009, then Figure (1.1) shows that the election is in constellation $iii_{(cyclic)}$ with (x, y) on the stapled line. Since $|A| = 5562$ and $|C| = |5532|$, condition (3.11) is clearly satisfied. By (3.10) and (3.9),

$$5395 \leq y \leq 8215, \text{ and} \\ 2684 \leq q \leq 2954$$

If q BCA-voters (are allowed to) cancel their ballots “ex-post”, B will be eliminated, but C kept as IRV-winner (with enough subsidiary votes from the remaining BCA-voters); if not, they get A.

With $x = 3801$, i.e. on the stapled line in Figure (1.1), also (3.4) is satisfied, and by (3.3) states

$$31 \leq k \leq 642$$

In Frome 2009, with counterfactual $(x, y) = (3801, 5600)$, k more CAB-voters would have spoiled the election for C and caused B to win, while q BCA-voters less would have meant victory for C.

Only some of the cyclic elections, i.e. in constellation $iii_{(cyclic)}$ can allow both a No-Show accident (condition on x) and itself be caused by a No-Show accident (condition on y).

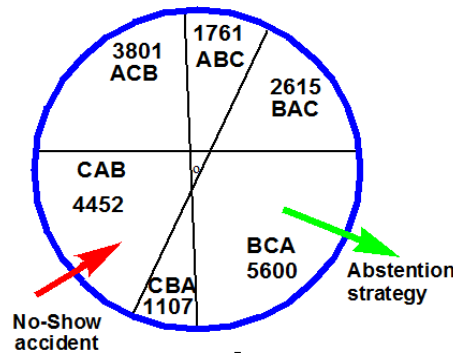


FIGURE (3.3) No-Show accident and abstention strategy are both possible

With $(x, y) = (3801, 5600)$, on the stapled line in Figure (1.1), the election is cyclic; thus, triangle T covers the circle center. T also covers a fraction 0.005049 of the circle area. The vote vector (1.3) is $(1761, 3801, 4425, 1107, 5600, 2615)$, and the constellation is $iii_{(cyclic)}$

If k voters *join* CAB, $31 \leq k \leq 642$, A is eliminated and B wins instead of A.

If q voters *leave* BCA, $2684 \leq q \leq 2954$, B is eliminated and C wins instead of A.

Realistic vote vectors with Perfect Pie-sharing are obtained by following straight lines, $x = 3801$ and $y = 5600$, until intersection with the middle curve in Figure (1.1).

EXAMPLE (3.2) Burlington's mayoral election 2009

The election in Burlington, Vermont, 2009, was in family \mathcal{B} . In the second last round, respectively

M (Montroll, democrat); W (Wright, republican); K (Kiss, progressive),

had standard rôles $C = M$; $B = W$; $A = K$. The election was in constellation vi , with vote vector

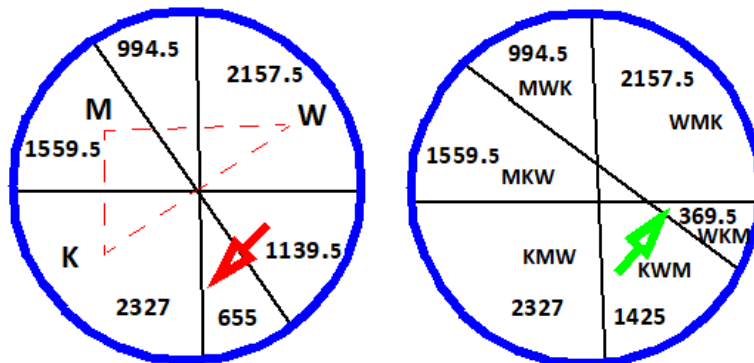
$$(3.12) \quad (|KWM|, |KMW|, |MKW|, |MWK|, |WMK|, |WKM|) \\ (655, 2327, 1559.5, 994.5, 2157.5, 1139.5)^5$$

It has repeatedly been claimed that the victory for $A = K$ was due to a No-Show accident, e.g. (Gierzynski 2009, 2011). Only the necessary condition (3.11) is satisfied. By (3.10), the claim is false, because y is too small by $2235.5 - 2157.5 = 78$ votes:

$$|B| = |W| = 3297, \quad |C| = |M| = 2554, \quad |A| = |K| = 2982, \quad y = |BCA| = 2157.5, \\ y < |B| - |C| + |A|/2 + 1 = 2235.5$$

However, the condition (2.14) for a non-monotonicity trap was satisfied in Burlington:

$$N/2 - |C| + 1 = 1863.5 \leq y \leq 2553 = |C| - 1$$



Before the trap effect; const. vi After the trap effect; const. iii

FIGURE (3.4) Pictogram of the Burlington vote vector (3.12) and trap effect

T covers a fraction 0.000098 of the circle area. The arrow in the first pictogram (the real election) indicates how g voters are trapped if they move from WKM to KWM, $744 \leq g \leq 1139$. The right hand pictogram shows the result for $g = 770$; T then covers a fraction 0.009807 of the circle, and the arrow shows the reverse effect, i.e. a trick effect (Pushover). Thus, the real vote vector (3.12) could have been the result of Pushover, but only from a very unrealistic start position.

⁵ Vote vector data appear with small and insignificant differences in different places. Here they are from a table in RangeVoting.org. Incomplete ballot rankings were accepted, and a ballot ranking that only contained X is in (3.12) counted as half a vote for XYZ and half a vote for XZY. Thus, by symmetrizing, 1289 ballots that supported W without showing a preference in {K, M}, contribute with 644.5 to both |WMK| and |WKM| in the vote vector (3.12). All ballots are then included in the picture of the political landscape, and without changing the tally result.

According to (2.10), g voters might then have moved from WKM to KWM, i.e. from BAC to ABC, walked together into the trap and activated it, thus causing the elimination of $B = W$ and a victory for $C = M$ in pairwise comparison with $A = K$,

$$(3.13) \quad 1 + |B| - |C| = 744 \leq g \leq 1139.5 = |B| - y$$

Thus, the Burlington vote vector is a trap position. If activated, a trap is a practical consequence of non-monotonicity, but it is hard to see what reason so many voters should have for a move from WKM to KWM; see Remark (2.2).

Still, in a “case study” of Burlington 2009, Ornstein and Norman (2014) conclude: *“The Burlington election offers a compelling illustration of monotonicity failure’s practical importance, but so detailed IRV ballot data are rare.”*

Burlington 2009 is compelling as an illustration of a theoretic possibility. So are many 3-candidate IRV-elections with data only from a standard IRV-tally (0.7). Even without any published details on (x, y) , a figure similar to Figure (1.1) and inequalities (3.4) and (3.10) always show the theoretical possibilities. Moreover, although x and y usually are unknowns in Australian IRV-elections, opinion polls and “how-to-vote” cards will often give a good idea on (x, y) , but a judgment of IRV should build on evidence on voter behavior as well as on theoretic possibilities.

By (3.13), 744 moves were necessary to produce a non-monotonic effect in Burlington 2009, and utterly unrealistic. The distance to the non-monotonic change is a practically important barrier against an accidental trap effect in 3-candidate elections in family \mathfrak{B} .

In a referendum 2010, Burlington repealed IRV, and went back to the common 2-day election, which people in many countries know e.g. from French presidential elections.

EXAMPLE (3.3) The 2-day election, a.k.a. TRS (Two Round System), allows a strategy akin to Pushover, but easier to see in the simpler setting of TRS: In France 2002, LePen (far right) squeezed out Jospin (center-left) on day one; on day two, incumbent Chirac (center-right) won a landslide victory with massive support from the left. How many alert Chirac-supporters voted LePen on day one in order to avoid the strong challenger Jospin on day two? ⁶

⁶ LePen was particularly suited as a tool for such strategic voting, because it was clear that he would receive only an insignificant number of new voters on day two. First, however, he caused elimination of all the small parties of the left. With IRV, vote transfers between the left wing parties would have made at least one of them strong enough to eliminate LePen and give the incumbent Chirac a strong opponent in a final IRV tally round.

(3.14) No-Show effects – a summary The third pictogram of Figure (1.2), constellation vi , is compatible with the standard tally of Frome 2009. It could have been caused by a No-Show accident, taking IRV-victory from C to A.⁷ Equivalently, if a suitable number q of voters in category BCA had abstained, or could cancel their ballots, then C could have won, since $y = |BCA| = 7062$ satisfies condition (3.10):

$$(3.15) \quad (|ABC|, |ACB|, |CAB|, |CBA|, |BCA| - q, |BAC|) \\ = (262, 5300, 4425, 1107, 7062 - q, 1153).$$

Here q must satisfy (3.9), i.e. $2684 \leq q \leq 5878$; after cancellation of q BCA-ballots, B is eliminated and C will defeat A with support from remaining BCA-voters. For a 3-candidate election in family \mathfrak{B} , the abstention strategy usually requires many participants, because the q -step path from constellation vi to constellation iii must pass another constellation.

In some realistic vote distributions in constellations vi or v , the voter group BCA would gain if a suitable number of their ballots were cancelled. When this happens, the accounting is real, and the case is fundamentally different from a claim that a monotonicity failure has occurred and had a practical consequence (Example (0.1)).

It seems unreasonable that q BCA-voters who enter in (3.15) prove insufficient to help C or B, just because of A's tiny 30-vote lead over C in top-ranks.⁸

Moreover, the election in (3.15), with $q = 0$, is close to Perfect-Pie-sharing according to the pictogram, and must be seen as realistic in a politically split society where most supporters of the Plurality winner B give strong subsidiary support to the Condorcet winner and "lesser evil" C on the opposite wing.

In a judgment of IRV, the fact that it always eliminates the Condorcet winner C in constellations v and vi , deserves attention even if criterion (3.10) is not satisfied. When it is, fairness to candidates A and C and fairness to supporters of Runner-up B are themes that get even more closely connected and urgent, but the present IRV-rules ignore $|BCA|$ and $|BAC|$.

⁷ It was the extra information, $x = 3801$, which showed that the real Frome election was in constellation iii or, possibly, $iii(cyclic)$, and certainly could have been the starting point for Pushover action and for a No-Show accident. Possibly, but unlikely, y satisfied (2.14), so that a trap also existed: Moves from BAC to ABC could then have let C win instead of A; see Figure (2.5). Possibly, but very unlikely, y satisfied (3.10); if so, then an abstention/cancellation strategy of BCA-voters could have let C win instead of A; see Figure (3.3).

⁸ The common dramatization of No-Show narratives with "new" voters entering is, hopefully, more didactic than misleading. Of course, a continued influx of new BCA-votes would eventually cause B to win. The problem is that $|BCA|$ is, temporarily, too small to elect B instead of A and too large to elect C instead of A: B prevents promotion of C to the final, but with q cancellations, q as in (3.9), the B-supporters could avoid A as IRV-winner.

4 Electoral basis, fairness and legitimacy

A ballot from voter v in any n -candidate preferential election may be reshaped as an $n \times n$ -matrix $\text{Con}(v)$ with 1 in position (X, Y) if voter v ranks X before Y , all other entries being 0. Two auxiliary column vectors are derived from $\text{Con}(v)$: In $\text{Plu}(v)$, entry $X = 1$ if X is top-ranked, while the other entries are 0. In $\text{Bor}(v)$, the entries give the Borda-points, $n-1, n-2, \dots, 0$.

EXAMPLE (4.1) If voter v is in the CAB-category in a 3-candidate election, then

$$\text{Plu}(v) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \text{Con}(v) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}; \quad \text{Bor}(v) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Summing over all v , the election in the second pictogram of Figure (1.2) gives these sums:

$$(4.1) \quad \text{Plu} = \begin{bmatrix} 5562 \\ 8215 \\ 5532 \end{bmatrix}; \quad \text{Con} = \begin{bmatrix} 0 & 9987 & 10040 \\ 9322 & 0 & 9477 \\ 9269 & 9832 & 0 \end{bmatrix}; \quad \text{Bor} = \begin{bmatrix} 20027 \\ 18799 \\ 19101 \end{bmatrix}$$

Condorcet and Borda

Condorcet methods ignore top-ranks: Plu in (4.1) shows that B is Plurality winner but this cannot be retrieved from Con , although $\text{Plu}(v)$ is trivially retrieved from $\text{Con}(v)$. Aggregation loses information. The Borda Count in Bor is an over-aggregation, since not just information on top-ranks, but also most information on pairwise comparisons is lost in Bor .

However, one connection survives the aggregation; we see from Bor that if there is a Condorcet winner, it has to be candidate A, who is the only candidate with above average Borda-score: In both $\{A, B\}$ and $\{A, C\}$, A is supported by $> 50\%$ of the voters, and Bor is the column sum of Con . Thus, (4.1) illustrates a fact well known:

$$(4.2) \quad \text{A Condorcet winner has above average Borda-score.}$$

The fact (4.2) allows special Condorcet methods defined through an elimination tally similar to IRV, but with Bor in the rôle of Plu : Calculate Bor and eliminate

$$(4.3) \quad \begin{array}{l} \text{either all candidates with average Borda-score or less (Nanson 1882)} \\ \text{or the candidate with the lowest Borda-score (Baldwin 1926).} \end{array}$$

Then calculate the new Bor , etc. By (4.2), a Condorcet winner survives all eliminations.

Eliminations based on top-ranks were proposed before Nanson (University of Melbourne) proposed his variation on the Condorcet theme (1882). STV (Single Transferable Vote) was for multi-seat elections, and the idea of one-by-one eliminations was well known to Nanson, who

also knew Ware’s suggestion to use it in single-seat elections. Nanson still preferred en-bloc eliminations, because they give a shorter tally process. When Baldwin suggested a more elaborate tally than Nanson, IRV, the single-seat version of STV, was also already in use in Australia (McLean 2002). Thus, Baldwin suggested a Condorcet method compatible with a familiar concept of one-by-one eliminations. This familiarity is not an explicit argument in his paper, but he argues that with his tally-technical guidelines, the extra time spent on one-by-one eliminations would be just a matter of minutes.

Plurality v Borda Count: Coarseness v Cruelty

A common kind of strategic voting is Compromise (0.3) in Plurality elections. The voter casts an *instrumental* vote for a tolerable candidate or “the lesser evil” among the candidates with a reasonable chance to win, instead of casting an *expressive* vote for the candidate that seems best suited for the office. Plurality elections expose voters to some coarseness. The decision can be painful. In both cases, the voter should expect some scolding. Either: *You waste your vote!* Or: *You commit favorite betrayal!* However, there is no insincerity behind instrumental voting; voters simply include electability among their ranking criteria, with high weight. With pre-election opinion polls, substantial application of Compromise shows up in real accounting:

EXAMPLE (4.2) In the UK general election of 2017, the opinion poll (by the market research company YouGov) and the election result one month later were as follows for “East of England”:

East of England 2017	Con	Lab	UKIP	LibDem	Green	Others
April 4th - May 5th	56%	19%	9%	12%	2%	1%
June 8th - election	54.6%	32.7%	2.5%	7.9%	1.9%	0.3%

The sample size was 1339. There is a clear decrease in two accounts for top-ranks: UKIP down by 6.5%, LibDem down by 4.1%, while Lab (Labour) went up by 13.7%. The unavoidable inaccuracies in opinion polls can only explain a part of the account changes. Moves from UKIP to Lab were common in England. In some constituencies, they sufficed to make the Lab candidate pass the Con (Conservative) candidate.

The Borda Count is a remedy worse than the Plurality ailment it should cure. In political elections, the urge to apply “Burying” (0.3) must be overwhelming: In a race with front-runners P and Q,

every voter who switches from PQRSTUUVW to PRSTUUVWQ makes up for seven others who rank Q first, P second. Attempting to trade part of their soul for power, they contribute insincere ranking in the pairs {Q, R}, {Q, S}, {Q, T}, {Q, U}, {Q, V}, and {Q, W}. Fear that a political enemy will use Burying to make Q defeat P motivates more voters to join the action. The switchers are under pressure from a “cruel” dilemma imposed by the Borda Count.

Mutually stimulated insincerity may hide a reality where a majority would prefer both P and Q to the declared winner, and be a step in the direction of anarchy.⁹

Balanced methods: Condorcet or IRV?

In elections less coarse, intimidating and polarizing than Plurality and less anarchistic and cruel than the Borda Count, the tally must process more ballot information than Plurality does, and not forget the top-ranks in the aggregation, as the Borda Count does. Condorcet methods and IRV follow different ways to pick a winner with a reasonable electoral basis.

Condorcet methods In all Condorcet methods the ballot rankings in all pairs influence the tally. However, if X wins a 3-candidate cycle, the supporters of X could have created the cycle by switching from XYZ to XZY, snatching the win from Condorcet winner Y, starting from a realistic vote vector (Perfect Pie-sharing). Different Condorcet methods give different opportunities for winning this way (Stensholt 2013).¹⁰

Some professional organizations actually use a Condorcet method. A Condorcet method is likely to function better in decisions based on judgment than in decisions based on interest.¹¹

⁹ On Borda’s suggestion, the French Academy of Sciences for a while used the already controversial Borda Count for its election of new members. The combined efforts of two other prominent members caused its repeal: P-S de Laplace and his younger associate N. Bonaparte (Szpiro 2010).

¹⁰ In Nanson’s and Baldwin’s methods, the elimination rule may also allow Pushover: In a cycle XYZX, ballot switches from XYZ to YXZ improves Y’s Borda-score and may cause elimination of Z instead of Y.

¹¹ In an election with fifteen candidates to the Wikimedia Foundation Board of Trustees 2008, a candidate quartet formed a cycle, i.e. two overlapping cyclic triples. Thus, 2 of $15 \cdot 14 \cdot 13 / 6 = 455$ triples were cyclic. The narrowest pairwise win in the five pairs involved (745 – 737) was in both cycles; a reversal in this pair would give a transitive tally preference. The quartet members all lost to the five first candidates and won against the last six. The aggregated 15×15 -matrix Con of (4.1) is published in RangeVoting.org.

Similar results were found by Tideman (Gehrlein 2006, p.47-48). A possible mechanism behind such fluke cycles is like this: Voters apply several criteria in their judgment and give them different weights. Roughly equal voter groups, I; II; III, emphasize criterion 1; 2; 3, respectively, but may agree that each criterion alone would give ranking XYZ; YZX; ZXY. This may cause a cycle XYZX in the middle range, with small margins among candidates who are strong according to one criterion and weak according to another.

In a No-Show accident, new ballots that rank X before Y cause Y to win instead of X. The accident is of the “strong” kind if X is top-ranked in the extra ballots, otherwise it is of the “weak” kind. The strong kind is impossible in IRV, since more top-ranks to X never can harm X. Obviously, it cannot harm a Condorcet winner in any Condorcet method either.¹²

The property that seems to limit the use of Condorcet methods in political elections is apparent from the candidate triangle ΔABC in pictograms of Perfect Pie-sharing cases, e.g. in Figure (1.2): The Condorcet winner is the candidate closest to the center. This gives the candidates a bizarre incentive for their election campaigns:

(4.4) Appear to the electorate as being the most central candidate!

Such a guideline hardly promotes a political clarification of contested issues for the voters. It is likely to have real impact even if formulations are more equivocal than (4.4). Marquette, Michigan, used the Nanson variation in city elections in the 1920s (McLean 2002), seemingly a unique reference in election literature. Melbourne University repealed Nanson’s method for Council elections in 1983. The reason given should not surprise anybody (McLean 2002):

“The reason for abandoning the Nanson system was that it was perceived to advantage inoffensive but not outstanding candidates as against those who attracted strong support.”

IRV and the B-supporters The eliminations in an IRV-tally makes Burying and its reverse impossible since no information on a ballot ranking after candidate X is available to the tally officials as long as X is still “hopeful”, i.e. not yet eliminated. A huge advantage of IRV over the Borda Count is its immunity to the “Burying” strategy (0.3). To varying degree, this is an advantage over the Condorcet methods too. However, there is a high price for this immunity.

The ERS (Electoral Reform Society) compares IRV with Plurality: *“For voters, there is less need for tactical voting, as voters can cast a vote for their favourite candidate without worrying that their vote will be wasted.”*¹³ In family \mathfrak{B} (0.9), A challenges Plurality winner B in the final tally round, and according to ERS, B-supporters should have no reason to worry about wasting their vote. The

¹² All Condorcet methods with ≥ 4 candidates sometimes allow No-Show accidents (Moulin 1988). The strong kind requires an election without a Condorcet winner. Moulin’s paper and later investigations are of considerable combinatorial interest, but they do not appear as connected to significant practical problems in Condorcet elections.

¹³ <https://www.electoral-reform.org.uk/voting-systems/types-of-voting-system/alternative-vote/>

ERS statement concerns the vote transfer rule, which is essential in IRV (and all other STV-methods). It is a central argument used by IRV-protagonists.

Still, voters who give the runner-up B top-rank in their original ballots, experience that their favorite finally loses, and that no other information from their ballots counts in the tally. As under Plurality, the BCA-voters might well have obtained a better result through “Compromise” (instrumental voting). However, in IRV it is less likely that they will be aware of need and opportunity for Compromise. Their ballots are just as wasted in IRV as votes for a loser are in a Plurality election. Voters have reason to worry. This is the price. Is it too high?

EXAMPLE (4.3) In Burlington 2009, constellation vi , see Example (3.2), (3.12) and Figure (3.4), B (= Wright) had 33% of the top-ranks already in round one. C (= Montroll) was a clear Condorcet-winner, but the tally ignored all subsidiary preferences of the B-supporters. As the central transfer rule of IRV is concerned, this is a decisive *disenfranchisement* of one large voter group. IRV’s treatment of the supporters of Plurality winner B was part of the background when Burlington repealed IRV in a referendum 2010.

Even abstention from voting may sometimes improve the result in IRV. The criterion is (3.10), and (3.8) tells how many BCA-ballots that must be *cancelled* in order to reach a constellation where C wins instead of A. This makes a difference in constellations v and vi . The BCA-voters will learn that a No-Show accident turned some of their own ballots into a “destructive surplus”, in fact making their votes worse than wasted.

EXAMPLE (4.4) In Frome 2009, A won, qualifying for the final with a narrow margin over C. In the 19309 ballots, the runner-up, B (= Boylan), had 7576 top-ranks from start, i.e. 39%, see Remark (2.1), but the subsidiary rankings from B’s supporters were ignored in the tally. The third pictogram of Figure (1.2), constellation vi , deserves particular attention. Although counterfactual, it is realistic and compatible with the standard IRV tally. C is Condorcet winner and faces elimination because of the disenfranchisement of all B-supporters. Thus, an IRV-tally ignores their massive subsidiary preference for C in $C v A$. Moreover, (3.10) is satisfied, i.e. $y = |BCA|$ is so large that it harms the BCA-voters; see Example (3.1) and Figure (3.3).

Eliminations and subsidiary rankings

Two important, but not compatible, principles for a preferential election method are:

Principle I: *Avoid the Burying strategy (0.3) and, equivalently, its reverse.* IRV achieves this goal with its sequence of eliminations based on top-ranks.

Principle II: *If no subsidiary rankings of a complete ballot counts, then the ballot must be one that ranks the election winner on top.* This is a central principle, which, if implemented, would have substantiated the claim that voters have no reason to worry that they waste their votes.

IRV follows Principle I consistently, but Examples (4.3) and (4.4) show that it is not even close to Principle II. However, rule (4.5) follows Principle II consistently, and still follows Principle I with eliminations based on top-ranks, until only 3 candidates, A, B, and C, remain:

(4.5) IRV/Condorcet *When three candidates remain in an IRV-tally, continue with a Condorcet method, either always or when some criterion is satisfied.*

Supporters of candidate X cannot help X to a place in the 3-candidate Condorcet final by changing their subsidiary rankings. Here we just consider a criterion in terms of $y = |BCA|$:

Choose $\lambda \in [0, 1]$ and define

$$(4.6) \quad \mu(\lambda) = (1-\lambda) \cdot \delta_c + \lambda \cdot (|B| - |C| + |A|/2 + 1)$$

Then continue with IRV if $y < \mu(\lambda)$, but with a chosen Condorcet method if $\mu(\lambda) \leq y$. Thus, $\lambda = 0$ means to use the Condorcet method always when 3 candidates remain. In non-cyclic cases, it makes a difference only in v and v_i , with a Condorcet winner last in top-ranks. With $\lambda = 1$, it is used only when, otherwise, A would win in a No-Show accident, see (3.10).

When the IRV elimination sequence reaches the stage (4.5), it has eliminated every other candidate X before any change after X in any ballot could influence the outcome. Therefore, Burying cannot help any fourth candidate X to replace A, B, or C in the final Condorcet tally.¹⁴

People are likely to have different opinions on what candidate that is most acceptable as a fair and legitimate election winner in constellation v or v_i . When y is just above the bound δ_c in (1.5) and makes C a Condorcet winner with narrow margin, it is still acceptable to argue that a clear advantage to A over C in top-ranks is more important. With $\lambda = 1$ in (4.6), A (= Kiss) will still win over Condorcet-winner C (= Montroll) in Burlington, Example (4.3), but Condorcet-winner C wins the election in Example (4.4) and a No-Show accident is avoided.

¹⁴ Theoretically, an action akin to Pushover may still, under favorable circumstances, let a fourth candidate D replace C (say) in the final. Most likely, however, D's prospects in a Condorcet final with {A, B, D} will not be good.

IRV in a split society If a voter group is a 60% majority, another a 40% minority, and transfers are internal in each, then one minority candidate reaches the 2-candidate final in the ordinary IRV tally, and the minority voters' ranking of majority candidates get ignored. A Condorcet final gives voters and campaigning candidates an incentive to cross a dividing line in a society divided by ethnicity, religion, ideology, or social class. To promote crossover voting, it is likely that some citizens will prefer to let a Condorcet final come earlier, while c electable candidates still remain in the race, $c > 3$, or when all remaining candidates have reached a certain share of the current top-ranks, e.g. 20%.

Two Condorcet methods; are they suitable in (4.5)?

With a Condorcet final, the possibility of defeating a Condorcet winner with strategic voting becomes an issue. The following discussion concerns Burying in 3-candidate elections, (0.3). Two Condorcet methods appear as a particularly natural choice in (4.5):

(4.7) IRV as cycle-break One natural idea is to return to IRV for a cycle-break if there is no Condorcet winner. Then A wins all cycles, and in constellations *i, ii, iii, iv*, the IRV-winner A is already Condorcet-winner; see Figure (0, 1). C is Condorcet winner in *v* and *vi*, but with rule (4.7), A wins by Burying, reducing x , and creating a cycle. Figure (1.1) illustrates that by decreasing x below δ_C , thus creating a cycle with a move from ACB to ABC, supporters of A can always snatch the IRV/Condorcet victory from C, i.e. Burying (0.3), and without any risk.

(4.8) Baldwin's Condorcet method Another natural idea is to apply Baldwin's variation (4.3) when three candidates remain: The only change is to use the Borda sum instead of top-ranks when three candidates remain; it is not necessary to check if there is a cycle.

Of the cycles in the NW of Figure (1.1), some are won by A, some by B, and some by C. The Borda ranking changes when one of three lines through the point $(x, y) = (3790, 4455)$ is crossed. Burying actions from C-supporters in constellation *iv* are visualized in a 3D generalization of Figure (1.1).¹⁵

¹⁵ The 3D-model is a rectangular box $|A| \times |B| \times |C| = (\delta_B + \delta_C) \times (\delta_C + \delta_A) \times (\delta_A + \delta_B)$ cut accordingly in 8 sub-boxes with edges of lengths in $\{\delta_A, \delta_B, \delta_C\}$, one for each constellation (Stensholt 2013). Two sub-boxes of volume $\delta_A \cdot \delta_B \cdot \delta_C$, at diagonally opposite corners, contain the cycles. Figure (1.1) is replaced by a layer of thickness δ_B . Constellation *vi* is always the smallest box, of volume $\delta_B \cdot \delta_A \cdot \delta_B$. The subsidiary votes from C-supporters, in accounts $|CAB|$ and $|CBA|$, determine which family, \mathcal{A} or \mathcal{B} , the election belongs to, Figure (0.1).

How to choose the Condorcet method Every cycle in $i_{(cyclic)}$ or $iii_{(cyclic)}$ may be created by voters who support a Condorcet runner-up and switch their second and third preference. A cycle in $iii_{(cyclic)}$ (say), may be created by supporters of A, B, or C starting from a constellation where their favorite is second in pairwise comparisons, i.e. from vi , iii , or ii , respectively. Thus, the cycle could be due to a successful Burying action for the candidate the cycle-break rule awards it to.

The target is a Condorcet winner, accepted as fair winner also by many opponents, so there is not likely to be much enthusiasm for joining an action. Still, the two cycle-break methods described above are significantly different.

If IRV is used for cycle-break in (4.5), only A can win with Burying, and the target must be C, but there are not many possibilities since relatively few 3-candidate IRV-elections are in constellation v or vi . However, if the election is reliably predicted to land in constellation v or vi , then the task for the A-supporters is quite easy: Just reduce x in Figure (1.1) and push the election into the adjacent set of cyclic constellations. When Burying is easy, it may be tempting for some A-supporters to apply it. If the tally shows a cycle, suspicion of a win by Burying may reduce the general confidence in the election method.

The Baldwin method gives many opportunities for the (Condorcet) runner-up to win with Burying, but this must happen at the polls, when constellations and other geometric conditions depending on $|A|$, $|B|$, and $|C|$ are still unknown. Baldwin awards some cycles to A, some to B, and some to C. On many occasions, post-election analysis with access to additional data will show that supporters of the runner-up missed an opportunity to defeat a Condorcet winner with the strategy of Burying. Most likely, one may then also see that their task was unrealistic for a reason similar to a reason that makes Pushover unrealistic in IRV; see Figure (2.1): Before election, one cannot know the lower and upper bounds for the number of actionists. A failed Burying attempt may well make the outcome worse.

5 A mathematical framework for preferential elections

Although some basic ideas in election theory appear in medieval writings, see e.g. (Colomer 2013), much work from the last two generations clearly belongs to a dual tradition from Duncan Black's focus on the structure of preference distributions (1948) and Kenneth Arrow's axiomatic approach (1950).

Arrow's conditions

Arrow adapted his work into a philosophic context, see e.g. Morreau (2014). Mathematically, it concerns a multivariate "social map" of relations

$$(5.1) \quad \Phi: (R_1, R_2, \dots, R_N) \rightarrow R$$

R is the social preference relation "at least as good as" over a set of $H \geq 3$ candidates, obtained by aggregating the ballot preferences R_j of voters $j \in \{1, 2, \dots, N\}$ over the same candidate set,

$$(5.2) \quad \{C_1, C_2, C_3, \dots, C_H\}$$

Let P_j and P [I_j and I] denote the associated strict preference [indifference] relations:

$$(5.3) \quad X P Y \text{ when } X R Y \text{ but not } Y R X, [X I Y \text{ when } X R Y \text{ and } Y R X], \text{ etc.}$$

Today, it is common to derive Arrow's result from the following set of axioms:

- 1) *UD "Unrestricted Domain"* Voter j chooses R_j as any complete and transitive ordering, and Φ accepts every combination of them for input.
- 2) *SO "Social Ordering"* R is a complete and transitive ordering of the candidates.
- 3) *IIA "Independence of Irrelevant Alternatives"* Whether $X P Y$, $Y P X$, or $X I Y$, depends exclusively on the restrictions of the R_j to $\{X, Y\}$; i.e. all ballot information about other candidates is irrelevant.
- 4) *WP "Weak Pareto condition"* If $X P_j Y$ for all j , then $X P Y$.¹⁶

Arrow's "*(im)possibility theorem*" states that if $H \geq 3$, then UD, SO, IIA, and WP together imply that one voter, d , is *dictator*, i.e.

$$(5.4) \quad \text{if } X P_d Y, \text{ then } X P Y.$$

Thus, if d has strict preferences, then $R = P_d$. A common, equivalent formulation is that five desiderata, viz. UD, SO, IIA, WP, and ND (Non-Dictatorship), are incompatible.

¹⁶ The strong Pareto condition says that if all voters say $X R_j Y$, and at least one of them says $X P_j Y$, then, in the social relation R , X is strictly preferred to Y , i.e. $X P Y$.

Social Choice theory is the study of social maps, both in welfare economics and voting theory. Arrow's impossibility theorem is basic in the welfare part of modern Social Choice theory, where WP is a natural requirement. However, it hardly says anything useful about voting methods.

The reason is WP, which becomes active only if there is a pair of candidates $\{X, Y\}$ where $X P_j Y$ for every voter $j \in \{1, 2, \dots, N\}$. The unanimity implies that Y gets 0 top-ranks, which must be extremely rare since Y, presumably, gets some top-ranks from those who nominated Y.

If, hypothetically, Y is last and X second last in every ballot ranking, then WP requires that $X P Y$. That is impossible in a single-seat preferential election method designed to distribute $H \geq 3$ candidates into one singleton class (elected) and one indifference class with $H-1$ candidates (not elected). Obviously, such a design is in conflict with WP, which here requires at least one singleton indifference class for Y, another one for X, and a third one for the winner.

Any preferential election designed to give an R with two indifference classes, must fail to satisfy WP. One may still hope that removal of WP will allow the construction of democratic, new, and useful preferential election methods that satisfy axioms UD, SO, IIA, and ND, and fail WP only.

Wilson's improvement Robert Wilson (1972) showed that such hope must be in vain. A useful impossibility result should say so. It makes no sense to include WP with the other conditions. Dropping WP, Wilson found that UD, SO, and IIA together imply that there is either a dictator d , or else one of a few other equally undemocratic arrangements. They are as follows:

- (AD) an *Anti-Dictator* a (if $X P_a Y$, then $Y P X$), or
- (TI) a *Total Indifference* ($X I Y$ for all X and Y), or more generally,
- Φ gives an "imposed" constant social preference R, sometimes called a "tradition", e.g. that R ranks the candidates by age, no matter how the ballots are.

Wilson's reduced axiom system is not logically equivalent to Arrow's, because removal of WP opens for an Anti-Dictator or an Imposed R (including TI). Thus, it may seem that the removal of WP makes no practical difference, but it does. In Wilson's words,

"... the fact remains, that Arrow's other conditions suffice to exclude all of the democratic social choice processes of interest".

WP is harmful to applications since Arrow's original version with WP keeps back this message from Wilson to constructors of preferential election methods.

In a survey article, Morreau (2014) brings together UD, WO, IIA, WP, and ND as the “canonical form” into which Arrows conditions have settled. Since this axiom set has not been through a normal pruning process, such “canonization” is still premature – long after Wilson’s result.

Very Weak Pareto condition Proofs of Arrow’s impossibility result still make some use of WP. However, it seems harmless to replace WP with the condition VWP:

(5.5) VWP “Very Weak Pareto” If $X P_j Y$ for all j , then $X R Y$.

UD, SO, IIA, and VWP suffice to imply either TI or Dictatorship. If there is no TI, Cassels’ short proof (1981) of Arrow’s original version works with minor modifications if VWP replaces WP. Arrow’s conclusion (5.4) follows and implies WP, but WP as axiom makes the result insignificant for elections. It is unlikely that seriously suggested preferential methods will violate VWP.

Arrow’s theorem soon received wide attention, and a theme in the debate was how Arrow’s conditions might be relaxed in order to allow more acceptable methods than Dictatorship. Wilson’s improvement shows that removal of WP clears the stage of one unnecessary prop, but more relaxation is required. It seems natural that this debate has had focus on IIA and UD.

However, the most important forms of strategic voting, including those of (0.3) considered above, would clearly not be possible if IIA was satisfied. The SO axiom requires that cycles do not occur in R . Thus, a basic problem is how to satisfy both SO and IIA. The Condorcet relation of pairwise comparisons satisfies IIA, but is not necessarily transitive. It turns out that with a certain restriction on UD, due to Duncan Black, it also satisfies SO.

Black’s Single-Peak Condition

Even before Arrow hit the stage, Black (1948) had shown that if the voters have a certain common feature in their cognition of the political landscape, then their ballot preferences naturally become coordinated such that the Condorcet relation of pairwise comparison is transitive.

List the H candidates from left to right in an *ordered H-tuple*, by a political or any other principle,

(5.6) $(C_1, C_2, C_3, \dots, C_H)$

Single-Peak for a single voter The ballot preference from voter j is said to satisfy the *Single-Peak Condition with respect to* (5.6) if it is complete, strict, and transitive, and moreover,

(5.7) for every triple $\{C_{u-1}, C_u, C_{u+1}\}$, $2 \leq u \leq H-1$, $C_u P_j C_{u-1}$, or $C_u P_j C_{u+1}$, or both.
Thus, voter j ranks the middle candidate C_u first or second in the triple.

This means that the ranking in j 's ballot falls away from the top-ranked candidate C_{peak} .

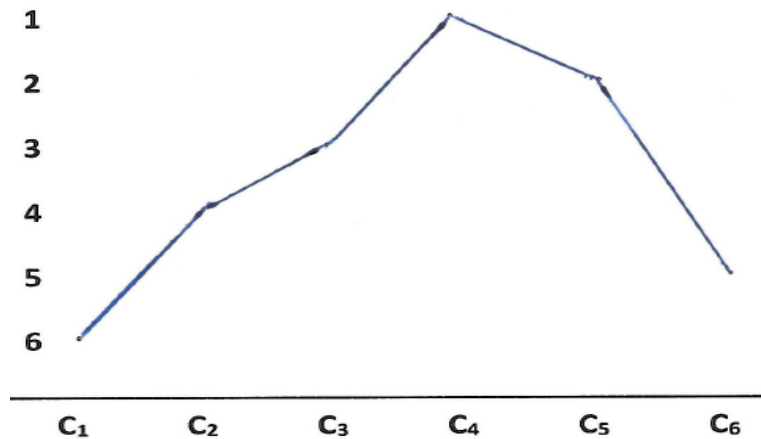


FIGURE (5.1) A single-peaked ballot preference with respect to (5.6)

The ranking $C_4 C_5 C_3 C_2 C_6 C_1$ is single-peaked with $C_4 = C_{peak}$. Such rankings are likely to be common if many voters perceive the candidates as located on a left-to-right axis. There are 2^{H-1} single-peaked rankings with respect to (5.6), because either C_1 or C_H comes last after one of 2^{H-2} possible single-peak rankings of the other $H-1$ candidates.

Figure (5.1) illustrates that if voter j has a single-peaked preference with respect to (5.6), then, after cancellation of any candidate, the shortened ballot ranking is single-peaked with respect to the shortened version of (5.6). This way every triple $\{C_r, C_s, C_t\}$, $r < s < t$, can be reached, and

(5.8) every voter ranks the middle candidate C_s first or second in the triple.

Single-Peak and the electorate Black required all ballot preferences R_j to be single-peaked with respect to (5.6). An intention was to present a structural feature of the preference distribution that avoids Condorcet cycles. In a Condorcet cycle of length > 3 , one may draw a chord and obtain a shorter cycle. Thus, it is enough to see that Black prevents 3-cycles:

Because of (5.8), in the notation of (0.1),

$$(5.9) \quad |C_r C_t C_s| = |C_t C_r C_s| = 0 \text{ if } r < s < t$$

Thus, in a pictogram for $\{C_r, C_s, C_t\}$, the chords meet at the circle periphery, the empty triangle T shrinks to a point (Perfect Pie-sharing), and T is so far away from the center as possible. On the

other hand, if a triple is cyclic, T must have positive area and cover the circle center, as in Figures (2.5) and (3.3). Thus, the condition (5.9) is an extreme over-fulfillment of Black's intention. With respect to avoiding a cycle, a single-peak case therefore has some robustness to perturbations.

In Figure (1.1), the only point that gives a single-peak election is $(x, y) = (|A|, |B|)$, i.e. the NE corner and end-point of the middle curve. However, continuation along the middle curve, keeps Perfect Pie-sharing, and cycles do not occur. Figure (1.1) illustrates that in a triple, Perfect Pie-sharing generalizes the Single-Peak Condition as a sufficient criterion for transitivity.

The Single-Peak Condition for the electorate is a tall order. It demands that all ballots are in a subset of only 2^{H-1} in the set of all $H!$ complete and strict rankings. However, a single-peak preference distribution is just a theoretical construction. Perturbations of the $H!$ -component vote vector will violate (5.9), but unless they are too large, they still keep each triple $\{C_r, C_s, C_t\}$ within constellation v or vi , and give a realistic approximation to a given single-peak distribution.

REMARK (5.1) The Single-Peak Condition gets attention in its own right in models of preference distributions where voters agree in their perception of a 1D-structure of the political landscape.

For tables of the frequencies of various combinations of anomalies in IRV with $H = 3$ candidates, see Smith (2010; revised 2016). Based on three probability distributions of the vote vector, a large number of elections are simulated. One distribution generates 3-candidate IRV-elections by randomly picking three candidate positions in the line segment $[0, 1]$ and then also a number of voters who rank the candidates by distance, i.e. according to (5.6), and submit one of the four single-peaked ballots. In 11.11% of the elections, the election lands at the NE corner in Figure (1.1) or an analogue, where condition (3.10) is satisfied.¹⁷

In Figure (1.1), most realistic (x, y) are in the 0.001-zone, but only if $y \geq 5395$ are they caused by a No-Show accident. Thus, the 11.11% is an upper bound for the frequency of real elections where a "win by cancellations" is possible. This bound is many times higher than the real frequency, because the NE corners are the only points visited in the simulation. Thus, it records a "hit" when

¹⁷ Obtained by adding the probabilities of anomaly combinations where a No-Show accident has happened: $2.3750 + 4.5139 + 1.7916 + 2.4305 \approx 11.11$. The simulation results on *monotonicity failures* count only elections where a non-monotonicity event *theoretically* may have happened.

just a small part of the large rectangle (in constellation v or v_i) could have been the result of a No-Show accident. Nevertheless, (3.11) and Figure (1.1) indicate that (3.10) is satisfied so often that rule (4.5) and (4.6) should be considered.

REMARK (5.2) Already for $H = 4$, it is difficult to assign in a meaningful way an “ideal point” to each voter on the same line as to the candidates: Assume voter 1 ranks $C_2C_3C_4C_1$, and voter 2 ranks $C_3C_2C_1C_4$. Both preferences are single-peaked, but $\{C_2, C_3\}$ shows voter 1 as more “leftist” than voter 2, while $\{C_1, C_4\}$ shows voter 2 as the most “leftist”.

6 Political and legal frameworks

For all sides to accept the outcome of a single-winner election as fair, it helps that the winner can claim support by more than 50% of the votes. This applies to elections for president, governor, mayor, and a single seat in a legislature. However, elections designed to produce such a winner (2-day election or IRV) are more expensive and time-consuming than Plurality elections (a.k.a. FPTP, “First-Past-The-Post”), and they may require special effort from the voters.

Presidential elections Many countries consider that a presidential election is important enough to justify the expense, the time, and the voters’ “cognitive costs”:

EXAMPLE (6.1) A table in “*New Handbook*”, i.e. *Electoral System Design: A New International IDEA Handbook* (Reynolds & al. 2005) shows that 78 countries use a 2-day election similar to the French method, see Example (3.3), while only 22 use the Plurality method. However, only Ireland uses IRV (a.k.a. AV, “Alternative Vote”) for presidential elections.¹⁸

Legislature elections Mostly, elections to a legislature belong to one of two categories. The table in *New Handbook* classify 72 cases of “*Proportionality*”, where the number of seats won by a party is roughly proportional to its number of votes nationwide; usually there are multi-seat constituencies where voters choose between party lists, but some countries use STV (Single Transferable Vote). IRV is the single seat version of STV.

There are 91 cases of “*Plurality/Majority*”. Most of them use Plurality or a 2-day election in single-seat constituencies. A Plurality tradition strongly stimulates the development or preservation of a two-party system (“*Duverger’s law*”). The winners’ electoral bases include, to varying degree, instrumental votes from voters whose alternative was to give an expressive vote to another candidate. Subsidiary preferences mean more in a 2-day election, but they do not appear in the ballots. A winning candidate may still be strongly committed to promoting the policy of one party. With a preferential method, they appear in the ballots, tallies find facts about subsidiary preferences, and campaigners are likely to think about the structure of their electoral basis.

¹⁸ A purely technical objection against IRV concerns the complication in tallying ballots in several constituencies. In Ireland, 43 constituencies count the top-ranks simultaneously, report the numbers to a central where addition is done, the candidate with the smallest number of top-ranks nationwide is identified, and the elimination is reported back to the constituencies before the next round. If one initial report from each constituency should be enough (e.g. like the matrix Con of (4.1) in a Baldwin election), it would, most likely, have to contain much more information, and make the process less transparent.

Mixed Member Proportional

“Mixed Systems” is a third category with 30 cases, defined in “New Handbook” as: “A system in which the choices expressed by voters are used to elect representatives through two different systems, one *proportional representation* system and one *plurality/majority* system.” Each constituency elects one candidate by plurality/majority.

Germany uses a kind of Mixed System, called MMP (“Mixed Member Proportional”), for electing the federal legislature (Bundestag). Each voter has two votes, both in the same ballot.

The first vote (“Erststimme”) counts in a Plurality election of one *district representative* from each of D constituencies; the voter supports just one candidate.

There is a total of S seats, $S > D$. With the second vote (“Zweitstimme”), the voter supports just one party. The Zweitstimme determine the value of S and distribute $S - D$ party seats to parties that pass a threshold criterion of support.

Notation The parties are Z_1, \dots, Z_t , of which Z_1, \dots, Z_k pass the threshold, $k \leq t$;

- (6.1) $\Lambda(Z_j)$ is the set of voters with Zweitstimme to party Z_j (in the whole nation);
 $\Gamma(A, Z_j)$ is the subset with Erststimme to candidate A (in the constituency of A).

Sometimes a party does not nominate a candidate in a constituency. Also in other cases, many voters *split their votes*: Their Erststimme then supports, perhaps instrumentally, a district candidate A not from the party Z_j that they support with their Zweitstimme. Thus, a district seat winner W has an *electoral basis*, $\text{Bas}(W)$, which is a disjoint union of all sets $\Gamma(W, Z_j)$, $1 \leq j \leq t$:

$$(6.2) \quad \text{Bas}(W) = \Gamma(W, Z_1) \cup \Gamma(W, Z_2) \cup \dots \cup \Gamma(W, Z_t)$$

This decomposition of $\text{Bas}(W)$ has no rôle in the present tally rules in MMP, but Erststimme and Zweitstimme appear in the same ballot, so it is available for an alternative tally considered here.

REMARK (6.1) It may happen that a district winner W is candidate for party Z_j with $k < j \leq t$. In this notation, an independent W belongs to an “imagined party” Z_j with $\Lambda(Z_j)$ empty.

The Zweitstimme are intended to achieve a nationwide distribution of seats “proportional” to the size $|\Lambda(Z_j)|$ of $\Lambda(Z_j)$, $1 \leq j \leq k$; here all S district- and party-seats are seen together. A suitable “accounting principle” is of the essence here, as it also is in the study of non-monotonic events in IRV; see Example (0.1).

The present accounting rule records a district winner W from party Z_j on an account for $\Lambda(Z_j)$:

(6.3) *Present accounting rule:*

W 's district seat counts as one full seat won by $\Lambda(Z_j)$ if W comes from Z_j , $1 \leq j \leq t$.

However, all voters in $\text{Bas}(W)$, (6.2), give one Erststimme to W , and (6.4) is an alternative to (6.3):

(6.4) *Alternative accounting rule:*

A fraction $|\Gamma(W, Z_j)| \cdot |\text{Bas}(W)|^{-1}$ of W 's seat counts as won by $\Lambda(Z_j)$, $1 \leq j \leq t$.

Accounting rules (6.3) and (6.4) give two different definitions of $D(Z_j)$, the number of district seats which the Erststimme tally records on $\Lambda(Z_j)$'s account, $1 \leq j \leq t$. The Zweitstimme tally gives a non-negative number $P(Z_j)$ of party seats to $\Lambda(Z_j)$, $1 \leq j \leq k$. The central requirement in MMP is (6.5):

(6.5) *Proportionality rule:*

$\rho =$ Zweitstimme per seat (district- and party-), is the same for all parties that pass the threshold:

$$\rho = |\Lambda(Z_j)| \cdot [D(Z_j) + P(Z_j)]^{-1}, 1 \leq j \leq k; \quad P(Z_j) = 0 \text{ for } k < j \leq t.$$

Geographical allocation of party seats and necessary round-offs to integers are not discussed here; solutions $(\rho, P(Z_1), \dots, P(Z_k))$ of (6.5) are in non-negative reals. In any solution, ρ may be reduced, and then properly increased $P(Z_j)$, $1 \leq j \leq k$, give another solution. Thus, since $P(Z_j) \geq 0$,

maximal ρ occurs when $P(Z_m) = 0$ for some party Z_m , where $1 \leq m \leq k$.

More explicitly, rewrite (6.5) and get

$$(6.6) \quad \rho^{-1} - D(Z_j) \cdot |\Lambda(Z_j)|^{-1} = P(Z_j) \cdot |\Lambda(Z_j)|^{-1} \geq 0, \quad 1 \leq j \leq k$$

Define ρ^* and m so that, with $j \in \{1, \dots, k\}$,

$$(6.7) \quad (\rho^*)^{-1} = \max_j D(Z_j) \cdot |\Lambda(Z_j)|^{-1} = D(Z_m) \cdot |\Lambda(Z_m)|^{-1}$$

Thus, party Z_m has most district seats per Zweitstimme. By (6.6) and (6.7),

$$\rho^{-1} \geq D(Z_m) \cdot |\Lambda(Z_m)|^{-1} = (\rho^*)^{-1};$$

(6.8) thus, (6.5) has no solution with $\rho > \rho^*$, and $\rho = \rho^*$ when $P(Z_m) = 0$;

There are $S = \sum_j [D(Z_j) + P(Z_j)]$ seats, $1 \leq j \leq t$, and $\rho = \rho^*$ gives the "critical" assembly size S^* :

$$(6.9) \quad S^* = \sum_j [D(Z_j) + P(Z_j)], \text{ where } P(Z_m) = 0$$

S is chosen as small as possible, provided that S_0 "ordinary seats" are filled. Thus,

$S = \max \{S_0, S^*\}$, and if $S^* > S_0$, then $S^* - S_0$ "extra-ordinary seats" are created.

Rules (6.3) and (6.4) define $D(Z_j)$ differently and may cause big difference in S ; see Example (6.2).

EXAMPLE (6.2) Germany uses accounting rule (6.3) and has $S_o = 2D = 598$ ordinary seats.

In the Bundestag election 2017, the district candidates from CSU ¹⁹ *together* got

3255487 Erststimme and 46 district seats, but only $|\Lambda(\text{CSU})| = 2869688$ Zweitstimme.

Presumably, some number u of ballots from $\Lambda(\text{CSU})$ also showed a split vote, $u \geq 0$, so that

$3255487 - (2869688 - u) = 385799 + u$ voters supported one of CSU's *district candidates* without also supporting *the party* CSU with their Zweitstimme.

Present rule If CSU should get y party seats too, $y \geq 0$, (6.3) and (6.5) give a bound for ρ ,

$$(6.10) \quad \rho = 2869688/(46 + y) \leq 2869688/46 \approx 62384.5 \text{ Zweitstimme per seat.}$$

The tally showed that all other parties in $\{Z_1, \dots, Z_k\}$ needed some party seats in order to get a ratio as low as 62384.5; $\text{CSU} = Z_m$, $y = 0$, and $\rho \approx 62384.5$ is the price in Zweitstimme per seat.

Together, Z_1, \dots, Z_k received a total of $\sum_j |\Lambda(Z_j)| = 44189959$ Zweitstimme.²⁰ All $D = 299$ district seats were won by candidates from parties that passed the threshold. It follows from (6.5) that

$$(6.11) \quad S = S^* = 44189959/\rho \approx 44189959/62384.5 \approx 708.4$$

With rules (6.3) and (6.5), and round-offs, the Bundestag got $S = 709$ members.

$\Lambda(\text{CSU})$ was the only Zweitstimme group without party seats.

Alternative rule Rule (6.4) records that $385799 + u$ voters used a split vote to support a CSU-candidate in the Plurality elections, $u \geq 0$. Ballots with Erststimme to a winning CSU candidate and Zweitstimme to party Z_j add up to give $\Lambda(Z_j)$'s fraction of the district seat.

Constituencies are roughly equal in size. An estimate of the sum of $\Lambda(\text{CSU})$'s own fractions of district seats is $46 \cdot (2869688 - u)/3255487 \approx (40.5487 - u \cdot 0.00001423)$ seats.²¹

Analogically to (6.10), if CSU should get y party seats, we have a bound:

$$(6.12) \quad \rho = (2869688 - u)/(40.5487 - u \cdot 0.00001423 + y) \leq (2869688 - u)/(40.5487 - u \cdot 0.00001423) \\ = 3255487/46 \approx 70771 \text{ Zweitstimme per seat.}$$

If all other parties need party seats to get a ratio ≤ 70771 , then $y = 0$. Since also $\rho = 3255487/46$ Erststimme per district seat, "the price is right" for all seats.

¹⁹ CSU is the smaller partner of the CDU/CSU coalition in the Bundestag work; in elections, CSU operates only in Bavaria and CDU only in the 15 other states.

²⁰ Parties that passed the threshold (CDU, CSU, SPD, AfD, FDP, Left, Green) got all district seats but received 44966765 Erststimme, i.e. more than $\frac{3}{4}$ million from ballots with Zweitstimme to a party that did not pass it.

²¹ CSU's candidates won all district seats in Bavaria, and thus no fraction of a district seat won by another party.

We check if CDU with 14030751 Erststimme and 185 district seats needs party seats: With ratio

$$14030751/185 = 75842 \text{ Erststimme per seat,}$$

it seems that CDU too needs some party seats to reach 70771, but this is not quite clear: Some of the 14030751 Erststimme to CDU were in constituencies not won by CDU, but CDU would also have obtained shares of some district seats won by candidates from other parties. However, it seems that only CSU is without party seats at the critical assembly size S^* : Analogically to (6.11),

$$(6.13) \quad S = S^* \approx 44189959/70771 \approx 624.4$$

Thus, with rules (6.4) and (6.5), and round-offs, $S = 625$ is a likely Bundestag size.

Proportional influence and assembly size With rule (6.4), a party whose candidates win many district seats, does not push the ratio ρ equally far down as it does with rule (6.3). Thus, the number of extra-ordinary seats decreases with rule (6.4). Example (6.2) shows a decrease from $709 - S_o = 111$ to $625 - S_o = 27$.

The 111 extra-ordinary seats in the 2017 election is not an extreme case with rule (6.3). When all parties are relatively small, there can be a much bigger demand for extra-ordinary seats:

EXAMPLE (6.3) If eight parties each have ca. 10% of the Zweitstimme, and one has 20%, spread evenly in all constituencies, the latter wins all D district mandates unless massive vote-splitting helps a smaller party. According to (6.3) and (6.5), the Bundestag gets $S = 5D = 1495$ seats.

With rule (6.3) the parties Z_1, \dots, Z_k are represented in proportion to the sizes $|\Lambda(Z_1)|, \dots, |\Lambda(Z_k)|$ of the groups who give them Zweitstimme, as though party Z_j owns the voter group $\Lambda(Z_j)$. With rule (6.4), $\Lambda(Z_j)$ is treated as a self-owning unit: The rule takes into account the variation inside the group in the district elections. Different voters in different constituencies adapt in different ways, e.g. to the usual problem in a Plurality election: Vote expressively or instrumentally?

Voters' satisfaction Accounting rule (6.3) fits with a view that only those $2869688 - u$ voters who gave both Zweitstimme and Erststimme to CSU in Example (6.2), got any "satisfaction" from a district winner they supported. However, voters who give their Zweitstimme to different parties form the electoral basis for a district winner; see (6.2): They share 46 "seat-units of satisfaction" in Example (6.2) already before the Zweitstimme tally. Moreover, a voter from $\Gamma(W, Z_j)$ where W is district winner, get more satisfaction than one from $\Gamma(L, Z_j)$ where L is a district loser.

Rule (6.3) is as if the 46 *district representatives* from CSU in Example (6.2) represent the 2689688 voters in $\Lambda(\text{CSU})$ who supported the CSU *party* with their *Zweitstimme*, and none of the 385799 + u other voters who really supported a CSU candidate with their *Erststimme*. Conceptually, the adoption of accounting rule (6.3) is somewhat surprising, since a district winner W from Z_j could even (also conceptually) have empty $\Gamma(W, Z_j)$ in (6.2).

With *Erststimme* and *Zweitstimme* appearing in the same ballot, rule (6.4) requires no change for the voters and it reflects their different degrees of satisfaction.

Preferential Erststimme

There are two basic ideas in the election of legislatures: An MP (Member of Parliament) represents either a constituency or a political party. The table in New Handbook reflects how most nations choose the one or the other. Some mixed-member systems are of the *parallel kind*; a party seat tally and a district seat tally are not connected. The district component usually prevents full proportionality, and an almost paradigmatic difference remains.

MMP is an ambitious and noteworthy attempt to combine a large district component with full proportionality (for parties who pass a threshold). As seen in Example (6.2) and Example (6.3), in an increasingly fragmentized political landscape the present accounting rule (6.3) and the proportionality rule (6.5) bring the number of seats out of control. The alternative accounting rule (6.4) brings down the seat number, but even in the not too bad 2017 election, 27 extraordinary seats would be required.

IRV fulfills a widely accepted 50%-criterion of fairness in single seat elections, and thereby promotes the legitimacy of district representatives in a way that Plurality elections cannot do.

The MMP setting is a very special political framework for a single-seat method: It creates a niche where IRV, potentially, also has another beneficial effect. The ratio $\rho^* = |\Lambda(Z_m)| \cdot |D(Z_m)|^{-1}$ is likely to get much higher. An IRV-tally of *ranked Erststimme* gives accounting rule (6.4) greater effect, since > 50% of the votes in a constituency support the district winner in the last tally round and spread “satisfaction” to accounts of different *Zweitstimme* groups; see (6.2).

In the Bundestag election, Example (6.2), there were about 7.4 million ballots in Bavaria. In order to win the 46 district seats with IRV, the CSU-candidates would have needed at least 3.7 million *Erststimme* after IRV eliminations, instead of the 3.26 million they got in the Plurality tally. Larger

parts of the district seats would have come on other parties' accounts, and S* would have been further reduced. It must be a worthwhile project to assess how IRV can fit in.

Cognitive costs with IRV

Burlington repealed IRV in a referendum 2010, 52% v 48%, and went back to a 2-day method. The background was the election in 2009, Example (3.2). The tally's neglect of subsidiary rankings from the supporters of the runner-up, Plurality winner Wright, may well have been decisive. However, in a 2-day election, the preference distribution in Burlington 2009 gives the same final round, and K still wins over W; see Example (3.2).²²

In 2011 UK decided in a referendum to keep Plurality elections to the House of Commons: 13013123 for Plurality; 6152607 for IRV. On that occasion, most voters had no personal experience with IRV, but might have expected that preparation of a full ballot ranking for IRV would mean a bigger effort than just making one choice for a Plurality election. The prospect of a new heavy burden may have meant more than IRV neglecting subsidiary rankings from supporters of the runner-up and more than repeated claims about non-monotonicity events.²³

In the aftermath of the Burlington election, Gierzynski (2009) emphasized such cognitive costs: *"If states in the US were to adopt IRV for all (or even some) of their elections, the situation would only be made worse. Instead of simply choosing the preferred candidate for president, senator, representative, governor, lieutenant governor, secretary of state, treasurer, and so on, the public would be asked to rank each candidate. Ranking each candidate in all these races means that the cognitive costs of voting would double, triple, or even quadruple."* Gierzynski emphasizes the importance of *"recognizing the limits to what a political system can ask of its citizens and*

²² With the preference distribution in Frome 2009, a 2-day election eliminates the IRV- and Condorcet-winner Brock; see Remark (2.1). A 2-day election may also cause a final round so lopsided that a second day to many will look like a waste of public resources, as in France 2002 (Chirac 82.21%, LePen 17.79%); see Example (3.3). A final round in a 2-day election may even occur between candidates R (far right) and L (far left), while any X from center/right to center/left is prematurely eliminated, but would have won landslide pairwise victories in {X, R} and {X, L}.

²³ Advice from party leaders were clearly important. To have a referendum was part of the agreement when Conservatives and Liberal Democrats formed their coalition government. The referendum campaign strained their relations. Only the Liberal Democrats wanted a change from Plurality to IRV. A common second choice for both Conservative and Labour voters, they often have the Condorcet winner and therefore win just by qualifying for the final round. Many voters must have thought that, in reality, the referendum was about introducing a lower threshold for MP status for Liberal Democrats than for other parties.

recognizing that adding complexity to an already complex ballot will disproportionately harm some groups of people more than others”.

However, Australia has experience in these matters since 1918, so there is some study material. Although citizens must vote or pay a fine, their burden is not necessarily heavy. Usually, a complete and strict ranking is required, but there is also a remedy to reduce the cognitive costs: Parties generally offer a “how-to-vote card” which recommends subsidiary preferences to their supporters.²⁴

This practice encourages negotiations on mutual support between neighbor parties. The voters’ burden may get lighter even until they hardly notice any. The parties’ production of how-to-vote cards may become mandatory and with a format regulated by law. Voters may then accept with one pencil mark the subsidiary rankings officially suggested by their chosen party. Acceptance may even be their default choice.

Litigation and constitutionality

The US presidential election in Florida 2000 (technically for the Florida seats in the Electoral College) ended officially with 2912790 for Bush, 2912253 for Gore. In such a close election, “*natural sortition*” decides; see also Example (2.1). However, with votes cast, the possibility of a decisive mistake in ballot handling or violation of tally rules raises more questions. Many remember the legal dispute under time pressure before inauguration.

Legal disputes on the constitutionality of the election rules themselves also attract attention, but mainly concern how an election method functions in a normal political landscape, not only in a terrain where natural sortition decides.

A constitution may demand a particular election method (Maine 2017) or a method with particular properties (Germany 2008).

(6.14) *The German Federal Constitutional Court 2008* Allocation of seats according to different proportionality conditions (e.g. to states according to size or to parties according to their

²⁴ Still, “donkey voting” is a recognized problem: Some voters rank the candidates by a simpler principle, e.g. in the order of appearance in the ballot paper. One remedy is “Robson rotation”; with a sufficient variation in the order of appearance in the printed ballots, donkey procedures will not systematically (dis)favor particular candidates.

Zweitstimme) is notoriously difficult (Balinsky and Young 1983). The distinction between district and party representation is an additional complication.

On July 3rd 2008, the court considered the fact that the election rules then used in Bundestag elections gave several violations of the Participation Criterion. Sometimes some increase (decrease) of Zweitstimme for a party would have decreased (increased) the number of seats won by that party. This was the so-called “Negatives Stimmgewicht”.²⁵ The ruling was to reduce this effect as far as possible. The remedy was to allow $S^* - S_0$ extra-ordinary seats, as required by the present rules (6.3) and (6.5).

(6.15) *Opinion of Maine Supreme Court 2017* On Election Day in Maine 2016, a majority recommended, in a referendum, a change from Plurality to IRV (there called “Ranked-Choice Voting”) for elections of governor, senators, and members in the House of Representatives. A citizen action had collected enough signatures to require a referendum, but the constitutionality of IRV had already become an issue. After amendments in 1847, 1875, and 1880, the state constitution required Plurality for Governor, House, and Senate. See e.g. Shepherd, referring to events of 1879-1880 (Bangor Daily News, January 20th-21st, 2016).²⁶

²⁵ Assume party Z_r from a total of S seats receives $D(Z_r)$ district seats based on Erststimme. A strictly proportional distribution of all S seats based on Zweitstimme, $S \geq D (= 299)$, will give it $Q(Z_r) = S|\Lambda(Z_r)| / \sum_j |\Lambda(Z_j)|$ seats: The difference $D(Z_r) - Q(Z_r)$ depends on S ; if positive, it is called the *overhang* for party Z_r .

“*Overhang mandates*” was a common concept in the political and legal discussions on *Negatives Stimmgewicht*. This unfortunate effect could be due to repercussions caused by interplay between the detailed rules for distribution of party seats between parties and member states, if increased support gave a party one more district seat.

For $S < S^*$, there must be overhang seats. At $S = S^*$, the last overhang disappears. At $S = 2D = 598$, the positive overhangs in 2017 totaled 46 seats. However, the number $S^* - 2D = 111$ of extra-ordinary seats was determined by the smallest number of Zweitstimme per district seat, 62384.5, obtained by $Z_m = \text{CSU}$. Only the difference, here $(111 - 46 = 65)$, is called *compensatory seats* (“*Ausgleichmandate*”), despite the fact that the 46 (unidentified) winners share 299 ordinary seats with all other district winners. Party list candidates sit in all 111 extra-ordinary seats.

²⁶ In the gubernatorial election on September 7th 1879 (for a term starting in January 1880), Republican candidate Daniel F. Davis obtained a plurality, but no majority. According to the state constitution, the new legislature, elected at the same time, should then elect governor. Local tallies showed comfortable majorities for the Republicans. Thus, it seemed clear that Davis would win.

However, a “Committee of Returns”, in the administration of Garcelon (Democrat, and still governor), had to scrutinize the tally reports. It disqualified many of the local results, and the Republican majority disappeared in both chambers.

The development of the next weeks culminated with a standoff in the capital Augusta between two armed groups. Each supported their own group of would-be senators and representatives who arrived to occupy the same seats. With his personal authority, Joshua Chamberlain dampened the conflict, persuaded armed people to go home, and gave the Supreme Court time to treat the case. Finally, Davis won. See Pullen (1999).

Next, the constitution was changed, and Plurality is still (2018) the method in all elections it regulates.

The state senate requested an opinion from the Maine Supreme Court. On May 23rd 2017, in an “*opinion of the justices of the supreme judicial court*”, the seven justices concluded without dissent: “*Yes, the Ranked-Choice Voting Act conflicts with the Maine Constitution.*”

The Maine legislature then voted to delay any implementation of the referendum result until 2021, and IRV-proponents got some time to propose necessary Constitutional amendments.²⁷

Minnesota Supreme Court 2009

In a referendum 2006, Minneapolis had recommended IRV for its municipal elections. After appeal, the claim that IRV was in conflict with Minnesota’s constitution came before the Supreme Court as:

Minnesota Voters Alliance, et al. (appellants) vs. The City of Minneapolis, et al. (respondents).

FairVote Minnesota, Inc. was intervenor-defendant. FairVote works for the introduction of STV (including IRV) in American elections. Claimed unconstitutionality of IRV/STV was a *facial* challenge, not just an *as-applied* challenge; if successful, the appellants would have struck out all intended use of the method. This looks as an uphill fight, but the appellants claimed a *precedent* in the same court from 1915. Thus, it was essential to compare the new with the older case, *Brown v Smallwood (1915)*: In 1915, the state’s Supreme Court declared another voting method, adopted in Duluth, to be unconstitutional. The Duluth method belongs to the wide family of *Bucklin methods*.²⁸ The rules appear from the tally as reported in the decision:

Duluth 1915	First Choice	Second Choice	1 st &2 nd Choice	Add'l Choice	1 st , 2 nd & Add'l Choice
Louisell	992	734	1,726	402	2,128
Norton	3,417	1,501	4,918	167	5,085
Smallwood	3,496	2,845	6,341	240	6,581
Windom	4,408	604	5,012	54	5,066
	12,313	5,684	17,997	863	18,860

A voter made a 1st choice, had the option to name another candidate for 2nd choice, and also to add more names for additional choice. The tally was in 3 rounds; no candidate got ½ of the 1st

²⁷ The Maine constitution does not regulate mayoral elections and from 2011, Portland elects its mayor with IRV. It does not regulate primary elections either; for its statewide primaries, Maine uses IRV from 2018.

²⁸ The idea is also in writings of Condorcet, but is likely to occur independently to many people. James W. Bucklin (1856-1902), of Grand Junction, Colorado, was a lawyer, politician, and supporter of Henry George’s economic theory, and he promoted the election idea in the US. It gained popularity during the “*Progressive Era*”.

choices; no candidate got $\frac{1}{2}$ of the 1st & 2nd choices; finally Smallwood won with most 1st, 2nd & add(itiona)l choices. The Duluth method is just one of many Bucklin variations. The number of rounds and the number of candidates to be chosen for round 1, 2, ... , may be varied.²⁹ The candidates, for short L, N, S, and W (incumbent), ran for a position as municipal judge.

Fifty-six years further back Central to the court's judgment in 1915 was its understanding of the state's Constitution from 1859. The Constitution was not explicit on voting methods. However, a majority took judicial notice of the assumed meaning of the word "vote" before the "Progressive Era" with its many reforms and proposals in electoral and other matters:

When the Constitution was framed, and as used in it, the word "vote" meant a choice for a candidate by one constitutionally qualified to exercise a choice. Since then it has meant nothing else.

The court considers two violations of the assumed meaning:

() It was never thought that with four candidates one elector could vote for the candidate of his choice, and another elector could vote for three candidates against him.*

*(**) The preferential system directly diminishes the right of an elector to give an effective vote for the candidate of his choice. If he votes for him once, his power to help him is exhausted. If he votes for other candidates he may harm his choice, but cannot help him.*

W, the incumbent, had a plurality of 1st choices, but most of the $992 + 3417 + 3496 = 7895$ voters with L, N, or S as 1st choice were clearly anti-W, since only $604 + 54 = 658$ of them gave 2nd or add'l choice to W. An anti-W voter would obviously not choose W at all, but could follow (*), give 1st, 2nd & add'l choice to {L, N, S}, and hope that one in the triple would beat W.

With its distinction between 1st, 2nd and add'l choices, a Duluth ballot may distinguish between four candidates just as a Borda ballot. With counting of approval points in each round, one must expect some similarity in properties of the two methods. In (**), the court points to the strategy now called "Burying"; (0.3) describes it with three candidates. Without being anti-Y, a voter has

²⁹ The table does not show data that allow counting with IRV or Condorcet methods. Each round is reminiscent of "Approval voting", where a ballot specifies two indifference classes, i.e. of approved and not approved candidates, (freely chosen), but there the candidate with most approvals wins. A Bucklin winner may be a candidate first to appear in the choice of 50% of the voters, but a visible feature in the tally table is that this is not enough in the Duluth variation: After round 2, Smallwood was already chosen by 6341 of the 12313 voters, but the tally went on.

reason to fear that 1st choice X will be harmed by a strong Y as 2nd choice, and therefore not choose Y at all, neither as 2nd nor add'l.

REMARK (6.2) The Borda Count urges voters to behave as described in (*) and (**) of the 1915 ruling. Confronted with the idea behind “Burying” (0.3), Borda is reported as saying “*My scheme is only intended for honest men*” (Black 1958). However, it is easy to see that the Borda Count has a serious weakness even when all voters are honest and well qualified to have an opinion: *Sports journalists elect their country’s “athlete of the year”*. Comparison of candidates in the same sport may be objective, based on results. Comparison of candidates from different sports is much more demanding and cannot be equally fine-tuned. Candidate A in sport I has 60% of the top-ranks, and B in sport II has 40%. However, in everybody’s mind, C in sport II is very close to B, just slightly behind B according to results. Thus, every ballot is ABC... or BCA... . The Borda Count gives every B-supporter a double vote in {A, B}, and B becomes Borda-winner.

A set of k candidates are “clones” if they occupy k consecutive ranks in every ballot (Tideman 1987). The Borda protagonist Michael Dummet (1998) was concerned about the *(dis)similarity effect*, where A loses to B because of dissimilarity between a small group of candidates similar to A and a larger group similar to B. Clones just make a special case suited for theoretical study. In the sports journalist case, A would win in round 1 with the Duluth method, but statement (*), indicates that the Minnesota Supreme Court in 1915 was alert to (dis)similarity effects.

REMARK (6.3) Smallwood was elected, with the Duluth method, on April 6th 1915 and installed in the office on May 3rd, before the Supreme Court, on July 30th 1915, decided that Smallwood was not entitled to the office because the method was unconstitutional. According to Field (1935), this led to new “... tangled situations ... as a result of conflicting claims to the salary ... ” in *Windom v Prince* (state ex. rel.), *Smallwood v Windom* (state ex. rel.), and *Windom v Duluth*.

Ninety-four years later In 2009, the same court summarized its finding in a “syllabus”:
Instant Runoff Voting as adopted in Minneapolis is not facially invalid under the United States or Minnesota Constitution, and does not contravene any principles established by this court in Brown v. Smallwood, 150 Minn. 492, 153 N.W. 953 (1915)

The court considered *“the burden that appellants contend IRV imposes on the right to vote”*. The eliminations in IRV means that no ballot counts as supporting several candidates at the same time, see statement (*) from 1915, and that Burying is impossible, see statement (**).

The last burden claimed in 2009 was, despite a wrong formulation, linked to non-monotonicity:

- *by creating the possibility that casting a vote for a preferred candidate may harm the chances for that candidate to win office.*

A candidate can never become IRV-winner through cancellation of some ballots where the candidate is top-ranked.³⁰ However, the court interprets this final claim as a complaint against the trap effect: *“... the final assertion relates to appellants’ argument that IRV burdens the right to vote because it is non-monotonic”*.

The court compared IRV to the 2-day *“primary/general”* election, used in Minneapolis before adoption of IRV, and still the alternative to IRV. Obviously, the 2-day election has many features in common with IRV, in particular strategic voting with an idea akin to Pushover; see Example (3.3), but it was not under attack.

Naturally enough, the appellants could not establish that non-monotonicity was a significant real-life effect and not just a possibility. *“Although it is apparently undisputed that the IRV methodology has the potential for a non-monotonic effect, there is no indication, much less proof, of the extent to which it might occur, and so there is no way to know whether the alleged burden will affect any significant number of voters.”*

The real-life effect requires a change in two *“accounts”*; see Example (0.1). It is not likely that anyone can prove to a court that 100 BCA-votes, say, cast at the polls, *“originally”* belonged to the category CBA, but see (2.19).

Election rules and Rule of law

Many anomalies are ubiquitous in the field of election rules, but they do not always concern the fundamental rights one should associate with the Rule of Law.

The Frome election was a *“trick”* position: The tally and a piece of additional information showed, in hindsight, that a suitable group of B-supporters might have moved from BCA to CBA, and let B

³⁰ In (many) Condorcet methods, some 3-cycles allow this *“strong”* No-Show paradox; then cancellation of some ballots with candidate X on top turns X into winner. In IRV only the *“weak”* kind occurs (section 3).

snatch A's IRV-victory with Pushover, as shown e.g. in Figure (2.5). Every trick position and a trap position obtained from it with Pushover (section 2) form a "non-monotonicity pair".³¹

(6.16) *The fallacy of the "rightful winner"* This is an argument that IRV always picks a "wrong candidate" in at least one of the two elections in a non-monotonicity pair. This purported result receives some attention, in particular because a Wikipedia definition of monotonicity, itself trivially but significantly fallacious, is a background:

"A ranked voting system is monotonic if it is neither possible to prevent the election of a candidate by ranking them higher on some of the ballots, nor possible to elect an otherwise unelected candidate by ranking them lower on some of the ballots (while nothing else is altered on any ballot)".

Trivially, up- or down-ranking of B is possible *only if something else is altered*, and that on the same ballot. Significantly, in IRV it is this concomitant, opposite move of C, which changes the next elimination and decides whether B will be challenged by A or C. B loses *despite* more top-ranks, *not because* of it; see Example (0.1). Alert readers may notice that a change in top-ranks can decide between A and C, but a fallacy repeated in places purported to inform about IRV, has a psychological effect. The "rightful winner argument" goes like this:

In a non-monotonicity pair, A wins the trick position, and B the trap position. Assume that B also is the "rightful winner" in the trap position. Then B is the "rightful winner" in the trick position too, because there B has more support.

The notion that an election has a "rightful winner" reminds of a medieval view; "voting was conceived as a way to reveal God's will and discover the truth" (Colomer 2013, p. 318). Other medieval ordeals served the same purpose, and a "voice of the people" may be equally difficult to discern.³² Rule of law should sustain fundamental standards of fairness not based on any nebulous ideas about "rightful winner" or "voice of the people".

³¹ Possibly, but rather unlikely, the unknown subsidiary votes of the B-supporters created the Condorcet cycle, constellation *iii_{cyclic}*, in Figure (2.5). If so, the real Frome election is a "trap" position and a "trick" position at the same time, with "non-monotonicity" partners in constellation *i_{cyclic}*; see Figure (2.2).

³² In a letter to Charlemagne, Alcuin made a distinction: 'And those people should not be listened to who keep saying the voice of the people is the voice of God, since the riotousness of the crowd is always very close to madness'.

(6.17) Election rules and their limitations The G-S impossibility theorem, (Gibbard 1973, Satterthwaite 1975), provides another background. It is linked to Arrow's theorem and is valid for any preferential single seat election method, with three or more candidates, such that

- 1) it always picks a unique winner;
- 2) every candidate will win with a suitable preference distribution.

The G-S concludes that if there is no *dictator* (i.e. a special voter who picks the winner), then preference distributions exist which allow *strategic voting*, but with *a very broad definition*.

In the sense of G-S, a voter who ranks X before Y, performs a successful strategic voting by changing the ballot in any way that makes X win instead of Y: The intention expressed by the first ballot is more effectively represented by the second ballot. With such a wide definition, the result of G-S is perhaps less surprising than the effort that a proof of G-S requires from a reader.

REMARK (6.4) Together, the definitions of Compromise and Pushover for $n = 3$ candidates in (0.3) are extended to $n > 3$: An actionist reshuffles the candidates ranked above the target Z, and one of them wins instead of Z. A very wide strategy definition is essential for G-S. It includes strategies that are not formulated anywhere and may work in election methods never defined. Just as IRV avoids Burying, one might hope to find a method that avoids both Burying and extended Compromise/Pushover without violating one of the "symmetry conditions".³³

Unfortunately, that is not the case: Assume that a 3-candidate preferential single seat election method treats both voters and candidates in symmetric ways ("*anonymity*" and "*neutrality*"), and that it avoids both Burying and Pushover. If the order by top-ranks is XYZ, then Z cannot beat X in the social ordering (Stensholt 2010). The consequences are quite annoying:

EXAMPLE (6.3) In constellation v and v_i , the Condorcet winner C is last in top-ranks. By Remark (6.4), Condorcet winner C can then only win if the method either violates a symmetry condition or allows one of the strategies Burying and Pushover. Sometimes the public may find it very unfair that IRV eliminates C. Consider e.g. the two elections in constellation v_i shown above: One is counterfactual, but realistic and compatible with the standard tally in Frome 2009, Figure (1.2), third pictogram; another is the real Burlington 2009 in Figure (3.4), first pictogram.

³³ As illustrated in Figures (2.1) and (2.2), Pushover and Compromise are closely related, but they are very distinct from Burying. One preferential method that obviously does not allow extended Compromise/Pushover lets the candidate with the smallest number of bottom-ranks win. Are there other preferential single seat election methods, seriously suggested, that never allow a voter group to gain by extended Compromise/Pushover?

A legal ban against rules that allow an annoying anomaly may have consequences that are even more annoying. Proposed election rules should be simple and understandable, but the Rule of law requires them also to protect and promote fundamental voting rights and people's notions of fairness. G-S shows that perfection, or at least an interpretation of it, is unattainable. That calls for a balanced assessment of pros and cons for the election rules rather than an unconditional red or green light for a method that allows Burying, Pushover, or No-Show. See (6.18) below.

(6.18) *Plurality and Compromise* Among all strategies, the Plurality method allows only "Compromise", which means a switch from *expressive* to *instrumental* voting. For protagonists of the Plurality method it is an obvious argument that Compromise is a "*sincere manipulation*" (Dowding and Van Hees 2008). The method contributes to shaping and keeping up a political landscape with two major parties (Duverger's law), which many see as a good thing. However, when and how a 2-party system develops into destructive polarization of a society remains a topic in political science and public debate.

In Plurality elections, existence of minor parties urges voters who would like to support them to consider the Compromise strategy. In a political landscape shaped by a Plurality tradition, opinion polls give voters information about the preference distribution, which is useful to those who consider voting instrumentally. One must expect that some voters experience external pressure and mental stress in their choice between expressive and instrumental voting.

In other Plurality elections, a voter may have no way to know what candidates that various alliances of other voters will support, and therefore be at a disadvantage.

Like all strategic voting, Compromise leaves no trace in the tally accounts, see Example (0.1), but there is abundant evidence that Compromise is important in Plurality elections. Still, opinion polls as in Example (4.2) reflect only voters who shortly before election decide to vote instrumentally. Is this only the tip of an iceberg? Those who, without qualms, always vote instrumentally to avoid wasting their votes, remain invisible.

(6.19) *Monotonicity failure on the balance scale* In the Minnesota Supreme Court decision 2009, the former decision of 1915 was understood to allow IRV, since, at any tally stage, only the top-ranks are available to the tally officials. The court recognized that IRV is non-monotonic in

the sense that trick- and trap-effects may occur, but the lack of proof from the appellants' side "*of the extent to which it might occur*", was decisive.

Could the appellants, with proper preparation, have proven for the court any case of real occurrence? In Frome 2009, constellation *iii*, h BCA-voters, $31 \leq h \leq 321$, could *possibly* have come from CBA and unfortunately walked into a non-monotonicity trap, but how to substantiate that it really happened? A more likely possibility was that f ACB-voters, $16 \leq f \leq 3801$, came from CAB and, with the Compromise strategy, fortunately avoided election of B; see Figure (1.3).

For a simple ballot change, one cannot assume "*everything else being equal*"; this phrase is just convenient in theories. Even if an unpleasant experience as in (2.19) really occurs, in the real world's context of Compromise and its reversal, of natural sortition etc, it is wrong to blame the trap effect.

(6.20) *The Participation criterion on the balance scale* The third pictogram of Figure (1.2), constellation *vi*, is realistic and compatible with the standard tally in Frome 2009; fortunately it is counterfactual. C wins clearly against B (56% v 44%) and crushingly against A (66% v 34%), while A's win over B is more modest (52% v 48%). With "Perfect Pie-sharing", nothing is "pathological" in the preference distribution. Similar situations must occur now and then in the penultimate round of real IRV-tallies. If q of the $y = 7062$ BCA-voters stay home, $2684 \leq q \leq 2954$, then C wins instead of A; see also Example (3.1), $y = 5600$, on the stapled line in Figure (1.1).

Many citizens will find A's victory unfair to Condorcet-winner C and particularly unfair to those BCA-voters who supported runner-up B from round 1 *and must see that a tally by the present rules ignores their overwhelming subsidiary support for C*. If it also became clear that the Participation criterion was violated (a suitable *reduction* of $|BCA|$ would change the IRV-winner from A to C) one should expect strong reactions. The alternative rule (4.5) lets C win.

Formulations in the Minnesota judgment 2009 do not distinguish between the No-Show accident and the non-monotonicity trap; this reflects a serious lack of preparation from the appellants. Therefore, it remains an open question what the Minnesota Supreme Court would have said if required to consider the fact that IRV violates the Participation criterion. Is this fact, in turn, a violation of the constitution or incompatible with the Rule of law?

The ruling of *The German Federal Constitutional Court* (6.14) concerned an anomaly, in a multi-seat context, but akin to the strong form of the No-Show paradox where candidate X may win through cancellation of some ballots with top-rank to X. According to Theorem (0.2), this cannot happen in the usual IRV. In a 3-candidate Condorcet final (4.5), depending on the chosen Condorcet variation, a number of extra ballots with the winner X on top may destroy the election for X, but obviously only if there already is a cycle.

(6.21) *The Burying strategy on the balance scale* A variation of IRV considered in (4.5) eliminates according to the Borda ordering (instead of the top-rank ordering) when exactly three candidates remain. The idea is to obtain more fairness, by avoiding both results like A's victory in the third pictogram of Figure (1.2) and a tally that always neglects the subsidiary rankings from supporters of the runner-up. However, when the final rounds are a 3-candidate Condorcet method, the price is that Burying sometimes must become possible.

The Bucklin methods have the Burying strategy in common with the Borda Count, but clearly not to an equally destructive extent. In a Baldwin method, it is difficult to target a Condorcet winner, since actionists need a reliable prediction and must accurately create a suitable cycle with the Condorcet winner, where their favorite wins. Moreover, potential actionists are likely to not be sufficiently motivated.

Would the 1915 Minnesota decision have worked as a precedent, and led the same court in 2009 to declare all preference methods unconstitutional if they allow Burying? Such a zealous interpretation of the 1915 decision will block a change to (4.5), and thus keep violations of the Participation criterion.

(6.22) *The ordeal of eliminations based on top-ranks* With three candidates, the IRV variation defined by (4.5) and restriction $\lambda = 1$ in (4.6) suffice to avoid No-Show. With $\lambda = 0$ (no restriction), monotonicity failures also disappear. In non-cyclic cases, the remedy is to keep the Condorcet winner, who is otherwise eliminated in constellations v and vi .

With $n \geq 4$ candidates, those anomalies will occur. In a political landscape with two major parties, the competition for third place may be similar to competition for victory in ordinary IRV. Thus, a No-show accident or a non-monotonic event is still possible. With hindsight and more data, an

analysis may sometimes show that a voter group could have changed their ballots and caused the elimination sequence to give a different final candidate triple, and a, for them, better result.

Voters ought to know that throughout the ordeal of eliminations, a ballot never harms its top-ranked candidate: Cancellation of the ballot cannot possibly help the candidate. Moreover, an attempt to gain through extended Compromise/Pushover may go very wrong and cause a worse result. Voters ought to be sceptic against attempts to organize strategic voting.

Without being perfect, the elimination ordeal of IRV finds, for the tally final, three worthy contenders who show strength by the top-ranks they obtain.

If $\lambda = 0$ in (4.6), there will always be a Condorcet final with three candidates. If $0 < \lambda \leq 1$, *the subsidiary rankings of the B-supporters are not neglected*; thus, all voters who did not give top-rank to the elected candidate, influence the tally through their subsidiary rankings. If $y = |\text{BCA}| \geq \mu(\lambda)$, there will also be a Condorcet final.

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