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Effects of Timing and Multiple Entries in Hotelling:

A One-sided and Two-sided Market Analysis

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The purpose of this paper has been to study how timing and multiple entries affect the equilibrium outcome in a Hotelling model for one-sided and two-sided markets. After reviewing relevant theory, we present two models: a location-cum-consumer-price model (one-sided market), and a location-cum-advertisement-price model (two-sided market). In the models, the firms, choosing locations on the Hotelling line when entering and then prices, either make their location-decision simultaneously, or one by one. These two models, both with and without sequential location-decisions, has been solved using numerical analysis. The equilibrium outcomes have been analyzed using the Herfindahl-Hirschman Index, a Locational Asymmetry Index, and by looking at consumer surplus loss due to transportation costs. We find that increasing the number of firms generally lead to a more socially optimal outcome. When allowing for sequential entry, some firms manage to take advantage of the locational structure. We observe, however, that it does not always pay to move first.

Acknowledgements

This thesis would not have been possible without the input and assistance of several persons affiliated with the Norwegian School of Economics.

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We also owe a large thanks to assistance professor Chang-Koo Chi, also in the Economics department. During a brief conversation, in a time when we were stuck looking for difficult solutions to a difficult problem, he revealed a solution hiding in plain sight, vastly simpler than those we had looked at. The scope of this thesis would have been greatly reduced had it not been for this conversation and the subsequent email-correspondence when implementing the solution.

In all our time spent in the Economics department, there has been a couple of other persons that have been influential for our thesis: Professor Fred Schroyen's mathematical expertise made us realize the difficulty of assuming anything else than a uniform distribution of consumers along the Hotelling line.

PhD candidate Timothy G.A. Wyndham had a simple, yet illuminating, input on the Hotelling model, which inadvertently loosened a writer's block.

In addition, we owe our thanks to the Norwegian School of Economics' administration, which granted us access to their best performing virtual desktop. Without it, this thesis would not have been feasible.

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1 Introduction

The Information Age has enabled consumers to cater to their deeper preferences for specific product varieties. The product variety range has increased, and firms offering products have also increased in number. This is a consequence of how the Internet has changed the competition structure between firms: how firms interact with customers, their cost-structure (Anderson, 2004; Brynjolfsson et al., 2006; Hinz et al., 2011), and how they generate revenue.

Firms can target the consumers' preferences by deciding on product characteristics. If firms choose differing characteristics, on otherwise similar products, we say the products are differentiated. A firm's decision on how much to differentiate determines the firm's and competitors' demand, which makes this decision an influential and important tool. The Internet has made endlessly differentiated products accessible for consumers at a keystroke's notice, making differentiation a central part of today's consumerism.

The Internet has also transformed the way a lot of firms choose to make money. From the traditional business model of charging consumers a price for goods and services, firms today has, to a larger degree than in the past, the opportunity to generate income through a mix between consumer prices and adverts, or solely through adverts.

Together with the choices of differentiation and revenue streams, the timing of firms' product announcement is an important tool in their strategic apparatus (Krishnan and T. Ulrich, 2001). In many industrial economics models, however, the timing of decisions is ignored because of the convenience of assuming decisions happen at the same time. Real-life decisions are generally thought of as occurring one after another Tirole (1988), but with simultaneous decisions, firms' ability to account for competitors' past decisions or future responses is overlooked. In the age of the Internet, where information flows freely, this seems like a particularly dubious assumption.

We believe these aforementioned decision tools play an important role in the competition outcome between firms. Accounting for these tools, and multiple firms, in models within the field of industrial organization is meaningful to gain knowledge on markets that grow in complexity due to technological developments.

The foundation of our analysis is d'Aspremont et al. (1979)'s version of Hotelling's linear city, where firms choose product characteristics across a subjective dimension when entering the market. Hotelling originally studied a duopoly with simultaneous entry, with firms competing in consumer prices. We employ two models based on his model, with modifications allowing for more firms to enter as well as firms entering sequentially. The first model is for a one-sided market, where firms have chosen the traditional business model. The second model is for a two-sided market, where firms have chosen to generate their income solely through advertisement.

There exist several articles that we have used as building blocks when developing our one-sided market model. Among these are Brenner (2005), who looks at simultaneous entry and Prescott and Visscher (1977); Lane (1980); Neven (1987); Economides et al. (2002), who study sequential entry, all with multiple firms.

Our analysis stands out with our extension to include two-sided markets, developing a model based on Dietl et al. (2013). We know of no other literature with the same approach as ours for two-sided markets.

We compare and characterize the market equilibria produced by both models, by using tools such as the Herfindahl-Hirschman Index, a Locational Asymmetry Index, and a welfare measurement for the consumers' surplus loss caused by transportation costs. Using these tools we make comparisons between traditional and two-sided markets, allowing for both simultaneous and sequential entry in each model.

Using Python, we solve the models as far as possible and then use numerical techniques to find and analyze the equilibrium outcomes. Numerical techniques were necessary since the models quickly became unsolvable analytically when including multiple firms. The runtime of the simulations has ranged from under a minute to exceeding one month and, although we were blessed with high-grade hardware, some simulations were not possible to complete.

The paper is structured as follows, in Chapter 2 we give an overview of Hotelling, two-sided markets, together with simultaneous and sequential games. In Chapter 3, we provide a formal mathematical presentation of our two models. In Chapter 4, we analyze the models. In Chapter 5, we give an exposition of the numerical analysis employed to solve the two models. In Chapter 6, we present, discuss and compare the equilibrium outcomes. Finally, in Chapter 7, we conclude.

2 Literature review

> (...) I announced one morning, without any previous warning, that in the future we were going to build only one model, that the model was going to be "Model T," and that the chassis would be exactly the same for all cars, and I remarked: "Any customer can have a car painted any colour that he wants so long as it is black"

- Henry Ford, Ford (1922)

Henry Ford, most famously known for lowering production costs immensely on automobiles,[1] was, as the quote suggests, not very fond of accommodating customer's wishes.[2] But however strong he opposed it, the introduction of his widely known car, the Model T, may inadvertently have created one of the largest markets in the world where appealing to wishes of its consumers is essential: The market for affordable cars.

This market aims to reach middle-class consumers, which means cars should be reasonably priced. Therefore, in Henry Ford's time, manufacturers stripped cars of any luxury and started to produce "en masse" to bring costs down. Today, there are several different classifications where we find affordable cars, such as with the compact cars, or the large and the small station wagons. For the affordable cars in these segments, the size and quality is largely the same from one manufacturer to another, and these markets has thus became homogeneous. What is left for each firm to vary in order to attract consumers is the look of the car and its price.

Some consumers may prefer gray instead of black cars. Perhaps a consumer prefers car seats made with nylon fabric rather than faux vinyl fabric. When eating a meal, some may prefer the taste of sweetness while others prefer umami, or, if reading a newspaper, some readers may prefer content favoring their political views rather than content reflecting opposite views. What is common for all these factors is that they are subjective to the individual. A taste

[1] He achieved this by standardizing Model T's design and introducing moving assembly line production in his factories.

[2] In his biography he refers to a customer's wishes with regards to style as "personal whims." We refrain from using Ford's expression, and are opting to use "preferences" instead.

or a political view cannot be ranked as being objectively better than another: although you may prefer to drive a yellow car, it will not drive any faster than a green car if the cars are otherwise identical.

Through these subjective preferences, firms can address whichever group of consumers they want by choice on product characteristics. By choosing black for the Model T, for example, the Ford Motor Company appealed to consumers with a preference for dark colored cars.[3] Whenever more than one firm is present in a market, and each offers a product on which they choose differing product characteristic, we say the firms product differentiate. It is called horizontal differentiation when these product characteristics appeal to consumers' subjective preferences.[4]

Consumers' preferences and firms' choice of differentiation can be represented by a simple model; Hotelling's linear city.

2.1 Hotelling's linear city

Harold Hotelling introduced the concept of spatial competition in a linear city in his seminal paper titled "Stability in Competition" from 1929. The simplest version of the model studies a location-cum-price[5] game where duopolists compete non-cooperatively on a line [0,1],[6] coined the Main Street, with consumers uniformly distributed along it.

As the nickname "Main Street" suggests, the line has traditionally had a geographical interpretation, where the consumers are said to be living on a street. Another well-established interpretation and how we interpret the line, is that it represents a product characterization

[3] The color, and to only have one, was out of necessity rather than choice. Ford Motor Company produced cars at such a pace that the drying of paint had become a bottleneck in the production. Because Ford did not enjoy watching paint dry, and wanted to keep the production rate as fast as possible, he opted for the paint which dried the quickest: black.

[4] A contrast to horizontal differentiation is vertical differentiation, which relates to characteristics that make products objectively better, something which is not discussed in this paper. It can, however, be easily incorporated into our models for future research.

[5] Location-cum-price is an expression used by Loertscher and Muehlheusser (2011)'s to describe a Hotelling model in a context where firms first choose locations along the line, then choose prices.

[6] Hotelling originally studied a line of length 35, but the choice of units is irrelevant for the analysis (Tirole, 1988).

space where the consumers have a variety of tastes (Tirole, 1988).[7] Because of how we have chosen to interpret the line, we mainly refer to the line as the Hotelling line or the unit interval, rather than Main Street.

To illustrate the model we continue with the colored-car example. Imagine that consumers prefer different colored cars on a gray-scale, where the gray-scale is represented with a line of fixed length stretching from white to black. This is our Hotelling line. We place the consumers along it according to which color they prefer the most.

In a duopoly, two car manufacturers compete on the line. They sell only one product, and must make two decisions: The first is where to place their product on the line to reach as many consumers as possible, and the second is to find what price they should charge given this location. Their decisions are entirely determined by the utilities of the consumers and the competing firm's decisions. How well they choose ultimately affects how much demand they will receive.

Generally, it is assumed that the consumers in the model, when buying a product, derive their utility from a common function, which is a function of the consumer's reservation price, the product's price and the consumer's transportation costs for the product. The reservation price, also known as a consumer's willingness to pay, is in our case assumed positive, constant across all consumers and large enough to guarantee that everyone buys at least one unit. Utility depends negatively on prices and transportation costs, and while the price-relation is linear, it may depend for transportation costs.

Although somewhat abstractly named, the term transportation cost originates from the Hotelling line's geographical interpretation. Here, transport costs refer to the cost a consumer incurs by traveling to a firm along the Main Street. In our interpretation, the cost represents a consumer's distaste for moving from their preferred product to a less preferred alternative. Relating to our car-example, it seems reasonable to assume people do not enjoy settling for colors other than their preferred one, and that the further away a product locates along the gray-scale, the more their utility decrease. This loss in utility is the consumer's

[7] The line can also, in some contexts, depict political views ranging from the left (0) to the right (1) Gabszewicz et al. (2002).

transportation cost.

In Hotelling's linear city, the concept of transportation cost is essential for firms to determine demand. To see how demand is derived in a more detailed, and perhaps easier to understand, way, see Chapter 3. In short, when locating along the line, transportation costs enable firms to find the location of an indifferent consumer. This is a consumer located between two firms, and, as the name implies, is indifferent of buying from either one since they receive the same utility from each product. Having found this consumer's location, the firms know that all other consumers located between them and the indifferent consumer's location will buy their product, and thus they have found their demand. Firms also know how the indifferent consumer's position is affected by product differentiation and price, making the firms able to take decisions accordingly. In Figure 1, we see the indifferent consumer, sometimes referred to as the marginal consumer, locating in the middle when firms locates at each their end-point of the line.

Note that the explanation above of how firms derive their demand has the implied assumption that consumers buy only one product, which is not necessarily true. We explain why we keep this assumption in more detail in Section 2.4. A small advantage of keeping it, however, is that by normalizing the number of consumers along the line to one, we can simply refer to a firm's demand as their market share.

As previously mentioned, a consumer's utility may or may not depend linearly on transportation costs. Hotelling assumed the transportation costs to be linear in his original model, i.e., tx where x is a location and t is cost per unit traveled. Hotelling then observed that this lead to an agglomeration of the two firms around the center of the Main Street (Hotelling, 1929). The observation has lead to the term "Principle of Minimum Differentiation" or Hotelling's law: it is rational for firms to imitate each others product differentiation decisions, leading to no differentiation of the products at all.

The result has since been disputed and was declared invalid by d'Aspremont et al. (1979). They showed that there would be no stable price equilibrium around the center, which comes as a result of a discontinuity in the price reaction function of each firm. The discontinuity is caused by firms not always gaining market shares in a smooth and continuous fashion by lowering prices. This is due to the firms operating in what is in effect Bertrand competition[8] when equilibrium locations are too close (Prescott and Visscher, 1977).

d'Aspremont et al. (1979) provided a modification of Hotelling's model by considering quadratic transportation costs, i.e. tx^2. The two firms were then shown to have incentives to increase the distance between each other, since increasing the distance to its neighbour gave larger market power and hence higher prices. This effect resulted in maximum differentiation, shown in Figure 1 - a contradiction to Hotelling's law. Their result emphasizes that product differentiation is beneficial for the firms, i.e., that it is rational for duopolists to divide the market into submarkets in order to gain a larger degree of market power. Economides (1986) later showed that d'Aspremont et. al.'s result was not valid for parameters of transportation costs, tx^α, $\alpha \geq 5/3$.

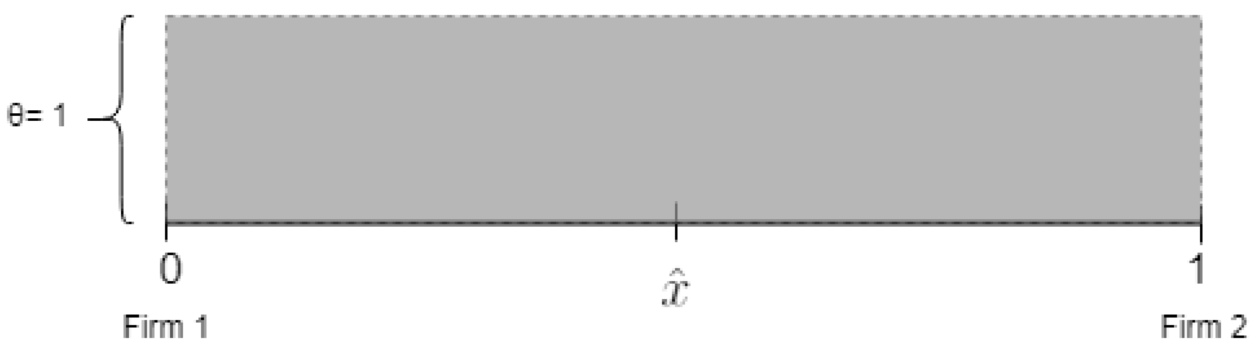


Figure 1: The Hotelling Line

The framework of Hotelling with maximum differentiation. $\hat{x}$ is the indifferent consumer, θ represents a uniform distribution of consumers, here normalized to 1.

2.2 Increasing the number of firms

Our thesis study Hotelling models where two or more firms compete along the unit interval. However, the models we have referenced thus far by Hotelling (1929) and d'Aspremont et al. (1979) are strictly concerned with duopolies, and provide limited insights to what hap-

[8]Bertrand competition is a type of price competition void of product differentiation. Since the products are homogeneous, the firm with the lowest price takes the whole market. The firms will thus undercut each other resulting in an iterative process where firms end up pricing to marginal cost. This equilibrium persists even with two firms, which has become known as the Bertrand paradox: the equilibrium of a duopoly with price competition and homogeneous goods are equal to that of a game of perfect competition (Tirole, 1988)

pens when a larger number of competitors are present. Fortunately, there are others that have expanded upon the Hotelling model: Brenner (2005); Neven (1987); Loertscher and Muehlheusser (2011); Economides et al. (2002). These authors' papers have largely, in one way or another, influenced our thesis.

Allowing more firms into the location-cum-price model with quadratic transportation costs reveals that neither Hotelling's law of minimal differentiation or d'Aspremont et al.'s result of maximum differentiation holds for more than two firms (Brenner, 2005). The result is something in-between. But, although the outcome of the model changes when moving away from the duopoly-case, the intuition of the Hotelling model remains the same as before.

For simplicity, we refer to the firms closest to the end-points of the line as corner firms, and those in-between as interior firms. The indifferent consumer we introduced in the previous section is by definition located between two products, so for each additional firm we add, another indifferent consumer appears. This implies that interior firms derive their demand by looking at two indifferent consumers, one on each side, while corner firms still only need to look at one.

To see why there is no maximum differentiation when more firms enter, we can reason around a case with three firms. Locating at the end-points, would make the corner firms very exposed. If the third firm established itself in the interior, it would capture half the market share simply by placing itself in the middle and offering the same price as the corner firms. This is a threat the corner firms cannot allow, and they will therefore locate a lot closer to the middle, and instead squeeze the middle firm.

The middle firm's response is to reduce its price, and since it is being squeezed from both directions, with nowhere to move, the firm experiences the intensity of the competition to a much larger degree than the corner firms. Including more firms, the increase in competition intensity would be stark because because interior firms would have to compete amongst themselves in addition to being squeezed. When squeezing the interior firms, a big advantage of being in the corner is that consumers located between the firm and the end-point do not have any other options than buying the corner firm's product. This is because they want a product that most closely resembles their own preferences. This market power the corner firm has over these consumers give the firms an incentive to set a higher price.

Our reasoning above builds on the assumption that firms' locations relative to each other are assigned, and that they choose product differentiation all at the same time. We return to the subject of decisions and their timing in Section 2.5.

2.3 Distribution of consumers along the Hotelling line

Although it makes the model more tractable and computationally feasible, the assumption that consumers are uniformly distributed in tastes is not very realistic. Nevo (2003)'s paper, on why the price-cost margin is so high in the cereal industry, and Berry et al. (1995)'s paper, studying demand and supply in a differentiated Automobile-market, are only two examples of several empirical studies where findings do not support uniform distributions.

Nevo wanted to see whether the high price-cost margins could be explained by the firms' product differentiation, their range of products, or by price colluding. His approach was to estimate brand-specific demand, dependent on product characteristics, price, and unobserved effects, and using this estimate to find price-cost margins without having to observe actual costs. As part of his analysis, Nevo finds that preferences vary dependent on demographics, and characteristics such as sogginess of a cereal brand may influence its demand. His finding suggests that the distribution of consumers are not uniform.

Berry et al. had a similar approach when modelling consumer's demand for automobiles, and similarly found indications that the distribution of consumers is non-uniform.

Loertscher and Muehlheusser (2011) examine, similar to us, a sequential Hotelling model, albeit with costly entry and non-uniform distribution.[9] They argue that existing theories' focus on the uniform distribution makes it problematic to study some of the more interesting scenarios in markets where there is a concentration of consumers, and posits this as a motivation for relaxing the assumption in their paper. Their model contribution addresses mainly monotone densities, but classes of densities that are symmetric at the midpoint of the

[9]Introducing entry costs in our model would warrant a natural extension of our analysis to study entry deterrence amongst firms. This is outside the scope of our thesis, and thus we simply assume that number of firms entering are exogenously given.

Hotelling line are, with some additional restrictions, also studied. They find that equilibrium locations can be determined independently of when these locations are occupied.

Even though we find Loertscher and Muehlheusser (2011)'s points valid, and their methodology for arriving at equilibrium locations intuitive, we have instead opted for Neven (1987)'s methodology, with an exogenous number of firms and a uniform distribution of consumers. The main reason using Neven's methodology and assumptions is that it is simpler to pursue equilibria, and verifying our results. We also find it difficult to argue that consumers are monotonically distributed for horizontally differentiated products.[10] This would be a density more likely to be found in markets where products are vertically differentiated along quality.

For simultaneous models, on the other hand, there are several papers that have studied spatial games with a non-uniform distribution of consumers. Neven (1986) finds that if the consumer density is concave and sufficiently concentrated, firms tend to locate closer together, even with quadratic transportation costs. Anderson et al. (1997) find the same results by analyzing a game with a log-concave consumer density function and also quadratic transportation costs. Regardless, since we have chosen to pursue equilibria with a uniform distribution in the sequential model, and because it makes the models more computationally feasible, we have chosen to stick with the uniformity-assumption.

An example of where it would be natural to employ a non-uniform distribution is in the analysis of long tail markets. These are a type of markets where the emergence of the internet, bringing with it its effective distribution channels and low-cost solutions, allow for deeper consumer preferences. These consumer preferences resemble that of a long tail distribution, ranking products against popularity and demand. The phenomenon can be illustrated by fitting a bell-curve to the Main Street, segmenting the market in Niche tastes and Mainstream tastes. This is shown in Figure 2 below.

[10] An exception may be a Hotelling model representing political views, where the population leans toward one extremity.

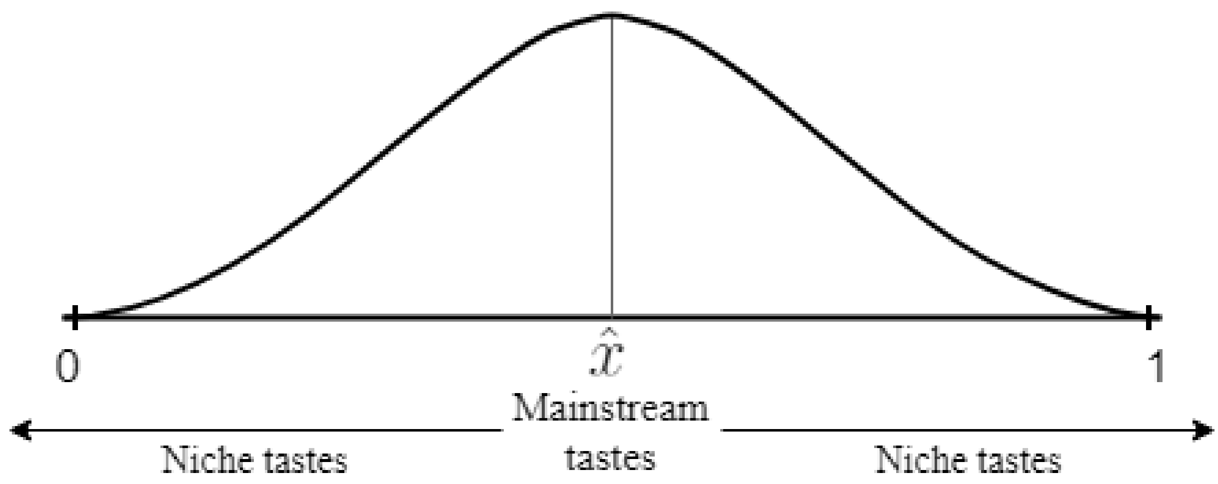


Figure 2: The Hotelling line with a bell curve distribution

An example of a long tail distribution of tastes where the most popular products are in the center of the Hotelling line, referred to as mainstream tastes, while the least popular products in the niche segment exist in the boundaries of the Hotelling line.

2.4 Two-sided markets

Senator Orrin Hatch to Mark Zuckerberg during the Senate hearing about the Facebook–Cambridge Analytica data scandal:

> "So, how do you sustain a business model in which users don't pay for your service?"
> "Senator, we run ads," Zuckerberg responded, smirking
> "I see," Hatch replied. "That's great."

- Transcript from the hearing published by The Washington Post (Bloomberg Government, 10.04.2018)

The theory of two-sided markets is a relatively new addition to the theory of industrial organization. What we so far have described as a firm, is in two-sided markets characterized as a platform. A platform accommodates two groups, the consumers and the advertisers, and acts as an intermediary between the group's interests. Similarly to a firm, platforms make a decision on where to place themselves on the line and pick a price, one price per group. Rochet and Tirole (2006) defines it more formally as "markets in which one or several platforms enable interactions between end-users and try to get the two (or multiple) sides "on board" by appropriately charging each side." The consumer's and advertiser's utility functions are interrelated and often this interrelation results in externalities for one side of the market, caused by the other group's use of the platform.

As is common in the literature on the Hotelling model, we assume consumers are single-homers while advertisers multi-home. The multi-homing-term originates from communication technology, and relates to connecting a host or computer network to multiple networks. The essence is kept in economics, where multi-homing agents are agents that can patronize more than one platform. In our case, the advertisers multi-home, while consumers are single-homers. Single-homing restricts the consumers to buy one product only. We get back to why we make this assumption later.

Simply put, consumers are interested in the product or content that the platform hosts, while the advertisers are interested in hosting their adverts on the platform for consumers to see. It is in the advertisers' interest that as many consumers as possible see their advertisements, thus their willingness to pay for an advert placement increases as number of consumers on the platform increases. This is because an increase in people seeing their advert will equivalently increase their expected number of sales. It is also often assumed that consumers' utility is affected negatively on the intensity of advertisement on a platform, which we refer to as advertisement aversion.

We expand the Hotelling model for these markets because we believe two-sidedness, alongside horizontal differentiation, is a common and important feature of markets in the age of the Internet.

In our models we assume platforms set consumer prices to zero, and only set advertisement prices. Search engines, such as Google, Yahoo, or Bing, may therefore be suitable example for our platforms. Search engines have built a business model on offering their services for free, generating their revenue through ads.[11] The consumers' and advertisers' interaction is illustrated in Figure 3. Consumers receive results to their inquiries without payment, but have to provide their attention to the search engine. Advertisers, seeking this attention, benefit

[11] We reckon a search engine demanding payment per search would not be particularly popular.

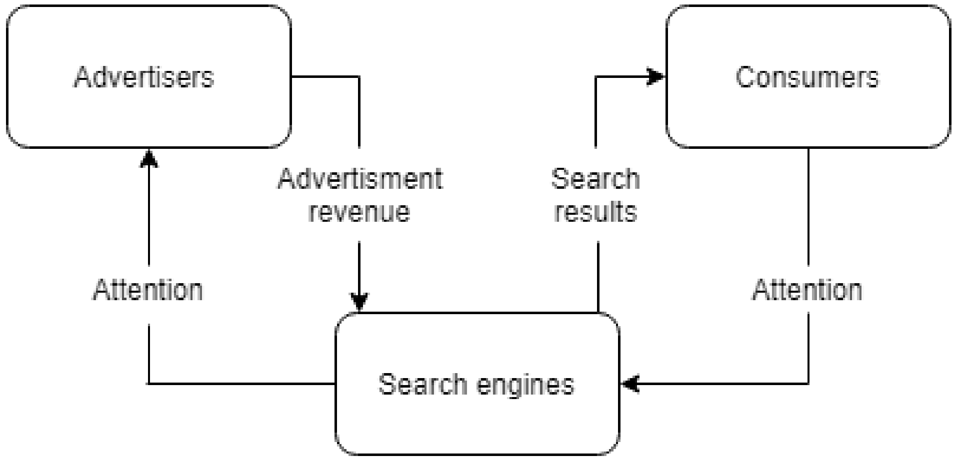


Figure 3: An illustration of two-sided markets in the case of search engines

from connecting with as many consumers as possible. If consumers dislike advertisement, then the advertiser's use of the platform causes a negative externality. The search engine is able to internalize this externality by charging the advertisers a higher price, capitalizing on the group's wish to interact with consumers, implying that advertisers subsidizes the consumer's use of the search engine.

Note that this internalization rests on the assumption that consumers single-home. Since consumers cannot be reached by the advertisers anywhere else in the market than through a single platform,[12] each platform is gifted a sort of monopoly power to exert over the advertisers. The multi-homing advertisers are willing to pay a higher price simply because each platform has a monopoly on supplying their consumers to them.

Choosing single-homing or multi-homing for one of both groups can therefore have a substantial effect on the outcome of a model. Armstrong (2006) studies three models for two-sided markets, and what we have described thus far is the model where one group is allowed to multi-home while the other single-homes. Armstrong refers to this as a competitive bottleneck model. He explains the following mechanism: whenever a single-homing group's use of platforms benefit the multi-homers, the platforms will intensify the price competition for single-homers, while charging the multi-homers a higher price. This is equivalent to the explanation of how search engines internalizes the externalities to charge advertisers a higher

[12] An assumption which may not be very realistic in the context of search engines, but which we use to illustrate.

price, in order to let consumers use the search engine free of charge.

The assumption that consumers single-home might be applicable for goods it is the norm to buy only one of, such as cars, or if consumption of one product mutually excludes another, such as shows broadcast at the same time on the television. However, it might be less applicable for other goods, such as newspapers or search engines. An alternative to the competitive bottleneck model, that Armstrong also suggests, assumes that both groups multi-home. This seems to be a credible assumption in a world in which people can visit several platforms on the internet in a moment of a keystroke, however, it may have its complications. Armstrong does not analyze this model, but reasons around why he did not find this case interesting.

If we assume multi-homing consumers, sufficiently low transportation costs, and that platforms have set their consumer prices to zero, all platforms receive a demand equivalent to the unit interval. Armstrong reasons that if it is the case that every consumer on the interval frequents all platforms, there is no reason for an advertiser to buy an advert placement on more than one platform. Hence, one of the groups end up single homing. If a platform increased its advertisement price when all platforms holds the entire market, the advertisers would simply pick any other platform. This makes platforms unable to charge the advertisers extra to subsidize/capitalize on the consumer's use.

Multi-homing consumers in a two-sided market has, however, been explored, and the results shows that Armstrong's initial reasoning towards multi-homing consumers may have been a bit imprecise. Anderson et al. (2012) tried to reconcile the standard two-sided market models with empirical findings. They attribute the discrepancies between theoretical predictions and empirical findings to assumptions regularly made for these kind of models, namely the assumption of no advertisement congestion and the assumption of single-homing. When discussing the latter, which we are interested in, they reference a paper by Anderson et al. (2016), where Armstrong's competitive bottleneck model is discussed. In this paper, they develop an incremental pricing model, where platforms value consumers that are exclusive to them more than the multi-homing consumers. Each platform's price to advertisers are then the value of their exclusive consumers plus an incremental value of the multi-homing consumers.

Albeit Anderson et al. (2016) present an interesting model, we ultimately chose to assume single-homing consumers. This was because of the large literature using this assumption, its simplicity to implement and computational feasibility for our models. We note that allowing for multi-homing consumers would be an interesting expansion of the models.

We have to now based this section on a premise that consumers dislike advertisement. But are consumers in reality advertisement averse? Many theoretical papers in industrial organization assume consumers have an aversion to advertisements, making it an established practice. However, whether advertisement causes disutility for consumers is an empirical question. It has been addressed in a few studies, but with contradicting results.

Kaiser and Song (2009), using data on German consumer magazines between 1992 and 2004, find little evidence in support of an overall dislike of advertisements. When analyzing several different magazine segments, like Women's magazines and Business and politics magazines, they found only one segment where readers disliked advertisements: adult magazines. In most other segments, they find that consumers appreciate advertisement. Common for most magazines is that the advertisements are more likely to be perceived as informative, while in the adult magazine's case they are considered rather uninformative.

An example often used to illustrate platforms is newspapers, where the assumption of disutility on adverts is widespread. Also here, the empirical results are mixed. Some studies that has taken a closer look at these markets present evidence that most European newspaper readers are advertisement averse (Sonnac, 2000), while American newspaper readers seemingly like advertisements (Rosse, 1980).

Through reasoning, we believe advertisements are more likely to cause consumers a disutility for larger amounts of ads, and that assuming disutility is therefore a reasonable element to include in our models. A consumer using a search engine might not be annoyed by one or two advertisements presented in the search results, but increasing the number of adverts may dissuade the consumer from continuing to use the platform.[13]

[13] Note that, as far as Google-searches go, adverts seem to appear in searches only when the adverts are relevant to your search. Advertisement disutility may therefore not play a large part for search engines since

2.5 Decisions and their timings

It has already been established that firms and platforms must make two decisions: choose a location along the Hotelling line and pick a price.[14] We have, however, refrained from mentioning anything about the timing of these decisions.

First and foremost, we assume that firms make their location-choice prior to picking price, an assumption which is regularly made for Hotelling's linear city (Tirole, 1988). The intuition is that prices are more often than not a flexible decision variable, and more flexible than picking product differentiation.

A second aspect of timing is how firms make their decisions relative to their competitors. Ignoring every mixed case that is in-between, we can assume either that all firms make their decision at the same time, or that they make their decision one after another.[15]

We develop two models in Chapter 3, one for one-sided markets, and the other for two-sided markets. In addition, for each model there are two versions: One version where the location-decisions are made simultaneously and another version where firms choose their location in sequence.[16] In the following subsections, we present literature that is relevant for each version.

2.5.1 Simultaneous games

Although not explicitly stated, the models of Hotelling (1929) and d'Aspremont et al. (1979), presented in Section 2.1, Hotelling's linear city, are examples of duopolies where the location-decision is made simultaneously.[17] However, since we are particularly interested in studying

the adverts are more likely to be perceived as informative.

[14] Additionally, firms also decide whether or not to enter. However, we assume that choice of entry and location happens at the same time, and we therefore take to say that entering the market and picking a location means the same thing.

[15] In the real world, sequential decision-making is often referred to as the most realistic assumption to build into a model (Tirole, 1988).

[16] In all our models, and each version of them, we assume prices are chosen simultaneously in the last stage of the model.

[17] Interestingly enough, the equilibrium outcome is the same regardless whether they decide on location at the same time or one after the other.

Hotelling models including more than two firms, we find Brenner (2005)'s paper to be especially relevant. One of our model versions is equivalent to his. This thesis has been heavily influenced by his paper, and the equilibria he found has been a great tool for verifying our own model's results.

Similarly to Brenner, we look for pure strategy equilibria and assume that each firm's location relative to one another is exogenously chosen. By exogenously chosen, we mean that firm 1 is assigned location x_1, firm 2 is assigned location x_2, firm 3 is assigned location x_3 and so on, where $x_1 \leq x_2 \leq x_3 \leq \ldots$. Thus we avoid a coordination problem between the firms. It is pointed out in his article that not providing any explanation to why or how the firms order themselves on the unit interval may be a shortcoming of his model. Note that this shortcoming is not present when the location-decision becomes sequential.

Brenner finds equilibria for up to nine firms, and points out that prices can be characterized as having a U-shaped structure. We find this remark to be correct for up to eight firms, however not for nine and ten firms, something which we show and discuss in our results in Chapter 6.

Similar to what we have found, Brenner notes that corner firms has a lot of market power, and that the distance between the corner firms and their closest neighbor is larger than distances between interior firms. He states that this is caused by an asymmetry in the price competition.

2.5.2 Sequential games

If we instead of assuming that locations are chosen at the same time, we assume locations are chosen sequentially along the Hotelling line, the intuition on how firms choose the best possible location changes substantially.

If the firms choose locations one by one, and assuming they have complete knowledge about the game, we say our model is a dynamic game of complete and perfect information. Dynamic games are games that have either repeated stages where each player observe the outcome of each stage before the next, and/or firms makes decisions in a sequential fashion and observe the previous actions of other players before making their choice.[18] In the scope of our thesis, we limit ourselves to discuss dynamic games in a sequential fashion.

Complete and perfect information implies that previous actions made by other players are observable and that the current player knows precisely how his decision affects the decisions of players coming after. The players' payoffs from each feasible combination of moves are thereby common knowledge. Based on the available information, each firm maximizes its payoff based on rational expectations of the actions of other players, or as Prescott and Visscher (1977) put it: "Each firm (...) recognizes that other potential entrants into the market are not unlike itself; no firm mistakenly considers itself a profit-maximizer in a world of fools."

Many economic problems are of such nature, and also popular leisure activities, like Chess and Go, are examples of games that fit the description of sequential games. Examples of economic problems are Stackelberg (1932), Rubsenstein's (1982) sequential bargaining model and Leontief's (1946) model of wages and employment in a unionized firm. Stackelberg games are some of the most known dynamic problems within industrial organization, and we therefore turn to it as a point of departure in describing the intuition behind dynamic games of complete information. Explaining Stackelberg here also makes it considerably easier to later explain the complexity of sequential location choice in a spatial model.

Consider a Stackelberg game with Cournot competition.[19] The players decide how many units they want to produce one after another, and the first player is called the Stackelberg leader. The remaining players, making their decisions later, are Stackelberg followers. The leader is rational, and knows how the followers' optimal decisions must affect his payoff at the end of the game. Therefore, when making a strategic decision s_0 at time 0, he will choose an optimal quantity that takes all of their optimal decisions into consideration.

The follower next in line observes the leader's choice and must then choose an optimal action

[18] Note that when the location-decisions happened simultaneously, the model could also considered dynamic, as firms would choose price after locations. However, we found it practical to later refer to our models with different timing as simultaneous and sequential games, so we conveniently ignore to mention this any further.

[19] Cournot competition is a type of competition where firms compete in quantity and where the goods are homogeneous (Tirole, 1988)

s_1 at time 1. When this follower now makes a decision, he is the leader over the remaining players, and needs to consider as the first player did how later players will respond to his quantity-choice. This process repeats for all remaining players, where player i picks an optimal action s_i at time i, which affects later players.

Knowing everyone's reaction to his decision, the first Stackelberg leader can ensure a first-mover advantage. However, this advantage depends on the leader's ability to commit to a decision at time 0 (Sargent and Stachurski, 2016). Absent said strategic commitment, the leader is not able to ensure any advantage.

The conclusion of first-mover advantage is sensitive to the type of competition the firms operate in. Above we assumed Cournot competition, where decisions variables are strategic substitutes. This results in the classical example where the leader can choose a higher quantity produced in order to soften the competition and thus gain an advantage over its peers. The result changes when looking at competition where decision variables are strategic complements. One example is price competition with differentiated goods. In a Stackelberg leader-follower framework, this type of competition results in a first-mover disadvantage (Shy, 1995).[20]

The intuition of leader/follower in the Stackelberg case is transferable to the case of a spatial model such as Hotelling's linear city. Unfortunately though, locating optimally along a line proves not to be as straight forward as choosing an optimal quantity. The reason for this is that the firms need to know how all the firms will position themselves relative to each other at the end of the game, to determine whether or not a location is ideal. Since this positioning is seemingly impossible to determine beforehand,[21] the sequential game will consist of a large tree of subgames. The first mover has to choose between as many subgames as there are possible positions, while the followers, as previous firms has made their locations, have to choose between possible positions that are left. In every subgame, the followers can deviate

[20]This is because it is rational for the follower to always slightly undercut the prices of the leader, thus resulting in a higher profit for the follower. The profits are however in general higher for both, compared to simultaneous entry, due to the slope of the reaction curves.

[21]At least in the case of uniformly distributed consumers. Loertscher and Muehlheusser (2011) was able to determine where firms would locate relative to each other when consumers were distributed according to monotonic density functions.

from the leader's plan, which indicates that there is not necessarily a first-mover's advantage, as in the Stackelberg game. Because it is difficult to give a quick and intuitive understanding of how the game works, we return to how optimal locations are found in our exposition of numerical methodology, in Chapter 5.

There exists some literature on sequential games of complete and perfect information in the setting of Hotelling's linear city, but the literature is far from exhaustive. One of the pioneering papers is Prescott and Visscher (1977). They considered an extension to the original Hotelling model by studying a game where firms set prices and quality, interpreting the [0,1] interval as waiting time offered to customers resulting from said quality. They looked at a game with three firms. They furthermore implemented some modifications to the original Hotelling model, which ensured continuity in the price reaction curves, avoiding the problem of Hotelling's original model.[22] Due to these modifications, the resulting equilibria were quite different from later literature.

Another central contributor to sequential entry in Hotelling's spatial game is Lane (1980), who considers a model where firms differentiate through bundling of product characteristics. This bundling lets firms differentiate across more than one dimension. They furthermore assume a Cobb-Douglas specification of consumer's utility function where the bundle of characteristics is assumed to be two relevant product characteristics. The bundling of product characteristics, together with the Cobb-Douglas utility function, produce results somewhat different to what we observe later.

They extend the analysis by looking at the number of firms being endogenously determined and illustrates how the number of firms is sensitive to fixed costs of entry. The introduction of fixed costs changes the dynamics of the game by

1. changing the cost-benefit analysis of entry. Firms will enter the market as long as they can make strictly positive profits. Firms who are indifferent between entering and staying outside the market are assumed not to enter (i.e., profits equal to zero). An equilibrium is thus reached when every firm manages to earn positive profits, and no

[22]Note that Prescott and Visscher (1977) published their paper two years before that of d'Aspremont et al. (1979), they thus developed an independent solution to the discontinuous price reaction functions problem

later firms would be able to cover their fixed costs if entering. In addition to that, no firm has incentives to unilateral change its price.

2. enabling firms to employ deterrence strategies, as fixed costs work as a natural barrier of entry. The effect of locations on prices, and in turn profits, gives incumbent firms incentives to engage in entry deterrence behavior. This is in effect done by incumbent firms choosing locations in such a way that it is not profitable for another firm to enter, and in doing so are able to hold a higher profit than if it had entered. Such behavior is a dominant force in Lane (1980)'s model. Furthermore, there is a public good aspect to deterrence since every incumbent firm benefits from it, and the sequence of entry determines the sharing of the burden of entry deterrence.

Lane (1980)'s positive profit condition reduces the strategy space of incumbent firms, since the price vectors are reduced such that no firms undercut their competitors. Lane thus assumes away the non-existence of price equilibria, described earlier. Furthermore, Lane obtains an interesting result when it comes to distribution of profits between firms when studying a triopoly. Even though the first firm to enter retains a first-mover advantage consistently, the same cannot be said about the second firm. Here, the first firm to enter locates at half of the unit interval, while the second and third entrants takes the positions as corner firms. However, the third firm manages to locate closer to the end-point of the Hotelling line and is exposed to less competition compared to the second firm, thus securing a higher profit in comparison. Thus, the problem of motivating the second firm to enter before the third firm arises. This is because the second firm lacks the incentive to enter the market when she knows there will be more profits to be earned later on. Consequently, firm two is inclined to play a waiting game.

Neven (1987) furthers the research of Lane, but develops a separate model, studying a one-dimensional product space in line with Hotelling's framework. Neven's model is very similar to the one we analyze later on. Neven studies the same problems as Lane: sequential entry with and without entry deterrence. However, Neven only looks at games where the number of firms is endogenously determined. Moreover, he puts no a priori restriction on price strategies. This lets him study a broader set of deterrence strategies. Neven's modified model and assumptions avoid the problem with motivating entry for the second firm when the number of firms equals three.

Neven suggests that the asymmetry in location choice fades as the number of firms increases. This makes sense since the first entrant's ability to capitalize on the asymmetric behavior of the later entrants must in some way be related to the number of possible location choices, determined by the number of firms allowed to enter. If the number of firms is determined by a non-negative profit condition, where costs are negligible, then this should result in little distance and high rivalry between firms on the unit interval, since every profitable location would be filled.

Economides et al. (2002) presents another paper looking at sequential entry and deterrence. However, their model model and approach are almost identical to Neven (1987). We base much of our solution methodology for sequential games on both Economides et al.'s and Neven's papers.

2.5.3 Multilevel optimization and Mathematical Programming with Equilibrium Constraints

The problems that needs to be solved in the sequential games becomes a lot more complex than in the simultaneous versions, and we therefore limit ourselves to studying models with three firms. Since we study games of a sequential form and employ numerical analysis later on in order to solve them, we present the mathematical theory related to solving such games here. This will serve as an important illustration of and building-block for our solution methodology, which we present in Chapter 5.

Multilevel optimization problems are hierarchical mathematical problems which have a subset of their variables constrained to be an optimal solution of other programs. When these programs are pure mathematical programs, we are dealing with bilevel programming(Vicente and Calamai, 1994). Bilevel programs are often modeled to represent the autonomous and possibly conflictual nature between two decision makers. In the latter sense, it is closely related to Stackelberg games described in section 2.5.2 (Colson et al., 2007).

Bilevel programs are very difficult optimization problems to deal with and belong,[23] in complexity theory, to the same class as the famous Traveling Salesman Problem (TSP) (Jeroslow, 1985).[24]

The continuous bilevel programming problem (BLPP) consists of two different spheres: the upper-level problem, with its upper-level variables and the lower level problem with its lower-level variables.

The general formulation of a bilevel programming problem (BLPP) is:

$$\min_{x \epsilon X, y} \quad F(x,y) \tag{1}$$

$$\text{subject to} \quad G(x,y) \leq 0 \tag{2}$$

$$\min_{y} f(x,y) \tag{3}$$

$$\text{subject to} \quad g(x,y) \leq 0 \tag{4}$$

$x \epsilon \mathbb{R}^{n_1}$ is called the upper-level variable and $y \epsilon \mathbb{R}^{n_2}$ is called the lower-level variable. Similarly, the functions F: $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \implies \mathbb{R}$ and f: $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \implies \mathbb{R}$ are called the upper-level and lower-level objective functions . The vector-valued functions G: $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \implies \mathbb{R}^{m^1}$ g: $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \implies \mathbb{R}^{m^2}$ form the upper-level and lower-level constraints respectively (Colson et al., 2007). The two problems are interrelated in that the upper-level problem clearly depends on the lower level decision variable, y, and vice-versa.

Even for the simplest linear bilevel problems, a polynomial algorithm is unlikely to be able to find global optimality (Deng, 1998). For this reason, bilevel problems are often transformed into a single-level optimization problem, due to lack of well-established solution procedures. One such method is transforming the lower level problem into an equilibrium condition rather than an optimization problem. This class of problems is coined Mathematical Programs with Equilibrium Constraints (MPEC). BLPPs and MPECs are so closely related that many

[23] Bilevel programs belong to the class NP-hard, for an explanation of what this entails, see Lundgren et al. (2010)

[24] TSP is a combinatorial optimization problem, where the problem is to find the shortest route traveled, for a given list of cities with pairwise distances, where every city has to be visited exactly once, and where the salesman must return to the origin city upon completion of the route.

chose to use the same terminology for the different set of problems (Colson et al., 2007). A classical approach in turning the BLPP into an MPEC is done by employing a Karush-Kuhn-Tucker (KKT) reformulation to the lower-level problem, turning the hierarchical optimization problem into a single-level optimization problem (Ye, 2006). This can be done under the assumption that the KKT conditions are necessary and sufficient for optimality in the lower-level problem Morales et al. (2014), the latter hinges on the convexity of the problem.

MPEC and KKT assume complementary constraints. We do something similar; however, we not make use of complementary constraints. Instead, we use first-order necessary conditions in the form of the Lagrange multiplier, which is a special case of the general KKT-specification. This approach does not technically fall under the definition of MPEC, but the similarities are several. For a detailed explanation of the solution methodology, see Chapter 5.

3 Model presentation

So far, we have explained the main components of the two location-cum-price models that we want to study,[25] one which models a one-sided market and the other modelling a two-sided market. In this chapter, we formally introduce the models and their building blocks. Our paper use these models to study how multiple entrants, and their timing of entering, affect each of model's equilibrium outcome.

For both models, our point of departure is Hotelling's linear city, where a one-dimensional product characteristic is represented by a [0,1] interval. The products the firms sell are equal in every respect, except their differentiation across the unit interval. Consumers are distributed uniformly along the line, with mass normalized to 1, and we base both our models on d'Aspremont et al. (1979)'s version of the linear city where consumers have quadratic transportation costs. We look for perfect Nash equilibria in pure strategies only.

The number of firms, denoted N, is exogenously given, while locations are determined en-

[25] Location-cum-price is an expression used by Loertscher and Muehlheusser (2011)'s to describe a Hotelling model in a context where firms first choose locations along the line, then choose prices.

dogenously. We denote the entrants' locations as $\mathbf{x} = (x_1, x_2...x_N)$.

Before continuing, we make a distinction in our use of the words position and location. Positions refer to firms' end-of-model placements along the line. Here, each firms' location must be seen relative to the others' locations. Location refer to a firm's value on the Hotelling line, and, although related, a firm's location is determined in the stage it enters, while a firm's position can only be determined when all locations have been found and can be seen relative to one another. As a technical note, positions determines which profit-function or FOC belong to what firm.

Say firm 1 enters at 0.6, firm 2 enters at 0.1 and firm 3 enters at 0.9. These numerical values are their locations, and using notation, they are written as $x_1 = 0.6$, $x_2 = 0.1$ and $x_3 = 0.9$. But, looking at these locations relative to one another, we can order them such that $0.1 < 0.6 < 0.9$. This gives their positions, which are: firm 2, firm 1, firm 3.

For the simultaneous games, the firms' positions are exogenously assigned to be: $x_1 \leq ... \leq x_i \leq ... \leq x_N$. Brenner (2005) notes that assigning positions is a short-coming of the model since there is no reason as to why firms entering simultaneously should agree to asymmetric equilibrium outcomes, but it is necessary to solve the simultaneous games. When allowing for sequential entry, however, firms choose their location one after another, and thus the firms' positions becomes endogenous. These positions could for example be similar to those in the previous paragraph.

3.1 The location-cum-consumer-price model

We call the model for the one-sided market the location-cum-consumer-price model, hereafter only referred to as LCCP. In this model firms first choose locations, then compete in consumer prices. We model both simultaneous and sequential location choices, and we start by discussing aspects that are common for both.

3.1.1 Consumers

The consumer's utility function specifies the utility level gained for being served a given product. We denote $\hat{x}$ as the consumers' preferred location along the one-dimensional product characteristic space, and x_i as the taste offered by a firm i. Furthermore, as mentioned previously, we assume quadratic transportation costs.

The common utility function can be written as:

$$U_i = v - p_i - t(x_i - \hat{x}_i)^2 \tag{5}$$

$$U_{i+1} = v - p_{i+1} - t(x_{i+1} - \hat{x}_i)^2 \tag{6}$$

v, is the consumers reservation price and is common for all consumers. Consumers are single-homers, and we assume that v is large enough so that every consumer purchases one product. p_i denotes the price of the product offered by firm i. t is the transportation cost by unit of distance. We can see from the utility function that consumer i has a peak in utility level when $x_i = \hat{x}_i$. That is, when the variety of the product offered coincides with the variety desired by the consumer. Moreover, which firm a consumer patronizes depends on the firm's closeness in variety and the price charged for the product.

We find a firm's marginal consumer who is indifferent between patronizing firm i and firm $i+1$ by:

$$U_i = U_{i+1} \tag{7}$$

$$\implies v - p_i - t(x_i - \hat{x}_i)^2 = v - p_{i+1} - t(x_{i+1} - \hat{x}_i)^2 \tag{8}$$

$$\implies \hat{x}_i = \frac{p_{i+1} - p_i}{2t(x_{i+1} - x_i)} + \frac{x_i + x_{i+1}}{2} \tag{9}$$

Using the above result, we find the aggregate demand facing firm $i \neq \{1, N\}$:

$$D_i = \int_{\hat{x}_{i-1}}^{\hat{x}_i} d\hat{x} = \frac{p_{i+1} - p_i}{2t(x_{i+1} - x_i)} + \frac{x_i + x_{i+1}}{2} - \Big[\frac{p_i - p_{i-1}}{2t(x_i - x_{i-1})} + \frac{x_{i-1} + x_i}{2}\Big] \tag{10}$$

The demand facing the first and the last firm respectively is:

$$D_1 = \int_0^{\hat{x}_1} d\hat{x} = \frac{p_2 - p_1}{2t(x_2 - x_1)} + \frac{x_1 + x_2}{2} \tag{11}$$

$$D_N = \int_{\hat{x}_{N-1}}^1 d\hat{x} = 1 - \left[\frac{p_N - p_{N-1}}{2t(x_N - x_{N-1})} + \frac{x_N + x_{N-1}}{2}\right] \tag{12}$$

3.1.2 Firms

Without loss of generality, we assume marginal and fixed costs normalized to zero. Firm i's profit function is as follows:

$$\Pi_i = p_i D_i \tag{13}$$

3.1.3 The timing of the games

Referring to the theory in section 2.5.1 and 2.5.2, we give here a more formal presentation of the timing of location-cum-consumer-price games. In the case of simultaneous entry, the game has the following two stages:

1. Firms enter the market simultaneously.
2. All firms simultaneously set prices p_i non-cooperatively.

In the case of sequential entry, the game has the following $N + 1$ stages:

1. In the first stage, firm one enters.

   .

   .

   .

2. In the N'th stage, the last firm enters.
3. In the N+1 stage, all firms simultaneously set prices p_i non-cooperatively.

3.2 The location-cum-advertisement-price model

Here, we present a model variant in which platforms compete in a two-sided market, as described in section 2.4. It is referred to as the location-cum-advertisement-price model, LCAP for short. Platforms first choose locations across the unit interval as they enter, then compete in advertisement prices. We base our model largely on the model presented by Dietl et al. (2013), but extend it to allow for multiple platforms and sequential entry. Dietl et al. (2013)'s model is ideal for this purpose due to its simplicity. We assume here that the platforms only generate revenue through advertisement. We furthermore assume that platforms set lump-sum prices: platforms charge the advertisers a single amount for their use of the platform. In our LCAP model, this reduces the difficulty of solving the model.

The notation and assumptions are largely in accordance with section 3.1. As before, we model both simultaneous and sequential location choices and start by discussing aspects that are common for both.

3.2.1 Consumers

We assume here that consumers are advertisement averse. The consumer's utility function:

$$U_i = v - \gamma a_i - t(x_i - \hat{x}_i)^2 \tag{14}$$

$$U_{i+1} = v - \gamma a_{i+1} - t(x_{i+1} - \hat{x}_i)^2 \tag{15}$$

γ is a measure of the consumers advertisement aversion. a_i is the number of ads from platform i. Notice that, if $\gamma = 1$, then there is a one-to-one relationship between a platform's advertisement and the consumer's disutility from being exposed to it.

The marginal consumer is defined as:

$$\hat{x}_i = \frac{\gamma(a_{i+1} - a_i)}{2t(x_{i+1} - x_i)} + \frac{x_i + x_{i+1}}{2} \tag{16}$$

Using (16), we can find the aggregate demand facing firm $i \neq \{1, N\}$:

$$D_i = \int_{\hat{x}_{i-1}}^{\hat{x}_i} d\hat{x} = \frac{\gamma(a_{i+1} - a_i)}{2t(x_{i+1} - x_i)} + \frac{x_i + x_{i+1}}{2} - \Big[\frac{\gamma(a_i - a_{i-1})}{2t(x_i - x_{i-1})} + \frac{x_{i-1} + x_i}{2}\Big] \tag{17}$$

The demand facing the first and the last firm respectively is as follows:

$$D_1 = \int_0^{\hat{x}_1} d\hat{x} = \frac{\gamma(a_2 - a_1)}{2t(x_2 - x_1)} + \frac{x_1 + x_2}{2} \tag{18}$$

$$D_N = \int_{\hat{x}_{N-1}}^{1} d\hat{x} = 1 - \Big[\frac{\gamma(a_N - a_{N-1})}{2t(x_N - x_{N-1})} + \frac{x_N + x_{N-1}}{2}\Big] \tag{19}$$

3.2.2 Advertisers

Advertisers belong to the group on the opposite end of the two-sided market from consumers, as described in section 2.4. We define advertisers as any firm producing goods and services that wishes to get in contact with consumers, using ads as a medium. We here assume, as Dietl et al. (2013), that each advertiser only places one ad on each platform, implying that the number of advertisers equals the number of ads. The advertiser's utility of placing an add at platform i is:

$$u_i = \beta D_i - R_i - \eta \tag{20}$$

β is the marginal contribution to profits from reaching one more consumer, while D_i is as before, demand. R_i is the lump-sum price charged by platform i for ad placements on the platform. η is the fixed cost of making an advertisement. η is assumed to be uniformly distributed along the [0,1] interval. We assume that advertisers multi-home and places an ad as long the utility of doing so is larger than zero, $u_a \geq 0$. Since each advertiser multi-homes, their demand for ads is independent of a specific platform's realized demand. The advertisers demand specification is:

$$a_i = \beta D_i - R_i \tag{21}$$

3.2.3 Platforms

We assume as before that for each platform, marginal and fixed costs are equal to zero. Platform i's profit function is as follows:

$$\Pi_i = a_i R_i \tag{22}$$

3.2.4 The timing of the games

The timing of the platforms' decisions in the LCAP-model is here formally presented. In the case of simultaneous entry, the game has the following two stages:

1. Platforms enter the market simultaneously.
2. All platforms simultaneously set advertisement prices R_i non-cooperatively.

In the case of sequential entry, the game has the following $N + 1$ stages:

1. In the first stage, platform one enters.

   .

   .

   .

2. In the N'th stage, the last platforms enters.
3. In the N+1 stage, all platforms simultaneously set advertisement prices R_i non-cooperatively.

4 Model analysis

In this chapter, we analyze the advertisement and consumer price models presented in chapter 3. In order to find sub-game perfect equilibria, we use backward induction to solve the models. The analysis starts at the end – the price stage. The models are similar in structure and only differ on the price stage. We, therefore, treat the models separately up to the location stage.

4.1 The price stage: consumer prices

The price stage is the last stage of the model, after locations has been determined. In this stage firms set prices simultaneously and non-cooperatively.

Before we dive into our analysis, we demonstrate the uniqueness of equilibria in the price stage. One can establish the uniqueness by proving that the best reply mapping is a contraction. This can be shown by employing the following relation:

$$\frac{\partial^2 \Pi_i}{\partial p_i^2} + \sum_{i \neq j} \left| \frac{\partial^2 \Pi_i}{\partial p_i \partial p_j} \right| < 0 \tag{23}$$

The proof of this relation can be found in Economides (1989). For the interior firms, it gives us:

$$\frac{-1}{2t(x_{i+1} - x_i)} - \frac{1}{2t(x_i - x_{i-1})} < 0$$

For the corner firms, we have:

$$\frac{-1}{2t(x_2 - x_1)} < 0$$
$$\frac{-1}{2t(x_{N-1} - x_N)} < 0$$

We have thus proven that the relation in Equation (23) holds for every combination of prices and profit functions and that it, therefore, exists a unique fixed point in the contraction constituting a unique equilibrium in the price stage (Moulin, 1984).

Result 1 *For the location-cum-consumer-price model (LCCP), there exists a unique equilibrium in the price setting stage for locations* $(x_1, ..., x_N)$.

Now that the uniqueness has been established, we solve the second-stage by using the first-order conditions of the firms, and solving simultaneously for prices:
The first-order conditions for firm $i \neq \{1, N\}$:

$$\frac{\partial \Pi_i}{\partial p_i} = \frac{p_i(x_{i-1} - x_{i+1}) + t(x_{i+1} - x_{i-1})(x_{i-1} - x_i)(x_i - x_{i+1})}{2t(x_{i-1} - x_i)(x_i - x_{i+1})} + \frac{(p_i - p_{i-1})(x_i - x_{i+1}) + (p_i - p_{i+1})(x_{i-1} - x_i)}{2t(x_{i-1} - x_i)(x_i - x_{i+1})} \quad (24)$$

For the corner firms:

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{2p_1 - p_2 + t(x_1 - x_2)(x_1 + x_2)}{2t(x_1 - x_2)} \quad (25)$$

$$\frac{\partial \Pi_N}{\partial p_N} = \frac{2p_N - p_{N-1} + t(x_N - x_{N-1})(2 - x_N - x_{N-1})}{2t(x_{N-1} - x_N)} \quad (26)$$

We then obtain prices as functions of locations, $p^*(\mathbf{x}) = (p_1^*(\mathbf{x}), ..., p_N^*(\mathbf{x}))$.

Economides et al. (2002) shows that equilibrium profits after the second stage for firm $i \neq \{1, N\}$ are:

$$\Pi_i^*(\mathbf{x}) = \frac{p_i^*(\mathbf{x})^2}{2t(x_{i+1} - x_i)} + \frac{p_i^*(\mathbf{x})^2}{2t(x_i - x_{i-1})} \quad (27)$$

For the corner firms:

$$\Pi_1^*(\mathbf{x}) = \frac{p_1^*(\mathbf{x})^2}{2t(x_2 - x_1)} \quad (28)$$

$$\Pi_N^*(\mathbf{x}) = \frac{p_N^*(\mathbf{x})^2}{2t(x_N - x_{N-1})} \quad (29)$$

From equation (28) and (29), we see that the corner firms' profits depend on the prices they set in squared terms, but also negatively on the difference between their location and their closest neighbors' location. In comparison, we see from equation (27) that the interior firms' profit function adds another term. The difference is a consequence of that the interior firms have two neighbors to compete with, while the corner firms compete directly with one neighbor only, and have complete control over the consumers from their location and the respective end-points on the unit interval.

4.2 The price stage: advertisement prices

Similar to the LCCP model we start in the price setting stage. However, this stage has a bit different solution methodology compared to the consumer price stage from before. In the advertisement price game, the platforms need to take into account the externalities that the groups cause each other by the groups shared use of the platforms. This is because the platforms' advertisement prices depends on these externalities. The internalization of the externalities is accounted for by inserting the advertisement demand, Equation (21), into the demand functions, Equations (17), (18) and (19), for platform $i \neq \{1, N\}$:

$$D_i = \frac{x_{i+1} - x_{i-1}}{2} + \gamma \left[\frac{\beta(D_{i+1} - D_i) + (R_i - R_{i+1})}{2t(x_{i+1} - x_i)} - \frac{\beta(D_i - D_{i-1}) + (R_{i-1} - R_i)}{2t(x_i - x_{i-1})} \right] \tag{30}$$

For the corner firms:

$$D_1 = \frac{x_2 - x_1}{2} + \gamma \left[\frac{\beta(D_2 - D_1) + (R_1 - R_2)}{2t(x_2 - x_1)} \right] \tag{31}$$

$$D_N = 1 - \left[\frac{x_N - x_{N-1}}{2} + \gamma \left[\frac{\beta(D_N - D_{N-1}) + (R_{N-1} - R_N)}{2t(x_N - x_{N-1})} \right] \right] \tag{32}$$

From the demand functions above, a few things are interesting to note. First, demand for the platforms depends positively on the difference in realized demand between the generic platform i, and its neighbor(s). This is because if consumers are advertisement averse, higher consumer demand on other platforms will, everything else equal, result in more ads on these platforms, which in turn increases demand on platform i (this is, of course, an iterative process). Second, consumer demand depends positively on the difference in the advertisement prices between platform i, and its neighbor(s). The intuition is similar to the previous observation; increasing R_i relative to platform i's neighbor(s), will lead to less advertisement demand on platform i and hence a lower advertisement level.

From Equation (30), (31) and (32) we see that all the demand functions directly or indirectly depend on all other demand specifications. We have thus a system of N equations, which we solve simultaneously for demand. Solving the equations for N platforms get increasingly complex and messy as N grows large. The solution we obtain for each platform's demand is equivalently messy, and we thus denote the solution to this system of equations as $D_i^{sol}(\mathbf{x}, \mathbf{R})$ for the generic firm i, and omit the specific solutions.

$D_i^{sol}(\mathbf{x}, \mathbf{R})$ is then inserted into the advertiser's demand function in Equation (21), denote this as $a_i^{sol}(\mathbf{x}, \mathbf{R})$. Finally, $a_i^{sol}(\mathbf{x}, \mathbf{R})$ is inserted into the Equation (22), the platform's profit function.

As before, we test for uniqueness in the price setting stage by proving that the best reply mapping is a contraction by using relation (23). It can be shown that this relation holds for any platform and any combination of platforms in the games we study. The proof can be found in the appendix.

Proposition 1 *For the location-cum-advertisement-price model and for the games comprised hereof, there exists a unique equilibrium in the price setting stage for locations* $(x_1, ..., x_N)$.

We derive the platform's equilibrium advertisement prices, $R^*(\mathbf{x}) = (R_1^*(\mathbf{x}), ..., R_N^*(\mathbf{x}))$, by solving the platform's first-order condition simultaneously for all platforms. Advertisement prices are determined by two conflicting effects: 1. Increasing the advertisement price leads to a direct effect of lower advertisement demand. 2. A higher price results in an indirect effect of higher consumer demand through lower advertisement volumes because consumers dislike advertisement. The platform thus seeks to balance these two opposing effects in the price setting stage.

Concerning equilibrium prices, there is a crucial difference between the consumer price and advertisement price game. In the LCAP model, platforms indirectly target consumer demand by regulating advertisement level. The way platforms do this is by price setting and location choices. Since prices are a function of the advertisers' non-negative profit condition and since the consumers are advertisement averse, causing demand to always be positive, prices are also always positive. The result is that the Bertrand paradox is not present in this model, even for equal location choices.

One of the essences of the model is that the platforms mitigate some of the customer disutility from increased ads, by setting high advertisement prices. We can see this since prices are increasing in the advertisement aversion parameter $\frac{\partial R_i^*(\mathbf{x})}{\partial \gamma} > 0$. This is owing to the fact that the subjective price of using a platform for the consumer is higher, everything else equal, for higher values of γ. This decreases the advertisement volume on the platform, and thus increases the advertisers willingess to pay for ads. This implies that platforms mitigate this higher subjective price, by charging advertisers extra for using the platform, resulting in less demand for ads in equilibrium.

Concerning prices, what we mentioned above to be a subjective price a consumer has to pay by way of disutility, is part of what is termed the generic price in industrial economics. The generic price consists in our examples of two parts: the subjective price and the consumer price. Each represented in isolation in each model. The term γa_i in consumer i's demand function quantify the subjective price. So, when $\gamma = 1$, the subjective price is equal to the advertisement level on the platform.

We also have that $\frac{\partial R_i^*(\mathbf{x})}{\partial \beta} > 0$. The intuition is that if the advertisers' marginal utility from advertising on the platform increase, then the platforms have incentives to charge advertisers a higher price in equilibrium.

4.3 The location stage - simultaneous entry

In this section and the next, we analyze the location stage for LCCP and LCAP for simultaneous and sequential entry. Since the general presentation of the location stages are largely the same, we do not present them independently for each model. To avoid confusion, in the following sections, we consistently use the word firm to denote both platforms operating in a two-sided market and firms operating in a one-sided market.

The location stage is the first stage of the simultaneous games, where firms choose locations along the Hotelling line. Since the price which firm i sets depends on its own location, the location of their closest neighbors and the prices they set, firm i's price depends indirectly on every other firm's location and prices. The latter is a central aspect of the specification of the sets of prices, $p^*(\mathbf{x})$ and $R^*(\mathbf{x})$.

Transforming this set of prices into functions of locations, substituting the expression into each respective profit function and finding equilibrium locations is mathematically complex, resulting in quite messy expressions. We have thus turned to numerical analysis, described in chapter 5 in order to solve the problem. The general solution methodology is to start from Equation (27), (28) and (29), take the partial derivative with respect to the firm's location and solving the resulting system of equations simultaneously for locations such that (33) is satisfied. For firm i:

$$\frac{\partial \Pi_i^*(\mathbf{x})}{\partial x_i} = 0 \tag{33}$$

As an interesting anecdote, we note that since we have two models with horizontal differentiation, where the initial choice of location is costless, firms are symmetric, consumers are uniformly distributed and decisions are happening simultaneously, we can expect our equilibrium locations to exhibit some form of symmetry.

The assumptions in this stage results in what we will soon observe to be a reflective symmetry. This means that by running an axis through the middle of the unit interval, one can demonstrate that every entrant $j \leq \frac{N}{2}$ is located the same distance away from the middle axis as entrant $k = N - j$, thus mirroring each other. This is equivalent to saying that for every pair j and k, both are located equally far away from their nearest endpoint of the unit interval: $x_j = 1 - x_k$. When the number of firms is an odd number, one firm will not belong to any pairs, and this firm must be paired with itself, thus laying on the axis we ran through the Hotelling line: $x_j = 1 - x_j \Leftrightarrow x_j = \frac{1}{2}$. If the locations has symmetry for every pair, the prices in each pair are equal.

Next, we use comparative statics to look at the effects duopolists have to take into account when deciding on where to locate. The discussion below will be helpful when we discuss and present our results later on.

LCCP: Comparative statics in the duopoly case

Every firm in the LCCP model are exposed to two opposing effects when moving along the unit interval. Moving away from the center of the unit interval has a decreasing effect on demand, but an opposite effect on their price. To illustrate this trade-off, we use comparative statics and the envelope theorem. We write firm 1's profit function in stage 1 as:

$$\Pi_1 = p_1(x_1, x_{,2})D_1(x_1, x_2, p_1(x_1, x_2), p_2(x_1, x_2))$$

We can thus decompose Equation (33) by taking the total differential as such:

$$\frac{d\Pi_1}{dx_1} = p_1\left[\underbrace{\frac{\partial D_1}{\partial x_1}}_{\text{Direct effect} > 0} + \underbrace{\overbrace{\frac{\partial D_1}{\partial p_2}}^{<0}\overbrace{\frac{dp_2}{dx_1}}^{>0}}_{\text{Strategic effect} < 0} \right]$$

There are two effects. The direct effect of moving towards the middle leads to more demand, and the strategic effect, leading to a lower price for firm 2, affecting firm 1's consumer demand negatively. Equilibrium locations are determined by which effects dominate the above equation.

LCAP: Comparative statics in the duopoly case

There exists a direct and a strategic effect for the duopoly in LCAP model as well. The effects are, however, not as clear-cut as in the one-sided market. We start with writing the profit function of platform 1 as:

$$\Pi_1 = R_1(x_1, x_{,2})(D_1^{sol}(x_1, x_2, R_1(x_1, x_2), R_2(x_1, x_2)) - R_1(x_1, x_{,2}))$$

We break up Equation (33) in a similar fashion as previously:

$$\frac{d\Pi_1}{dx_1} = R_1\Bigg[\underbrace{\frac{\partial D_1^{sol}}{\partial x_1}}_{\text{Direct effect} > 0} + \underbrace{\overbrace{\frac{\partial D_1^{sol}}{\partial R_2}}^{0>}\overbrace{\frac{dR_2}{dx_1}}^{(1.,2.)}}_{\text{Strategic effect}}\Bigg]$$

When locations are symmetric we get the following intervals:

1. < 0 for $0 < x_1 < 0.4389$
2. > 0 for $0.4389 < x_1 < 0.5$

The direct effect is the same as in LCCP: moving closer to the center increases demand. The strategic effect is, however, different. Here, we see that demand is decreasing in platform 2's price. The sign hinges on the discussion in section 4.2. We also observe that the second part of the term's sign varies with values of x_1. This means that, for the values of x_1 presented in point 1., the second term of the above equation is positive, and negative for the interval in point 2. In point 1, we thus have that platform 2's price is decreasing in increasing values of x_1. This is because when platform 1 moves closer to the middle, the relocation leads to a consumer demand stealing effect on platform 2. This demand stealing effect decreases advertisers' willingness to pay for ads on platform 2, resulting in platform 2 setting a lower advertisement price. With symmetrical locations, this effect is of course decreasing as platform 1's location nears the middle.

Another related effect in place is the way platform 1's location choice affects the consumers disutility from being exposed to ads. This is follows from the consumer demand stealing effect, which causes higher advertisement levels on the platform. Moreover, observing that the area between the platform's location and the nearest endpoint of the unit interval is controlled solely by the platform, then the area must be a representation of the platform's market power. For high degrees of market power, i.e. large areas controlled solely by the platform, there will also be, everything else equal, a large volume of advertisement on said platform. Consequently, if the platform with a high market power moves closer to their competitor, the consumer demand stealing effect would eventually be offset by the consumers distaste for advertisement. This effect works, cetris paribus, towards a higher advertisement price set by platform 2 since the advertisers' willingness to pay for ads increases.

In point 1, the consumer demand stealing effect seems to be dominating. For point 2, the offsetting effect caused by advertisement aversion seems to be dominating.
Based on our above discussion, we state the following proposition:

Proposition 2 *The strategic effect for the duopolists in LCAP depends on the on location choices of the platforms. For firm 1, the strategic effect is positive for the interval* $0 < x_1 < 0.4389$*, and negative for* $0.4389 < x_1 < 0.5$*.*

The comparative statics analysis studied above for LCCP and LCAP changes when considering more than two firms, yet the fundamental insight of how firms react to each others' locational choices, remains the same. Including more firms, will necessarily lead to more strategic effects that each firm has to balance, which changes how they adapt. The latter will become apparent when we present our results.

4.4 The location stage - sequential entry

In the case of sequential entry, the location games constitute the first part of the game and comprise what is essentially N different but dependent stages. There are several similarities in this part and the first stage in the case of simultaneous entry. Many of the points in Section 4.3 are therefore valid here. One is that there is a reciprocity in how firms affect each other's locational choices and prices. The sequential nature of the game results, however, in different behavior of the firms in response to this reciprocity. For example, the first firm who enters the market is assumed to have perfect and complete information about how its locational choice affects the remaining player's optimal response, and in turn, its payoff.

Since the firm has perfect foresight, it can use this fact when optimally choosing a location. The remaining players act with a similar foresight. However, they consider the locational choice of the preceding firms as given.

Let $[1, ...i, ...N]$ denote the position in the sequence in which a firm enters the market, and let $[1, ...j, ...N]$ denote the spatial ordering along the unit interval. $\mathbf{y}^* = (y_1^*, ..., y_i^*, ..., y_N^*)$ denotes the equilibrium locations contained in the set of location choice variables $\mathbf{x}^* = (x_1^*, ..., x_j^*, ..., x_N^*)$; the following multilevel optimization problem thus incorporates the sequentiality of the game:

$$\begin{aligned} &\max_{y_1} \quad \Pi_1^*(\mathbf{x}) \\ &\text{subject to} \quad \max_{y_2} \Pi_2^*(\mathbf{x}) \\ &\qquad\qquad\qquad \ddots \\ &\qquad\qquad\qquad\qquad \text{subject to} \quad \max_{y_N} \Pi_N^*(\mathbf{x}) \end{aligned}$$

5 Exposition of numerical methodology

The models are programmed and solved using Python. The code can be found in the appendix. Packages used are mainly Sympy, Numpy and Scipy. This section is for the particularly interested reader, curious of how the models are solved using Python, or has an unwarranted love for grid searches.

Finding equilibrium locations for simultaneous models is straight forward when using software. Most programming languages have packages that are able to find the zero/root of a function using numerical solvers, and Python is no different with its Scipy package. Since we solve these models through backwards induction, we utilize Scipy's fsolve-function on the first stage FOCs, which are dependent only on locational variables. The solver returns an array of values for these variables, which are our equilibrium locations that ensure that FOCs equals zero.

Although finding locations is easy when arriving at the first stage, adding more firms into the model causes it to be increasingly difficult even getting to that point. Because of the increasing complexity of the algebraic expressions that needs to be solved, time and hardware quickly becomes constraints. To simplify the models, we therefore set the transportation costs, t, advertisement aversion, γ, and the advertiser's marginal contribution of reaching an additional consumer, β, to 1.

To put the increasing time it takes to solve the models into perspective: while it takes seconds to solve the LCCP model for three firms, and a few minutes for four, it took 11 days to arrive at an equilibrium when solving for ten firms. The main culprit for this is the Sympy package's solve-function. Before arriving at where equilibrium locations are found numerically, a set of FOC-equations in the second stage must be solved with respect to prices. Sympy stores these equilibrium prices. The FOC-equations' complexity is exponentially increasing with the number of firms and, since we are unable to increase Python's/the solver's use of CPU, the processing time is exponentially increasing.

Additionally, the solver seizes a lot of the computer's memory. For the LCCP model this is fine, but memory becomes a bottleneck when running the solver in the LCAP model. Even with 32 GB RAM available,[26] the latter is unsolvable with more than four platforms. The reason is that this model utilizes two solvers, rather than one as in the LCCP-case: the first is to derive explicit functions of demand while the second solves FOCs to find equilibrium prices. When attempting to solve the model with a fifth platform, the program ran for over a month until the solvers had consumed all available memory on the server, and the kernel stopped running. The sets of equations proved too demanding.

Someone more competent in programming may circumvent the memory-constraint by programming a specialized solver, however, we do not believe additional platforms in this relatively simple model will add much more insight. We are thus satisfied with the four-platform equilibrium we find.

Using the root-method to arrive at an equilibrium is appropriate only if firms make decisions simultaneously. When the models are sequential, each firm needs to consider the choices of

[26]We owe gratitude to the Norwegian School of Economics' administration for making their best performing virtual desktop available for us. It would not have been possible getting this far without it.

later entrants, as well as considering how their choices are affected by endogenous positions. Thus, a different approach is needed for working out these models.

Optimizing for more than two firms is difficult, as it entails multilevel optimization. Here, as previously mentioned in Section 2.5.3, the first firm optimizes its profit, subject to the second firm's FOC, which optimizes its profit subject to the third firm's FOC, and so on. Because the profit functions/FOCs are tied to positions, however, and not firms, we cannot know beforehand how they should optimize. Instead, to infer what is the best location strategy, firms optimize for every possible positions they can be in at stage N. Recall how we made the distinction between location and positions in Chapter 3.

On the first stage, there are $N!$ number of different ways the firms can be organized/positioned at the end of stage N. The first firm will optimize for every permutation to consider all its possible optimal outcomes, and it wants to choose the location that yields the highest possible profit. However, a condition is that those entering later have no wish to deviate from the position that the first firm has optimized for.

To see if anyone wants to deviate, assume that a first mover evaluates one of its N! optimal locations: the second mover, deciding whether or not it wants to deviate, performs its own optimizations. It does this $\frac{N!}{1!} - 1$ -number of times.[27] The denominator simply says how many firms have previously made their decision, limiting the number of possible ways the firms can be positioned at stage N. The second firm deviates if it finds that it receives a higher profit elsewhere than for those positions the first firm optimized according to. If not deviating, it is the third firm's turn to check, checking the $\frac{N!}{2!} - 1$ remaining permutations positions can take. If it does not deviate, the next firm will do the same, a process continuing until it is firm number N's turn to decide, considering $\frac{N!}{(N-1)!} - 1 = N - 1$ permutations. The total number of optimizations that has to be made to determine whether or not a first mover has found a potential equilibrium is:

$$N! + \frac{N!}{1!} - 1 + ... + \frac{N!}{(N-1)!} - 1$$

[27] Minus one because one of the second firm's positions has implicitly been optimized for in the first firm's optimization, and needs not be considered when checking if the firm wants to deviate.

or

$$N! + N!(\frac{1}{1!} + \frac{1}{2!} + ... \frac{1}{(N-2)!})$$

If there are more than one optimal location passing the "no-deviation"/rationality-condition, the first firm will choose the one yielding the highest profit. This is the model's equilibrium solution.

As we discussed in Section, 2.5.3, solving a multilevel optimization problem is difficult. However, there are methods to avoid a lot of the complexity.[28] A very mechanical method, and which has been the simplest for us to implement, is performing a grid search, which can remove the need for optimization entirely. This is partially how we have chosen to solve the sequential models.

If a firm's optimal location exist, it is a location with a rational number on the unit interval [0,1], numbers of which there are infinitely many. Evaluating this many locations is not feasible, but fortunately it is sufficient to limit ourselves to a finite number. Using a grid search, the number of a firm's potential locations along the unit interval is limited by values that are equidistant apart, forming a grid. We refer to the distance as coarseness, and denote it as the coarseness-parameter, c. The smaller this parameter is, the finer the grid, and the more locations a firm needs to evaluate. We have chosen to use $c = 0.001$.

The number of locations in a grid is expressed as: $j = (\frac{1}{c} + 1)$, where the 1 in the numerator is the length of the Hotelling line. Through an iterative process, the firm evaluates its profit for each location, increasing location value by c for each iteration. After running through the entire interval, the firm has found exactly where they receive the highest profit.

Putting one firm on a grid is simple enough, but doing a grid search for K-number of firms, all entering sequentially, is a time-consuming process. The time it takes increases exponentially for each added k since grids belonging to later firms are contingent on previous firms' grids. In other words, for every new point a past firm evaluates on its grid, the next firm has to run through its entire grid. Thus, the total number of locations that need to be evaluated is

[28] A thank you to Chang-Koo Chi, assistant professor at NHH, that pointed out that there existed vastly simpler methods than those we were looking at.

$b = j^K$,[29] where we remember that j was the number of grid-locations. The reason that grids are dependent on past movers has to do with positions: all locations must be known for each firm to know which position's profit function they are evaluating.

Alternatively, it is possible to combine multilevel optimization and grid search. The first, second, third, up to the K-th firm would then be placed on grids, while the $N - K$ remaining firms find their optimal location through optimization. The $K + 1$-th firm must optimize $\frac{N!}{K!}$ times for every new location that K evaluates along its grid. If $N - K = 2$, we can reduce the multilevel optimization to a bilevel optimization problem. It is then possible, and simple, using for example Python's Scipy-package to optimize for the two remaining firms. The second to last and last firm has to optimize $\frac{N!}{K!} = (N)(N-1)$ and $N-1$ number of times, respectively. We believe combining the methods can, if the package doing the optimization is relatively efficient, save time overall. This is because optimization needs to be performed relatively few times. However, time saved running the code may be outweighed by the time it takes to write the code.

In our thesis, we use the combination of grid search and optimization for finding equilibria. We limit ourselves to study markets with two and three entrants. The bilevel optimization for two entrants was straight forward using the Scipy-package. For three entrants, we put the first firm/platform on a fine grid, with $c = 0.001$, which means it evaluates $j = \frac{1}{0.001} + 1 = 1001$ points along the Hotelling line.

For each grid-point the first entrant is on, the second firm/platform must optimize $\frac{3!}{1!} = 6$ times to see where it wants to place itself relative to the first firm. After picking the optimum resulting in the highest profit, the second entrant checks whether the third entrant wants to deviate or not. The third entrant optimizes $\frac{3!}{2!} - 1 = 2$ times to check, and if it wants to deviate, the positions the second firm evaluated is not an equilibrium. If it does not deviate, it is a potential equilibrium. The first entrant then jumps to the next point on the grid, and the same process is rinse and repeat for the other 1000 iterations. When finished, the model will have evaluated $1001 * 6 * 2 = 12012$ positions. At the end, the first firm/platform is left

[29] Assuming the programmer has not put additional bounds on the grids in any way. An example of a bound we have used is halving the first firm's grid by only looking at grid-points between 0.5 and 1. We can do this since we know that the only other equilibrium will be symmetric in the equilibrium locations.

with a set of potential equilibria, and it simply picks the one ensuring it the highest profit.

In the later stage of writing our thesis, we spotted a potential weakness of our code for the sequential game. Once the third entrant choose to deviate, we never stop to consider the second entrant's next-best position as a potential equilibrium. We are uncertain whether this affects the equilibrium outcomes or not.

As a final retrospective note in this chapter, it would have been relatively easy to make a version of our code that was easily scalable in number of firms, using only grid search. This would have reduced the risk of error substantially.

6 Results & Discussion

In this chapter, we start by clarifying which conditions ensure our equilibrium outcomes, and then we specify some equilibrium measurements that we use to assess our results. Afterward, we present and discuss the results of the LCCP model and the LCAP model, respectively. We conclude the chapter by comparing the overall results for the games.

6.1 Conditions for maxima

The following shows what conditions are necessary to ensure that our equilibrium outcomes are in fact local maximas. For a more detailed explanation than the one presented here, see Lundgren et al. (2010).

The first-order conditions for firm i:

$$\frac{\partial \Pi_i^*(\mathbf{x})}{\partial x_i} = 0 \tag{34}$$

A local maximum is guaranteed by verifying that the first-order and second-order conditions of the profit functions are fulfilled. The former is fulfilled if the relation (34) is binding. The latter holds if the second derivative of the profit function with respect to the firm's locations is negative, that is:

$$\frac{\partial^2 \Pi_i^*(\mathbf{x})}{\partial x_i^2} < 0 \tag{35}$$

It can be shown numerically that for every equilibrium outcome we obtain in section 6.3 and 6.4, the relation (34) and (35) holds. The proof can be obtained by the authors upon request.

6.2 Specification of equilibrium measurements

We use several different tools to measure effects from competition and locational asymmetry to analyze the equilibrium results.

In order to measure market concentration of firms in our games, we make use of the Herfindahl-Hirschman Index (HHI). HHI is the sum of the squared market shares for each competeting firms (Herfindahl, 1950), defined as follows:

$$HHI = \sum_{i=1}^{N} s_i^2 \tag{36}$$

We also use measurements proposed by Economides et al. (2002): a measurement of consumer surplus loss (CSL) caused by transportation costs, minimal consumer surplus loss (CSL^0), potential market area for each firm (m_i) and a locational asymmetry index (LAI).

CSL is a measure to quantify and compare the consumer surplus loss due to transportation costs. It is a natural supplement to our analysis since we wish to compare how the number of firms/platforms changes the consumer surplus. CSL for each firm i, is defined as follows:

$$CSL_i = \int_0^{\hat{x}_i - x_i} ts^2 ds + \int_0^{x_i - \hat{x}_{i-1}} ts^2 ds = \frac{t[(\hat{x}_i - x_i)^3 + (x_i - \hat{x}_{i-1})^3]}{3} \tag{37}$$

$$CSL(N) = \sum_{i=1}^{N} CSL_i \tag{38}$$

Referring to chapter 3, $\hat{x}_i$ is as before the location of the marginal consumer, while x_i is the location of the firm. Note that $\hat{x}_{i-1}$ is 0 for the left corner firm, while $\hat{x}_i$ is 1 for the right corner firm. CSL(N) is then an aggregate measure of the consumer surplus loss due to consumers' transportation costs, traveling from their most preferred variety, to the variety offered by the firm.

In order to compare the $CSL(N)$ measurement, we use the minimal consumer surplus loss defined as:

$$CSL^0(N) = \frac{1}{12N^2} \tag{39}$$

Equation (39) is based on the fact that, seen from a central planner's viewpoint, seeking to maximize the welfare of the consumers, the optimal location choice for the firms is $\frac{1}{N}$ apart from its neighbors, while the corner firms locates $\frac{1}{2N}$ distance from the boundary (Eaton and Lipsey, 1975). The reasoning behind this is that, in our models, maximizing consumer surplus is the same as minimizing transportation costs. This is hinges on our assumption that the market is covered.

The potential market area is the market area a firm obtains from competing in our models if the generic prices (see section 4.2) set by the firms are equal. For firm $i \neq \{1, N\}$, it is defined as:

$$m_i = \frac{x_{i+1} - x_{i-1}}{2} \tag{40}$$

For the corner firms:

$$m_1 = \frac{x_2 + x_1}{2} \tag{41}$$

$$m_N = 1 - \frac{x_N + x_{N-1}}{2} \tag{42}$$

The potential market area indicates locational advantages that a firm has vis-à-vis other firms when generic prices are equal. If for example firm i has a higher potential market area, compared to firm j, then this indicates a more favorable position for firm i at the outset.

Since firm i will, if generic prices are equal, gain a higher profit compared to firm j. The potential market area is also the main ingredient in the locational asymmetry index. Which is defined as:

$$LAI = \sum_{i=1}^{N} m_i^2 \tag{43}$$

LAI shares many similarities with HHI; we can see from the demand functions for the location-cum-consumer-price game that when the prices are set equal, realized demand equals m_i, causing the indices to be equivalent. The same is true when looking at the demand specifications for the location-cum-advertisement-price game; if the advertisement aversion parameter, γ, is set equal to zero, and or realized demand and advertisement prices are equal, the demand collapses to m_i. By comparing HHI to LAI, we can indirectly measure to which degree the firm's asymmetric location choices and/or locational advantages are transformed into increased market shares.

6.3 Results from the location-cum-consumer-price game

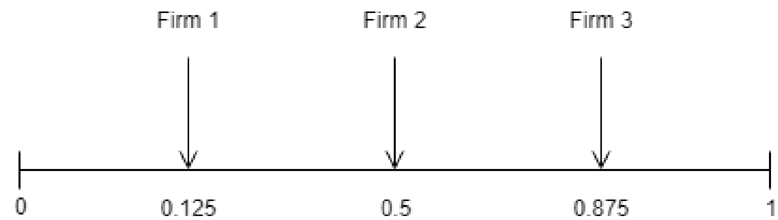
For both the location-cum-consumer-price and the location-cum-advertisement-price models, we first present the results for simultaneous entry, and then the version that assumes sequential entry. In this section our focus lays on the LCCP model.

To illustrate our results, we present the equilibrium outcomes as location choice on the Hotelling line. We also include more detailed tables for each of the cases studied, containing information on prices, profits, HHI, LAI and more. We compare the equilibrium outcomes for all the cases studied in LCCP using HHI and LAI. Also, we extend this approach to compare the consumer surplus loss for each case with the minimal consumer surplus loss.

6.3.1 Results from the simultaneous game

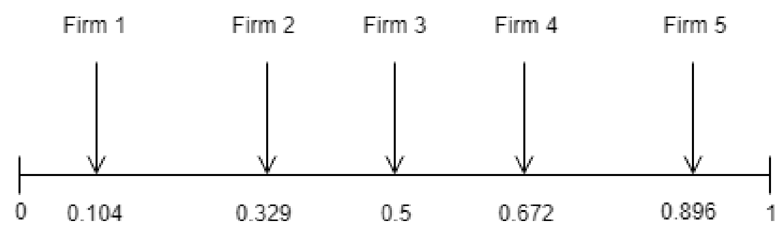
Below, we have visualized the equilibrium locations for two to ten firms as depicted on the unit interval.

N=2



N=3

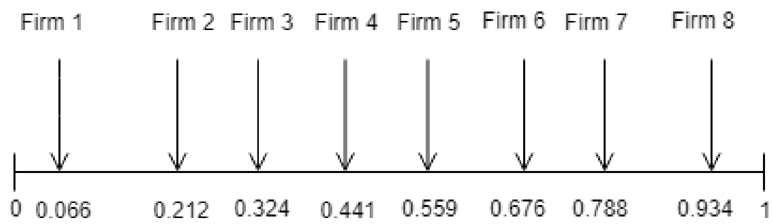
N=4



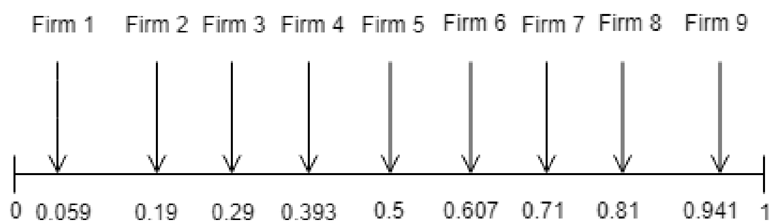
N=5

N=6

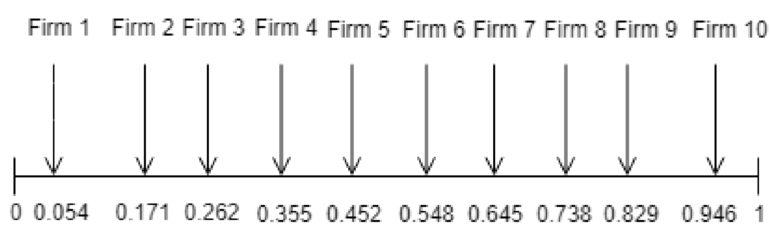
N=7



N=8



N=9



N=10

Figure 5: LCCP: Equilibrium locations for two to ten firms

From the figures above, we see d'Aspremont et al. (1979)'s result of maximum differentiation when $N = 2$, where each firm locates as far away from each other as possible on the Hotelling line. This is, in fact, an understatement, since we have limited the firms' locational choice to the unit interval; in a duopoly with our model set-up, firms wish to locate outside the Hotelling line to -0.25 and 1.25 respectively.

When adding more firms, however, the story of maximum differentiation changes. In general, for there to be maximum differentiation, corner firms should locate at the boundaries, while the interior firms would need to locate equidistantly away from its neighbors. To fulfill the requirement of maximum differentiation for three firms, they would have had to locate at 0, 0.5 and 1 on the unit interval, but instead they locate at 0.125, 0.5 and 0.875. As long as $N > 2$ there is no maximum differentiation. This is because the strategic effect of product differentiation becomes less important for the firms in for the oligopoly cases studied relative to the duopoly case.

Generally, for $N > 3$ in the cases studied, the interior firms agglomerate around the middle, while the firms closest to the boundary of the Hotelling line, still are spaced further away from the rest of the firms, but not entirely at the boundary as in the duopoly case. When N grows large, however, the corner firms seems to be pushed further and further to the periphery of the line, as more firms agglomerate in the middle and the location pattern becomes tighter overall.

Equilibrium outcome for two firms

To study the cases and to discuss the equilibrium outcomes in more detail, we here introduce tables that provide more detailed information.

Table 1: Simultaneous entry: detailed table of the equilibrium outcome for two competing firms

Firm number	Price	Profit	Market share	Potential market area
1	1	0.5	0.5	0.5
2	1	0.5	0.5	0.5
HHI:	0.5	$CSL(2)$:	0.08333	
LAI:	0.5	$CSL^0(2)$:	0.02083	

In the duopoly case, the locations are symmetric. This implies that price, profit, and market share are the same for both firms. What more, because prices are equal, the obtained market shares are equivalent to the potential market areas, which means that HHI and LAI are identical.

We here observe that the consumer surplus loss due to transportation costs (CSL) is 0.08333, this is the highest consumer surplus loss if firms differentiate. The reason is that the firms cannot move further apart given the bounds of the Hotelling line. This CSL serves as a useful reference point for later analysis.

Equilibrium outcome for three firms

Table 2: Simultaneous entry: detailed table of the equilibrium outcome for three competing firms

Firm number	Price	Profit	Market share	Potential market area
1	0.203	0.055	0.271	0.3125
2	0.172	0.0787	0.4633	0.375
3	0.203	0.055	0.271	0.3125
HHI:	0.362	$CSL(3)$:	0.01139	
LAI:	0.3359	$CSL^0(3)$:	0.0093	

In the triopoly case, profits, prices, and market shares are no longer equal for all firms. Prices follow a V-shaped distribution, where prices of the corner firms are the same, and higher than for Firm 2 that is in the middle of the unit interval. Firm 2 takes a lower price because it is exposed to higher competition compared to the corner firms, who enjoys a higher market power. The profits follow an inverted V-shaped distribution where the interior firm enjoys the highest profit. This is a consequence of Firm 2 managing to expand on its locational advantage, evident from the potential market area, and capture higher market shares by setting a lower price relative to its neighbours. This causes a higher variance in realized market shares compared to potential market area. We thus get a different result than from the duopoly case, namely, HHI>LAI.

Compared to the duopoly game, the aggregate CSL has dropped significantly, being just 13.66% of the value in the duopoly case and, while still above the aggregate CSL^0 measurement, the difference is quite small.

Equilibrium outcome for four firms

Table 3: Simultaneous entry: detailed table of the equilibrium outcome for four competing firms

Firm number	Price	Profit	Market share	Potential market area
1	0.1066	0.02089	0.195	0.26
2	0.0714	0.02176	0.3034	0.24
3	0.0714	0.02176	0.3034	0.24
4	0.1066	0.02089	0.195	0.26
HHI:	0.261	$CSL(4)$:	0.00763	
LAI:	0.2504	$CSL^0(4)$:	0.005208	

In the quadropoly case, the structure of the game is very similar to that of three firms. The V-shaped distribution of prices in the case of the triopoly has now developed into a U-shaped distribution, which persist up to the case of nine firms. The profits and market shares, on the other hand, are distributed in an inverse U-shape. Interestingly, potential market areas are now lower for the interior firms compared to the boundary firms. While the realized market shares for the interior firms are higher than the potential market areas, the opposite is true for the boundary firms. This indicates that while the interior firms are initially at a disadvantage, they manage to set their prices so low relative to the corner firms' prices, that they capture a large portion of the corner firms' potential market area. Overall, the advantage the interior firms manages to create is more than enough to offset their initially disadvantageous position, creating a higher variance in market shares compared to the potential market area. This results in the familiar HHI>LAI. Furthermore, $CSL(4) > CSL^0(4)$, and $CSL(4)$ is reduced to 9.16% of $CSL(2)$.

Equilibrium outcome for five firms

Table 4: Simultaneous entry: detailed table of the equilibrium outcome for five competing firms

Firm number	Price	Profit	Market share	Potential market area
1	0.0715	0.0113	0.1585	0.2165
2	0.0452	0.0105	0.2323	0.198
3	0.0371	0.0081	0.218	0.171
4	0.0452	0.0105	0.2323	0.198
5	0.0715	0.0113	0.1585	0.2165
HHI:	0.2057	$CSL(5)$:	0.00521	
LAI:	0.2014	$CSL^0(5)$:	0.0033	

In the quintopoly case, the structure of the equilibrium outcomes that has persisted for $N > 2$ has now changed. Similarly to prices, profits now follow a U-shaped distribution. The reason the profit distribution is now U-shaped is that the rivalry between firms is higher for the interior firms, and increasingly so when nearing the middle of the Hotelling line. The increasing rivalry for the interior firms comes as a consequence of firm 2 and 4 squeezing firm 3, shortening its potential market area. This again results in a higher price competition effect since firm 3 lowers its price in response to being located in the middle of the cluster, to dampen the reduced demand effect caused by its squeezed position. This effect is also present, though to a lesser extent, for firm 2 and 4, who are also exposed to competition from the corner firms.

The U-shaped distribution of potential market area we saw in the case of four firms remain. However, the link between realized market shares has become fuzzier. While the lowest market shares still belong to the corner firms, they do not increase persistently as one nears the middle of the unit interval. This hinges on the discussion in the previous paragraph. Instead realized market shares are the highest for firms 2 & 4, and lower for firm 3. The difference between realized market shares and the potential market area is, as before with four firms, due to the U-shaped distribution of the prices, meaning the firms who manage to under-price their closest neighbors will capture parts of their neighbors' potential market area.

We note that the trend for our equilibrium measurements persists; $HHI > LAI$ and $CSL(5) > CSL^0(5)$ and $CSL(5)$ is 6.25% of $CSL(2)$.

Equilibrium outcome for nine and ten firms

In the cases of six, seven, and eight firms, the equilibrium structure is the same as with five firms. We thus refrain from repeating the observations already made. The tables concerning the cases six, seven and eight firms can be found in the appendix.

Table 5: Simultaneous entry: detailed table of the equilibrium outcome for nine competing firms

Firm number	Price	Profit	Market share	Potential market area
(1),(9)	0.0237	0.0022	0.0929	0.1245
(2),(8)	0.015	0.002	0.132	0.1155
(3),(7)	0.0118	0.0014	0.1153	0.1015
(4),(6)	0.0113	0.00121	0.1075	0.105
(5)	0.01134	0.001207	0.1064	0.107
HHI:	0.1132	$CSL(9)$:	0.00141	
LAI:	0.1118	$CSL^0(9)$:	0.001029	

Table 6: Simultaneous entry: detailed table of the equilibrium outcome for ten competing firms

Firm number	Price	Profit	Market share	Potential market area
(1),(10)	0.0193	0.0016	0.0821	0.1123
(2),(9)	0.0122	0.0015	0.1197	0.104
(3),(8)	0.0096	0.001	0.1046	0.092
(4),(7)	0.00925	0.000899	0.0972	0.095
(5),(6)	0.00928	0.000894	0.0958	0.0965
HHI:	0.10123	$CSL(10)$:	0.001112	
LAI:	0.10053	$CSL^0(10)$:	0.000833	

Concerning the case of nine and ten firms, it is interesting to note is that the distribution of prices is no longer fully U-shaped. This result is different to what Brenner (2005) got with when studying nine firms. This is most likely because Brenner operated with less decimals than we do.

The reason the distribution is no longer fully U-shaped is that the firms closest to the middle, Firm 5 in the nine firm case and Firm 5 and 6 in the ten firm case, set a higher price than their neighbors. This is a result of these aforementioned middle firms enjoying lower competition since they are spaced farther away from their neighbors compared to the distance between their neighbors and their neighbors' neighbors. This causes the prices to not be fully U-shaped.

Comparison between the equilibrium outcomes

Here we turn to compare the results and sum up our main findings in our analysis of the LCCP simultaneous game. We have observed for $10 \geq N > 2$ that firms choose neither maximum nor minimum differentiation, but something in between, varying with the firm's location. A common feature of all the games presented above is that profits and prices are decreasing in the number of firms entering. The intuition for this is straightforward, increasing the number of firms decreases the distance between each firm and increases the rivalry and competition between them. This result is based on an implicit assumption of no demand-growth. [30]

The increase in the rivalry between firms spills over to the market shares, and naturally the HHI measurement. This is also an intrinsic characteristic of the LAI - increasing number of firms decreases the potential market area of each firm in equilibrium. In Figure 6, we compare HHI and LAI. In every case where the number of firms is $N > 2$, we see that HHI remains larger than LAI. This implies that some firms always manage to expand their market share over their potential market area by price setting, causing a higher variance in the market shares than in the potential market area. It is evident though that the effects of price setting decrease as more firms enter the market since HHI and LAI converge towards each other.

An interesting observation is that while the market share of the average firm decreases with the number of firms entering the market, the division of the whole pie depends on the case studied. We see from Table 2 that the middle firm has the highest profit and market share. The same is true for Table 3 where the two middle firms have the highest profits and market shares. For games where $N > 4$, the link between market shares and profits becomes fuzzier.

[30] The number of consumers in our models is constant and along with the assumption that all consumers buy one product from one firm only, there is no total demand-growth in the market. Had there been growth, it could have mitigated some of the increase in competition intensity, but since this is not the case, the increase in firms is particularly noticeable in prices.

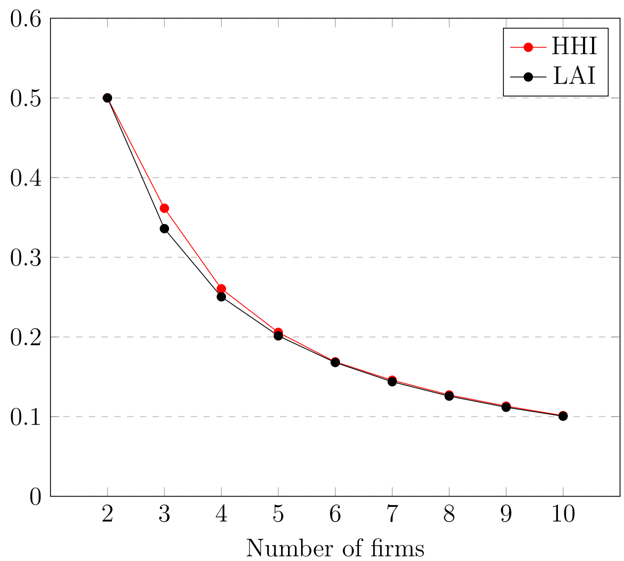


Figure 6: LCCP: A comparison between HHI and LAI

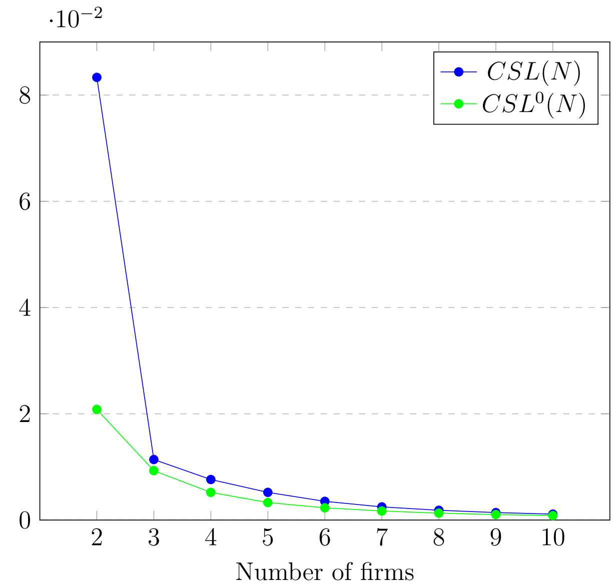


Figure 7: LCCP: A comparison between $CSL(N)$ and $CSL^0(N)$

For firms 5 to 10, the general pattern, profit-wise, is that the corner firms enjoys the highest profits, and decreasing as one nears the middle. The intuition hinges on the rationale from before, the firms closer to the boundary enjoys higher market power - less competition - than those closer to the middle. This implies that the strategic effect of product differentiation becomes relatively more important compared to the demand effect as N grows large.

When it comes to our measurement of consumer surplus loss, the story follows in similar lines; it is decreasing in the number of firms competing, $CSL(N)$ and $CSL(N)^0$ converges towards each other and both converge to zero as N grows towards infinity. This is because the consumers have to travel less to purchase a product when more locations are occupied along the Hotelling line. This effect can also be shown by comparing $CSL(N)$ to $CSL(N)^0$. We can see from Figure 7 that, while the initial difference between the two measurements is substantial, they tend to converge towards each other as N increases.

6.3.2 Results from the sequential game

In this section, we present and discuss our results in the LCCP model for sequential entry with two and three firms.

Equilibrium outcome for two firms

Figure 8: LCCP: Sequential entry - N=2

Table 7: Sequential entry: detailed table of the equilibrium outcome for two competing firms

Firm	Price	Profit	Market share	Pot. mrk. area
1	1	0.5	0.5	0.5
2	1	0.5	0.5	0.5
HHI:	0.5	$CSL(2)$:	0.08333	
LAI:	0.5	$CSL^0(2)$:	0.02083	
$\frac{CSL(2)}{CSL^{sim}(2)}$:	1	$\frac{CSL(2)}{CSL^0(2)}$:	4	

For two firms, the optimal location is the same as in the simultaneous case. This result may come as a surprise, but is based on the fact that the firms experience the same trade-offs as with simultaneous entry. Namely that if firm 1 chooses x_1 so that $x_1 \epsilon [0, 0.5)$, then firm 2's optimal response is to set $x_2 = 1$, since $\frac{\partial \Pi_2^*(\mathbf{x})}{\partial x_2} > 0$ for all values of $x_2 \epsilon (0.5, 1]$. Knowing this, firm 1's optimal choice in the first stage is to set $x_1 = 0$, since $\frac{\Pi_1^*(\mathbf{x})}{\partial x_1} > 0$ for all values of $x_1 \epsilon [0, 0.5)$.

Equilibrium outcome for three firms

Figure 9: LCCP: Sequential entry - N=3

Table 8: Sequential entry: detailed table of the equilibrium outcome for three competing firms

Firm	Price	Profit	Market share	Pot. mrk. area
1	0.18766	0.088	0.4691	0.40744
2	0.2524	0.0688	0.2728	0.3428
3	0.1819	0.0469	0.258	0.24979
HHI:	0.3611	$CSL(3)$:	0.01477	
LAI :	0.3459	$CSL^0(3)$:	0.0093	
$\frac{CSL(3)}{CSL^{sim}(3)}$:	1.297	$\frac{CSL(3)}{CSL^0(3)}$:	1.588	

In the case of three firms, we see that the equilibrium outcome has changed significantly compared to the duopoly case. Firm 1 locates at 0.574 in between Firm 2 & 3 located at 0.1115 and 0.9264 respectively. Comparing these location choices to that of the simultaneous case, we can see that Firm 1 uses its perfect foresight and the rational expectations of Firm 2 & 3 to create an asymmetric location setting, by locating over one half of the unit interval. Using the games asymmetry to its advantage, Firm 1 manages to expand its market share over its potential market area. This causes higher variance in the market shares of the firms resulting in the familiar HHI>LAI in the LCCP set-up. Firm 1 accomplishes this by setting its price just above Firm 3, but below Firm 2. Furthermore, profits, ranked by order of magnitude, coincides with the order of entry, meaning $\Pi_1 > \Pi_2 > \Pi_3$. $CSL(3)$ is naturally above CSL^0. $CSL(3)$ is 17.72% of the duopoly case. However, comparing it to the simultaneous game with three firms, CSL becomes 29.7% higher when allowing for sequential entry.

6.4 Results from the location-cum-advertisement-price game

In this section, we present and discuss the results from our location-cum-advertisement-price model in a similar fashion as we did with the LCCP model. We employ the same measurements as before and we begin by first studying games of simultaneous entry. Afterward, we present the results obtained by allowing firms to enter sequentially, and discuss these as well.

6.4.1 Results from the simultaneous game

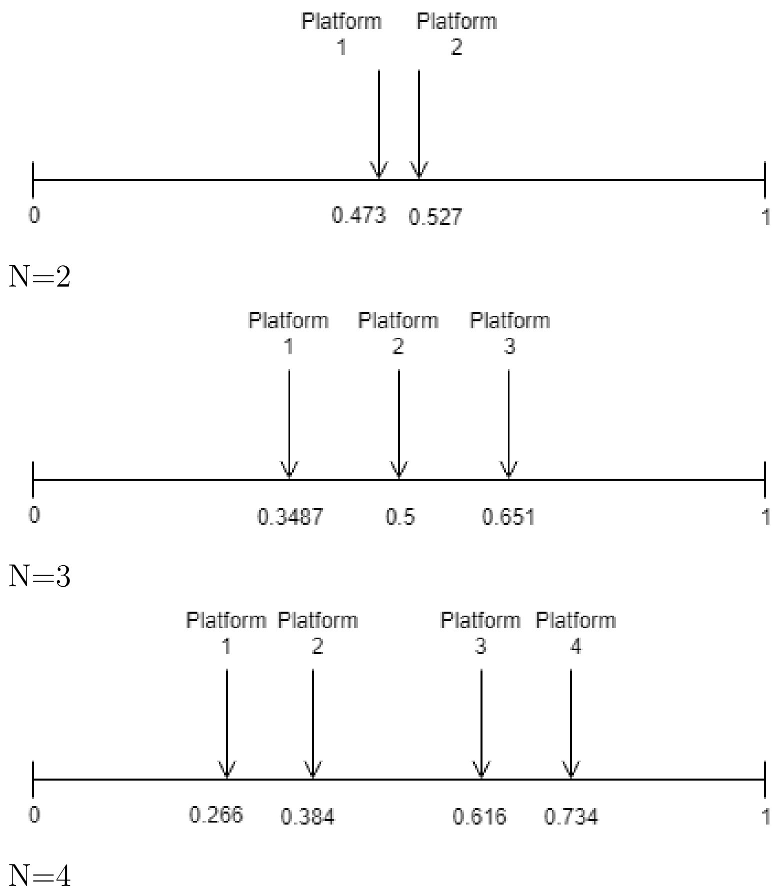


Figure 10: LCAP: Equilibrium locations for two to four platforms

From the figures above, we see that the locational equilibrium outcome in the simultaneous case has changed significantly compared to LCCP. When $N = 2$, we see that maximum differentiation no longer prevails. On the contrary, the platforms locates closely together around the middle of the Hotelling line. This follows from the effects we studied in the section 4.2; prices and profits are to a large extent determined by demand captured and this effect is so dominating that it pulls the platforms close to each other. This pull-effect is still present when $N = 3$, and here we can see that its neighbors squeeze the median platform. For $N = 4$, the platforms are located at a greater distance than before; there are now two "groups" of platforms, where each platform belonging to the group is located near each other.

Do we observe minimum or maximum differentiation? Based on the points section 6.3.1, it is evident that when $N > 2$ we do not observe maximum differentiation. Furthermore, minimum differentiation in the spirit of Hotelling (1929) constitutes identical location choices. When $N = 2$, we observe that the platforms have located very closely together, yet, they are not located at the exact same points. Even here, there exists some degree of differentiation. The latter is a standard feature of all the cases studied; there exists some degree of differentiation between platforms in equilibrium. Moreover, differentiation increases as N gets larger.

Equilibrium outcome for two platforms As before, we will now discuss the equilibrium outcomes in more detail.

Table 9: Simultaneous entry: detailed table of the equilibrium outcome for two competing platforms

Platform	Price	Profit	Market share	Pot. mrk. area	Advertisement level
1	0.328	0.0564	0.5	0.5	0.172
2	0.328	0.0564	0.5	0.5	0.172
HHI:	0.5	$CSL(2)$:	0.0708		
LAI:	0.5	$CSL^0(2)$:	0.02083		

In the duopoly case, the locations are symmetric. This implies that the prices, profits, advertisement shares and market shares are identical. What more is that the potential market area equals obtained market shares, which means that the HHI collapses to mirroring LAI. These observations are the same as before; however, comparing the results, we can see that both prices and profits are significantly lower than that in LCCP. This hints at that, in our model set-up, the solely advertisement business model is less profitable compared to that of the traditional consumer price model.

We observe that $CSL(2) = 0.0708$ which is higher than $CSL^0(2)$: there is too little differentiation in equilibrium. It is, however, interesting to note that the $CSL(2)$ in the advertisement price case is lower than in the consumer price case. As noted before, a CSL of 0.083333 is the highest possible. Moreover, it can only be obtained by maximum or minimum differentiation. In the LCAP duopoly case, we thus see that there is a more optimal differentiation compared to the LCCP case, though not by much.

Equilibrium outcome for three platforms

Table 10: Simultaneous entry: detailed table of the equilibrium outcome for three competing platforms

Platform	Price	Profit	Market share	Pot. mrk. area	Adv. level
1	0.2037	0.026	0.3313	0.4244	0.1276
2	0.22	0.019	0.30636	0.1512	0.08636
3	0.2037	0.026	0.3313	0.4244	0.1276
HHI:	0.3344	$CSL(3)$:	0.03067		
LAI:	0.3844	$CSL^0(3)$:	0.0093		

In the triopoly case, profits are distributed in a V-shape: the corner firms posses a locational advantage and platforms with the broadest market reach gain the highest profits. Advertisement prices are here distributed in an inverted V-shape, with the center platform setting the highest ad price. The advertisement price distribution follows from the V-shaped distribution of the subjective prices, defined in 4.2. This is because the advertisers have a higher willingness to pay for ads on platform 2, caused by a low advertisement volume but relatively high consumer demand.

We see that platform 2 expands its market shares above it's potential market area, and opposite for the corner platforms. This is due to that advertisement level is higher on the other platforms, which is a consequence of a higher prices by platform 2, but also due to the corner platforms' market reach. Thus, the locational disadvantage of platform 2 is countered by a distaste for ads which works as a self-regulating mechanism in the market. The market shares are thus partially evened out and we get that LAI>HHI. This is the opposite from what we observed in the LCCP game; the platforms are unable to take advantage of their favorable locations to gain higher market shares.

We can also observe that $CSL(3) > CSL^0(3)$, where $CSL(3)$ has dropped by 43.32% from the duopoly case. This change is less than what we observed in the three firm LCCP case. This is because the platforms still agglomerate around the center, resulting in a less change in differentiation strategy and causing it to be too little differentiation in equilibrium.

Equilibrium outcome for four platforms

Table 11: Simultaneous entry: detailed table of the equilibrium outcome for four competing platforms

Platform	Price	Profit	Market share	Pot. mrk. area	Adv. level
1	0.1832	0.01429	0.2612	0.325	0.078
2	0.1758	0.0111	0.239	0.175	0.06314
3	0.1758	0.0111	0.239	0.175	0.06314
4	0.1832	0.01429	0.2612	0.325	0.078
HHI:	0.251	$CSL(4)$:	0.01477		
LAI:	0.272	$CSL^0(4)$:	0.005208		

In the quadropoly case, we have that profits, market shares, advertisement levels and potential market areas are U-shaped in distribution. Prices are also distributed in a U-shape. This stands in contrast with the three platforms case. The introduction of another platform has resulted in a fiercer competition for advertisement levels in the middle. Point 2 of the strategic effect, studied in 4.3, between platform 2 and 3 seems to be negatively dominating, resulting in lower prices for the middle platforms and the "grouping" in the locational pattern.

Again we have that HHI<LAI this is because the interior platforms manage to increase their market shares over potential market areas, at the cost of the locational advantage of the corner firms. $CSL(4)$ is here also higher than $CSL^0(4)$, and it has dropped by 20.86% compared to the duopoly case. This is more than what we observed in LCCP.

Comparison between the equilibrium outcomes

Before we conclude this section, we will now compare the different cases and point out some main points. From the locational choices of the platforms, it is evident that the platforms choose neither maximum nor minimum differentiation, this is similar to LCCP for firms $N > 2$. The difference lies however in the nuances: in LCAP, the platforms chooses overall less differentiation than the corresponding cases in LCCP. The dynamics of the game is, however, not the same. Consumers dislike advertisement, and a point of differentiation between platforms is thus the advertisement level. The advertisement level on each platform can be decomposed into the consumer demand part and the advertisement price set by the platform. Platforms with a higher price will capture more consumers, cetris paribus, since they will have fewer ads. Platforms can also secure higher potential market areas by being located strategically; this will increase their market power vis-à-vis the advertisers since doing so will increase their consumer demand, everything else equal. This will also lead to higher levels of advertisement on the platform. These opposing effects thus determine the equilibrium outcome.

Opposite to LCCP, LAI remains larger than HHI for the cases studied here when $N > 2$. This result comes despite that the market reach in equilibrium is more extensive for the corner platforms, caused by the corner platforms squeezing the interior platforms. The latter producing very uneven potential market areas at the heart of the games. However, the platforms with a locational advantage do not manage to increase their output over their potential market area. This is due to the self-regulating mechanism of consumers' advertisement aversion. It is interesting to note here, that LAI is higher for LCAP than LCCP overall. The latter part is due to that firms will not locate as closely as they do in the advertisement game since this means a direct consumer price reduction for the firms. Thus no firm obtains the same locational advantage as the corner platforms in LCAP. We can see from Figure 11 that the LAI continually remains higher than the HHI during all the games studied, $N > 2$ excluded. We also note that, as was the case with LCCP, both the LAI and HHI converge towards each other.

The result observed in LCCP regarding CSL persists to an extent here, which can be seen from figure 11: the initial difference between $CSL(N)$ and $CSL^0(N)$ is substantial, however, as N grows, the difference dissipates.

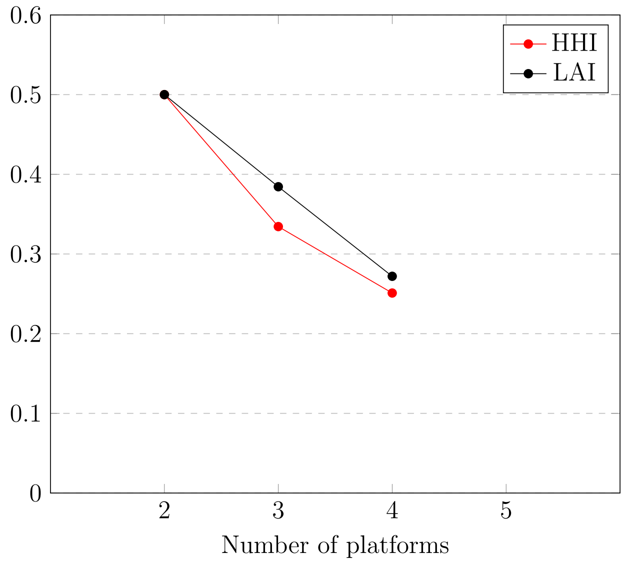


Figure 11: LCAP: A comparison between HHI and LAI

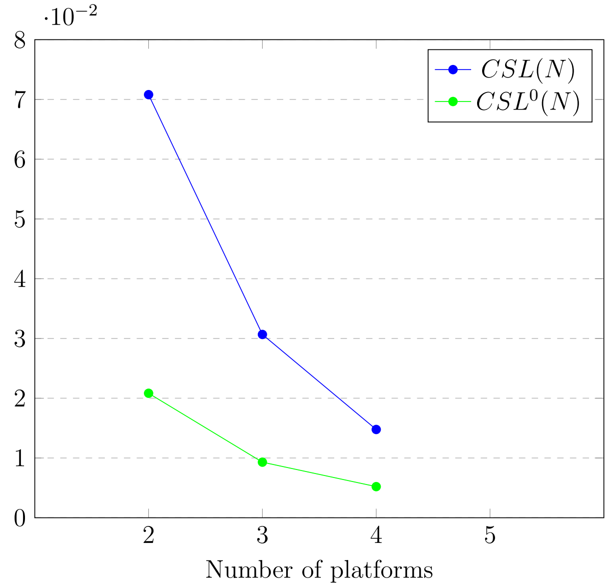


Figure 12: LCAP: A comparison between $CSL(N)$ and $CSL^0(N)$

6.4.2 Results from the sequential game

In this section, we present and discuss our results for games withd sequential entry for two and three platforms.

Equilibrium outcome for two platforms

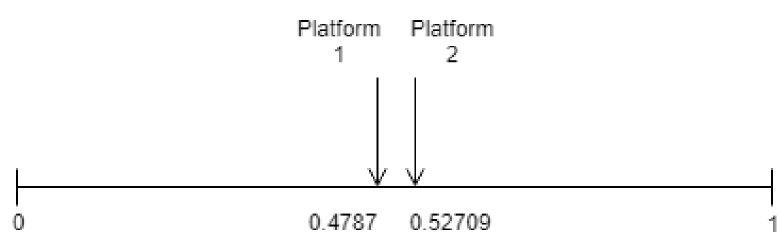


Figure 13: LCAP: Sequential entry - N=2

Table 12: Sequential entry: detailed table of the equilibrium outcome for two competing platforms

Platform	Price	Profit	Market share	Pot. mrk. area	Adv. level
1	0.32805	0.05645	0.50013	0.5029	0.17208
2	0.32852	0.05629	0.49986	0.49708	0.17134
HHI:	0.5	$CSL(2)$:	0.07184		
LAI:	0.500017	$CSL^0(2)$:	0.02083		
$\frac{CSL(2)}{CSL^{sim}(2)}$:	1.01469	$\frac{CSL(2)}{CSL^0(2)}$:	3.4489		

For two platforms in the sequential game, the optimal location choice is very similar to that of the variant with simultaneous entry. We get, however, a marginally different result in favor of platform 1. We observe here that, while platform 2 has a higher price, platform 1 gets a higher market share, implying a higher advertisement level and consequently a greater profit. Platform 1 succeeds with this by locating marginally closer to the center of the unit interval pushing platform 2 slightly away, thus capturing a higher potential market area. We see, however, that platform 2 mitigates some of this effect by setting a higher price, resulting in HHI to be slightly smaller than LAI.

We also have that CSL is marginally larger than CSL in the simultaneous game – 1.469% higher.

Equilibrium outcome for three platforms

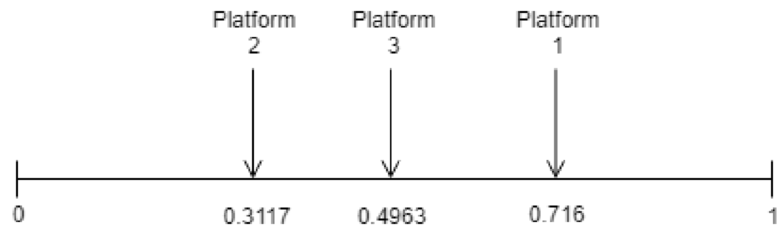


Figure 14: LCAP: Sequential entry - N=3

Table 13: Sequential entry: detailed table of the equilibrium outcome for three competing platforms

Platform	Price	Profit	Market share	Pot. mrk. area	Adv. level
1	0.2271	0.026	0.3416	0.3938	0.11451
2	0.2315	0.02624	0.3449	0.40399	0.1134
3	0.222	0.0203	0.3135	0.2022	0.0915
HHI:	0.3337	$CSL(3)$:	0.02037		
LAI:	0.3592	$CSL^0(3)$:	0.0093		
$\frac{CSL(3)}{CSL^{sim}(3)}$:	0.66416	$\frac{CSL(3)}{CSL^0(3)}$:	2.19		

In the case of three platforms, the equilibrium outcome has changed significantly compared to the duopoly case. Platform 1 locates at 0.716, while the second entrant, platform 2, locates at 0.3117. Platform 3 locates between them at 0.4963, making the first and second entrants corner firms. A striking observation is that, contrasted to the sequential game in the one-sided market with three firms, profits can no longer be ranked by order of entry. In this case, we have that $\Pi_2 > \Pi_1 > \Pi_3$. The reason is that platform 2, located closer to platform 3 than what platform 1 is, has a higher demand in the left periphery from its location. This is contrasted to platform 1, which is located closer to the end of the Hotelling line. Platform 2 thus sets a higher price, has less advertisement on its platform and gets a higher market share than platform 1. Platform 3, on the other hand, gets the lowest profit. This is because it occupies the middle position, which is, as we observed in the simultaneous game, in general, the most unfavorable position.

Even though the difference in profits between platform 1 & 2 is small, it still implies some interesting implications as noted in section 2.5.2: if platform 1 can get a higher profit from entering after platform 2, what incentive does it have to enter the market first? Consequently platform 1 could be inclined to play a waiting game.

HHI is here smaller than LAI; this is an intriguing observation since one might expect a more similar outcome to that of the LCCP sequential game. The reason is, however, that the market shares even out as a result of consumers dislike for advertisement.

We observe, however, that the difference between LAI and HHI in LCAP is higher in the simultaneous game compared to the sequential variant, which means that platform 1 and 2, to a larger degree, manages to use the asymmetry in their favor, compared to the simultaneous game.

We also have that $CSL(3)$ is larger than $CSL(3)^0$, this is, however, to a less extent than in the simultaneous game. Here we can see that $CSL(3)$ is, in fact, 66.416 % of the $CSL(3)$ when the platforms located simultaneously. This is a consequence of platforms locating wider apart than in the game of simultaneous entry, thus resulting in a more socially optimal location pattern.

6.5 Discussion of equilibrium outcomes

We have hitherto seen that allowing for timing in the games, independently, has a substantial impact on the equilibrium outcomes.

Comparing the main overall points made in the previous discussions, we first consider the LAI and HHI measurements. In the LCCP model, we observed that the HHI were consistently larger than the LAI for $N > 2$. This is contrary to what we observed in the LCAP model, where the LAI was continuously higher than the HHI in the same cases. The reasoning behind this hinges on that the firms with advantageous locations in the one-sided market were able to capitalize on this advantage to obtain higher market shares. The same is not true in the LCAP model. Here, the corner firms managed to obtain advantageous locations and thus create very uneven potential market areas causing a high LAI. Yet due to the self-regulating mechanism in the market — consumers' advertisement aversion — market shares evened out in equilibrium.

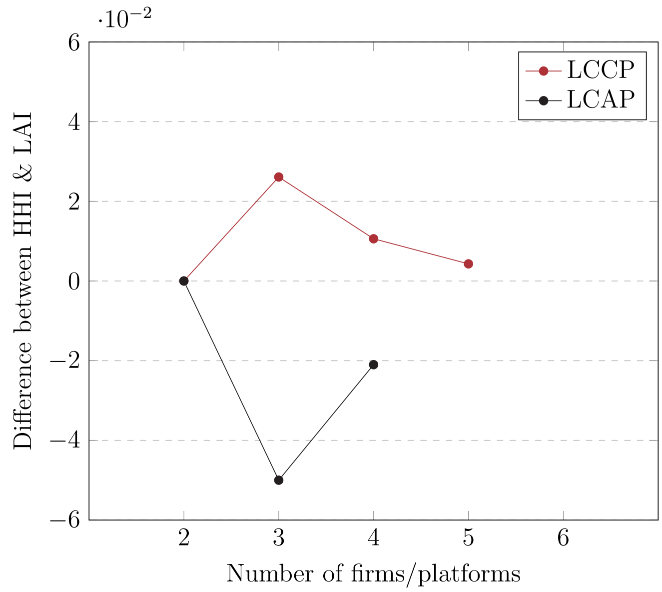


Figure 15: Simultaneous entry: A comparison between LCCP and LCAP based on differences between HHI & LAI

Comparing the consumer surplus loss in the simultaneous games, we can see from Figure 16 that, initially, both LCCP and LCAP produce a substantial consumer surplus loss. For LCCP it is caused by too much differentiation in equilibrium in the duopoly and triopoly case and too little when $10 \geq N > 3$. For LCAP it is caused by too little differentiation for all the cases studied. We can also see that CSL in LCCP is somewhat higher than in LCAP. This observation is, however, reversed in the next cases. Here, the difference between CSL^0 and CSL for LCCP drops nearly to zero, while the difference remains substantially higher for LCAP. This indicates that the dynamics of LCAP leads to a higher consumer surplus loss, caused by too little differentiation overall due too similar location choices. The model setup of LCCP, on the other hand, results in nearly socially optimal location choices for $N > 2$.

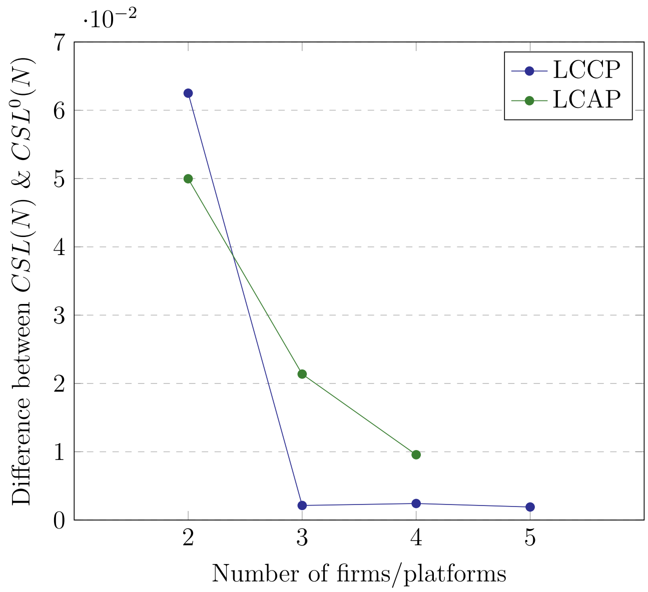


Figure 16: Simultanous entry: A comparison between LCCP and LCAP based on differences between $CSL(N)$ & $CSL^0(N)$

Using figure 16 we see a pattern present in both models: CSL converges to the optimal consumer surplus as number of firms/platforms are allowed to enter the market. This is an innate attribute of the measurement: more firms in the market must necessarily lead to fewer compromises on the desired and the offered taste for the consumer. We thus state the following conjecture:

Conjecture 1 *For both LCCP and LCAP, the consumer surplus loss due to transportation costs converges to social optimum as more firms/platforms are allowed to enter the market.*

When it comes to differences between simultaneous entry and sequential entry for LCAP and LCCP, a few things are interesting to note. First, we have seen that for LCCP, the duopoly outcome was identical when varying the timing of entry, this was not the case for LCAP. In the latter case, we observed that the outcome was marginally different from the simultaneous game. Here, platform 1 managed to use the timing to its advantage. The result changed, however, when we increased the number of platforms to three. We then observed that platform 1 was at a disadvantage from platform 2, meaning that platform 2 managed to obtain a slightly higher profit than platform 1, which gave rise to some motivation issues – a result similar to that observed by Lane (1980) when studying a sequential LCCP game for

three firms. Since we get conflicting results from the duopoly to the three platform case, and since this problem has not been studied previously, we can unfortunately not say anything about the result of studying more than three platforms. It could, however, very well be that, as was the case with Lane (1980), the distribution of profits in the three platform case is unique in the sense that it does not generalize for four platforms and above.
Comparing sequential entry in LCCP, we observed that, for three firms, we get the desired distribution of profits, $\Pi_1 > \Pi_2 > \Pi_3$. In the LCCP case, we can say with certainty, based on reviewing Economides et al. (2002), that profits will be distributed according to the order of entry, either such that a firm has a higher profit than subsequent followers, or that the profits are equal. This will remain at least to the extent that the leader and subsequent followers manage to take advantage of the asymmetric location choices, an advantage which is, of course, decreasing in N.

Second, comparing the profits of the games, we have studied and varying the timing, for LCCP, the accumulated profit is higher in the sequential game (not considering $N = 2$). This is true, also when reviewing the cases studied by Economides et al. (2002). When $N = 2$ in LCAP, the overall profits are in fact, lower than in the simultaneous game. This result is contradicted when $N = 3$. We can, therefore, not say anything general about the profits in LCAP without further observations.

Lastly, we have in LCCP, that the consumer surplus loss in the sequential game is higher than in the simultaneous game. We note here that, based on Economides et al. (2002), this difference decreases, and for $N = 5$, is, in fact, lower for the sequential game. This shows that for games up to $N = 5$ a social planner would prefer the simultaneous structure (excluding $N = 2$, where he should be indifferent). For LCAP, comparing the cases studied, the social planner would prefer the simultaneous game for $N = 2$, while for $N = 3$, he would prefer the sequential game.

7 Conclusion

The purpose of this paper has been to study how multiple entrants, and their timing of entering, affects the equilibrium outcome of our two models. Both models were based on Hotelling's linear city to allow entering firms to differentiate their products horizontally. Consumers single-homed, were uniformly distributed and had quadratic transportation costs à la d'Aspremont et al. (1979). The first of our models, the location-cum-consumer-price model, looked at one-sided markets, while the other, the location-cum-advertisement-price model, looked at two-sided markets. In the latter model, firms, referred to as platforms, had two user groups: consumers and advertisers, and we assumed platforms to solely set advertisement prices. We modified both our models to accommodate multiple firms/platforms.

We characterized the equilibrium outcomes by some equilibrium measurements, which were the *Herfindahl-Hirschman Index* (HHI), a *Locational Asymmetry Index* (LAI) and a measurement for *Consumer Surplus Loss* (CSL) due to transportation costs. These measurements enabled us to properly compare the models when we included a different number of firms and/or changed the timing of firms' entry into the market. Our findings suggest that varying the type of market, number of entries into the market and allowing for timing of entry has a substantial effect on the equilibrium measurements.

When adding more entrants to our models, our equilibrium measurements generally decreases, which is an innate characteristic of how the tools work in the Hotelling model. HHI and LAI decrease simply because more firms sharing the market means each firm gets a smaller piece of the market pie. Also, the decrease in consumer surplus loss is intuitive; an increase in the number of differentiated firms leads to consumers having to make smaller compromises, overall reducing consumers' transportation costs.

In the cases where entrants entered the markets simultaneously, we looked at differences in the rate at which our equilibrium measurements decreased as the number of entrants increased. In the one-sided market, we simulated for up to ten firms, but because of computational constraints, we were only able to simulate for up to four platforms in the two-sided market. When comparing the models up to four firms, we found that the consumer surplus loss in the one-sided market quickly converged towards social optimum, especially when moving from the duopoly to the triopoly case. In the two-sided market, on the other hand, although the loss was smaller for the duopoly, it converged slower to social optimum when increasing number of entrants. We found this to imply that a one-sided market produces a more socially optimal outcome when there are more than two entrants.

In the one-sided market with more than two entrants, HHI was consistently larger than LAI. This implies that some firms managed to gain higher market shares by adjusting their price optimally. In the two-sided market, we observed the opposite: HHI was consistently lower than LAI in cases of more than two entrants, meaning that platforms in disadvantageous positions invariably managed to increase their output beyond what was indicated by their potential market areas. The latter result is caused by the consumers' advertisement aversion working as a self-regulating mechanism in the market.

In equilibrium, firms have to balance the direct and strategic effect(s). Looking at duopolies, the firms in the one-sided market faced a trade-off between the two effects, where the strategic effect was unambiguously dominating, resulting in maximum differentiation. In the two-sided market, however, the pull of the strategic effect became more ambiguous. From an interval starting at an end-point to a given point close to the center of the Hotelling line, both the direct effect and the strategic effect pull towards the middle. Between the given point and the middle, the strategic effect pulls in the opposite direction, which is opposite of the direct effect. Moreover, the strategic effect became dominating closer to the center, thus avoiding minimum differentiation.

Concerning differentiation in the other oligopoly cases, we observed neither maximum nor minimum differentiation when increasing the number of entrants above two. In the one-sided market, increasing the number of entrants on the unit interval did, not surprisingly, lead to overall less differentiation. In the two-sided market, however, the platforms seemed to pull away from each other as more platforms entered the market. This was an interesting result, but, unfortunately, we were unable to study a two-sided market with more than four firms and thus unable to see if this increase in differentiation would persist.

We also studied what happened when firms and platforms entered the two markets sequentially. We limited ourselves to study markets with two and three firms. When firms/platforms entered simultaneously, the location-order of the entrants along the unit interval was exogenously given, but in making the firms enter sequentially, their order was made endogenous.

In the duopoly case, we found the one-sided market equilibrium to be identical to when firms entered simultaneously. For the two-sided market equilibrium, which was almost identical to its simultaneous counterpart as well, the first firm was able to gain a marginally higher profit than the second firm. As a consequence, the results of sequential entry in a duopoly were much the same as the simultaneous versions.

In the triopoly case, the effects of sequential entry came clearly to light. In the one-sided market, the first firm located itself slightly to the side of the middle, creating an asymmetric situation for the next two firms. This was key in getting the largest market share and seizing the highest profit, which granted the firm a first-mover's advantage.

In the two-sided market, on the other hand, the first entrant had a first-mover's disadvantage. Its optimal location was towards one of the end-points, where it knew the second platform would locate towards the other end-point, and they both knew that the third platform would then have to locate between them.

The equilibrium outcome showed that the second firm received a higher profit than the first firm. This has implications for the incentives of being the first platform to enter, as the platform moving first might be inclined to play a waiting game.

Studying the equilibrium measurements, the two-sided market caused a higher consumer surplus loss than the one-sided market because the two-sided market had too little differentiation in equilibrium. We saw that the HHI was higher than LAI in the one-sided market, while in the two-sided market it was the opposite. Both indexes were, in the latter market, lower than for the one-sided market. Since the number of cases we have studied is limited, we cannot say whether or not these results would persist with additional firms.

With firms and consumers utilizing the Internet as a marketplace, it is important to develop models that better capture this new environment. We have not seen other papers develop a product differentiation-model in a two-sided market with more than two platforms, and we are excited to contribute to the literature. We believe our model for two-sided markets has been a step in the right direction to reflect some of the Internet's complexity.

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Appendix A: Tables and Proofs

LCCP - simultaneous entry: equilibrium outcome for six firms

Table 14: Simultaneous entry: detailed table of the equilibrium outcome for six competing firms

Firm number	Price	Profit	Market share	Potential market area
1	0.0513	0.0069	0.1342	0.183
2	0.0323	0.0063	0.1912	0.1695
3	0.0253	0.0043	0.1719	0.1475
4	0.0253	0.0043	0.1719	0.1475
5	0.0323	0.0063	0.1912	0.1695
6	0.0513	0.0069	0.1342	0.183
HHI:	0.1688	$CSL(6)$:	0.00352	
LAI:	0.168	$CSL^0(6)$:	0.0023	

LCCP - simultaneous entry: equilibrium outcome for seven firms

Table 15: Simultaneous entry: detailed table of the equilibrium outcome for seven competing firms

Firm number	Price	Profit	Market share	Potential market area
1	0.0383	0.0044	0.1134	0.158
2	0.0242	0.0041	0.1694	0.1465
3	0.019	0.0028	0.1468	0.1295
4	0.0181	0.0025	0.1394	0.132
5	0.019	0.0028	0.1468	0.1295
6	0.0242	0.0041	0.1694	0.1465
7	0.0383	0.0044	0.1134	0.158
HHI:	0.1457	$CSL(7)$:	0.00248	
LAI:	0.1438	$CSL^0(7)$:	0.0017	

LCCP - simultaneous entry: equilibrium outcome for eight firms

Table 16: Simultaneous entry: detailed table of the equilibrium outcome for eight competing firms

Firm number	Price	Profit	Market share	Potential market area
1	0.03	0.003	0.1	0.139
2	0.019	0.0028	0.1473	0.129
3	0.015	0.0024	0.16	0.1145
4	0.014	0.0017	0.1214	0.1175
5	0.014	0.0017	0.1214	0.1175
6	0.015	0.0024	0.16	0.1145
7	0.019	0.0028	0.1473	0.129
8	0.03	0.003	0.1	0.139
HHI:	0.1272	$CSL(8)$:	0.00184	
LAI:	0.1258	$CSL^0(8)$:	0.0013	

Proof of uniqueness of equilibria in the price stage of LCAP

Here we present the expressions for derived from using relation (23). For any of the expressions below, considering values along the unit interval and respecting the spatial ordering, (23) holds.

Duopoly

For platform 1 and 2, relation (23) gives:

$$\frac{-\frac{3\beta}{2}\gamma+2t(x_1-x_2)}{\beta\gamma-tx_1+tx_2}$$

Triopoly

Platform 1:

$$\frac{-4\beta^2\gamma^2-6\beta\gamma t x_3+8t^2x_2^2+2x_1(4\beta\gamma t-4t^2x_2+4t^2x_3)+2x_2(-\beta\gamma t-4t^2x_3)}{3\beta^2\gamma^2+4\beta\gamma t x_3-4t^2x_2^2+4t^2x_2x_3+x_1(-4\beta\gamma t+4t^2x_2-4t^2x_3)}$$

For platform 2:

$$\frac{-4\beta^2\gamma^2-6\beta\gamma t x_3+8t^2x_2^2-8t^2x_2x_3+2x_1(3\beta\gamma t-4t^2x_2+4t^2x_3)}{3\beta^2\gamma^2+4\beta\gamma t x_3-4t^2x_2^2+4t^2x_2x_3+x_1(-4\beta\gamma t+4t^2x_2-4t^2x_3)}$$

For platform 3:

$$\frac{-4\beta^2\gamma^2-8\beta\gamma t x_3+8t^2x_2^2+2x_1(3\beta\gamma t-4t^2x_2+4t^2x_3)+2x_2(\beta\gamma t-4t^2x_3)}{3\beta^2\gamma^2+4\beta\gamma t x_3-4t^2x_2^2+4t^2x_2x_3+x_1(-4\beta\gamma t+4t^2x_2-4t^2x_3)}$$

Quadropoly

Here we set β, γ and t equal to 1 to ease notation.

For platform 1:

$$\frac{x_1(-8x_2x_3+8x_2x_4+8x_2+8x_3^2-8x_3x_4-8x_4-6)+x_2^2(8x_3-8x_4-8)+x_2(-8x_3^2+8x_3x_4+6x_3+2x_4)-6x_3^2+x_3(6x_4+2)+4x_4+2.5}{x_1(4x_2x_3-4x_2x_4-4x_2-4x_3^2+4x_3x_4+4x_4+3)+x_2^2(-4x_3+4x_4+4)+x_2(4x_3^2-4x_3x_4-4x_3+1)+4x_3^2+x_3(-4x_4-1)-3x_4-2}$$

For platform 2 and 4:

$$\frac{x_1(-8x_2x_3+8x_2x_4+8x_2+8x_3^2-8x_3x_4-2x_3-6x_4-4)+x_2^2(8x_3-8x_4-8)+x_2(-8x_3^2+8x_3x_4+8x_3-2)-6x_3^2+x_3(6x_4+2)+4x_4+2.5}{x_1(4x_2x_3-4x_2x_4-4x_2-4x_3^2+4x_3x_4+4x_4+3)+x_2^2(-4x_3+4x_4+4)+x_2(4x_3^2-4x_3x_4-4x_3+1)+4x_3^2+x_3(-4x_4-1)-3x_4-2}$$

For platform 3:

$$\frac{x_1(-8x_2x_3+8x_2x_4+6x_2+8x_3^2-8x_3x_4-6x_4-4)+x_2^2(8x_3-8x_4-6)+x_2(-8x_3^2+8x_3x_4+8x_3-2x_4-2)-8x_3^2+x_3(8x_4+2)+4x_4+2.5}{x_1(4x_2x_3-4x_2x_4-4x_2-4x_3^2+4x_3x_4+4x_4+3)+x_2^2(-4x_3+4x_4+4)+x_2(4x_3^2-4x_3x_4-4x_3+1)+4x_3^2+x_3(-4x_4-1)-3x_4-2}$$

Appendix B: Code

Simultanoeus LCCP-model

```
# Importing packages
import sympy as sp
import numpy as np
from scipy.optimize import minimize, fsolve, root
import itertools
# Defining symbols:
n = 3 #Number of firms
x = sp.symbols('x1:{}'.format(n+1))
p = sp.symbols('p1:{}'.format(n+1))
t = sp.symbols('t')
# Indifferent consumer
ICL=[]
ICR=[]
# With quadratic Transportation costs:
j = 0
while j <= (n-1):
    if j == 0:
        Right = (p[j+1]-p[j])/(2*t*(x[j+1]-x[j])) + (x[j+1] + x
           [j])/2
        ICR.append(Right)
        ICL.append(0) # Simply to add an extra element
    elif j == (n-1):
        Left = (p[j]-p[j-1])/(2*t*(x[j]-x[j-1])) + (x[j] + x[j
           -1])/2
        ICR.append(0) # Simply to add an extra element
        ICL.append(Left)
    else:
```

```
        Right = (p[j+1]-p[j])/(2*t*(x[j+1]-x[j])) + (x[j+1] + x[j])/2
        Left = (p[j]-p[j-1])/(2*t*(x[j]-x[j-1])) + (x[j] + x[j-1])/2
        ICR.append(Right)
        ICL.append(Left)
    j += 1
# Demand:
Dem=[]
# With uniform distribution:
j = 0
while j <= (n-1):
    if j == 0:
        Dem.append(ICR[j])
    elif j == (n-1):
        Dem.append(1 - ICL[j])
        break
    else:
        Dem.append(ICR[j] - ICL[j])
    j += 1
# Profit functions:
Profit = []
j = 0
while j <= (n-1):
    Profit.append(p[j]*Dem[j])
    j += 1
# First order conditions & Equilibrium prices
FOC = []
j = 0
while j <= (n-1):
```

```
        FOC.append(sp.diff(Profit[j],p[j]))
        j += 1
Nash = sp.solve(FOC, p)
# Inserting equilibrium-prices into profit functions:
EqProfit = []
j = 0
while j <= (n-1):
        EqProfit.append(sp.factor((Profit[j].subs(Nash)).subs(t,1))
            )
        j += 1
# FOC and Response functions:
FOC = []
j = 0
while j <= (n-1):
        FOC.append(sp.diff(EqProfit[j],x[j]))
        j += 1

# Simultanoeus game solved:

# Lambdifying the set of FOC:
FOCLambdified = sp.lambdify((x), FOC)
# Making a tuple with initial guess
x0 = tuple()
j = 1
while j <= n:
        x0 += ((1/(n+1))*(j+1),)
        j += 1
# Defining function to use in finding the root
def fun(x):
        return FOCLambdified(*tuple(x))
```

```
# Finding roots of the set of FOC-functions, and storing the results
res=fsolve(fun, x0) #Fsolve, using previously defined function and initial guess
# Printing equilibrium locations
print(res)
```

Simultaneous LCAP-model

```

# Demand: Can run the program with or without non-uniform distribution of consumers.
import sympy as sp
import numpy as np
from scipy.optimize import minimize, fsolve
import itertools

# Defining symbols
n = 3  # Number of firms
x = sp.symbols('x1:{}'.format(n + 1)) # Locations
R = sp.symbols('r1:{}'.format(n + 1)) # Lump-sum advertisement prices
a = sp.symbols('a1:{}'.format(n + 1)) # Advertiser's demand (amount of advertisement)
y = sp.symbols('y1:{}'.format(n + 1)) # Demand
#t = sp.symbols('t')
#beta = sp.symbols('beta')
#phi=sp.symbols("phi")
t = 1
beta = 1
phi = 1
```

```
# Indifferent consumer
ICL = []
ICR = []
# Quadratic Transportation costs
j = 0
while j <= (n - 1):
    if j == 0:
        Right = (phi*a[j + 1] - phi*a[j]) / (2 * t * (x[j + 1]
            - x[j])) + (x[j + 1] + x[j]) / 2
        ICR.append(Right)
        ICL.append(0)
    elif j == (n - 1):
        Left = (phi*a[j] - phi*a[j - 1]) / (2 * t * (x[j] - x[j
             - 1])) + (x[j] + x[j - 1]) / 2
        ICR.append(0)
        ICL.append(Left)
    else:
        Right = (phi*a[j + 1] - phi*a[j]) / (2 * t * (x[j + 1]
            - x[j])) + (x[j + 1] + x[j]) / 2
        Left = (phi*a[j] - phi*a[j - 1]) / (2 * t * (x[j] - x[j
             - 1])) + (x[j] + x[j - 1]) / 2
        ICR.append(Right)
        ICL.append(Left)
    j += 1
# Demand
Dem = []
# Uniform distribution
j = 0
while j <= (n - 1):
    if j == 0:
```

```
        Dem.append(ICR[j] - y[j])
    elif j == (n - 1):
        Dem.append(1 - ICL[j] - y[j])
        break
    else:
        Dem.append(ICR[j] - ICL[j] - y[j])
    j += 1
# Advertiser's demand
Adv = []
j = 0
while j <= (n-1):
    Adv.append(beta*y[j]-R[j])
    j += 1
# Implementing the advertiser's demand function in the consumer's demand
Dem_a = [] #Dem_a is a dummy list
TempDict=dict(zip(a,Adv))

j = 0
while j <= (n-1):
    Dem_a.append(Dem[j].subs(TempDict, simultaneous=True))
    j += 1
# Solving demand for y
Dem_clear = sp.solve(Dem_a, y)
# Inserting Dem_clear into the advertiser's demand function (List: Adv):
AdvDem = []
j = 0
while j <= (n-1):
    AdvDem.append(Adv[j].subs(Dem_clear))
```

```
    j += 1
# Profit functions
Profit = []
j = 0
while j <= (n-1):
#     Profit.append(p[j]*Dem[j] + R[j]*a[j])
    Profit.append(R[j]*AdvDem[j])
    j += 1
# Finding FOC on lump sum advertisement price (second stage)
FOC = []

j = 0
while j <= (n-1):
    FOC.append(sp.diff(Profit[j], R[j]))
    j += 1
# Equilibrium lump sum advertisement price
EqR = []
EqR = sp.solve(FOC, R)
# Inserting equilibrium price into profit functions
EqProfit = []
j = 0
while j <= (n-1):
    EqProfit.append(Profit[j].subs(EqR))
    j += 1
# FOC - first stage
FOC = []
j = 0
while j <= (n-1):
    FOC.append(sp.diff(EqProfit[j], x[j]))
    j += 1
```

```
# Simultaneous game solved:

# Lambdifying the set of FOC:
FOCLambdified = sp.lambdify((x), FOC)
# Making a tuple with initial guess
x0 = tuple()
j = 0
while j <= (n-1):
    x0 += ((1/(n+1))*(j+1),)
    j += 1
# Defining function to use in finding the root
def fun(x):
    return FOCLambdified(*tuple(x))
# Finding roots of the set of FOC-functions, and storing the
    results
res=fsolve(fun, x0)
res=fsolve(fun, x0)# Fsolve, using previously defined function
    and initial guess
#Printing equilibriums locations
print(res)
```

Sequential extension applicable for both LCCP and LCAP models

```
# Sequential optimization for n = 3
# Functions needed:
# Function for second firm's maximization problem
def EquilProfit(TempX):
    return -1*Tempest[e](*tuple(TempX))
# Function for third firm's maximization problem
def DevFun(Place):
    TempDick = dict(zip(x, xDev))
```

```
    maxIt = -1*EqProfit[Place].subs(TempDick)
    maxItLambdi = sp.lambdify((x[Place]), maxIt)
    consThird = ({'type' : 'ineq', 'fun' : lambda x: (1 - x)},
                 {'type' : 'ineq', 'fun' : lambda x: (x - 0)})
    if j != ThirdGuess():
        maxItresVar = -1*minimize(maxItLambdi, x0 = ThirdGuess
           (), constraints = consThird, tol = 0.00000001).fun
    else:
        x0 = ThirdGuess() - 0.01
        print("Yeehaw")
        maxItresVar = -1*minimize(maxItLambdi, x0 = x0,
           constraints = consThird, tol = 0.00000001).fun
    if np.isnan(maxItresVar) == True:
        return 0
    else:
        return maxItresVar
# Functions for the constraints
def funFOC(TempX):
    return TempFOC[q](*tuple(TempX))
def TsaDude(TempX):
    if k == 0:
        return (1 - TempX[1]), (TempX[1]-TempX[0]), (TempX[0] -
            j), (j - 0)
    elif k == 1:
        return (1 - TempX[1]), (TempX[1] - j), (j - TempX[0]),
           (TempX[0] - 0)
    else:
        return (1 - j), (j - TempX[1]), (TempX[1]-TempX[0]), (
           TempX[0] - 0)

```

```
# Functions for initial guesses
# For the second firm's optimization problem
def initial(j):
    if k == 0:
        inx2 = float(j) + j/4
        inx3 = float(j) + j/2
        TempTuple = (inx2, inx3)
        return TempTuple
    elif k == 1:
        inx1 = float(j) - j/4
        inx3 = float(j) + j/4
        TempTuple = (inx1, inx3)
        return TempTuple
    else:
        inx1 = float(j) - j/2
        inx2 = float(j) - j/4
        TempTuple = (inx1, inx2)
        return TempTuple
# For the third firm's optimization problem
def ThirdGuess():
    if res[keyindex].x[q] < res[keyindex].x[e]:
        if res[keyindex].x[e] < j:
            if Place == 1:
                x0_3 = (res[keyindex].x[e] + j)/2
            if Place == 2:
                x0_3 = (j + 1)/2
        elif (res[keyindex].x[q] < j) and (j < res[keyindex].x[e]): # 3rd firm: x1. 1st firm: x2 = j, 2nd firm: x3
            if Place == 1:
                x0_3 = (j + res[keyindex].x[e])/2
```

```
            if Place == 2:
                x0_3 = (res[keyindex].x[e] + 1)/2
        else: # 3rd firm is placed in the middle, 2nd firm: x3,
            1st firm: x1
            if Place == 0:
                x0_3 = (0 + j)/2
            if Place == 2:
                x0_3 = (res[keyindex].x[e] + 1)/2
    else:
        if res[keyindex].x[e] > j: # Firm 3 is placed at x3, 2
           nd firm x2 and 1st firm x1
            if Place == 1:
                x0_3 = (res[keyindex].x[e] + j)/2
            if Place == 0:
                x0_3 = (0 + j)/2
        elif (j > res[keyindex].x[e]) and (res[keyindex].x[q] >
            j): # Firm 3 is placed at x3, 2nd firm x1 and 1st
           firm x2
            if Place == 1:
                x0_3 = (res[keyindex].x[e] + j)/2
            if Place == 0:
                x0_3 = (0 + res[keyindex].x[e])/2
        else:# 3rd firm is placed in the middle, 2nd firm: x1,
           1st firm: x3
            if Place == 0:
                x0_3 = (0 + res[keyindex].x[e])/2
            if Place == 2:
                x0_3 = (j + 1)/2
    return x0_3

```

```
# The nested loops for solving the game:
FinFirstFirmProfit = {} # Final equilibrium is
AlmostThere = {} # Used for temporary storage of possible equilibriums determined by second firm
SoClose = {} # Used for temporary storage of possible equilibriums where third firm does not deviate
ThirdFirmNoDeviationProfit = {} # For storing the third firm's profit where it chooses not to deviate

j = 0
while j <= 1: # Loop for coarseness
    res = {} # For each j, we store all results for 2nd firm's optimization (up to six results per iteration)
    TempRes = {} # For each j, we store all function-values from the results(For easier handling)
    FirstFirmProfit = {} # For each j, we store the first firm's profit
    Firm3Profit = {} # For each j, we store 3rd firm's profit
    Firm3DeviationProfit = {} # We store what is the maximum profit 3rd firm can achieve by deviating.
    k = 0 # First firms' position
    while k <= (n-1): # First firm loops through the firms' possible positions to "try" the coarseness
        TempList = []
        TempListFOC = []
        for i in EqProfit: # Loop to substitute the coarseness values the first firm tries out into profits
            TempList.append(i.subs(x[k], j))
        for i in FOC: # Substituting the coarseness into the FOC for each k-position
```

```
            TempListFOC.append(i.subs(x[k], j))
        TempX = []  # Temporary list of x-locations for second and third firm, where x[k] is removed
        l = 0
        while l <= (n-1):
            if l != k:  # Choosing positions which currently are not occupied by the first firm
                TempX.append(x[l])  # Appending these positions to the temporary list
            l += 1
        Tempest = [] # Temporary list to put lambdified expressions: profits
        TempFOC = [] # Temporary list to put lambdified expressions: FOC
        TempPro = [] # This is a list to correct for a mistake later in the code
        g = 0
        while g <= (n-1): # For choosing positions
            if g != k: #  Choosing functins which currently do not belong to the first firm
                Tempest.append(sp.lambdify((TempX), TempList[g])) # Lambdifying expressions
                TempFOC.append(sp.lambdify((TempX), TempListFOC[g])) # Lambdifying expressions
                TempPro.append(TempList[g]) # Lambdifying expressions
            g += 1
        TempX = tuple(TempX)
        # Here we need a new "massive loop" to run the optimization for both remaining firms
```

```
        maxDeviationProfit = {} # The most the third firm can receive by deviating
        e = 0 # Second firms' choice
        while e <= len(Tempest)-1: # Tempest is lambdified profit-function
            q = 0
            while q <= 1: #Picking which position Tempest should be constrained upon: if q = 0, the third firm is located to the left
                if Tempest[q] != Tempest[e]:
                    keyindex = x[k], TempX[e], TempX[q], j, k
                    x0 = initial(j)
                    cons = ({'type': 'eq', 'fun': funFOC},
                            {'type': 'ineq', 'fun': TsaDude}) # Criterion that 0<x1<x2<x3<1
                    res[keyindex] = minimize(EquilProfit, x0, constraints = cons, tol = 0.0000000001) # Magic happens here
                    if res[keyindex].success == True:
                        # Second firm's profit
                        TempRes[keyindex] = res[keyindex].fun
                        # First firm's profit
                        TempDict = dict(zip(TempX, res[keyindex].x))
                        FirstFirmProfit[keyindex] = TempList[k].subs(TempDict)
                        # Third firm's profit at 2nd firm's maximizing positions
                        Firm3Profit[keyindex] = TempPro[q].subs(TempDict)
```

```
# 3rd firm's profit if deviating:
                        if Firm3Profit[keyindex] > 0:
                            maxItres = {} # For temporary storing profits the third firm can receive by deviating
                            if res[keyindex].x[q] < res[keyindex].x[e]:
                                if res[keyindex].x[e] < j: # Firm 3 is placed at x1, 2nd at x2 and 1st at x3
# Deviating one to the right: 3rd firm deviates to the right of 2nd firm, left of 1st firm
                                    Place = 1
                                    xDev = (res[keyindex].x[e], x[Place], j)
                                    maxItres[keyindex, Place] = DevFun(Place)
# Deviating two to the right: Firm 3 deviates to the right of 2nd and 1st firm
                                    Place = 2
                                    xDev = (res[keyindex].x[e], j, x[Place])
                                    maxItres[keyindex, Place] = DevFun(Place)
                                elif (res[keyindex].x[q] < j) and (j < res[keyindex].x[e]): # 3rd firm: x1. 1st firm: x2 = j, 2nd firm: x3
# Deviating one to the right: 3rd firm deviates to the right of 1st firm, left of 2nd firm
```

```
                            Place = 1
                            xDev = (j, x[Place], res[keyindex].x[e])
                            maxItres[keyindex, Place] = DevFun(Place)
# Deviating two to the right:
                            Place = 2
                            xDev = (j, res[keyindex].x[e], x[Place])
                            maxItres[keyindex, Place] = DevFun(Place)
                        else: # 3rd firm is placed in the middle, 2nd firm: x3, 1st firm: x1
# Deviating to the left -> x3
                            Place = 0
                            xDev = (x[Place], j, res[keyindex].x[e])
                            maxItres[keyindex, Place] = DevFun(Place)
# Deviating to the left -> x1
                            Place = 2
                            xDev = (j, res[keyindex].x[e], x[Place])
                            maxItres[keyindex, Place] = DevFun(Place)
                    else: # Here: res[keyindex].x[q] > res[keyindex].x[e]:
                        if res[keyindex].x[e] > j: # Firm 3 is placed at x3, 2nd
```

```
                                        firm x2 and 1st firm x1
# Deviating one to the left (to the middle)
                                        Place = 1
                                        xDev = (j, x[Place], res[keyindex].x[e])
                                        maxItres[keyindex, Place] = DevFun(Place)
# Deviating two to the left
                                        Place = 0
                                        xDev = (x[Place], j, res[keyindex].x[e])
                                        maxItres[keyindex, Place] = DevFun(Place)
                                    elif (j > res[keyindex].x[e]) and (res[keyindex].x[q] > j): # Firm 3 is placed at x3, 2nd firm x1 and 1st firm x2
# Deviating one to the left
                                        Place = 1
                                        xDev = (res[keyindex].x[e], x[Place], j)
                                        maxItres[keyindex, Place] = DevFun(Place)
# Deviating two to the left
                                        Place = 0
                                        xDev = (x[Place], res[keyindex].x[e], j)
                                        maxItres[keyindex, Place] = DevFun(Place)
                                    else:# 3rd firm is placed in
```

```
the middle, 2nd firm: x1, 1st firm: x3
# Deviating to the left
                                    Place = 0
                                    xDev = (x[Place], res[keyindex].x[e], j)
                                    maxItres[keyindex, Place] = DevFun(Place)
# Deviating to the right
                                    Place = 2
                                    xDev = (res[keyindex].x[e], j, x[Place])
                                    maxItres[keyindex, Place] = DevFun(Place)
# Only keeping the potential equilibriums where 3rd firm does not deviate:
                            if Firm3Profit[keyindex] > maxItres[max(maxItres, key = maxItres.get)]:
                                ThirdFirmNoDeviationProfit[keyindex] = Firm3Profit[keyindex]
                q += 1
            e += 1
        k += 1
# Loop for sorting which positions the second firm prefers
    for i in FirstFirmProfit:
        if i == min(TempRes, key=TempRes.get): #
            AlmostThere[min(TempRes, key=TempRes.get)] = FirstFirmProfit[i], res[i]
```

```
# Loop for sorting which of the positions, sorted in above loop, corresponds to a position on which the third firm does not deviate
    setAlmostThere = set(AlmostThere)
    setThirdFirmNoDeviationProfit = set(ThirdFirmNoDeviationProfit)
    for i in setAlmostThere.intersection(setThirdFirmNoDeviationProfit):
        SoClose[i] = AlmostThere[i], ThirdFirmNoDeviationProfit[i]
    j += 0.01 #Coarseness parameter
FinFirstFirmProfit[max(SoClose, key=SoClose.get)] = SoClose[max(SoClose, key=SoClose.get)]

# Printing found equilibrium
print(FinFirstFirmProfit)
```