Strategic Insider Trading Equilibrium with a non-fiduciary market maker

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Strategic Insider Trading Equilibrium with a non-fiduciary market maker

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Abstract

The continuous-time version of Kyle's (1985) model is studied, in which market makers are not fiduciaries. They have some market power which they utilize to set the price to their advantage, resulting in positive expected profits. This has several implications for the equilibrium, the most important being that by setting a modest fee conditional of the order flow, the market maker is able to obtain a profit of the order of magnitude, and even better than, a perfectly informed insider. Our model also indicates why speculative prices are more volatile than predicted by fundamentals.

KEYWORDS: Insider trading, asymmetric information, strategic trade, price distortion, non-fiduciary market maker, bid-ask spread, linear filtering theory, innovation equation

Mathematics Subject Classification 2010: 60G35, 62M20, 93E10, 94Axx

1 Introduction

The continuous-time version of Kyle's (1985) model is studied, in which market makers are not fiduciaries. One important feature of a real securities

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market that remained unexplained in Kyle's analysis is the existence of a bid-ask spread. Kyle focuses on a continuous-time auction model in which trade takes place in a risky and a riskless asset among three kinds of agents. A single insider has access to perfect, private observation of the ex post liquidation value of the risky asset at the end of the horizon. Uninformed noise traders trade randomly. Market makers set prices and clear the markets after observing the quantities traded by others.

The market maker in the standard model has substantial market power, yet does not exploit this to his own advantage when setting the price; the market maker is assumed to be a fiduciary acting in the best interest of market participants. We question the realism of this assumption, and instead allow the market maker some degree of monopoly power in which she can perturb prices to her advantage after observing the order flow.

These issues were addressed in a recent paper (Aase and Gjesdal (2017)), in the setting of a one-period model. It is of interest to extend this analysis to several periods, which we do in this paper, where we consider a continuous-time extension of their model.

It turned out to be a non-trivial task to introduce continuous trading within this framework. For example does the stochastic differential equation for the total order flow contain a mean zero 'innovation' term in addition to a generalized mean reverting term of the Ornstein-Uhlenbeck type. This require special treatment using filtering theory. A key quantity here, as in the one-period model, turns out to be the trading intensity process of the insider.

We formulate the problem as a stochastic differential game, and find the Nash equilibrium using the stochastic maximum principle. In our case, this may be simplified somewhat by the Bellman approach, reducing the number of adjoint variables to be determined. By the use of variational calculus, however, we manage to find an integral equation for the insider's trading strategy, which we can treat by numerical methods. We present graphs of the trading intensity as a functions of time, which gives an interesting illustration of the time development of the trades.

Despite the technical difficulties, we can confirm most of the main findings in the one-period model: Also our analysis shows that for only a moderate perturbation of the price, the profits of the market maker may exceed that of the perfectly informed insider. In the paper we can moreover illustrate the time profiles of these profits. Our analysis can serve as one explanation of why so much wealth tends to end up in the financial industry, an obvious question many have posed after the 2008-financial crisis.

A regulatory authority (the SEC) is introduced to limit the distortion of prices. In our model this limits the degree to which the market maker can perturb the price, and results in an equilibrium in which the insider maximizes profits and the market maker trades fees. Even if the regulatory constraint limits the market maker's degree of price distortion, still the market maker's profit may exceed that of the perfectly well informed insider. This happens for reasonable degrees of price distortions, a concept developed in the paper.

Our pricing functional is nonlinear, which seems like a popular topic in itself in parts of the extant literature, together with "model uncertainty" and similar issues. In our model the nonlinearity stems from a specific economic assumption, namely that the market maker trades fees. As we know, in neoclassical equilibrium theory prices are linear for a variety of reasons, among others to avoid arbitrage possibilities, which is not an issue here.

There is a rich literature on the one period model, as well as on discrete time insider trading, e.g., Holden and Subrahmanyam (1992), Admati and Pfleiderer (1988), and others, all adding insights to this class of problems. Glosten and Milgrom (1985) present a different approach, containing similar results to Kyle. Before Kyle (1985) and Glosten and Milgrom (1985) there is also a huge literature on insider trading in which the insider acts competitively, e.g., Grossman and Stiglitz (1980).

The approach of this article is to study the continuous-time model directly, not as a limiting model of a sequence of auctions, and use the machinery of infinitely dimensional optimization, directional derivatives (or calculus of variations) and filtering theory to solve the problem. The stochastic maximum principle in the setting of differential game theory, as well as the Bellman approach appear in two appendices.

We are able to find the dynamic price of the risky asset, the various profit paths of the participants, all in terms of the insider's trading intensity process. The latter we show satisfies an integral equation, that can be solved by an iterative procedure. This we illustrate numerically, by graphs of the the trading intensity, the profits of the agents, and the other key variables developed in in the paper, all as functions of time.

The paper is organized as follows: The model is explained in Section 2. The analysis of the continuous time model starts in Section 3, where the mean, the variance and the covariances of the order flow y is derived in Section 3.1, with preliminary expressions for the profit functions of the insider and the market maker. In Section 4 the insider's optimization problem is treated in Theorem 4.1, resulting in expressions for the various profit processes, as well as the other dynamic quantities of interest. In Section 5 we suggest how the regulator's problem may be solved, in Section 6 we introduce a measure of dynamic price informativeness in the market, and present numerical illustrations. Section 7 presents various graphs and computations, which illustrate the key quantities in the paper, from which conclusions can be drawn. In Section 8 we provide some suggestions for further research, and Section 9 concludes. The paper also contains four appendices.

2 The Model

At time T there is to be a public release of information that will perfectly reveal the value of an asset; cf. fair value accounting. Trading in this asset and a risk-free asset with interest rate zero is assumed to occur continuously during the interval $[0, T]$. The information to be revealed at time T is represented as a signal $\tilde{v}$, a random variable which we interpret as the price at which the asset will trade after the release of information. This information is already possessed by a single insider at time zero. The unconditional distribution of $\tilde{v}$ is assumed to be *normal* with mean $\mu_{\tilde{v}}$ and variance $\sigma_{\tilde{v}}^2$.

In addition to the insider, there are noise (liquidity) traders, and risk neutral market makers. The noise traders are unable to correlate their orders to the insider's signal $\tilde{v}$. Thus the noise traders have random, price-inelastic demands. All orders are market orders and the net order flow is observed by the market maker. We denote by z_t the cumulative orders of noise traders through time t. The process z_t is assumed to be a Brownian motion with mean zero and variance rate σ_t^2, i.e., $dz_t = \sigma_t dB_t$, for a standard Brownian motion B defined on a probability space (Ω, P). As Kyle (1985) and Back (1992) we assume that B is independent of $\tilde{v}$. We let x_t be the cumulative orders of the informed trader, and define

$$y_t = x_t + z_t \qquad \text{for all } t \in [0, T] \tag{2.1}$$

as the total orders accumulated by time t.

The market maker only observes the process y, so he cannot distinguish between informed and uninformed trades. Let $\mathcal{F}_t^y = \sigma(y_s; s \leq t)$ be the information filtration of this process. The risk neutral market maker, assumed to have some degree of monopoly power, sets the price p_t at time t as follows

$$p_t = E[\tilde{v} + u_t | \mathcal{F}_t^y] := m_t + E[u_t | \mathcal{F}_t^y], \tag{2.2}$$

where $m_t = E[\tilde{v}|\mathcal{F}_t^y]$ is the "fair value", and $u_t = k_t y_t$ for $k_t \geq 0$ a deterministic function satisfying $k_t \to 0$ as $t \to T$. We assume that $k_t = (T-t)\kappa$, where κ is a non-negative constant set by the market maker. Clearly $E[u_t|\mathcal{F}_t^y] = k_t y_t$. The market maker, the insider and the noise traders all know the probability distribution of $\tilde{v}$.

We assume that the insider's market order at time t is of the form

$$dx_t = (\tilde{v} - p_t)\beta_t dt, \qquad x_0 = 0, \tag{2.3}$$

where $\beta \geq 0$ is some deterministic function. This form of the market order follows from the discrete time formulation of the problem, assuming the insider maximizes profits, in which case (2.3) follows from the first order condition; x_t does not depend on p_t since x_t is submitted before p_t is set by the market makers.

Assumption (2.3) is consistent with Kyle (1985).[1] The function β_t is called the *trading intensity* on the information advantage $(v - p_t)$ of the insider.

The basic assumptions behind this result is (i) profit maximization by the insider, where it is shown in Aase and Gjesdal (2016) that this result still holds when the market maker sets the price as we have assumed in (2.2) above, and (ii) the insider does not condition the quantity he trades on price. Here the insider chooses quantities ("market orders") instead of demand functions ("limit orders").

Assumption (2.2) is our deviation from the standard model.[2] Below we explain why this price setting leads to a positive expected profit for the market maker.

The stochastic differential equation for the total order y_t is

$$dy_t = (\tilde{v} - m_t)\beta_t dt - k_t\beta_t y_t\, dt + \sigma_t\, dB_t. \tag{2.4}$$

Aside from the first mean zero 'innovation' term, the equation shows that y_t has the structure of a (generalized) mean reverting Ornstein-Uhlenbeck process, oscillating around this mean zero term.

Let us denote by $S_t(\beta) = E\{(\tilde{v} - p_t)^2\}$ and by $\gamma_t(\beta) = E\{(\tilde{v} - m_t)^2\}$. Usually the assumption is made that $\lim_{t\to T^-} p_t = p_T = \tilde{v}$ a.s. This assumption seems natural, ensuring that all information available has been incorporated

[1] The finite variation property of x is assumed by Kyle (1985), and an equilibrium where this is the case is found by Back (1992).

[2] An alternative would be to assume that the market maker is risk averse, which would lead to a different model.

in the price at the time T of the public release of the information, at which time a price spread cannot be sustained.

In Aase et. al. (2012a) $m_t = p_t$ for all $t \in [0, T]$, and it was there demonstrated that $p_t \to \tilde{v}$ as $t \to T^-$, and $S_t(\beta) \to 0$ as $t \to T^-$ as a consequence of the insider following his optimal trading strategy. Here we find it natural to simply assume this, as was done in e.g., Kyle (1985), so that $p_t - m_t \to 0$ as $t \to T^-$, and both converge to $\tilde{v}$, since $k_t \to 0$ by assumption.

Denote the insider's wealth by w and the investment in the risk-free asset by b. The budget constraint of the insider can best be understood by considering a discrete time setting, of which the continuous-time model is the limit (in an appropriate sense). At time t the agent submits a market order $x_t - x_{t-1}$ and the price changes from p_{t-1} to p_t. The order is executed at price p_t, in other words, $x_t - x_{t-1}$ is submitted *before* p_t is set by the market makers. The investment in the risk-free asset changes by $b_t - b_{t-1} = -p_t(x_t - x_{t-1})$, i.e., buying stocks leads to reduced cash with exactly the same amount. Thus, the associated change in wealth is

$$b_t - b_{t-1} + x_t p_t - x_{t-1} p_{t-1} = x_{t-1}(p_t - p_{t-1}). \tag{2.5}$$

In other words, the accounting identity for the wealth dynamics is of the same type as in the standard price-taking model, except for one important difference; while, in the rational expectations model, the number of stocks in the risky asset at time t depends only on the information available at this time, so that both the processes x and p are adapted processes with respect to the same filtration, here the order x depends on information available only at time T for the market makers (and the noise traders).

As a consequence of (2.5) we obtain the dynamic equation for the insider' wealth w_t^I as follows

$$w_t^I = w_0^I + \int_0^t x_s dp_s \tag{2.6}$$

This is not well-defined as a stochastic integral in the traditional interpretation, since p_t is $\mathcal{F}_t^y$-adapted, and x_t is not. Thus it needs further explanation. However, since we assume that the strategy of the insider has the form (2.3) for some deterministic continuous function $\beta_t > 0$, then a natural interpretation of (2.6) is obtained by using *integration by parts*, as follows:

$$
\begin{aligned}
w_t^I &= w_0^I + x_t p_t - \int_0^t p_s dx_s \\
&= w_0^I + p_t \int_0^t (\tilde{v} - p_s)\beta_s ds - \int_0^t p_s(\tilde{v} - p_s)\beta_s ds \\
&= w_0^I + \int_0^t (\tilde{v} - p_s)^2 \beta_s ds - \int_0^t (\tilde{v} - p_t)(\tilde{v} - p_s)\beta_s ds. \qquad (2.7)
\end{aligned}
$$

Alternatively, one might obtain (2.7) by interpreting the stochastic integral in (2.6) as a *forward integral.* See Russo and Vallois (1993), Russo and Vallois (1995, 2000) for definitions and properties and Biagini and Øksendal (2005) for applications of forward integrals to finance.

Similarly we can find the market maker's profit from his price setting operations: His wealth w^M from these operations is

$$
w^M = w_0^M + (p_0 - p_1)y_0 + (p_1 - p_2)y_1 + \cdots
$$

When the total initial order $y_0 > 0$, the market maker has to sell to clear the market and accordingly sets the price p_0 a bit higher than he would have done if he were a fiduciary. Similarly, if $y_0 < 0$ she must buy to clear the market, so he sets the price p_0 a bit lower than he would if he sets the price fairly. Continuing this practice in every period, he will end up with a positive expected profit, simply because the profit he would have obtained by being fair has zero expectation[3].

Consider the situation where the total initial order $y_0 > 0$. Because of the mean reverting nature of y towards zero, it is more likely that $y_1 < y_0$ than the other way around. By the price setting mechanism used by the market maker, it is more likely that $p_1 < p_0$ than the opposite, in which case the market maker's profit is positive. A similar reasoning holds when $y_0 < 0$, in which case the market maker buys from the other participants at time zero, and sells the stock in the market at time one at the price p_1 he sets then, based on $y_1 - y_0$. Thus, in expectation the market maker's profit is positive.

Notice that the market maker takes some 'overnight' risk, in that, when he must sell to the other participants at time t, he sets the price p_t which he sells for, and the next day he sets the price p_{t+1}, based on the order $y_{t+1} - y_t$,

[3] One may think of trade as "synthetic" in that only money changes hands, based on dynamics of the underlying stock.

at which he buys in the market the stock that he 'delivered' the day before. By the price setting mechanism, more likely than the other way he profits from this operation. If he were a fiduciary, he would go even in 'the long run'. Here as a non-fiduciary, in expectation his profit is positive.

By going to the continuous time limit, his wealth at time t is

$$w_t^M = w_0^M - \int_0^t y_s dp_s = w_0^M - p_t y_t + \int_0^t p_s dy_s + [p, y]_t, \tag{2.8}$$

where $[p, y]_t$ is the quadratic covariance process of p and y. Unlike the corresponding expression for the insider, this integral is well-defined in the traditional interpretation, since p_t is $\mathcal{F}_t^y$-adapted, and so is of course y_t.

Finally, the noise traders' profit is

$$w_t^N = w_0^N + \int_0^t z_s dp_s = w_0^N + z_t p_t - \int_0^t p_s dz_z - [p, z]_t. \tag{2.9}$$

The stochastic integral $\int_0^t z_s dp_s$ is well-defined in the traditional meaning since z_t is $\mathcal{F}_t^B$-adapted, p_t is $\mathcal{F}_t^y$-adapted and $\mathcal{F}_t^y \supset \mathcal{F}_t^B$, and hence, by integration by parts, so is the latter stochastic integral in (2.9).

Since $y_t = x_t + z_t$ and x is of bounded variation, $[p, y]_t = [p, z]_t$ for all t. Since this is a pure exchange economy, it follows that the sum of the profits is zero with probability one, or, $w_t^I + w_t^M + w_t^N = w_0^I + w_0^M + w_0^N$ a.s.

3 Some basic analysis.

Returning to the stochastic process for the total order at time t, y_t, its representation is given by (2.4), which we repeat here

$$dy_t = \big(\tilde{v} - E(\tilde{v}|\mathcal{F}_t)\big)\beta_t dt - k_t \beta_t y_t \, dt + \sigma_t \, dB_t.$$

This is a Gaussian process consisting of an Ornstein-Uhlenbeck type process, with a normally distributed "innovation" term added to its drift term, the first term on the right-hand side in the above stochastic differential equation.

In order to analyse this model for the total order, we start by rewriting this equation as follows:

$$dy_t + y_t k_t dt = (\tilde{v} - m_t)\beta_t dt + \sigma_t dB_t.$$

If we define

$$\tilde{y}_t := y_t \exp\left(\int_0^t k_s \beta_s ds\right) \tag{3.1}$$

and use Ito's lemma, we obtain the following

$$d\tilde{y}_t = (\tilde{v} - m_t)\beta_t \exp\left(\int_0^t k_s \beta_s ds\right)dt + \sigma_t \exp\left(\int_0^t k_s \beta_s ds\right)dB_t. \tag{3.2}$$

Clearly $\mathcal{F}^{(y)} = \mathcal{F}^{(\tilde{y})}$ and hence

$$m_t = E[\tilde{v}|\mathcal{F}^{(y)}] = E[\tilde{v}|\mathcal{F}^{(\tilde{y})}]. \tag{3.3}$$

Therefore we may regard (3.2) as the innovation process of an "observation process" $\hat{y}_t$ defined by

$$d\hat{y}_t = \tilde{v}\beta_t \exp\left(\int_0^t k_s \beta_s ds\right)dt + \sigma_t \exp\left(\int_0^t k_s \beta_s ds\right)dB_t; \quad \hat{y}_0 = 0. \tag{3.4}$$

For this to hold, we need to verify that

$$\mathcal{F}_t^{(\tilde{y})} = \mathcal{F}_t^{(\hat{y})}. \tag{3.5}$$

Suppose (3.5) is proved. Then

$$m_t = E[\tilde{v}|\mathcal{F}_t^{(\hat{y})}]$$

is the filtered estimate of v given the observations $\tilde{y}_s; s \leq t$.

By Theorem 12.1 in [18] or Theorem 6.2.8 in [19], the filter m_t is given by the SDE

$$\begin{aligned} dm_t &= \frac{\gamma_t \beta_t \exp(\int_0^t k_s \beta_s ds)}{\sigma_t^2 \exp(2\int_0^t k_s \beta_s ds)} \Big[d\hat{y}_t - \beta_t \exp\left(\int_0^t k_s \beta_s ds\right) m_t dt\Big]; \quad t \geq 0; \\ m_0 &= E[\tilde{v}], \end{aligned} \tag{3.6}$$

where $S_t = S_t^{(\beta)} = \gamma_t(\beta) + k_t^2 V(t)$, where $V(t) = E(y_t^2)$, and $\gamma_t(\beta)$ solves the Riccati equation

$$\begin{aligned} d\gamma_t &= -\frac{\beta_t^2 \gamma_t^2}{\sigma_t^2}; \quad t \geq 0 \\ \gamma_0 &= E[(\tilde{v} - E[\tilde{v}])^2]. \end{aligned} \tag{3.7}$$

Thus we have a controlled state process

$$(\hat{y}_t, m_t, \gamma_t)$$

given by the equations (3.4),(3.6) and (3.7).

Rewriting the system in terms of (y_t, m_t, γ_t) we obtain the following set of equations

$$dy_t = (\tilde{v} - m_t - k_t y_t)\beta_t dt + \sigma_t dB_t; \quad y_0 = 0 \tag{3.8}$$

$$dm_t = \frac{\gamma_t \beta_t}{\sigma_t^2}[(\tilde{v} - m_t)\beta_t dt + \sigma_t dB_t]; \quad m_0 = p_0 = E[\tilde{v}] \tag{3.9}$$

$$d\gamma_t = -\frac{\beta_t^2 \gamma_t^2}{\sigma_t^2}; \quad \gamma_0 = E[(\tilde{v} - E[\tilde{v}])^2]. \tag{3.10}$$

The expected profits are

$$J^M(k,\beta) := w_0^M + E\Big(\int_0^T k_t y_t\,(k_t y_t + m_t - \tilde{v})\beta_t dt - \int_0^T y_t^2\, dk_t\Big) \tag{3.11}$$

$$J^I(k,\beta) := w_0^I + \int_0^T E[(\tilde{v} - m_s - k_s y_s)^2]\beta_s ds. \tag{3.12}$$

for the market maker and for the insider, respectively.

Let us now return to the problems of the previous section and calculate the profits of various participants in this economy. Towards this end we first need expressions for the mean, the variance and the covariances of the market order process y.

3.1 The variance and covariances of the process y.

We start with the variance. Based on the expression in (3.3), we proceed as follows. From equation (3.1) we have by Itô's lemma

$$d(\tilde{y}_t)^2 = 2\tilde{y}_t d\tilde{y}_t + \frac{1}{2}2(d\tilde{y}_t)^2 =$$

$$2y_t^2\big[(\tilde{v} - m_t)\beta_t \exp\big(\int_0^t k_s\beta_s ds\big)dt + \sigma_t \exp\big(\int_0^t k_s\beta_s ds\big)dB_t\big] +$$

$$\sigma_t^2 \exp\big(2\int_0^t k_s\beta_s\big)dt.$$

From this we deduce that

$$E\big[\tilde{y}_t^2\big] = E\Big[\int_0^t 2y_s^2\Big[(\tilde{v}-m_s)\beta_s \mathrm{exp}\big(\int_0^s k_u\beta_u du\big)ds$$

$$+\sigma_s \exp\big(\int_0^s k_u\beta_u du\big)dB_s\big] + \sigma_s^2 \exp\big(2\int_0^s k_u\beta_u du\big)ds\Big] =$$

$$\int_0^t \sigma_s^2 \exp\big(2\int_0^s k_u\beta_u du)ds.$$

Observe that

$$E\big((\tilde{v}-m_t)y_t^2\big) = E(\tilde{v}y_t^2) - E\big(E(\tilde{v}|\mathcal{F}_t^y)y_t^2\big) = 0$$

since $E\big(E(\tilde{v}|\mathcal{F}_t^y)y_t^2\big) = E\big(E(y_t^2\tilde{v}|\mathcal{F}_t^y)\big) = E(\tilde{v}y_t^2)$. Hence

$$\exp\big(2\int_0^t k_u\beta_u du\big)E\big[y_t^2\big] = \int_0^t \sigma_s^2 \exp(2\int_0^s k_u\beta_u du\big)ds$$

or

$$E\big[y_t^2\big] = e^{-2\int_0^t k_r\beta_r dr}\int_0^t \sigma_u^2\, e^{2\int_0^u k_r\beta_r dr}du. \tag{3.13}$$

This expression will be useful below. We use the notation $V(t) := E(y_t^2)$ for all $t \in [0,T]$. [4]

Moving to the covariances $E(y_t y_s)$ for any $s > t$, we proceed as follows. Here we use the notation

$$e(t) := e^{\int_0^t k_r\beta_r dr}.$$

For $s > t$,

$$d_s(\tilde{y}_s\tilde{y}_t) = (\tilde{v}-m_s)\beta_s e(s)\tilde{y}_t ds + \sigma_s e(s)\tilde{y}_t dB_s.$$

Integrating this from t to s, we get

$$E[(\tilde{y}_s - \tilde{y}_t)\tilde{y}_t] = E[\int_t^s (\tilde{v}-m_r)\beta_r e(r)\tilde{y}_t dr + \int_t^s \sigma_r e(r)\tilde{y}_t dB_r] =$$

$$\int_t^s E[(\tilde{v}-m_r)\tilde{y}_t]\beta_r e(r)dr + 0 = 0,$$

[4]The theory leading to the result in (3.13) may be linked to a deeper result in filtering theory. For details, see Appendix 4.

since

$$E[(\tilde{v}-m_r)\tilde{y}_t] = E[E[(\tilde{v}-m_r)\tilde{y}_t|\mathcal{F}_t^y] = E[\tilde{y}_t E[\tilde{v}-m_r|\mathcal{F}_t^y]] = 0.$$

The latter equality follows from $E[(\tilde{v}-m_r)|\mathcal{F}_t^y] = E[E[(\tilde{v}-m_r)|\mathcal{F}_r^y|\mathcal{F}_t^y]] = 0$, since the inner conditional expectation is zero. We obtain for $s > t$

$$E[\tilde{y}_s\tilde{y}_t] = E[(\tilde{y}_s - \tilde{y}_t)\tilde{y}_t] + E[\tilde{y}_t^2] = E[\tilde{y}_t^2].$$

Using the definition of $\tilde{y}$, we have that

$$E[y_s y_t] = E[y_t^2]e^{-\int_t^s k_r\beta_r dr}$$

Combining this with our above result (3.13), we conclude that

$$E[y_s y_t] = e^{-(\int_0^s k_r\beta_r dr + \int_0^t kr\beta_r dr)}\int_0^t \sigma_u^2 e^{2\int_0^u k_r\beta_r dr}du, \quad \text{for } s > t. \tag{3.14}$$

For $s = t$ we obtain the result in equation (3.13).

Figure 1 illustrates a graph of the covariance function $C(s,t;\kappa) := E[y_s y_t]$ when $\kappa = 0.045$ for $s, t \in [0, 10]$.

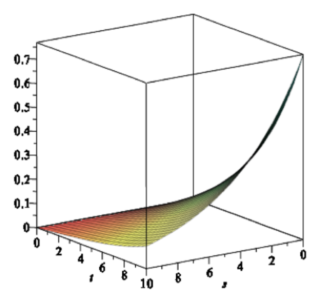

Fig. 1: The covariance function $C(s,t)$ of y when $\kappa = 0.045$.

The base case parameters are $\sigma_t = \sigma = 0.20$, a constant for all $t \in [0,T]$. Also $\gamma_0 = \sigma_{\tilde{v}}^2$, where $\sigma_{\tilde{v}} = 0.30$, and we have chosen $T = 10$. (Here we have anticipated a bit, and used the optimal value of the trading intensity β_t of the insider appearing in Section 4 below.)

3.1.1 The mean of y

We will also need the mean $E(y_t)$ of the process y for any t. Starting with the equation

$$y_t = y_0 + \int_0^t (\tilde{v} - E(\tilde{v}|\mathcal{F}_s)\beta_s ds - \int_0^t k_s\beta_s y_s ds + \int_0^t \sigma_s dB_s,$$

and letting $E(y_t) := \bar{y}_t$, where $\bar{y}_0 = y_0$, by taking expectation in the above equation we obtain

$$\bar{y}_t = y_0 - \int_0^t k_s \beta_s \bar{y}_s ds$$

or

$$\frac{d}{dt}\bar{y}_t = -\beta_t k_t \bar{y}_t$$

which is an ordinary, linear differential equation in $\bar{y}_t$, with initial condition $\bar{y}_0 = y_0$, the unique solution of which is

$$E(y_t) = y_0 e^{-\int_0^t k_s \beta_s ds}.$$

In our model $y_0 = 0$, which implies that $E(y_t) = 0$ for all $t \in [0, T]$. Thus $E(p_t) = E(m_t) + k_t E(y_t) = E(m_t) = E(\tilde{v}) = \mu_{\tilde{v}}$, so the price p_t has the correct expectation at all times.

3.2 The profit of the insider

Returning to the insider, from equation (2.7) giving the wealth w_t of the insider at any time t, since

$$\int_0^T E[(\tilde{v} - p_T)(\tilde{v} - p_t)]\beta_t dt = 0$$

by our assumption that $p_t \to p_T = \tilde{v}$, his task is to find the trading intensity β_t which maximizes the expected terminal wealth

$$E[w_T^I] = w_0^I + \int_0^T E[(\tilde{v} - m_t - k_t y_t)^2]\beta_t dt := J^I(k, \beta). \tag{3.15}$$

Later, when we consider the net profit at any time $t \in [0, T]$, we will use the notation $p_I(t, \kappa)$ for the insider's net profit by time t, so that $J^I(k, \beta) - w_0^I := p_I(T, \kappa))$ with this notation. Similarly for the market maker.

The dilemma for the insider is that an increased trading intensity at some time t will reveal more information about the value of $\tilde{v}$ to the market makers and hence induce a price p_t closer to $\tilde{v}$, which in turn implies a reduced insider information advantage. On the other hand she has to trade in order to make any profit at all.

First observe that

$$E\big((\tilde{v} - m_t)y_t\big) = E(\tilde{v}y_t) - E\big(E(\tilde{v}|\mathcal{F}_t^y)y_t\big) = 0$$

since $E\big(E(\tilde{v}|\mathcal{F}_t^y)y_t\big) = E\big(E(y_t\tilde{v}|\mathcal{F}_t^y)\big) = E(\tilde{v}y_t)$, a result similar to the one obtained above, with y_t instead of y_t^2.

By the definition of $\gamma_t(\beta) = E(\tilde{v} - m_t)^2$, we then obtain the following

$$J^I(k,\beta) = w_0^I + \int_0^T \beta_t(\gamma_t(\beta) + k_t^2 V_t)dt \tag{3.16}$$

since the cross term vanishes by by the above observation. Using the expression for $V(t) := E(y_t^2)$ given in (3.13), we obtain the following

$$J^I(k,\beta) = w_0^I + \int_0^T \beta_t\big(\gamma_t(\beta) + k_t^2 e^{-2\int_0^t k_s\beta_s ds}\int_0^t \sigma_s^2 e^{2\int_0^s k_r\beta_r dr}ds\big)dt. \tag{3.17}$$

The insider will now maximize this expression in the trading intensity process β, for a given price perturbation process k by the market maker.

Before we address this problem, we want to find the profit of the insider *at any time* $t \in [0,T]$, which will allow us to observe the relative time performance of the two profit functions of interest.

Towards this end, let us go back to the expression for the insider's profit at time t given in (2.7). Taking expectation in this equation we obtain

$$E(w_t^I) = w_0^I + \int_0^t E(\tilde{v} - p_s)^2\,\beta_s\,ds - \int_0^t E(\tilde{v} - p_t)(\tilde{v} - p_s)\,\beta_s\,ds =$$

$$w_0^I + \int_0^t (\gamma_s(\beta) + k_t^2 V(s))\,\beta_s\,ds - \int_0^t E(\tilde{v} - m_t - k_t y_t)(\tilde{v} - m_s - k_s y_s)\,\beta_s\,ds,$$

where the second term follows from (3.17). Consider the last term. The integrand can be written

$$\begin{aligned} E(\tilde{v} - m_t - k_t y_t)(\tilde{v} - m_s - k_s y_s) = E(\tilde{v} - m_t)((\tilde{v} - m_s) - \\ k_s E((\tilde{v} - m_t)y_s) - k_t E((\tilde{v} - m_s)y_t) + k_t k_s E(y_t y_s). \end{aligned} \tag{3.18}$$

The second expectation on the right-hand side is

$$\begin{aligned} E((\tilde{v} - m_t)y_s) = E[E[(\tilde{v} - m_t)y_s|\mathcal{F}_s^y]] = \\ E[y_s E(\tilde{v} - m_t|\mathcal{F}_s^y)] = E[y_s E[E(\tilde{v} - m_t|\mathcal{F}_t^y)]|\mathcal{F}_s^y] = 0 \end{aligned}$$

by standard iterated expectations, since $E(\tilde{v} - m_t|\mathcal{F}_t^y) = 0$, as shown before.

Notice that $s \leq t$ in these computations. The third expectation on the right-hand side of (3.18) is

$$E((\tilde{v} - m_s)y_t) = E[E[(\tilde{v} - m_s)y_t|\mathcal{F}_s^y]]$$
$$= E[E[E(y_t(\tilde{v} - m_t)|\mathcal{F}_s^y]|\mathcal{F}_t^y] = E[y_t E[E(\tilde{v} - m_s|\mathcal{F}_s^y)]|\mathcal{F}_t^y] = 0,$$

where the second equality above follows from a not so standard, but rather obvious, iterated expectation result (see e.g., Tucker (1967), Th 6, Ch 7), and again, because $E(\tilde{v} - m_s|\mathcal{F}_s^y) = 0$, the result follows.

It remains to compute the first expectation on the right-hand side of (3.18). It follows from Theorem 3.1 in Aase and Øksendal (2018) that

$$E(\tilde{v} - m_t)((\tilde{v} - m_s) = \gamma_t(\beta).$$

The last term in (3.18), the covariance, we have already computed in Section 3.1. Since here $t \geq s$, we rewrite this formula accordingly, namely as

$$E[y_t y_s] = e^{-(\int_0^t k_r\beta_r dr + \int_0^s kr\beta_r dr)} \int_0^s \sigma_u^2 e^{2\int_0^u k_r\beta_r dr} du, \quad \text{for } t \geq s. \tag{3.19}$$

This means that the insider's profit at any time t in $[0, T]$ is given by

$$E(w_t^I) = w_0^I + \int_0^t (\gamma_s(\beta) + k_s^2 V_s)\beta_s ds - \gamma_t(\beta) \int_0^t \beta_s ds - k_t \int_0^t k_s E(y_t y_s) ds.$$

Observe that as $t \to T$, this profit converges to the expression in (3.16), since both $\gamma_t(\beta) \to 0$ and $k_t \to 0$ then. By use of (3.19) the insider's profit can be written

$$\begin{aligned} E(w_t^I) = w_0^I &+ \int_0^t (\gamma_s(\beta) + k_s^2 V_s)\beta_s ds - \gamma_t(\beta) \int_0^t \beta_s ds \\ &- k_t e^{-\int_0^t k_r\beta_r dr} \int_0^t \big(e^{-\int_0^s k_r\beta_r dr} \int_0^s \sigma_u^2 e^{2\int_0^u k_r\beta_r dr} du\big) k_s ds. \end{aligned} \tag{3.20}$$

The problem of finding the optimal value of the insider's trading intensity β_t, and the corresponding expression for the profit fundtion is relegated to Section 4 below.

3.3 The profit of the market maker

The market maker's expected profit is:

$$J^M(k,\beta) := E(w_T^M) = w_0^M - E\big(\int_0^T y_t\,dp_t\big) =$$

$$w_0^M - E\big(\int_0^T k_t y_t\,dy_t + \int_0^T y_t^2\,dk_t\big) =$$

$$w_0^M - E\big(\int_0^T k_t y_t\,(\tilde{v} - m_t - k_t y_t)\beta_t dt + \int_0^T y_t^2\,dk_t\big) =$$

$$w_0^M + \int_0^T k_t^2(E y_t^2)\beta_t dt + \kappa \int_0^T E y_t^2\,dt.$$

The third equality follows since m is a martingale, the fourth since B_t is a F_t^y-martingale, and the last equality follows since y_t is orthogonal to $(\tilde{v}-m_t)$, and the Fubini theorem. Thus we have that this profit can be written

$$J^M(k,\beta) = w_0^M + \int_0^T \big(k_s^2 V(s)\beta_s + \kappa V(s)\big)ds. \tag{3.21}$$

Notice that the profit of the market maker at any time $t \in [0,T]$ is simply

$$E(w_t^M) = w_0^M + \int_0^t \big(k_s^2 V(s)\beta_s + \kappa V(s)\big)ds. \tag{3.22}$$

Using the expression (3.13) for $V_s = E(y_s^2)$, we obtain the following expression for this profit:

$$J^M(k,\beta) = w_0^M + \int_0^T \Big((k_s^2 e^{-2\int_0^s k_r\beta_r dr} \int_0^s \sigma_u^2 e^{2\int_0^u k_r\beta_r dr} du)\beta_s + \kappa(e^{-2\int_0^s k_e\beta_r dr} \int_0^s \sigma_u^2 e^{2\int_0^u k_r\beta_r dr} du)\Big)ds. \tag{3.23}$$

Consider the latter profit. The last term on the right-hand side increases without bounds as $k_t = (T-t)\kappa$ increases without bound for any given t, i.e., as the constant $\kappa \to \infty$. Surely k_t goes to zero as t goes to T, but the constant κ can in principle be set arbitrarily large by the market maker, since she simply decides the value of this constant once and for all. Also we know that β_t decreases with κ, but this effect more or less cancels out since the two exponentials where β enters are of different signs.

Likewise, the second term on the right-hand side, $\int_0^T k_t^2(Ey_t^2)\beta_t dt$, also possesses this property, despite the fact that here β enters linearly (in addition to its exponential dependence).

This is illustrated numerically in Figure 2. The base case parameters are the same as in Figure 1, where the horizon is $T = 10$. (Again we have anticipated a bit, and used the optimal values of the function β_t appearing is Section 4 below.)

Using the notation for the net profit of the market maker

$$p_M(t,\kappa) := \int_0^t \Big((k_s^2 e^{-2\int_0^s k_r\beta_r dr} \int_0^s \sigma_u^2 e^{2\int_0^u k_r\beta_r dr} du)\beta_s \\ + \kappa(e^{-2\int_0^s k_e\beta_r dr} \int_0^s \sigma_u^2 e^{2\int_0^u k_r\beta_r dr} du)\Big) ds.$$

at the intermediate time $t \leq T$, the upper graph is the net, terminal profit $p_M(T,\kappa)$ as a function of κ, while the the lower graph shows the net profit $p_M(t,\kappa)$ accumulated at the intermediate time $t = 2$ as a function of κ.

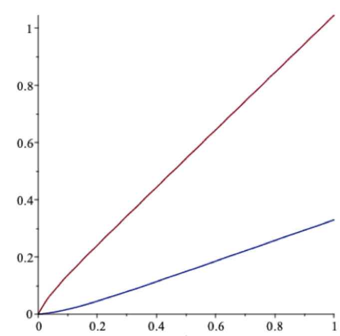

Fig. 2: The profit functions of the market maker as a function of κ.

As a result, this model displays similar properties to the one-period model, and a regulator is therefore introduced to limit the price perturbation caused by the market maker trading fees.

This set-up does not become a game between the insider and the market maker in the usual meaning of game theory, in that only the insider maximizes an objective, while the market maker trades fees that depend on the stochastic order flow, i.e., she sets the price to the best of her knowledge, and then adds the fee conditional on observing the order flow. In some sense, the market maker is not strategic in the ordinary interpretation of this term.

As the market maker obtains more information from the order flow, she lets this information be reflected in the price p_t. The introduction of trading fees reduces the informational contents of the true value of the asset in the price. The market maker may be assumed to set κ to the maximum value allowed by the regulator, or alternatively, by her own conscience, supposing she practices restraint in order to keep the markets open, whichever gives the smallest value of κ. It is in the interests of the market maker that the market does not break down, in which case she does not make any profits at all, and may also face legal issues. It is, after all, the market maker's task to operate such that the markets function.

The problem of relating the parameter κ to observables in the market is treated in Section 5 below.

Since this is a pure exchange economy, the profit of the noise traders is given by

$$J^N(k,\beta) = w_0^I + w_0^M + w_0^N - J^I(k,\beta) - J^M(k,\beta).$$

They will loose in this market.

4 The insider's problem

We now address the optimization problem of the insider. In our framework he is to determine the trading intensity β_t by which he trades at each time $t \in [0,T]$. We assume he determines this intensity such that his profit $J^I(k,\beta)$ is maximized, taking k as given. Vi have that

$$dp_t = dm_t + d(k_t y_t) = dm_t - \kappa y_t dt + k_t dy_t,$$

since the function k_t is of bounded variation. From filtering theory (see e.g., Kalman (1960), Davis (1977-84), Kallianpur (1980) or Øksendal (2003), Ch 6) we know that the corresponding conditional expected value $m_t = E(\tilde{v}|\mathcal{F}_t^y)$ is given by

$$dm_t = \frac{\beta_t \gamma_t(\beta)}{\sigma_t^2} dy_t.$$

Furthermore the square error function $\gamma_t(\beta) = E(\tilde{v} - m_t)^2$ satisfies the Ricatti equation

$$\frac{d}{dt}\gamma_t(\beta) = -\frac{\beta_t^2}{\sigma_t^2}\gamma_t^2(\beta),$$

which has the solution

$$\gamma_t(\beta) = \frac{\sigma_{\tilde{v}}^2}{1 + \sigma_{\tilde{v}}^2 \int_0^t \tilde{\beta}_s^2 ds}, \tag{4.1}$$

where $\tilde{\beta}_t := \frac{\beta_t}{\sigma_t}$. Here $\gamma_0 = E(\tilde{v} - E\tilde{v})^2 = \sigma_{\tilde{v}}^2$. Accordingly, the insider's problem is to solve the following

$$\sup_\beta \int_0^T \Big(\frac{\sigma_{\tilde{v}}^2 \beta_t}{1 + \sigma_{\tilde{v}}^2 \int_0^t \frac{\beta_s^2}{\sigma_s^2} ds} + \beta_t k_t^2 e^{-2\int_0^t k_s \beta_s ds} \int_0^t \sigma_s^2 e^{2\int_0^s k_r \beta_r dr} ds \Big) dt. \tag{4.2}$$

We find it natural to use directional derivatives, or equivalently, variational calculus to solve this problem. Based on this we have the following:

Theorem 4.1. *The optimal trading intensity β_t of the insider satisfies the following integral equation*

$$\beta_t = \frac{\sigma_t^2}{2\int_t^T \gamma_s(\beta)^2 \beta_s ds} \Big(\gamma_t(\beta) - V(t)\big(k_t^2 + 2k_t \int_t^T \beta_s k_s^2 e^{-2\int_t^s k_r \beta_r dr} ds\big)\Big), \tag{4.3}$$

where $V(t)$ is the variance process of the order flow y_t.

Proof: The proof can be found in Appendix 1.[5]

This integral equation can be solved iteratively, which we demonstrate in Section 7 below.

When $\kappa = 0$, the trading intensity is seen to be

$$\beta_t^0 = \frac{\sigma_t^2 \gamma_t(\beta)}{2\int_t^T \gamma_s(\beta)^2 \beta_s ds}, \quad (\kappa = 0.) \tag{4.4}$$

This can further be reduced to the following simple expression (see Aase et.al (2012a,b))

$$\beta_t^0 = \frac{\sigma_t^2 \Big(\int_0^T \sigma_s^2 ds\Big)^{\frac{1}{2}}}{\sigma_{\tilde{v}} \int_t^T \sigma_s^2 ds}, \quad \text{when } \kappa = 0. \tag{4.5}$$

[5]The problem may alternatively be formulated in terms of a stochastic differential game between the insider and the market maker, in which case we make use of the stochastic maximum principle. This leaves three adjoint variables (co-variables with shadow price interpretations) to be determined. Alternatively we can formulate the problem as a dynamic programming problem and use the Bellman approach. In this case this leaves us with the indirect profit function to be determined. We indicate these two formulations later (Appendix 2 and 3), without going all the way to the the bitter end.

When $\sigma_t = \sigma$ for all $t \in [0, T]$, where σ is a positive constant, this finally reduces to the Kyle (1985)-solution.

5 The regulator's problem

To limit the distortion of prices, a regulatory authority (the SEC) imposes an upper bound on price volatility. This is by and large the same as limiting the conditional expected degree of price distortion (see Aase and Gjesdal (2018)). In our model this limits the market maker's freedom to set prices. The market maker in our model is not really strategic, is risk neutral but exercises a certain degree of monopoly, as explained earlier. The regulator is introduced to mitigate this.

As in the standard model, informed traders realize what the market maker is up to, and take his behavior into account when deciding their own trade. Noise traders just trade. In this situation the market maker can make unbounded profits taking advantage of noise traders, which would not make sense. To avoid this outcome the regulator is introduced.

It is well acknowledged that insider trading increases the volatility of an asset. Also price perturbations caused by the market maker's trading fees increase the volatility. This can be utilized by the regulator, who can suspend the stock from further trading based on observing volatility over a certain acceptable, preset limit. A measure of volatility we consider as the basis for the regulator's ability to monitor the market.

The decision variable κ of the market maker has so far "no dimension", meaning that it is not an observable quantity in the market. We therefore seek a connection between this variable and and an observable quantity. This is an important step in the analysis, because it allows us to see if the market maker can really outperform the perfectly well informed insider in terms of profits at *reasonable* levels of trading fees, i.e., at levels where the regulator has not suspended the security.

From our expressions for the profit functions of the participants, we notice that as κ increases, the market maker's profit grows without limits, see e.g., Figure 2, and eventually it will dominate the profit function of the insider. The interesting question is then if this takes place at an *acceptable* level of price perturbations, set by the regulator.

With this in mind, we would like to develop a connection between the decision variable κ and total volatility. Consider the quantity $\text{var}(p_t) =$

$E(p_t - E(\tilde{v}))^2$. It is closely connected to the mean square deviation

$$S_t(\beta) = E(p_t - \tilde{v})^2 = \gamma_t(\beta) + k_t^2 E(y_t^2).$$

This latter quantity, or its square root, we assume can be observed by the regulator, who will then compare this to the corresponding term $\gamma_t(\beta^0)$ based on no price distortions by the market maker.

Recall the following definitions. $S_t(\beta) = E(\tilde{v} - p_t)^2$ and $\gamma_t(\beta^0) = E(\tilde{v} - m_t)^2$ where $m_t = E(\tilde{v}|\mathcal{F}_t^y)$. The function $\gamma_t(\beta^0)$ corresponds to the expected square deviation between the true value of the asset and the fiducial price m_t, provided the trading intensity β_t^0 is used in the computation of the latter quantity. S_t is the expected square deviation between the true value of the asset and the actual price that the market maker sets, in the case where she trades fees, as explained. Naturally, $S_t(\beta)$ is larger than $\gamma_t(\beta^0)$, and increasingly so as the market maker's decision variable κ increases.

This leads us to introduce the following quantity in relative terms (rv = relative volatility)

$$rv(t,\kappa) := \frac{\sqrt{S_t(\beta)}}{\sqrt{\gamma_t(\beta^0)}}, \quad t \in [0,T],\ \kappa \geq 0. \tag{5.1}$$

Our assumption that $S_t := S_t(\beta)$ is observable by the regulator also means that $rv(t,\kappa)$ is observable.

From our previous results $S_t(\beta) = \gamma_t(\beta) + k_t^2 V_t(k)$, where $V(t)$ depends on k_t and is given by equation (3.13), which is

$$V(t) = E(y_t^2) = e^{-2\int_0^t k_s \beta_s ds} \int_0^t \sigma_s^2 e^{2\int_0^s k_r \beta_r dr} ds,$$

and from (4.1) we have that

$$\gamma_t(\beta^0) = \frac{\sigma_{\tilde{v}}^2}{1 + \sigma_{\tilde{v}}^2 \int_0^t (\tilde{\beta}_s^0)^2 ds},$$

where $\tilde{\beta}_t^0 := \frac{\beta_t^0}{\sigma_t}$, and $\gamma_0 = E(\tilde{v} - E\tilde{v})^2 = \sigma_{\tilde{v}}^2$. Using these relations, $rv(t,\kappa)$ can be written

$$rv(t,\kappa) = \left(\frac{\gamma_t(\beta)}{\gamma_t(\beta^0)} + \frac{k_t^2}{\gamma_t(\beta^0)} e^{-2\int_0^t k_s \beta_s ds} \int_0^t \sigma_s^2 e^{2\int_0^s k_r \beta_r dr} ds \right)^{\frac{1}{2}}. \tag{5.2}$$

When $\kappa = 0$ we see from this expression that $rv(t, 0) \equiv 1$.

Here rv will give information about the degree to which the market maker trades fees. When $\kappa = 0$, the function $rv(t, 0)$ is constant through time and identically equal to 1 as noticed. As κ increases from zero, $rv(t, \kappa)$ will rise above 1, and indicate the percent-wise increase from the situation with fiducial trade, at every $t \in [0, T]$.

For example, when $rv(t, \kappa) = 1.20$, for some t and κ, the actions of the market maker has increased the volatility of the asset by 20% relative to the situation with fiducial price setting, where $\kappa = 0$. Thus the quantity rv seems like a reasonable measure of the degree of fee trading, in the hands of a regulator.

In this situation the map from κ to rv, and in particular its *inverse* mapping, will serve as a guide for acceptable values of κ, given a certain level of rv set by the regulator. This inverse mapping is illustrated in Section 7.5 below[6]. The market maker can, on the other hand, use this mapping to exercise restraint in setting κ in order to keep markets open.

6 A measure of price informativeness

We now derive a measure of the informativeness in prices, which is of particular interest when the prices are distorted. Consider the quantity

$$\iota(t, \kappa) := 1 - \frac{\text{var}(\tilde{v}|p_t)}{\text{var}(\tilde{v})}.$$

When the price carries no private information about the true value of the asset at some time t and some level of distortion κ, the conditional variance equals the unconditional variance, and consequently $\iota(t, \kappa) = 0$ at this point (t, κ). When the price equals the value of the asset, the conditional variance equals 0 and $\iota = 1$ at this point, in which case all the private information is reflected in the price. Consequently $0 \leq \iota(t, \kappa) \leq 1$ for all time points $t \in [0, T]$ and for all $\kappa \geq 0$.

Because of the joint normal assumption,

$$\text{var}(\tilde{v}|p_t) = \text{var}(\tilde{v})(1 - \rho^2_{\tilde{v}, p_t})$$

[6]In this section numerical illustrations can also be found, see in particular the two first rows in Table 2.

where $\rho_{\tilde{v},p_t}$ is the correlation coefficient between $\tilde{v}$ and p_t. Consequently $\iota(t,\kappa) = \rho^2_{\tilde{v},p_t}$ for all (t,κ).

In order to find this measure of price informativeness, we need to compute the quantities $\text{cov}(\tilde{v}, p_t)$ and $\text{var}(p_t)$. To this end, we first consider the covariance. Since $p_t = m_t + k_t y_t$,

$$\text{cov}(\tilde{v}, p_t) = \text{cov}(\tilde{v}, m_t) + k_t \text{cov}(\tilde{v}, y_t).$$

The first term on the right-hand side can be written

$$\text{cov}(\tilde{v}, m_t) = E(\tilde{v} m_t) - E(\tilde{v})E(m_t) = E(E(\tilde{v} m_t | \mathcal{F}_t^y)) - \mu_{\tilde{v}}^2 =$$

$$= E(m_t E(\tilde{v}|\mathcal{F}_t^y)) - \mu_{\tilde{v}}^2 = E(m_t^2) - (E(m_t))^2 = \text{var}(m_t),$$

since m_t is $\mathcal{F}_t^y$-measurable, where $m_t = E(\tilde{v}|\mathcal{F}_t^y)$ and $E(m_t) = \mu_{\tilde{v}}$. Furthermore,

$$\gamma_t(\beta) = E(\tilde{v} - m_t)^2 = E(\tilde{v}^2) - E(m_t^2) = \text{var}(\tilde{v}) - \text{var}(m_t),$$

by a similar type of conditioning as above. The last equality follows since $E(\tilde{v}) = E(m_t) = \mu_{\tilde{v}}$. Since we already have an expression for $\gamma_t = \gamma_t(\beta)$, see equation (4.1), we now have an expression for $\text{var}(m_t)$, and hence $\text{cov}(\tilde{v}, m_t) = \text{var}(m_t) = \sigma_{\tilde{v}}^2 - \gamma_t(\beta)$ by the above.

The term $\text{cov}(\tilde{v}, y_t)$ is calculated as follows: First notice that by iterated expectations $\text{cov}(\tilde{v}, y_t) = \text{cov}(m_t, y_t)$. From the Kalman filter equation we have

$$dm_t = \frac{\beta_t \gamma_t(\beta}{\sigma_t^2} dy_t,$$

see equation (3.9), and from the binormality between m_t and y_t and the corresponding projection theorem we obtain the following connection

$$\rho_{m_t,y_t} = \frac{\beta_t \gamma_t(\beta)\sigma_{y_t}}{\sigma_t^2 \sigma_{m_t}},$$

were ρ_{m_t,y_t} is the correlation coefficient between m_t and y_t, $\sigma_{y_t} := \sqrt{V(t)}$ and $\sigma_{m_t} := \sqrt{\text{var}(m_t)} = \sqrt{\sigma_{\tilde{v}}^2 - \gamma_t(\beta)}$.

We have then shown that

$$\text{cov}(\tilde{v}, p_t) = \sigma_{\tilde{v}}^2 - \gamma_t(\beta) + k_t \rho_{m_t,y_t} \sqrt{\sigma_{\tilde{v}}^2 - \gamma_t(\beta)} \sqrt{V(t)},$$

where we have formulas for all the terms on the right-hand side of this equation.

It remains to find the variance of p_t. Again, from $p_t = m_t + k_t y_t$ it follows that

$$\text{var}(p_t) = \text{var}(m_t) + k_t^2\text{var}(y_t) + 2k_t\text{cov}(m_t, y_t) = \sigma_{\tilde{v}}^2 - \gamma_t(\beta) + k_t^2 V(t) + 2k_t\rho_{m_t,y_t}\sqrt{\sigma_{\tilde{v}}^2 - \gamma_t(\beta)}\sqrt{V(t)}. \tag{6.1}$$

Putting all this together, we have

$$\rho_{\tilde{v},p_t} = \frac{\sigma_{\tilde{v}}^2 - \gamma_t(\beta) + k_t\rho_{m_t,y_t}\sqrt{\sigma_{\tilde{v}}^2 - \gamma_t(\beta)}\sqrt{V(t)}}{\sigma_{\tilde{v}}\sqrt{\text{var}(p_t)}} \tag{6.2}$$

where $\text{var}(p_t)$ is given above in (6.1). From this the informativeness ι in prices is given by

$$\iota(t,\kappa) = (\rho_{\tilde{v},p_t})^2, \tag{6.3}$$

where $\rho_{\tilde{v},p_t}$ is calculated using (6.2) and (6.1).

Table 1 illustrates the time development of the informativeness in the market. For a given value of the price distortion parameter κ we notice that the informativeness increases with time, see the last row in Table 1.

In the same table we have also computed some of the other key quantities that is used in the computation of the measure of informativeness $\iota(t,\kappa)$, such as the correlation coefficients $\rho_{m_t,y_t}(t,\kappa)$ and $\rho_{\tilde{v},p_t}(t,\kappa)$ and the variances of the price p_t and the 'fair' price m_t. While $\iota(t,\kappa) = \rho_{\tilde{v},p_t}(t,\kappa)^2$ and thus $\rho_{\tilde{v},p_t}(t,\kappa)$ represents an equivalent measure of information as $\iota(t,\kappa)$, the correlation coefficient $\rho_{m_t,y_t}(t,\kappa)$ is a measure connected to the 'fair' value m_t instead of the market price p_t, but computed with the value of β_t where the insider has optimally adjusted to the actual distortion of the price.

This measure tells us how closely correlated the 'fair' price m_t is to the order flow y_t. From the table we notice that there can be a high correlation, as in our example, which throws some new light on the market maker's advantage in observing the order flow.

In the example this measure decreases with time up to a certain value t^*, then increases after that, so that here the advantage is highest in the beginning and towards the end of the trading interval.

Since the correlation $\text{cov}(\tilde{v}, m_t) = \text{var}(m_t)$, this covariance also increases slowly with time in our example, which is reasonable since more information about the true value of the asset becomes available with time, hence the increase in the corresponding correlation coefficient $\rho_{\tilde{v},m_t} = \sqrt{\text{cov}(\tilde{v}, m_t)}/\sigma_{\tilde{v}}$. At the same time this shows that $\gamma_t(\beta)$ decreases slowly with time, again for the same reason: One knows more about the true value as time increases, and the present measure of uncertainty (the mean square error) then naturally decreases.

Finally, we illustrate the time development of the variance of the price p_t and of the 'fair' price m_t. The latter one has just been explained. For the former two different effects are in force: One is that $V(t)$ increases with time (see Section 7.3 below) and what has just been shown, that $\text{cov}(\tilde{v}, m_t)$ also increases with time. The other is that k_t decreases with time, see the expression (6.1) for the $\text{var}(p_t)$. In our example the first effect weakly dominates.

Cont. model:										
t	1	2	3	4	5	6	7	8	9	10
$\rho_{m_t,y_t}(t,\kappa)$	.96	.92	.87	.84	.81	.79	.78	.79	.83	1.00
$\rho_{\tilde{v},p_t}(t,\kappa)$	.31	.44	.53	.61	.67	.73	.79	.84	.91	.99
$\text{var}(p_t)$	.02	.04	.05	.06	.07	.07	.08	.08	.08	.09
$\text{var}(m_t)$	.009	.02	.03	.03	.04	.05	.06	.06	.07	.09
$\iota(t,\kappa)$	.10	.19	.28	.37	.45	.54	.62	.71	.83	1.00

Table 1: The quantity $\iota(t,\kappa)$ as a function of time ($\kappa = 0.035$).

The shape of $\iota(t,\kappa)$ as a function of the price distortion parameter κ for a given value of time t is illustrated in the next section, see the last row of Table 2 below. Naturally we expect that $\iota(t,\kappa)$ decreases with κ for any given value of t.

7 Illustrations of the theoretical results

7.1 General

Based on the integral equation (4.3) for trading intensity of the insider, we now present a few illustrations.

First, we indicate how to deal with this integral equation. This equation can be transformed to a differential-integral equation if one so pleases, but we choose to work with the version we have, where we use an iterative procedure.

As a first step we suggest to use a trial solution, β_t^0 say, on the right-hand side of this equation, and then find the first approximation, β_t^1, given by the left hand side as a function of this initial solution. A reasonable candidate for the trial solution is of course the solution when $\kappa = 0$, which we have in closed form (see (4.5)). Next one continues this procedure, where $\beta_1(t)$ becomes the new trial solution in the next step, and so on until convergence. This, of course, requires the right numerical tools.

7.2 The trading intensity of the insider

In Figure 3 below we illustrate the time development of the trading intensities of the insider obtained this way, where the upper curve is when $\kappa = 0$ and the lower curve is for $\kappa = 0.045$. The other base case parameters are the same as in Figure 1. We choose $T = 10$ here as well[7].

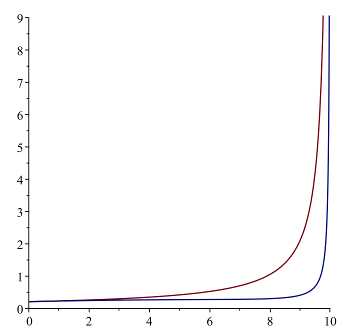

Fig. 3: The insider's trading intensities as functions of t.

Note that in both cases the insider intensifies her trade towards the horizon. Also, it is reasonable that the lower graph corresponds to the insider's trading intensity when the market maker perturbs the price. The insider, knowing what the market maker is up to, now trades more softly. However, towards the end she picks up trading again, since then the market maker trades fees to a less and less extent as t approaches T.

This type of analysis represents an interesting extension of the analysis in Aase and Gjesdal (2018), who treat the one period model. In that model the intensity is graphically represented as a decreasing, convex function of the decision variable κ (but there is no time development).

[7]Here and in most of the numerical computations we do not go beyond two rounds in the iterations indicated above.

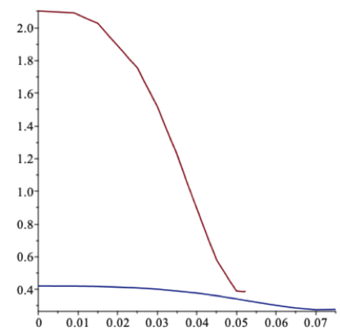

Fig. 4: The insider's trading intensities as functions of κ.

In Figure 4 we present two graphs of $\beta(t, \kappa)$ in the present continuous model as a function of κ for given t, for two different points in time: $t = 5$ and $t = 9$, where the upper curve is for $t = 9$. These are also seen to be decreasing. Unlike for the one-period model, these graphs are concave for small values of κ and then becomes convex as κ increases. That these graphs are decreasing in κ is in accordance with the results from the one period model: The insider trades more softly the more the market maker perturbs the price.

7.3 The variance function of the order flow y

We have already indicated a graph, Figure 1, illustrating the covariance function of the order flow. Here we study the time development of the variance function $V(t)$, as this quantity enters many of the key expressions in this theory. The base case parameters are the same as before.

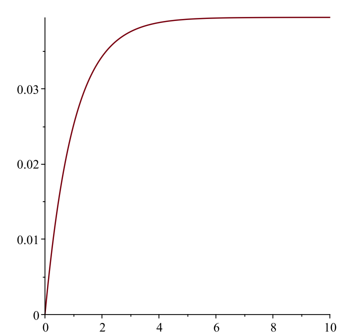

Fig. 5: The variance function $V(t, \kappa)$ as a function of t.

The variance of the total order flow y_t is seen to increase sharply in the beginning, and then flattens out at around $t = 3$ when $\kappa = 0.24$. It is well known from empirical studies that the volatility of the price increases as more relevant information enters the market, since this causes trade to increase. A reasonable model of insider trading should reflect this, and for our model this is illustrated in Figure 5.

However, note that these variances decrease with κ for any given point t in time. This is illustrated in Figure 6 for $t = 1, 5, 9$ as κ run from 0 to 0.5. The more the market maker trades fees, the less the insider trades and the lower the variance of the order flow y_t.

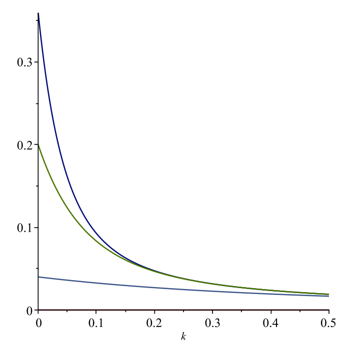

Fig. 6: The variance function $V(t, \kappa)$ as a function of κ.

The upper curve in Figure 6 corresponds to $t = 9$, then $t = 5$ and the lowest curve is for $t = 1$.

7.4 The two parties' net profits

Now we come to the important part, namely the net profit functions of the two key participants as functions of time. These are illustrated in figures 7 and 8.

In Figure 7 we consider the dynamics of the two net profits as functions of of time, for a given value of the parameter κ. We notice that both profit functions naturally start out low and then increase with time. When $\kappa = 0.045$ the insider's profit function cross the market maker's profit from below at around $t = 7$, and then ends up as the highest of the two at the final time $T = 10$[8] .

[8] The profit function of the insider is computed at discrete times only, due to the large number of computations required for a continuous plot.

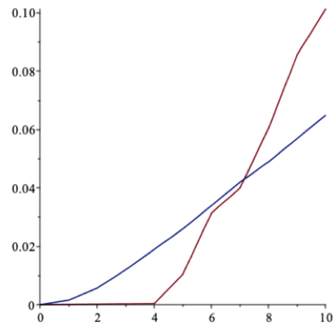

Fig. 7: The two net profits as functions of time. $\kappa = 0.045$.

In Figure 8 however, where $\kappa = 0.07$, the market maker's profit is seen to dominate from the start, and also ends up highest at the final time $T = 10$. When the market maker further increases κ, her profit function will increase for each value of t, while the profit of the insider will decrease for each t compared to the levels in Figure 8, (but will still be an increasing function of t for each given value of κ, as in the figures 7 and 8). As a consequence, the market maker outperforms the perfectly well informed insider from about $\kappa = 0.06$ onwards.

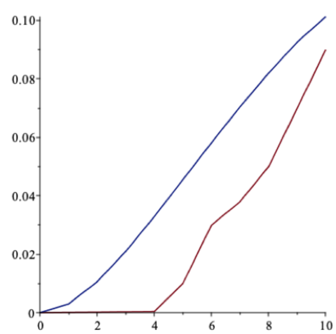

Fig. 8: The two net profits as functions of time. $\kappa = 0.07$.

7.5 When does the market maker's profit dominate?

We now illustrate the regulator's problem through some graphs of the market observable quantity rv. In the first figure we plot $rv(t, \kappa)$ as functions of κ for some given values of time t.

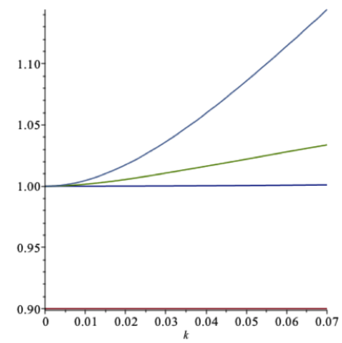

Fig. 9: $rv(t, \kappa)$ as a function of κ when $t = 0.01, 3.6$ and 9.0.

The lowest curve in Figure 9 corresponds to $t = 0.01$, the next lowest to $t = 9.0$ and the highest curve to $t = 3.6$. To interpret this figure, imagine that the regulator uses the rule that rv should correspond to less than 15% increase from the fiducial, ideal situation. This means that the market maker should not increase κ beyond 0.07, at least based on the the three time points in this illustration.

Since this line of reasoning is valid only for some particular values of t, in the next figure we study rv's time development:

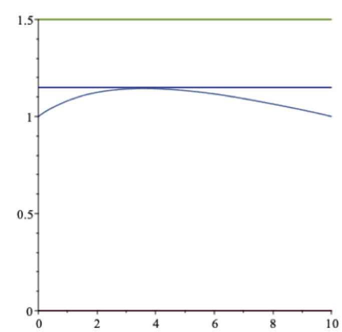

Fig. 10: $rv(t, \kappa)$ as a function of t when $\kappa = 0.07$.

Figure 10 illustrates the time picture of $rv(t, \kappa)$ for $\kappa = 0.07$, for $t \in [0, T]$ (the curved graph). The horizontal line tangent to the curve $rv(t, 0.07)$ is at level 1.15, corresponding to 15% maximal deviation from fiduciary trade. From the figure it follows that the regulator will keep the market open all the time.

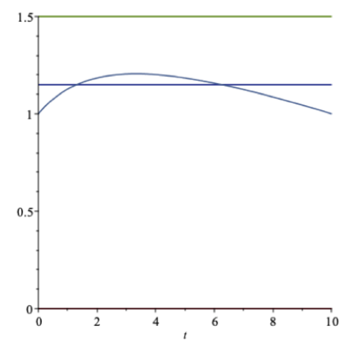

Fig. 11: $rv(t,\kappa)$ as a function of t when $\kappa = 0.09$.

Figure 11 illustrates the same time picture of $rv(t,\kappa)$ as in the previous figure, but now for $\kappa = 0.09$. With a 15%-regulation rule still in charge, the market would then be shut down and the stock suspended at around $t = 1.2$, but allowed to reopen at around $t = 6.5$, and then kept open for the rest of the period. Accordingly, the market maker would be wise to consider a lower value of κ to keep her reputation as a decent professional.

On the other hand, if the regulator used a 21%-rule, the market would stay open all the time for $\kappa = 0.09$.

The manner in which we find the *inverse* map from rv to κ is seen to proceed as follows: For a given value of rv^*, say $rv^* = 1.15$, we solve in t and κ the following inequality: $\sup_\kappa(\sup_t rv(t,\kappa)) \leq d^*$. Then the 'optimal' value of κ, call it κ^*, is the one which satisfies this inequality with equality sign: $\sup_t rv(t,\kappa^*) = d^*$. In Table 2 we illustrate this connection for some values of rv^*.

Cont. model: $t = 9$					
rv^*	1.03	1.05	1.08	1.15	1.21
κ^*	.025	.035	.045	.070	.090
$p_M(9,\kappa^*)$	.037	.049	.060	.087	.100
$p_I(9,\kappa^*)$	.126	.118	.100	.055	.045
$\iota(9,\kappa^*)$	.88	.83	.64	.45	.30

Table 2: The connection between $\kappa^*$and rv^* with associated net profits.

In Table 2 the insider has the highest net profits, denoted $p_I(t,\kappa)$, for lower values of rv^*, and this profit is decreasing with k^*. For larger values of rv^*, the market maker's profit is the largest of the two. The last row in

Table 2 illustrates the informativeness $\iota(t,\kappa)$ in the market as a function of κ when $t = 9$. It decreases as κ increases. Distorting prices is not informative to the other market participants.

In the above illustrations in the last two figures (figures 10 and 11) it is the market maker who has the highest profits of the two parties. This highlights one of the the main ideas in this paper: In real life we know that market makers actually do not set prices in an entirely fiducial manner, but rather determine prices in such a way that they make money. In the introduction we explained why this behavior is possible and likely to take place. This is in line with observed behavior in several financial markets, in particular those of the over-the-counter type that we have in mind.

For a modest fee conditional on the order flow, the market maker is able to obtain a profit of the order of the magnitude, or even better than, a perfectly informed insider, showing, among other things, the advantage of observing the order flow. This, we conjecture, may be one explanation why so much money tends to end up in the financial sector of the economy.

8 Suggestions for further research

In the discrete time paper of Aase and Gjesdal (2017) a situation is analyzed where the market maker has private information as well. This could also be of interest to analyze in the present setting. There is supposed to be no information flow between the market making department and the investment department of large financial institutions. But these 'Chinese Walls' - as they are known as - may not be entirely 'watertight'.

One could analyze the situation where the market maker's information is public, which can be used to determine the effects of information asymmetry.

In particular, price volatility is shown to increase with informed market maker in the one period model. This is an important aspect of the effect of privileged information on security prices, which may explain the price/dividend puzzle, a feature that could be extend to the time-continuous model (the world is, after all, time-continuous).

9 Conclusions

The dynamic auction model of Kyle (1985) is studied, allowing market makers to maximize profit within regulatory limits by charging time varying, stochastic fees. This has several implications for the equilibrium, the most striking one, perhaps, is that by trading fees to a relatively modest degree, the market maker is able to outperform a perfectly informed insider in terms of profits.

The dynamic aspects of this are highlighted and analyzed in the paper, and illustrated both numerically and graphically by examples. The analytical challenges turned out to be several, in particular the determination of the optimal trading intensity of the insider, when the market maker perturbs the price. The solution was presented in Theorem 4.1. Based on this and other results, we were able to discuss a wide variety of dynamic problems, like finding the profit processes of the two parties, the insider and the market maker, the stochastic properties of the order flow, and the informativeness in the market as time goes, all quantities as a functions of time and the degree of relative price perturbations. We also indicated how a regulator can monitor the market by observing a dynamic measure of relative price volatility, relative to the corresponding measure with fiducial price setting. This gives a convenient, observable connection between price volatility and price perturbation.

10 Appendix 1. Proof of Theorem 4.1 by Variational Calculus.

We now want to solve problem (4.2) by use of directional derivatives, or calculus of variations. Towards this end, let $\mathcal{A}$ be the family of all continuously differentiable functions $\beta : [0,T] \to R$ such that

$$\int_0^t \frac{\beta_s^2}{\sigma_s^2} ds < \infty \quad \text{for all} \quad t < T.$$

By this method we choose an arbitrary function $\xi_t \in \mathcal{A}$, a sufficiently rich set in this regard, and define the real function g by

$$g(x) = J^I(k, \beta + x\xi); \quad x \in R.$$

Then, assuming β is optimal, g is maximal at $x = 0$ and hence the first order condition of maximality at β is

$$0 = g'(0) = \frac{d}{dx} J^I(k, \beta + x\xi)|_{x=0} =$$

$$\frac{d}{dx}\Big(\sigma_{\tilde{v}}^2 \int_0^T \Big(1 + \sigma_{\tilde{v}}^2 \int_0^t \frac{(\beta_s + x\xi_s)^2}{\sigma_s^2} ds\Big)^{-1} (\beta_t + x\xi_t) dt\Big)|_{x=0} +$$

$$\frac{d}{dx}\Big(\int_0^T \Big((\beta_t + x\xi_t) k_t^2 e^{-2\int_0^t k_s(\beta_s + x\xi_s)ds} \int_0^t \sigma_s^2 e^{2\int_0^s k_r(\beta_r + x\xi_r)dr} ds\Big) dt\Big)|_{x=0} =$$

$$(10.1) \quad \int_0^T \big(\gamma_t(\beta) - 2\int_t^T \gamma_s^2(\beta)\beta_s ds\big)\frac{\beta_t}{\sigma_t^2}\,\xi_t\, dt +$$
$$\int_0^T \big(k_t^2 e^{-2\int_0^t k_s\beta_s ds} \int_0^t \sigma_s^2 e^{2\int_0^s k_r\beta_r dr} ds\big)\xi_t\, dt +$$
$$\int_0^T \big(\beta_t k_t^2 e^{-2\int_0^t k_r\beta_r dr}(-2\int_0^t k_s\xi_s ds)\int_0^t \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) dt +$$
$$\int_0^T \big(\beta_t k_t^2 e^{-2\int_0^t k_r\beta_r dr}\int_0^t \sigma_s^2 e^{2\int_0^s k_r\beta_r dr}(2\int_0^s k_r\xi_r dr) ds) dt = 0, \quad \forall \xi \in \mathcal{A}.$$

The second line on the left-hand side of (10.1) follows just as in Aase et. al (2012c), which presents a simple proof of the case $k = 0$.

Consider the last two lines, and start with the third integral in (10.1). By changing the order of integration between s and t, we obtain

$$\int_0^T \big(\beta_t k_t^2 e^{-2\int_0^t k_r\beta_r dr}(-2\int_0^t k_s\xi_s ds)\int_0^t \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) dt =$$

$$-2\int_0^T \int_s^T \big(\beta_t k_t^2 e^{-2\int_0^t k_r\beta_r dr}(\int_0^t \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) dt\big) k_s\xi_s ds =$$
$$-2\int_0^T \int_t^T \big(\beta_s k_s^2 e^{-2\int_0^s k_r\beta_r dr}(\int_0^s \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) ds\big) k_t\xi_t dt.$$

Next consider the fourth and last integral in (10.1). The inner integral given by

$$\int_0^t \sigma_s^2 e^{2\int_0^s k_r\beta_r dr}(2\int_0^s k_r\xi_r dr) ds) dt,$$

can be rewritten as

$$2\int_0^t (\int_s^t \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) k_s \xi_s ds.$$

This we now insert in the fourth term in (10.1), which gives

$$2\int_0^T \int_t^T \big(\beta_s k_s^2 e^{-\int_0^t k_u\beta_u du} (\int_t^s \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) ds) k_t \xi_t dt.$$

Putting all this together, the first order condition now takes the form

$$\begin{aligned}(10.2)\quad 0 = \frac{d}{dx} J^I(k,\beta + x\xi)|_{x=0} = \\ \int_0^T \Big(\gamma_t(\beta) - 2(\int_t^T \gamma_s(\beta)^2 \beta_s ds) \frac{\beta_t}{\sigma_t} + k_t^2 e^{-2\int_0^t k_s\beta_s ds} \int_0^t \sigma_s^2 e^{-2\int_0^t k_r\beta_r dr} ds \\ - 2k_t \int_t^T \beta_s k_s^2 e^{-2\int_0^s k_r\beta_r dr} (\int_0^s \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) ds \\ + 2k_t \int_t^T \beta_s k_s^2 e^{-2\int_0^t k_u\beta_u du} (\int_t^s \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) ds\Big) \xi_t dt, \quad \forall \xi \in \mathcal{A}.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}(10.3)\quad \gamma_t(\beta) = 2\frac{\beta_t}{\sigma_t} \int_t^T \gamma_s(\beta)^2 \beta_s ds + k_t^2 e^{-2\int_0^t k_s\beta_s ds} \int_0^t \sigma_s^2 e^{-2\int_0^t k_r\beta_r dr} ds \\ - 2k_t \int_t^T \beta_s k_s^2 e^{-2\int_0^s k_r\beta_r dr} (\int_0^s \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) ds \\ + 2k_t \int_t^T \beta_s k_s^2 e^{-2\int_0^t k_u\beta_u du} (\int_t^s \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) ds.\end{aligned}$$

Accordingly

$$\begin{aligned}(10.4)\quad \beta_t = \frac{\sigma_t^2}{2\int_t^T \gamma_s(\beta)^2 \beta_s ds} \Big(\gamma_t(\beta) - k_t^2 e^{-2\int_0^t k_s\beta_s ds} \int_0^t \sigma_s^2 e^{-2\int_0^t k_r\beta_r dr} ds \\ - 2k_t \int_t^T \beta_s k_s^2 e^{-2\int_0^s k_r\beta_r dr} (\int_0^t \sigma_r^2 e^{2\int_0^r k_u\beta_u du} dr) ds\Big).\end{aligned}$$

Hence, the optimal trading intensity of the insider, $\beta_t, t \in [0,T]$, is given by the integral equation (10.4).

This may be simplified by using the expression for the variance process $V(t)$ in (3.13). The result is

$$\beta_t = \frac{\sigma_t^2}{2\int_t^T \gamma_s(\beta)^2 \beta_s ds}\Big(\gamma_t(\beta) - V(t)\big(k_t^2 + 2k_t \int_t^T \beta_s k_s^2 e^{-2\int_t^s k_r \beta_r dr} ds\big)\Big). \tag{10.5}$$

which is equation(4.3) in Theorem 4.1. $\square$

11 Appendix 2. Optimization via Pontryagin and Nash.

We start from the system expressed in terms of (y_t, m_t, γ_t) in (3.8)-(3.10), which are

$$dy_t = (\tilde{v} - m_t - k_t y_t)\beta_t dt + \sigma_t dB_t; \quad y_0 = 0 \tag{11.1}$$

$$dm_t = \frac{\gamma_t \beta_t}{\sigma_t^2}[(\tilde{v} - m_t)\beta_t dt + \sigma_t dB_t]; \quad m_0 = E[v] \tag{11.2}$$

$$d\gamma_t = -\frac{\beta_t^2 \gamma_t^2}{\sigma_t^2}; \quad \gamma_0 = E[(\tilde{v} - E[\tilde{v}])^2]. \tag{11.3}$$

The performance functionals are

$$J^M(k, \beta) := w_0^M + E(\int_0^T k_t y_t \,(k_t y_t + m_t - \tilde{v})\beta_t dt - \int_0^T y_t^2 \, dk_t) \tag{11.4}$$

$$J^I(k, \beta) := w_0^I + \int_0^T E[(\tilde{v} - m_s - k_s y_s)^2]\beta_s ds. \tag{11.5}$$

Problem 11.1. *We want to find a Nash equilibrium (k_t^*, β_t^*) for the the two performance functionals J^M, J^I. In other words, we want to find (deterministic) control processes k_t^*, β_t^* such that*

$$\sup_{k_t} J^M(k_t, \beta_t^*) = J^M(k_t^*, \beta_t^*) \tag{11.6}$$

and

$$\sup_{\beta_t} J^I(k_t^*, \beta_t) = J^I(k_t^*, \beta_t^*). \tag{11.7}$$

This is a *stochastic differential game.*

Recall our assumption

$$k_t = \kappa(T-t); \quad 0 \le t \le T, \tag{11.8}$$

for some constant $\kappa \in [0, K]$, where K is a fixed constant (in principal set by the regulator).

Then the performance functionals take the following forms

(11.9)

$$J^M(\kappa, \beta) := w_0^M + E[\int_0^T \{\kappa(T-t)y_t\,(\kappa(T-t)y_t + m_t - \tilde{v})\beta_t + \kappa y_t^2\}dt]$$

(11.10)

$$J^I(\kappa, \beta) := w_0 + \int_0^T E[(\tilde{v} - m_t - \kappa(T-t)y_t)^2]\beta_t dt.$$

To study Problem 3.1 we use the stochastic maximum principle. Thus we define two *Hamiltonians* H^M and H^I by

$$\begin{aligned} H^M(t, y, m, \gamma, \kappa, \beta, p, q) &= -\kappa(T-t)y(v - m - \kappa(T-t)y)\beta + \kappa y^2 \\ &\quad + (v - m - \kappa(T-t)y)\beta p_1 + \sigma_t q_1 + \frac{\gamma\beta}{\sigma_t^2}(v-m)\beta p_2 \\ &\quad + \frac{\gamma\beta}{\sigma_t} q_2 - \frac{\beta^2\gamma^2}{\sigma_t^2} p_3 \end{aligned} \tag{11.11}$$

and

$$\begin{aligned} H^I(t, y, m, \gamma, \kappa, \beta, p, q) &= (v - m - \kappa(T-t)y)^2\beta + (v - m - \kappa(T-t)y)\beta p_1 \\ &\quad + \sigma_t q_1 + \frac{\gamma\beta}{\sigma_t^2}(v-m)\beta p_2 + \frac{\gamma\beta}{\sigma_t} q_2 - \frac{\beta^2\gamma^2}{\sigma_t^2} p_3. \end{aligned} \tag{11.12}$$

The BSDE's for the adjoint processes $(p_i^M, q_i^M); i = 1, 2, 3$, associated to H^M are

$$\begin{cases} dp_1^M(t) &= [\kappa(T-t)\beta_t(v - m_t - 2\kappa(T-t)y_t) - \kappa^2(T-t)^2 y_t\beta_t + \\ & \quad 2\kappa y_t + \kappa(T-t)\beta_t p_1^M(t)]dt + q_1^M(t)dB_t; \quad 0 \le t \le T \\ p_1^M(T) &= 0 \end{cases}$$

$$\begin{cases} dp_2^M(t) &= -[-\kappa(T-t)y_t\beta_t - \beta_t p_1^M(t) - \frac{\gamma_t\beta_t^2}{\sigma_t^2}p_2^M(t)]dt + q_2^M(t)dB_t; 0 \leq t \leq T \\ p_2^M(T) &= 0 \end{cases}$$

$$\begin{cases} dp_3^M(t) &= -[\frac{\beta_t^2(v-m_t)}{\sigma_t^2}p_2^M(t) + \frac{\beta_t}{\sigma_t}q_2^M(t) - \frac{2\beta_t^2\gamma_t}{\sigma_t^2}p_3^M(t)]dt + q_3^M(t)dB_t; 0 \leq t \leq T \\ p_3^M(T) &= 0 \end{cases}$$

The BSDE's for the adjoint processes $(p_i^I, q_i^I); i = 1, 2, 3$, associated to H^I are

$$\begin{cases} dp_1^I(t) &= -[2(v - m_t - \kappa(T-t)y_t)(-\kappa(T-t))\beta_t - \kappa(T-t)\beta_t p_1^I(t)]dt \\ &\quad +q_1^I(t)dB_t; \quad 0 \leq t \leq T \\ p_1^I(T) &= 0 \end{cases}$$

$$\begin{cases} dp_2^I(t) &= -[2(v - m_t - \kappa(T-t)y_t)(-\beta_t) - \beta_t p_1^I(t) - \frac{\gamma_t\beta_t^2}{\sigma_t^2}p_2^I(t)]dt \\ &\quad +q_2^I(t)dB_t; 0 \leq t \leq T \\ p_2^I(T) &= 0 \end{cases}$$

$$\begin{cases} dp_3^I(t) &= -[\frac{v-m_t}{\sigma_t^2}\beta_t^2 p_2^I(t) + \frac{\beta_t}{\sigma_t}q_2^I(t) - \frac{2\beta_t^2\gamma_t}{\sigma_t^2}p_3^I(t)]dt + q_3^I(t)dB_t; 0 \leq t \leq T \\ p_3^I(T) &= 0 \end{cases}$$

According to the maximum principle for stochastic differential games (see e.g. [20] , Theorem 2.1 and Theorem 2.3) the problem of finding a Nash equilibrium for the two performances $J^M(\kappa, \beta), J^I(\kappa, \beta)$ can (under some conditions) be reduced to finding a Nash equilibrium for the two Hamiltonians H^M, H^I. Thus we proceed to maximize $H^M(\kappa, \beta)$ with respect to κ for each given β, and then to maximize $H^I(\kappa, \beta)$ with respecty to β for each κ:

For each t the map

$$\kappa \mapsto H^M(t, y_t, m_t, \gamma_t, \kappa, p^M(t), q^M(t))$$

is convex and therefore it achieves its maximum $\kappa = \hat{\kappa}$ at the boundary, i.e. when

$$\hat{\kappa} = 0 \text{ or } \hat{\kappa} = K.$$

Here K is the maximum allowed by the regulator.

The map

$$\beta \mapsto H^I(t, y_t, m_t, \gamma_t, \kappa, \beta, p^I(t), q^I(t))$$

has a critical point when

(11.13)

$$\beta = \hat{\beta}(t) = -\frac{\left[(v - m_t - \kappa(T-t)y_t)^2 + (v - m_t - \kappa(T-t)y_t)p_1^I(t) + \frac{\gamma_t}{\sigma_t}q_2(t)\right]\sigma_t^2}{2[\gamma_t(v - m_t)p_2^I(t) - \gamma_t^2 p_3(t)]}$$

We conclude the following:

Theorem 11.2. *Suppose (κ^*, β^*) is a Nash equilibrium for Problem 3.1. Then*

$$\kappa^* = 0 \text{ or } \kappa^* = K,$$

and the optimal β^* is given in feedback form by

(11.14)

$$\beta^*(t) = \frac{(v - m_t^* - \kappa^*(T-t)y_t^*)^2 + (v - m_t^* - \kappa^*(T-t)y_t^*)(p_1^I)^*(t) + \frac{\gamma_t^*}{\sigma_t}(q_2^I)^*(t)}{2[\frac{\gamma_t^2}{\sigma_t^2}(p_3^I)^*(t) - \frac{\gamma_t^*}{\sigma_t^2}(v - m_t^*)(p_2^I)^*(t)]}$$

where $y_t^*, m_t^*, \gamma_t^*, (p_1^I)^*(t), (p_2^I)^*(t), (p_3^I)^*(t), (q_2^I)^*(t)$ are the system values corresponding to the controls κ^*, β^*.

12 Appendix 3. The Bellman approach.

It may be instructive to see what the dynamic programming approach gives in the present situation. In particular, this may throw some light on the interpretations of the adjoint variables in Theorem 3.2. In doing so, we take into account our previous remarks made just prior to equation (2.4) in Section 2.2, which tells us to focus on the insider's profit only, since the market maker does not act strategically, he only trades 'fees'.

Let us for short use the notation $x_t = (y_t, m_t, \gamma_t)$ for the system. The performance functional is given in (3.5), and the maximal profit of the insider is

(12.1) $$J^I(x) = w_0^I + \sup_\beta E\Big[\int_0^T \Big(\tilde{v} - m_s - k_s y_s\Big)^2 \beta_s ds\Big].$$

With $J^I(x,t)$ equal to the optimal wealth remaining at time t in state x, the Bellman equation can be written

$$\sup_\beta\Big\{(v-m_t-k_t y_t)^2\beta_t + L^\beta(J^I(x,t))\Big\} = 0, \tag{12.2}$$

where

(12.3)

$$L^\beta(J^I(x,t)) = \frac{\partial J^I(x,t)}{\partial t} + (v-m_t-k_t y_t)\beta_t\frac{\partial J^I(x,t)}{\partial y} + \frac{\gamma_t\beta_t}{\sigma_t^2}(v-m_t)\beta_t\frac{\partial J^I(x,t)}{\partial m}$$
$$-\frac{\beta_t^2\gamma_t^2}{\sigma_t^2}\frac{\partial J^I(x,t)}{\partial\gamma} + \sigma_t\frac{\partial^2 J^I(x,t)}{\partial y^2} + \sigma_t\frac{\partial^2 J^I(x,t)}{\partial m^2} + 2\sigma_t\frac{\partial^2 J^I(x,t)}{\partial y\partial m}.$$

Let us first address the maximization problem in (12.2). The first order condition in β_t can be written

$$(v-m_t-k_t y_t)^2 + (v-m_t-k_t y_t)\frac{\partial J^I(x,t)}{\partial y} + 2\beta_t\frac{\gamma_t}{\sigma_t^2}(v-m_t)\frac{\partial J^I(x,t)}{\partial m}$$
$$-2\beta_t\frac{\gamma_t^2}{\sigma_t^2}\frac{\partial J^I(x,t)}{\partial\gamma} = 0.$$

This gives the optimal β_t^* in terms of the function $J^I(x,t)$ (i.e., its partial derivatives) as follows

$$\beta_t^* = \frac{(v-m_t-\kappa(T-t)y_t)^2 + (v-m_t-\kappa(T-t)y_t)\frac{\partial J^I(x,t)}{\partial y}}{2\left[\frac{\gamma_t^2}{\sigma_t^2}\frac{\partial J^I(x,t)}{\partial\gamma} - \frac{\gamma_t}{\sigma_t^2}(v-m_t)\frac{\partial J^I(x,t)}{\partial m}\right]}. \tag{12.4}$$

It remains to determine the function $J^I(x,t)$.

Comparing this expression for the optimal trading intensity of the insider with the corresponding expression in (3.14) derived using the stochastic maximum principle for stochastic differentiable games, we notice that the adjoint variables $p_i^I(t)$, $i=1,2,3$ and $q_2^I(t)$ can be expressed as follows

$$(p_1^I)^*(t) = \frac{\partial J^I(x,t)}{\partial y}, \qquad (p_2^I)^*(t) = \frac{\partial J^I(x,t)}{\partial m},$$
$$(p_3^I)^*(t) = \frac{\partial J^I(x,t)}{\partial\gamma}, \qquad (q_2^I)^*(t) = 0.$$

These relationships give us more insight into how to interpret these adjoint variables, namely for the $p_i^I(t)$'s as marginal profits with respect to the state variables of the problem at any time $t \in (0,T)$. It also says that the adjoint variable $(p_2^I)^*(t)$ has no diffusion term, so this is a finite variation process.

Notice that it is not normally the case that the adjoint variables in the stochastic maximum principle can be interpreted as 'shadow prices' as we can here. When the state variables have different volatilities, or driven by different Brownian motions, this will in general no longer be true (see e.g., Yong and Zhou (1999)).

The insider's indirect utility function, $J^I(x,t) = \int_t^T \beta_s(\gamma_s(\beta) + k_s^2 V_s)ds$, can be written as a function of the state x variable and time t as indicated by this notation, and from a conjectured functional form we may attempt to solve the Bellman equation, and proceed to a solution for the trading intensity β. The result of this can in its turn be used to address the problem of Appendix 2. However, here we choose to stop and leave this for future research.

13 Appendix 4. A connection to filtering theory.

The results of Section 3.1 can alternatively be derived using filtering theory as follows: We first consider the process y for $k = 0$. Then $dy_t = (\tilde{v} - E(\tilde{v}|\mathcal{F}_t^y)\beta_t + \sigma_t dB_t$. From filtering theory (see Allinger and Mitter (1981)) we then know that y generates the same filtration as $\hat{y}$, i.e., $\mathcal{F}_t^{\hat{y}} = \mathcal{F}_t^y$, and that $\tilde{y}$ defined by $d\tilde{y}_t := \frac{1}{\sigma_t} dy_t := db_t$ is a Brownian motion with respect to the information filtration $\mathcal{F}_t^y$. [9]

Employing this result to our situation when $k \neq 0$, , we obtain that

$$\frac{1}{\sigma_t}\{dy_t + k_t \beta_t y_t dt\} := db_t$$

for an $\mathcal{F}_t^y$-Brownian motion b_t. We may express the total order process y as follows

$$dy_t = -k_t \beta_t y_t dt + \sigma_t db_t.$$

We now employ standard results for Gaussian processes to find $\mu_t := E(y_t)$ and $V(t) := E(y_t^2)$ for all $t \in [0,T]$. Using Karatzas and Schreeve (1985), we

[9] The result by Allinger and Mitter proved a long-standing conjecture by Kailath.

have that $\mu(t) = E(y_t) = 0$ for all t provided $y_0 = 0$, and the following first order non-homogeneous ordinary linear differential equation for the variance $V(t) = E(y_t^2)$,

$$\frac{dV(t)}{dt} = -2k_t\beta_t V(t) + \sigma_t^2, \quad V(0) = 0$$

which has the solution

$$V(t) = E(y_t^2) = e^{-2\int_0^t k_s\beta_s ds} \int_0^t \sigma_s^2 e^{2\int_0^s k_r\beta_r dr} ds. \tag{13.1}$$

This is (3.13).

Acknowledgments

References

[1] Aase, K. K. and F. Gjesdal (2017). *"Insider trading with non-fiduciary market makers."*. (Last version 2018) Discussion Paper FOR 8, Norwegian School of Economics.

[2] Aase, K. K. and B. Øksendal (2018). *"Strategic Insider Trading in Continuous Time: A New Approach."* Discussion Paper FOR, Norwegian School of Economics.

[3] Aase, K., Bjuland, T. and Øksendal, B. (2012a). "Partially informed liquidity traders." *Mathematics and Financial Economics* 6, 93-104.

[4] Aase, K., Bjuland, T. and Øksendal, B. (2012b). "Strategic Insider Trading Equilibrium: A Filter Theory Approach." *Afrika Matematika* 23(2),145-162.

[5] Aase, K., Bjuland, T. and Øksendal, B. (2011). "An anticipative linear filtering equation." *Systems & Control Letters* 1-4.

[6] Admati, A. R. and Pfleiderer, P. (1988). "A Theory of Intraday Patterns: Volume and Price Variability." *The Review of Financial Studies* 1, 1, 3–40.

[7] Allinger, D. F. and Mitter, S. K. (1981). "New Results on the Innovations Problem for Non-Linear Filtering". *Stochastics* 4, 339–348.

[8] Back, K. (1992). "Insider Trading in Continuous Time". *The Review of Financial Studies* Vol. 5, No 3, 387-409.

[9] Biagini, F., and Øksendal, B. (2005). "A general stochastic calculus approach to insider trading". *Appl. Math. Optim.* 52, 167–181.

[10] Davis, M. H. A. (1977). *Linear Estimation and Stochastic Control.* Chapman and Hall.

[11] Davis, M. H. A. (1984). *Lectures on Stochastic Control and Nonlinear Filtering.*. Tata Institute of Fundamental Research, Bombay.

[12] Eide, I. B. (2007). *"An equilibrium model for gradually revealed asymmetric information"*. Preprint, University of Oslo 6/2007.

[13] Glosten, L. R., and Milgrom, P. R. (1985). "Bid, Ask and Transaction Prices in a Specialist Market with Hetrogeneously Informed Traders". *Journal of Financial Economics*, 14, 71–100.

[14] Grossman, S. J., and Stiglitz, J. E. (1980). "On the Impossibility of Informationally Efficient Markets". *American Economic Review*, 70 , 393–408.

[15] Holden, C. W. and Subrahmanyam, A. (1992). "Long-Lived Private Information and Imperfect Competition". *The Journal of Finance*, XLVII, 1 247–270.

[16] Kallianpur, G. (1980). *Stochastic Filtering Theory.* Springer.

[17] Kalman, R. E. (1960). "A new approach to linear filtering and prediction problems". *J. Basic Engineering* D 82, 35-45.

[18] Karatzas I., and S. Schreve (1988). *Brownian motion and stochastic calculus.* New York: Springer-Verlag.

[19] Kyle, A. S. (1985). "Continuous Auctions and Insider Trading". *Econometrica* Vol.53, No. 6, 1315–1336.

[20] Liptser, R. S. and Shiryaev, A. N.: Statistics of Random Processes II. Springer 1978.

[21] Øksendal, B. (2013). *"Stochastic Differential Equations."* 6th Edition. Springer

[22] Russo, F., and Vallois, P. (1993). "Forward, backward and symmetric stochastic integration". *Probab. Theory Related Fields* 97, 403–421.

[23] Russo, F. and Vallois, P. (1995). "The generalized covariation process and Itô formula". *Stoch. Process. Appl.* 59, 81–104.

[24] Russo, F. and Vallois, P. (2000). "Stochastic calculus with respect to continuous finite quadratic variation processes". *Stoch. Stoch. Rep.* 70, 1–40.

[25] Tucker, H. G. (1967). *A graduate course in probability.* Academic Press, New York and London.

[26] Yong, J. and X. Y. Zhou (1999). *"Hamiltonian Systems and the HJB Equations"*. Springer-Verlag, New York, Inc.

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