Solution Algorithms for Optimal Buy-Back Contracts in Multi-period Channel Equilibria with Stochastic Demand and Delayed Information

BY Reza Azad Gholami, Leif K. Sandal and Jan Ubøe

DISCUSSION PAPER

NHH

Institutt for foretaksøkonomi
Department of Business and Management Science

FOR 10/2019

ISSN: 1500-4066

September 2019

Solution Algorithms for Optimal Buy-Back Contracts in Multi-period Channel Equilibria with Stochastic Demand and Delayed Information

Reza Azad Gholami* Leif Kristoffer Sandal† Jan Ubøe†

Department of Business and Management Science, NHH Norwegian School of Economics

July 2019

Abstract

We analyze the problem of time-dependent channel coordination in the face of uncertain demand. The channel, composed of a manufacturer and a retailer, is to address a time-varying and uncertain price-dependent demand. The decision variables of the manufacturer are wholesale and (possibly zero) buy-back prices, and those of the retailer are order quantity and retail price. Moreover, at each period, the retailer is allowed to postpone her retail price until demand uncertainty is resolved. In order to place emphasis on the price-decadent nature of demand, we embed a class of memory effects in demand structure, such that current demand at each period demand is affected by pricing history as well as current price. The ensuing equilibria problems, thus, become highly nested in time. We then propose our memory-based solution algorithm which coordinates the channel with optimal buy-back contracts at each period. We show that, contrary to the conventional belief, too generous buy-back prices may not only be suboptimal to the manufacturer, but also decrease the expected profit for the retailer and thus for the whole channel.

Keywords: stochastic optimization; bilevel programming; game theory; channel coordination; buy-back contracts; price postponement; pricing theory; contract theory
JEL Classification: C61, C73, D81, D47

*Department of Business and Management Science, NHH Norwegian School of Economics/ Helleveien 30, 5045 Bergen, Norway/ E-mail: reza.gholami@nhh.no

†Department of Business and Management Science, NHH Norwegian School of Economics

1 Introduction

Vertical competition between upstream and downstream vendors in decentralized channels may in general be detrimental to the aggregate profit obtained by each individual and by the whole channel. Thus, devising different contracts between the two levels of decision making to align their respective objective functions has been the subject of extensive research. A contract is said to coordinate the decentralized channel if the set of optimal acts prescribed by it constitutes a Nash equilibrium (Cachon 2003). Such coordinating solutions are usually sought within bilevel optimization problems where the upstream vendor's objective function is optimized in the outer level (i.e. treated as the leader problem) and that of the downstream is solved in the inner level.

Decentralized supply channel members usually aim at maximizing their revenue by addressing a time-varying and uncertain demand. Conventionally, in such a setting, the downstream and upstream suppliers are referred to as retailers and wholesalers (manufacturers), respectively. Usually it is the retailer who is in direct contact with the uncertain demand, while the upstream vendor may sense the demand uncertainty only through the order quantity that she receives from the retailer. A coordinating buy-back contract aims to find an optimal solution within which the upstream supplier also shares the risks stemming from market uncertainty for the final objective of increasing individual and channel profits.

We consider a channel composed of two members, a manufacturer and a retailer. The channel is to address an uncertain and time-dependent demand for a perishable commodity at different times. We divide the time frame into n discrete *periods*, and solve the coordination problem such that the set of decision variables obtained for each period $k \in \{1 \cdots n\}$ will result in the optimality of the holistic revenue from 1 to n. The commodity produced at each period is perishable and must be sold in that period; it cannot be stored to be supplied at the next periods. Thus, at the end of each period, the unsold items are to be salvaged at a lower price, or if possible, bought back by the manufacturer. In our bilevel programming problem, the retailer's decision variables are the retail price per product unit and the order quantity that she sends to the manufacturer. The retailer, thus, may incur a loss if her order quantity exceeds the actual demand. The risks stemming from demand uncertainty may cause the retailer to order a lower amount of products to the manufacturer, thus causing a lower profit margin for the manufacturer.

In our model, the manufacturer's decision variables are the wholesale and buy-back prices per unit of the product. The manufacturer may offer a non-zero buy-back price for the retailer's unsold items to incentivize a higher order quantity. By doing so, the manufacturer shares the risks caused by demand uncertainty in the hope of receiving a higher order quantity and securing a higher profit margin. However, while not providing a buy-back contract may cause the retailer to order conservatively, too generous a buy-back price may also cause the manufacturer worse off. Thus finding an optimal buy-back price becomes a vital task for the manufacturer and the whole channel.

In addition to the buy-back contracts which aim at sharing the risks stemming from demand uncertainty between both the channel members, there are a variety of approaches utilized by decentralized channels to decrease such risks. Price postponement by the retailer is one of such approaches which allows the retailer, at each period, to postpone her price until demand uncertainty at that period is resolved.

In our paper, we combine optimal buy-back coordination problem with a price postponement approach and find a general equilibrium solution which coordinates the channel at different times.

1.1 Coordinated Buy-back Contracts

In his seminal paper, Pasternack (1985) solves the problem of coordinating a buy-back contract[1] in a static (single-period) setting. The stochastic demand in his model is considered to have a general price-dependent probability density function. Song et al. (2008) analyze the single-period buy-back contract in a decentralized channel with the assumption that the profit function of each channel member is unimodal. In their model, too, the decision variables of the manufacturer are wholesale and buy-back prices. They find out conditions under which the manufacturer's profit becomes independent of demand uncertainty. In the single-period buy-back coordination analyses by Yao et al. (2008) and Wei and Tang (2013), the channel members compete in a Stackelberg framework. In the model offered by the latter, the only price-setting member is the manufacturer and the retailer sets only the order quantity. Li et al. (2012) divide the single-period coordination problem into two steps. In the first step, the Stackelberge bilevel optimization problem is solved and in the second step, a buy-back contract is added to the channel decision making process. Gümüş et al. (2013) extend their own buy-back contract coordination problem into two periods. In their model, they consider a uniformly distributed demand for durable products in online markets. For a short summary of the literature on channel coordination problem with buy-back contracts, see Nan and Fang (2016) and numerous sources therein.

The task of finding a set of optimal buy-back prices at different times, in multi-period time setting, becomes challenging due to the fact that the elastic demand at each period is influenced by the pricing history, thus making the ensuing bilevel problem highly nested in time. In order to put emphasis on this nestedness, in Section 2.1, we introduce our memory effects. These memory effects are embedded within the structure of the uncertain

[1] The buy-back contracts are sometimes referred to as *return policies* in the literature.

demand and carry the effects of previous pricing on current and future demands.

In Section 4.4, we embed these memory-based demand structure in the bilevel programming setting and propose a general solution for the ensuing equilibria problems at each time in the time interval between the first and nth periods. The results of this section are brought in Theorem 4.4.

In the buy-back contracts coordinated in Section 4.4, all decision variables are to be set at the beginning of each period. That is, at the beginning of each period k, the manufacturer offers her optimal wholesale and (possibly zero) buy-back prices to the retailer, who then finds and sets her own optimal retail price and order quantity, accordingly.

1.2 Buy-back Contractus with Retail Price Postponement

There are a variety of other contracts in which the retailer is allowed to postpone her decision variables on order quantity and retail price until demand uncertainty is resolved (Cheng et al. 2010). Van Mieghem and Dada (1999) have analyzed price postponement scenarios where suppliers allow haggling about the final price. The final price is thus not fixed and negotiable after the customers place their orders. According to their analysis, one advantage of price postponement is that the profit margin can be adjusted after demand uncertainty is resolved. Price postponement strategies have been used in online commerce and car dealership (Granot and Yin, 2008, Cheng et al. 2010).

Garnot and Yin (2008) solve the single-period problem of channel coordination with retail price postponement. In their setting, the uncertain demand is purely multiplicative. They analyze the effect of vertical competition and different contracts (including buy-back contracts) between channel members on the profit obtained by the whole channel and each individual member. Xu and Bisi (2011) study a price postponement scenario in a single-period newsvendor model with wholesale price-only contract. They, too, consider purely multiplicative or additive structures for the uncertain demand and make a series of assumptions about demand distribution which assure the unimodality of ensuing profit functions for the two channel members.

In Section 5, we coordinate a buy-back contract in which, at each period, the retailer postpones her decision on retail price until the demand uncertainty at that period is resolved. We analyze the effects of price postponement on the profit obtained by each channel member and the whole channel, in sections 5.2 and 5.3. The solution to the problem of coordinating multi-period buy-back contracts with price postponement is offered in Theorem 5.1.

We refer to the results of Section 5 as closed-loop solutions because they provide the retailer with delayed extra information about the demand uncertainty which, in turn, is used to enhance her decision variables obtained by solving the open-loop problem of Section 4.4.

Having proposed our analytic solution methods in sections 4 and 5, in Section 6 we implement the theoretical results in a few example scenarios and provide numerical results constituting equilibria at each scenario.

2 Stochastic Demand Structure

In this section we propose a demand structure for a perishable good in a dynamic, i.e. time-dependent, and price-dependent framework. The time scope is divided into n (possibly infinite) discrete intervals referred to as periods. We assume all the model variables and parameters to remain unvaried within each period. The supply channel members have to solve their overall bilevel profit optimization problems while addressing this demand at each period. Thus, having introduced our general demand structure, in the subsequent sections, we will embed it into various profit-optimization games to coordinate the channel accordingly.

We consider the dynamic and price-dependent demand at each period $k \in \{1, \cdots, n\}$ to be of the following additive-multiplicative form.

$$D_k = \tilde{\mu}_k(\mathbf{r}_k) + \tilde{\sigma}_k(\mathbf{r}_k)\,\epsilon_k \tag{1}$$

where r_k is the retail price at k, $\mathbf{r}_k = [r_1, \cdots, r_k]$ is the vector of the entire retail price history up to period k, $\tilde{\mu}_k(\cdot)$ and $\tilde{\sigma}_k(\cdot)$ are deterministic functions of $\mathbf{r}_k$ and time (period k), and ϵ_k is the stochastic variable at k.

We normalize the stochastic variable ϵ_k such that $\mathrm{E}[\epsilon_k] = 0$ and $\mathrm{Var}[\epsilon_\mathrm{k}] = 1$. We also assume that the density function for ϵ_k and its cumulative distribution function, $f_{\epsilon_k}(\cdot)$ and $F_{\epsilon_k}(\cdot)$ respectively, are known over its support $[\underline{\epsilon}_k, \overline{\epsilon}_k]$. Furthermore, we assume $F_{\epsilon_k}(\underline{\epsilon}_k) = 0$ and $F_{\epsilon_k}(\overline{\epsilon}_k) = 1$. Moreover, we assume that F_{ϵ_k} is invertible on the support interval and denote the resulting inverse cumulative distribution function (quantile function) by $F^{-1}_{\epsilon_k}(\cdot)$.

It is readily observable that the coefficient of variation of demand as presented in (1) depends on both the vector of retail prices and time. In a purely additive demand structure where at each period $D = \tilde{\mu}(r) + c\epsilon$ (c a constant), the volatility of demand is independent of both time and prices. Whereas in a multiplicative demand model $D = \tilde{\mu}(r)\zeta$, $\mathrm{E}[\zeta] = 1$. In the purely multiplicative demand model–which due to its numerical tractability has been widely used in the literature–the coefficient of variation of demand turns out to become a constant (i.e. 1). Thus the multiplicative model is equivalent to a special case in our model where the mean and standard deviation of demand uncertainty (ϵ) are equal.[2] Both these features are restrictive and undesirable (Young 1978).

[2]We find this assumption too strong and not always justifiable.

The dependence of the coefficient of variation of demand to time and retail prices in an additive-multiplicative model will play a key role in defining and generalizing the memory structure we will introduce in the subsequent section.

2.1 Memory Effects

In a market with elastic demand structure, the pricing history may affect the behavior of strategic buyers at present and in the future. For example, strategic customers aware of the repetitive patterns of pricing at previous seasons, may postpone their purchase with the hope of getting a lower price. The influence of previous prices on the customers' purchase decision in a market with an elastic demand structure can be generalized and systematized as demand memory.

In our model, we embed a class of functional forms in the demand structure in (1) such that they carry the effects of past pacing on current and future demand. We refer to these functional forms as memory functions and denote them by $\Phi_k(\mathbf{r}_{k-1})$.

As discussed in section 2, for the sake of generality, we consider the coefficient of variation of demand at each period to be a function of retail price.

$$CV_{D_k} = CV_{D_k}(r_k) \tag{2}$$

In this paper, we limit our analysis to the case where previous prices scale the level of the current demand.

$$\begin{aligned} &D_k(\mathbf{r}_k) = \Phi_k(\mathbf{r}_{k-1}) d_k(r_k) \\ &\text{where } d_k(r_k) = \mu_k(r_k) + \sigma_k(r_k)\epsilon_k \end{aligned} \tag{3}$$

Thus, from (3) and (1) we observe that

$$\begin{aligned} \widetilde{\mu}_k(\mathbf{r}_k) &= \Phi_k(\mathbf{r}_{k-1})\mu_k(r_k) \\ \widetilde{\sigma}_k(\mathbf{r}_k) &= \Phi_k(\mathbf{r}_{k-1})\sigma_k(r_k) \end{aligned} \tag{4}$$

which in turn satisfies (2).

Moreover, the structure of the memory functions must be such that at period $k+1$, the memory retains the information stored in the pervious periods' memory functions while being affected by the most recently observed piece of information, which is r_k. This feature can be obtained by

$$\frac{\Phi_{k+1}(\mathbf{r}_k)}{\Phi_k(\mathbf{r}_{k-1})} = \phi_{k+1}(r_k) \tag{5}$$

We refer to ϕ_ks as memory elements and allow them to have different functional forms at differing periods, adding to the level of non-autonomy of the ensuing equilibrium problems.

With the memory structure described so far, we will have

$$\begin{aligned}\Phi_k(\mathbf{r}_{k-1}) &= \prod_{i=1}^{k} \phi_i(r_{i-1}) \\ \Phi_1(\cdot) &= \phi_1(\cdot) = 1\end{aligned} \tag{6}$$

In Section 6.1, we analyze and introduce general functional forms, compatible with economic contexts, as memory functions.

3 Open-loop and Closed-loop Equilibria problems

Having introduced our memory-based demand structure, in the subsequent sections we embed it into different profit optimization problems for the supply channel. The supply channel is composed of two members, a manufacturer and a retailer. We consider the two channel members in a Stackelberg bilevel optimization framework in which the manufacturer is the leader and the retailer is the follower. Each channel member has to solve her own profit optimization problem while being subject to the optimality of the other player's solution as a constraint. The channel is to address the uncertain demand discussed earlier for a perishable commodity.

At the beginning of each period, the manufacturer sets the wholesale price, w_k, per unit and offers a buy-back price per unsold unit, $b_k \geq 0$, to the retailer. The retailer then solves her own optimization problem accordingly, and orders an amount of q_k of the commodity to the manufacturer and sets the retail price r_k per unit.

In this paper, we classify the ensuing coordination problems into two major classes, based on the behavior of the retailer at the beginning of each period. In the first class of the equilibrium problems, at the beginning of each period, the retailer having solved her optimization problem does not postpone her declaration of the price to the market. We refer to this class as open-loop coordination problems or no-postponement problems, interchangeably.

In the second class, the retailer, after receiving the manufacturer's decision variables at the beginning of each period, solves the open-loop problem to find out the optimal retail price and order quantity. She then orders an amount of q_k items to the manufacturer. However, she postpones her decision on r_k until demand uncertainty at that period ϵ_k is resolved. She will then use the demand uncertainty as a rectifying feedback signal to improve her open-loop prices both locally in time and also for the rest of the periods. That is, after observing ϵ_k she solves her equilibria problems anew to find a new set of retail prices from that period onward: r_is, $i \in \{k, \cdots, n\}$. We refer to this class of problems as closed-loop or price-postponement coordination problems.

The first and second class of coordination problems will be analyzed in sections 4 and 5, respectively.

4 An Open-loop Coordinated Model without Postponement

The assumption of accessibility of demand distribution as a form of a priori knowledge is commonplace in the existing literature; see for instance Cachon (2003), Pasternack (2008), and Kim et al (2015). In our coordination analysis, we assume that both the channel members possess an a priori knowledge about the distribution of the demand uncertainty at each period, i.e. f_{ϵ_k} and F_{ϵ_k} are known. Notice that the distribution of the noise term at each period is independent of the retail price.

Moreover, in the no-postponement coordination analysis, we assume that both the retailer and the manufacturer are risk-neutral, that is each channel member tries to maximize her respective expected profit over the course of the n periods. At the beginning of each period the manufacturer sets the wholesale price and also may offer a buy-back price to the retailer. The retailer then finds her optimal order quantity and retail price accordingly.

It should be noted that the channel under study is considered to be a segment of a more complete market, such that a segmentation of the pool of customers are addressed by it. The market demand structure, in general, is an aggregation of the individual demands from possibly heterogenous consumers who may be affected by the supply of competing products from other vendors. This feature is embedded in D through the choice of $\mu_k(k, r)$ and $\sigma_k(k, r)$. Therefore, although the manufacturer and the retailer in our model are basically monopolistic suppliers, the model considers competition via demand structure.

We denote the ensuing open-loop equilibria variables by w_k^*, b_k^*, q_k^*, and r_k^*. It should be noted that the equilibria states are the results of solving the bilevel optimization problem over the whole span of periods from 1 to n, ($n \to \infty$ in the infinite-horizon analysis).

If the amount of the ordered items exceeds the demand at a period k, that is if $q_k^* > D_k$, the retailer can salvage the unsold items at a price of s_k. However, because the commodity is perishable, the retailer cannot restore the unsold items at the end of each period and thus will not be able to supply them to market in the next period.

Moreover, the manufacturer may offer the retailer with the additional buy-back price $b_k^* \geq 0$ per unit for surplus items. Note that the existence of a buy-back contract ($b^* > 0$) in a decentralized channel does not necessarily mean that the unsold items will be physically sent back to the manufacturer (Cachon 2003). The manufacturer, in general, may provide the retailer with a non-zero credit for any unsold item at the end of a period, in order to incentive a higher order quantity. Obviously $r > b + s$.

4.1 Coordination in a Single Period (Static) Problem

For illustration purposes, we start the open-loop coordination of the decentralized channel by solving it in a single-period horizon. In section 4.2 we generalize the solution to cover multi-period coordination problems as well.

4.1.1 Model Framework and Solution Procedure

Model Variables and Parameters

$w =$ wholesale price per unit, (decision variable)

$r =$ retail price per unit, $r > w$ (decision variable)

$q =$ quantity of items to be supplied to the market, (decision variable)

$D =$ actual uncertain demand

$c_m =$ manufacturing cost per unit, $c_m < w$ (given parameter)

$c_r =$ retailer's marginal cost per unit, $c_r < r - w$ (given parameter)

$s =$ salvage price per unit

$b =$ buy-back price per unit

$\pi^m =$ manufacturer's profit

$\pi^r =$ retailer's profit

Since this is a single-period problem, we have dropped the time index k. The general demand expression in (1) can now be recast as below

$$D = \mu(r) + \sigma(r)\epsilon \tag{7}$$

where $\mu(r)$ and $\sigma(r)$ are given functions of the mean and standard deviation of the uncertain demand.

Each player's profit is then obtained as follows.

$$\begin{aligned} \pi^r &= r\min(D, q) + s(q - D)^+ - c_r q - wq + b(q - D)^+ \qquad (8) \\ &= (r - s - b)\min(D, q) + (s + b - c_r - w)q \end{aligned}$$

$$\pi^m = (w - c_m)q - b(q - D)^+ = (w - c_m - b)q + b\min(D, q) \tag{9}$$

Because the two channel members are risk-neutral, their objective functions will be the expected values of the profits, optimized in a bilevel framework.

$$\begin{aligned}
\max_q \mathrm{E}[\pi^r(r,q,b,w)] &\quad \text{to obtain} \quad q^*(r,b,w) \\
\max_r \mathrm{E}[\pi^r(r,b,w)] &\quad \text{to obtain} \quad r^*(b,w) \\
\max_b \mathrm{E}[\pi^m(b,w)] &\quad \text{to obtain} \quad w^*(b) \\
\max_w \mathrm{E}[\pi^m(w)] &\quad \text{to obtain} \quad w^* \to q^*, r^*, b^*
\end{aligned} \tag{10}$$

The first two equations in the bilevel optimization problem (10) constitute the inner (follower) problem and the rest are the outer (leader) problem. For notational simplicity, we denote the expected values of the profits in the single-period problem as follows.

$$\begin{aligned}
\overline{\pi}^r &:= \mathrm{E}[\pi^r] \\
\overline{\pi}^m &:= \mathrm{E}[\pi^m].
\end{aligned} \tag{11}$$

Proposition 4.1.
Assume that ϵ has a continuous distribution, supported on the interval , with density $f_\epsilon > 0$ a.e. on its support $[\underline{\epsilon}, \overline{\epsilon}]$, and a corresponding quantile function F_ϵ^{-1}. Then the equilibrium state decision variables to the single-period bilevel optimization problem in (10) are obtained from the closed-form expression in (12) and numerical solutions to (13) and (14).

$$q^*(r,b,w) = \mu(r) + \sigma(r)\, F_\epsilon^{-1}\left(\frac{r-w-c_r}{r-s-b}\right) \tag{12}$$

$$\max_r \overline{\pi}^r(r,b,w) \tag{13}$$

$$\max_{b,w} \overline{\pi}^m(b,w) \tag{14}$$

where

$$\overline{\pi}^r(r,b,w) = (r-w-c_r)\,\mu(r) + (r-s-b)\,\sigma(r)\int_{\underline{\epsilon}}^{z} t f_\epsilon(t)dt \tag{15}$$

$$\textit{where} \quad z(r,w) = F_\epsilon^{-1}\left(\frac{r-w-c_r}{r-s-b}\right)$$

$$\overline{\pi}^m(w) = \mu\left(r^*(w)\right)\Big(w-c_m\Big) + \sigma\left(r^*(w)\right)\Big[\Big(z^*(w)\Big(w-c_m-\frac{r^*-w-c_r}{r^*-s-b}\Big) + b\int_{\underline{\epsilon}}^{z^*} t f_\epsilon(t)dt\Big] \tag{16}$$

$$\textit{where} \quad z^*(w) = F_\epsilon^{-1}\left(\frac{r^*-w-c_r}{r^*-s-b}\right) \tag{17}$$

Proof
See Appendix 1.

Remark 4.2. Notice that of all the four decision variables, only q^* can be formulated in a closed-form expression. The rest of the optimal decision variables must be obtained numerically by two-level (follower-leader) optimization processes. These features, as we will see, will be inherited by the corresponding sets of optimal decision variables in the multi-period coordination problem.

Remark 4.3. The condition that ϵ is supported on an interval with $f_\epsilon > 0$ a.e. on its support is required to ensure that F_ϵ is invertible. If F_ϵ is not invertible, it is possible that the retailer's expected profit is maximized at several order quantities between which the retailer is indifferent. Different order quantities lead to different profits for the manufacturer, but the manufacturer lacks an instrument to ensure that the retailer chooses order quantities that are optimal for the manufacturer.

Example 1.
In this example, we have been using the following Cobb-Douglas functions for the mean and standard deviation of demand. The structure of demand expression is such that as time goes by, the absolute value of price elasticity of demand increases. However, because this is a single-period coordination problem, we have $k = 1$.

$$\mu(r, k) = \frac{1000}{r^{2+0.1(k-1)}} \quad \sigma(r, k) = 0.1\mu(r, k) + \frac{100}{r^3} \tag{18}$$

The given parameters are as below.

$$c_m = 3,\ c_r = 0,\ s = 1 \tag{19}$$

In this example, we coordinate a supply channel facing the uncertain demand structure given in (18). The manufacturer offers the retailer with a fixed buy-back price b. We solve the coordination problem with different values of offered buy-back prices and analyze the effect of buy-back price on the channel partners' expected profits. Note that in this example, we treat each value of b as a given parameter in the bilevel optimization problem.

Figures 1 and 2 illustrate the expected profits for the channel members in different coordinated buy-back contracts. We observe that the highest expected profit for the manufacturer is obtained when she offers a buy-back price of 1.51. While too generous buy-back prices, for obvious reasons, are detrimental to the manufacturer's expected profit, a buy-back price of zero is also suboptimal. This is due to the fact that a non-zero buy-back price encourages the retailer to opt for a higher order quantity, which in turn may increase the manufacturer's expected profit. Thus, optimization of the manufacturer's objective function with respect to offered buy-back prices seem necessary.

We also observe that the retailer's expected profit does not monotonically increase with buy-back prices. This is because an increase in the offered buy-back price is usually accompanied by an increase in w^*.

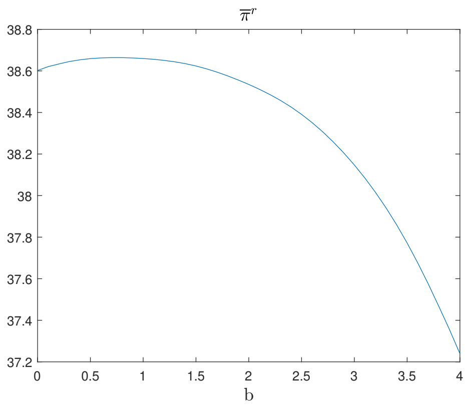

Figure 1: Retailer's expected profit versus b

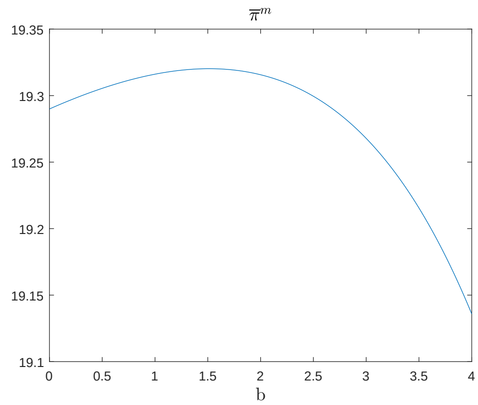

Figure 2: Manufacturer's expected profit versus b

4.2 The Dynamic (multi-period) Equilibria

In this section, we propose a general solution to the multi-period version of the bilevel optimization problem discussed in Section 4.1. We denote the manufacturer's and retailer's local-in-time profits at period k by π_k^m and π_k^r, respectively.

The retailer's price optimization problem is formulated as below.

$$\max_{\mathbf{r}_k} \overline{\Pi}^r = \max_{\mathbf{r}_k} \mathrm{E}\Big[\sum_{k=1}^{n} \alpha_k \mathrm{E}[\pi_k^r | D_1, \cdots, D_{k-1}]\Big] \tag{20}$$

In (20), α_k is a given discounting factor at period k. Time-dependent discounting factors enable the model to cover a higher level of non-autonomy as they allow for different period lengths.

The retailer's optimization problem must be solved in tandem with that of the manufacturer within a Stackelberg framework. We formulate the manufacturer's price optimization problem as below.

$$\max_{\mathbf{w}_k} \overline{\Pi}^m = \max_{\mathbf{w}_k} \mathrm{E}\Big[\sum_{k=1}^{n} \alpha_k \mathrm{E}[\pi_k^m | D_1, \cdots, D_{k-1}]\Big] \tag{21}$$

We start our analysis by studying the retailer's price optimization problem in (20). Without loss of generality, we can consider $\mathrm{E}[\pi_k^r]$ (the local-in-time expected profit) as a function of demand mean and standard deviation at each k. However, according to the multi-period demand expression in (1), $\tilde{\mu}_k$ and $\tilde{\sigma}_k$ depend on the whole history of retail prices $\mathbf{r}_k$. This will make the optimization problem in (20) highly nested in time.

In addition, analogous to the single-period bilevel problem, since the retailer is the follower, her optimal decision variables will first be determined as functions of the decision variables of the leader. That is, for instance, at each period k when the bilevel optimization algorithm proceeds to the outer level optimization problem, the optimal price of the retailer will be of the functional form $r_k^*(w_k, b_k)$, and will not be determined numerically until the outer problem is solved, i.e. until w_k^* and b_k^* have been found and substituted in the expression for r_k^*. This bilevel structure will also add to the complexity of the problem.

4.3 General Equilibrium Solution

Using backward induction method, we begin the solution of the multi-variable nested optimization problem by analyzing the final period. The only profit expression in (20) which depends on r_n is $\mathrm{E}[\pi_n^r]$. Thus maximization of the entire multi-variable sum,$\overline{\Pi}^r$, with respect to r_n is equivalent to maximization of only the single-variable $\mathrm{E}[\pi_n^r]$ with respect to r_n.

$$\max_{r_n} \overline{\Pi}^r \equiv \max_{r_n} \mathrm{E}[\pi_n^r] \tag{22}$$

Moreover, at period n all of the previous decision variables and demands have become common knowledge. Therefore given $\mathbf{r}_{n-1}^*$ and $\mathbf{D}_{n-1} = [D_1, \cdots, D_{n-1}]$ and assuming that the mapping $r_n \mapsto \mathrm{E}[\pi_n^r|\mathbf{D}_{n-1}]$ has a global maximum, this global maximum can be expressed as a function of the previous retail prices and demand history.[3]

$$r_n^* = r_n^*(\mathbf{r}_{n-1}, \mathbf{D}_{n-1}) \tag{23}$$

Now the backward induction method proceeds to period $n-1$ where having r_n^* as expressed in (23) enables us to conclude that maximization of $\overline{\Pi}^r$ with respect to r_{n-1} is equivalent to maximization of $\alpha_{n-1}\mathrm{E}[\pi_{n-1}^r] + \alpha_n \mathrm{E}[\pi_n^r]$ with respect to r_{n-1}. The resulting r_{n-1}^* will be a function of $(\mathbf{r}_{n-2}^*, \mathbf{D}_{n-2})$. Inserting this new function into (20) and iterating the same procedure backward in time, we obtain the vector $\mathbf{r}_n^*$.

4.4 Memory-based Equilibrium Solution

The general construction outlined in Section 4.3 becomes highly nested both in time and solution level. This section should be regarded as an attempt to propose an analytically constructed and numerically efficient solution method to the ensuing nested equilibria problems in a multi-period time setting. In this section, we introduce our memory-based equilibrium solution method utilizing the memory-based demand structure. The importance of our memory-based demand scheme lies in the structure it will create when embedded inside the expressions for the manufacturer's and retailer's expected profits.

[3] It is still a function of the manufacturer's decision variables as well. This is because at this time, all these procedures are happening within the lower level (follower) solution algorithm.

We begin by embedding the memory effects into the additive-multiplicative demand structure. From (1) and (4), it is straightforward to see that

$$D_k(k, \mathbf{r}_k) = \tilde{\mu}_k(\mathbf{r}_k) + \tilde{\sigma}_k(\mathbf{r}_k)\,\epsilon_k = \Phi_k(\mathbf{r}_{k-1})\,\Big[\mu_k(r_k, k) + \sigma_k(r_k, k)\,\epsilon_k\Big] \tag{24}$$

We refer to the expression in (24) as memory-based demand structure. Substituting this demand structure into the retailer's expected profit at period k we obtain the following.

$$\begin{aligned} \mathrm{E}[\pi_k^r] &= (r_k - w_k - c_{r_k})\,\widetilde{\mu}_k(\mathbf{r}_k) + (r_k - s_k - b_k)\,\widetilde{\sigma}_k(\mathbf{r}_k)\int_{\underline{\epsilon}_k}^{z_k} t f_\epsilon(t)dt \\ &= \overbrace{\Big[(r_k - w_k - c_{r_k})\mu_k(r_k) + (r_k - s_k - b_k)\,\sigma_k(r_k)\int_{\underline{\epsilon}_k}^{z_k} t f_\epsilon(t)dt\Big]}^{:=\overline{\pi}_k^r} \cdot \Phi_k(\mathbf{r}_{k-1}) \\ &\text{where} \quad z_k(r_k, w_k) = F_\epsilon^{-1}\Big(\frac{r_k - w_k - c_{r_k}}{r_k - s_k - b_k}\Big) \end{aligned} \tag{25}$$

We refer to $\overline{\pi}_k^r$ as scaled expected profit for the retailer at k. Thus (25) can be simplified as below.

$$\mathrm{E}[\pi_k^r] = \overline{\pi}_k^r \cdot \Phi_k(\mathbf{r}_{k-1}) \tag{26}$$

Note that the in the single-period case, where $\Phi(\cdot) = 1$, the expression in (26) will turn into $\mathrm{E}[\pi^r] = \overline{\pi}^r$ which is consistent with (11). Similarly, we can calculate the manufacturer's expected profit at k as below.

$$\begin{aligned} \mathrm{E}[\pi_k^m] = \Bigg\{ &\mu_k\left(r_k^*(w_k)\right)\left(w_k - c_{m_k}\right) + \sigma_k\left(r_k^*(w_k)\right)\Big[(z_k^*(w_k)\Big(w_k - c_{m_k} - \frac{r_k^* - w_k - c_{r_k}}{r_k^* - s_k - b_k}\Big) \\ &+ b_k \int_{\underline{\epsilon}_k}^{z_k^*} t f_\epsilon(t)dt\Big]\Bigg\} \cdot \Phi_k(\mathbf{r}_{k-1}^*) \\ &\text{where} \quad z_k^*(w) = F_\epsilon^{-1}\Big(\frac{r_k^* - w_k - c_{r_k}}{r_k^* - s_k - b_k}\Big) \end{aligned} \tag{27}$$

Analogously, we refer to the term inside the curly brackets in (27) as the scaled expected profit for the manufacturer at k and denote it by $\overline{\pi}_m^r$. Whence (27) is simplified as below.

$$\mathrm{E}[\pi_k^m] = \overline{\pi}_k^m \cdot \Phi_k(\mathbf{r}_{k-1}^*) \tag{28}$$

Using the result of Proposition 4.1, the numerical value for the optimal order quantity at k is obtained from the following closed-form expression.

$$q_k^* = \Phi_k(\mathbf{r}_{k-1}^*)\,\Big[\mu_k(r_k^*) + \sigma_k(r_k^*)\,F_{\epsilon_k}^{-1}\Big(\frac{r_k^* - w_k^* - c_{r_k}}{r_k^* - b_k^* - s_k}\Big)\Big] \tag{29}$$

It is important to note that in general, the argmax of the expected profit in a specific period k for either supplier, i.e. the result of $\max\limits_{r_k, m_k} \mathrm{E}[\pi_k^{r,m}]$ is not equal to the value of the

kth optimal decision variable for that supplier when the objective function is the whole expected profit within the periods 1 to n. In other words, in general

$$\max_{r_k,b_k,w_k} \mathrm{E}[\pi_k^{r,m}] \not\equiv \max_{r_k,b_k,w_k} \overline{\Pi}^{r,m} \tag{30}$$

Our purpose is to find the results of the LHS of (30) – those decision variables which, considering the effect of the pricing in the past on current and future demand, manipulate the demand such that the highest amounts of expected profits for each decision maker over the time interval between 1 and n.

Thus the following nested bilevel constrained optimization problems must be solved throughout the periods from 1 to n.

$$\begin{aligned}\max_{\mathbf{r}_n} \overline{\Pi}^r = \max_{\mathbf{r}_n} \mathrm{E}\Big[&\overline{\pi}_1^r(r_1,b_1,w_1) + \cdots + \alpha_k \Phi_k(\mathbf{r}_{k-1})\, \overline{\pi}_k^r(r_k,b_k,w_k) \\ &+ \cdots + \alpha_n \Phi_n(\mathbf{r}_{n-1})\, \overline{\pi}_n^r(r_n,b_n,w_n)\Big] \quad \text{The inner level optimization}\end{aligned} \tag{31}$$

$$\begin{aligned}\max_{\mathbf{w}_n,\mathbf{b}_n} \overline{\Pi}^m(\mathbf{w}_n,\mathbf{b}_n) = \max_{\mathbf{w}_n,\mathbf{b}_n} \mathrm{E}\Big[\sum_{k=1}^{n} \alpha_k \Phi_k(\mathbf{r}_k^*) \overline{\pi}_k^m(w_k,b_k)\Big] \quad \text{The outer level optimization} \\ \text{s.t.} \quad 0 \le b_k < w_k - c_{m_k} \quad \forall k \in \{1,\cdots,n\}\end{aligned} \tag{32}$$

Analogous to the single-period bilevel optimization, the optimal decision variables obtained from the inner (follower) optimization problem will be functions of the variables of the outer optimization problem. Additionally, for the manufacturer's problem, the feasible domain must be searched for couples of b_k, w_k at each period.

According to the procedure proposed in Section 4.3, at each level, the nested n-variable problem should be decoupled into n single-variable optimization problems. Similar to the observation in that section, it is evident that the variable r_n appears only in the final discounted profit term. Thus utilizing the backward induction method, we begin the optimization from the final period.

$$\max_{r_n} \overline{\Pi}^r(\mathbf{r}_n) \equiv \max_{r_n} \overline{\pi}_n^r(r_n) \tag{33}$$

In order to develop this approach, at each period k we define J_k^r as the discounted expected value of the profit obtained from that period onward, i.e. within the time interval $\{k,\cdots,n\}$. each period k we define J_k^r as the discounted expected value of the profit obtained from that period onward, i.e. within the time interval $\{k,\cdots,n\}$.

$$J_k^r = \alpha_k \Phi_k(\mathbf{r}_{k-1}) \overline{\pi}_k^r(r_k) + \cdots + \alpha_n \Phi_n(\mathbf{r}_{n-1}) \overline{\pi}_n^r(r_n) \tag{34}$$

We observe that in this structure, beginning from the last period, the variable r_k in Π^r appears for the first time in the expression for J_k^r. Having solved the RHS of (33) we obtain r_n^* and proceed to the previous period $n-1$. Going further backward in time, we can generalize this procedure as shown in (35) and (36), given that $\alpha_1 = 1$ and $\Phi_1(\cdot) = 1$.

$$\begin{aligned} J_k^r =& \alpha_k \Phi_k(\mathbf{r}_{k-1}) \bigg(\overline{\pi}_k^r(r_k) + \phi_{k+1}(r_k) \Big[\frac{\alpha_{k+1}}{\alpha_k} \overline{\pi}_{k+1}^r + \frac{\alpha_{k+2}}{\alpha_k} \phi_{k+2}(r_{k+1}) \overline{\pi}_{k+2}^r \\ & + \cdots + \frac{\alpha_n}{\alpha_k} \overline{\pi}_n^r \prod_{i=k+2}^{n} \phi_i(r_{i-1}) \Big] \bigg) \end{aligned} \tag{35}$$

$$\max_{r_k} J_k^r$$

$$J_k^r = \alpha_k \underbrace{\Phi_k(\mathbf{r}_{k-1})}_{\text{price history}} \overbrace{\left(\overline{\pi}_k^r(r_k) + \phi_{k+1}(r_k) \underbrace{\left[\frac{\alpha_{k+1}}{\alpha_k} \overline{\pi}_{k+1}^r(r_{k+1}^*) + \cdots + \frac{\alpha_n}{\alpha_k} \overline{\pi}_n^r(r_n^*) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*) \right]}_{\mathcal{F}_k^r = \text{ expected (future) values, given at } k\text{th period}} \right)}^{\mathfrak{J}_k^r(r_k)}$$

$$\max_{r_k} J_k^r \tag{36}$$

In general, we define $\mathcal{F}_k^r$, the scaled expected future profit within $\{k+1, \cdots, n\}$ and $\mathfrak{J}_k^r$, the scaled expected profit within $\{k, \cdots, n\}$, as below.

$$\mathcal{F}_k^r \coloneqq \frac{1}{\alpha_k} \sum_{j=k+1}^{n} \prod_{i=k+2}^{j} \phi_i(r_{i-1}^*) \cdot \alpha_j \overline{\pi}_j^r(r_j^*) \tag{37}$$

$$\mathfrak{J}_k^r(r_k) \coloneqq \overline{\pi}_k^r + \phi_{k+1}(r_k) \mathcal{F}_k^r \tag{38}$$

As it is demonstrated in (36), when the backward induction process reaches the kth period, the scaled profit expected to gain in the future denoted by $\mathcal{F}_k^r$ has been determined and is treated as a constant. We also observe the following relationship between J_{k+1}^r and $\mathcal{F}_k^r$, the resolved future expected earning when the backward induction reaches k with.

$$\mathfrak{J}_{k+1}^r(r_{k+1}^*) = \frac{\alpha_k}{\alpha_{k+1}} \mathcal{F}_k^r \quad 1 \leq k < n \tag{39}$$

Note that, unlike $\mathcal{F}_k^r$ and $\mathfrak{J}_{k+1}^r$, J_{k+1}^r includes the entire pricing history $\Phi_k(\mathbf{r}_{k-1})$ and hence is not known at k. In fact, J_k^rs are not resolved until the backward induction reaches $k = 1$. The effect of the past represented by $\Phi_k(\mathbf{r}_{k-1})$, though not yet determined by backward induction, is factorized in (36) such that it only scales the expected profit from k onward. Therefore, we will have:

$$\max_{r_k} \overline{\Pi}^r(\mathbf{r}_n) \equiv \max_{r_k} J_k^r(\mathbf{r}_k) \equiv \max_{r_k} \mathfrak{J}_k^r(r_k) \tag{40}$$

Combining (36) and (39) we can summarize the retailer's part of the multi-period bilevel optimization in the following recursive procedure.

$$\begin{aligned} &\mathcal{F}_n^r = 0 \quad \text{no future earning after } n \\ \max_{r_k} \mathfrak{J}_k^r(r_k) = \max_{r_k} &\Big[\overline{\pi}_k^r(r_k) + \phi_{k+1}(r_k)\mathcal{F}_k^r\Big] \quad k = n, \cdots, 1 \text{ (backward)} \;\rightarrow \text{ yields } r_k^* \\ &\mathcal{F}_{k-1}^r = \frac{\alpha_{k-1}}{\alpha_k}\mathfrak{J}_k^r(r_k^*) \quad k = n, \cdots, 2 \text{ (backward)} \end{aligned} \tag{41}$$

From the procedure outlined in (41) it is readily observable that, in general, the holistic optimal retail prices (r_k^*s) are not the optimizers of individual $\overline{\pi}_k^r$s. The only situation where $r_k = argmax(\overline{\pi}_k^r)$ is when $\phi_{k+1} = C_k$, where C_k is a constant. A scenario in which all the memory elements are constants, will create identical repeated games at different periods.

It goes without saying that the same procedure can be applied and obtain equilibria results if the channel was comprised of one supplier (thus constituting a centralized channel). The only difference in a bilevel setting is that the optimal results obtained by solving the inner problems will be functions of the variables of the outer problems, i.e. $r_k^* = r_k^*(b_k, w_k)$. When the leader's optimization problems are solved, i.e. when b_k^*s and w_k^*s are found, the follower can find numerical values to her optimal results.

In decoupling the nested n-variable optimization problem of the retailer into n single-variable problems, we did not make any assumption about the level of the optimization problem. Thus the same scheme can be applied twice to the manufacturer's optimization problems to decouple them into $2n$ single-variable ones. Once to obtain $b^*(w)$s and next to find numerical results for w_k^*s.

We state the final results of this section in the following two theorems.

Theorem 4.4.
Let n be the number of periods and assume that the uncertain demand at period k is given by

$$D_k(\boldsymbol{r}_k) = \Phi_k(\boldsymbol{r}_{k-1})\Big(\mu_k(r_k) + \sigma_k(r_k)\epsilon_k\Big) \tag{42}$$

where

$$\Phi_1(\cdot) = \phi_1(\cdot) = 1, \quad \Phi_k(\boldsymbol{r}_{k-1}) = \prod_{i=1}^{k}\phi_i(r_{i-1})$$

and where ϵ_ks are continously distributed with $\mathrm{E}[\epsilon_k] = 0$ *and* $\mathrm{Var}[\epsilon_k] = 1$ *for all k. with $f_{\epsilon_k} > 0$ a.e. on their supports. If for each k the single-period Stackelberg problem below has an equilibrium at r_k^*, b_k^* and w_k^**

$$\begin{aligned} \mathfrak{J}_k^r &= \overline{\pi}_k^r + \phi_{k+1}(r_k)\mathcal{F}_k^r \\ \mathfrak{J}_k^m &= \overline{\pi}_k^m + \phi_{k+1}(r_k)\mathcal{F}_k^m \end{aligned} \tag{43}$$

where $\mathcal{F}_k^r$ and $\mathcal{F}_k^m$ are found recursively from:

$$\begin{aligned} &\mathcal{F}_n^r = 0, \quad \mathcal{F}_k^m = 0 \\ \mathcal{F}_{k-1}^r = \frac{\alpha_{k-1}}{\alpha_k} \mathfrak{J}_k^r(r_k^*),& \quad \mathcal{F}_{k-1}^m = \frac{\alpha_{k-1}}{\alpha_k} \mathfrak{J}_k^m(w_k^*, b_k^*), \quad k = n, \cdots, 2 \end{aligned} \tag{44}$$

and

$$\begin{aligned} \overline{\pi}_k^r &= (r_k - w_k - c_{r_k})\mu_k(r_k) + (r_k - s_k - b_k)\,\sigma_k(r_k) \int_{\underline{\epsilon}_k}^{z_k} t f_\epsilon(t) dt \\ \overline{\pi}_k^m &= \mu_k\left(r_k^*(w_k)\right)\left(w_k - c_{m_k}\right) + \sigma_k\left(r_k^*(w_k)\right)\Big[\left(z_k^*(w_k)\Big(w_k - c_{m_k} - \frac{r_k^* - w_k - c_{r_k}}{r_k^* - s_k - b_k}\Big)\right. \\ &\qquad + b_k \int_{\underline{\epsilon}_k}^{z_k^*} t f_\epsilon(t) dt\Big] \end{aligned} \tag{45}$$

then the bilevel (Stackelberg) optimization problem

$$\begin{aligned} \overline{\Pi}^r &= \sum_{k=1}^{n} \alpha_k \mathrm{E}[\pi_k^r] = \sum_{k=1}^{n} \alpha_k \Phi_k(\boldsymbol{r}_{k-1}) \overline{\pi}_k^r \\ \overline{\Pi}^m &= \sum_{k=1}^{n} \alpha_k \mathrm{E}[\pi_k^m] = \sum_{k=1}^{n} \alpha_k \Phi_k(\boldsymbol{r}_{k-1}^*) \overline{\pi}_k^m \end{aligned} \tag{46}$$

has an equlibrium at $\boldsymbol{r}_n^* = [r_1^*, \cdots, r_n^*]$, $\boldsymbol{b}_n^* = [b_1^*, \cdots, b_n^*]$, and $\boldsymbol{w}_n^* = [w_1^*, \cdots, w_n^*]$. The optimal order quantity at k is then calculated as below.

$$q_k^* = \Phi_k(\boldsymbol{r}_{k-1}^*) \Big[\mu_k(r_k^*) + \sigma_k(r_k^*)\, F_{\epsilon_k}^{-1}\Big(\frac{r_k^* - w_k^* - c_{r_k}}{r_k^* - b_k^* - s_k}\Big)\Big] \tag{47}$$

Next, we prove that the results of Theorem 4.4 are subgame perfect.

Proposition 4.5.
The equilibrium obtained in Theorem.4.4 is subgame perfect. That is, subsets of the equilibrium results covering the time interval between an arbitrary period j and n, i.e. $[r_j^*, \cdots, r_n^*]$, $[b_j^*, \cdots, b_n^*]$, and $[w_j^*, \cdots, w_n^*]$ and, a fortiori, their resulting $[q_j^*, \cdots, q_n^*]$ will also constitute an equilibrium for the corresponding subgame of the original problem, covering that time-interval:

$$\begin{aligned} J_j^r &= \alpha_j \Phi_j(\boldsymbol{r}_{j-1}) \overline{\pi}_j^r(r_j) + \cdots + \alpha_n \Phi_n(\boldsymbol{r}_{n-1}) \overline{\pi}_n^r(r_n) \\ J_j^m &= \alpha_j \Phi_j(\boldsymbol{r}_{j-1}^*) \overline{\pi}_j^r(w_j) + \cdots + \alpha_n \Phi_n(\boldsymbol{r}_{n-1}^*) \overline{\pi}_n^r(w_n) \end{aligned} \tag{48}$$

Proof. (By induction)
We have to prove that if $\{r_j^*, \cdots, r_n^*\}$, $\{r_j^*, \cdots, r_n^*\}$, and $\{w_j^*, \cdots, w_n^*\}$ are subsets of the equilibrium results for $[\overline{\Pi}^r, \overline{\Pi}^m, 1 : n]$, then they also constitute an equilibrium for $[J_j^r, J_j^m, j : n]$.

Beginning from the final period, we analyse the two agents' equilibrium problem. In the expressions for both J^r_k and $\overline{\Pi}^r$ the variable r_n appears in $\overline{\pi}^r_n(r_n)$ only. The same logic is applicable to the manufacturer's solution procedure.

$$\max_{r_n} J^r_k \equiv \max_{r_n} \overline{\pi}^r_n \equiv \max_{r_n} \Pi^r$$

$$\max_{w_n} J^m_k \equiv \max_{b_n, w_n} \overline{\pi}^m_n \equiv \max_{b_n, w_n} \Pi^m$$

Thus, at n the conclusion is obvious. The rest of the proof for an arbitrary k, $j < k < n$ has been argued in detail within the discussion resulting in (40). □

We will use the subgame perfection of the open-loop equilibrium in Section 5 in the analysis of the closed-loop equilibrium in a price-postponement scenario.

5 Coordination with Price Postponement: A Closed-loop Model

In this section, we analyse a closed-loop variant of the problem, in which the retailer postpones the announcement of retail price until after the demand uncertainty has been resolved. We use essentially the same notations for the model variables and parameters as those in Section 4. We use $\hat{r}_k$, and $\hat{q}_k$ to denote the optimal retail price and order quantity, respectively.

In a price-postponement scenario, the two players start with the open-loop equilibrium solution procedures and obtain $\mathbf{r}^*_n$, $\mathbf{b}^*_n$, $\mathbf{w}^*_n$, and $\mathbf{q}^*_n$. At the beginning of the first period the manufacturer sets b^*_1 and w^*_1, then the retailer orders $\hat{q}_1 = q^*_1$. But the retailer postpones the announcement of the retail price $\hat{r}_1$ until after she observes $\hat{\epsilon}_1$. In sections 5.1 we solve the equilibrium problems for each player to obtain the optimal post-observation decision variables at an arbitrary period k.

Furthermore, since in the price-postponement scenario the entire demand is not necessarily addressed by the retailer, for the sake of generality we must also consider a (possibly time-dependent) salvage price for the retailer, and a buy-back contract between the two agents.

Similar to the open-loop equilibrium settings, in a retail-price postponement scenario, the manufacturer who is the leader imposes the following constraint on the buy-back price to ensure a non-negative profit at each period.

$$0 \leq b_k < w_k - c_{m_k} \tag{49}$$

While in the ex-ante analysis of the no-postponement equilibria states, we used the dynamic programming method known as *backward induction*, here in the ex-post analysis of price-postponement scenario we use a forward induction approach. Thereby, we incorporate the newly-revealed information in the form of feedback signals into the decision-making process. This is due to the fact that the retailer now changes future demand by her postponement.

5.1 Post-observation Bilevel Optimization

In our analysis of the retail price-postponement scenario, we divide the decision-making process into two steps. First, at the beginning of each period k, both the retailer and the manufacturer solve the expected profit optimization (equilibrium) problem in a Stackelberg framework within the time interval $\{k, \cdots, n\}$. The manufacturer then declares the equilibrium wholesale price and offers a (possibly zero) buy-back price, then the retailer submits her order quantity to the manufacturer. However, the retailer does not declare her retail price to the market. Instead, she postpones doing so until after she observes demand uncertainty.

In the second step, having observed $\hat{\epsilon}_k$, the retailer incorporates this new information and solves the equilibrium problem anew while considering the manufacturer's response for the next periods. That is, after observing $\hat{\epsilon}_k$ the retailer tries to find optimal retail prices within $\{k, \cdots, n\}$ while being subject to the optimality of the wholesale prices within $\{k+1, \cdots, n\}$. The equilibrium solution will provide the retailer with her post-observation optimal retail price vector $[\hat{r}_k, \cdots, \hat{r}_n]$. Then she declares the first element of her newly found optimal price vector, $\hat{r}_k$, to the market.

We begin the analysis of the equilibrium problem from the first period and using forward induction reasoning delineate a general optimization procedure for all periods. At the first step in the first period, both the retailer and the manufacturer solve the equilibrium problem aimed at maximizing their own respective expected holistic (throughout entire time interval between periods 1 and n) profits while subject to the optimality of the other player's solution. Thus they obtain the results of the open-loop equilibrium, i.e. $\{\mathbf{r}_k^*, \mathbf{b}_k^*, \mathbf{q}_k^*, \mathbf{w}_k^*\}$. Therefore at $k = 1$ the manufacturer proceeds with declaring w_1^* and the retailer orders q_1^* items. However, instead of declaring r_1^* to the market, the retailer waits for the uncertainty of demand, ϵ_1 to be resolved. In the second step and after observing $\hat{\epsilon}_1$, the retailer (and the manufacturer) solve the following equilibrium problem to obtain the optimal retail prices.

$$\max_{\mathbf{r}_n} \Pi^r$$

$$\max_{\{(b_2,w_2),\cdots,(b_n,w_n)\}} J_2^m$$

where

$$\begin{aligned} \Pi^r &= \pi_1^r(r_1, b_1^*, w_1^*, q_1^*) + \cdots + \alpha_k \Phi_k(\mathbf{r}_{k-1})\,\overline{\pi}_k^r(r_k, b_k, w_k, q_k) \\ &\quad + \cdots + \alpha_n \Phi_n(\mathbf{r}_{n-1})\,\overline{\pi}_n^r(r_n, b_n, w_n, q_n) = \pi_1^r + J_2^r \\ J_2^m &= \alpha_2 \Phi_2\big(\hat{r}_1(w_1)\big)\,\overline{\pi}_2^m(w_2) + \cdots + \alpha_n \Phi_n\big(\hat{r}_1(w_1), \cdots, \hat{r}_{n-1}(w_{n-1})\big)\,\overline{\pi}_n^m(w_n) \\ \text{and } \pi_1^r &= (r_1 - s_1 - b_1)\min \overbrace{\Big(\mu_1(r_1) + \sigma_1(r_1)\hat{\epsilon}_1\Big)}^{D_1(r_1)} + (s_1 + b_1^* - c_{r_1} - w_1^*)q_1^* \end{aligned} \tag{50}$$

Note that the only difference between the retailer's problem expression in (50) and the one in (34) is in the first term, where the expected value of the profit in the first period $\overline{\pi}_1^r$ is replaced by the real profit π_1^r. Thus the retailer, having observed $\hat{\epsilon}_1$, tries to find the vector of optimal retailer prices $\hat{\mathbf{r}}_n$ to optimize the sum of her real profit at the first period π_1^r and the expected (discounted) profits in the future J_2^r.

To solve (50) we use the backward induction reasoning again. Starting with the retailer's problem in the last period n we observe that in order to obtain $\hat{r}_n$ from (50) the retailer has to solve (33) once again. This means that $\hat{r}_n(w_n)$ equals $r_n^*(w_n)$ which was obtained in the pre-observation optimization. In general, going backward in time from period n to 2, the retailer will face the exact same optimization problems as the ones in the pre-observation analysis, i.e. $\hat{r}_k(w_k) = r_k^*(w_k)$, $k \in \{2, \cdots, n\}$. However, when the backward induction reaches the first period, it will face the only term in the objective function which is different from the corresponding one in (34), i.e. π_1^r. Thus, in general the optimal $\hat{r}_1$ is different from r_1^*.

$$\max_{r_1} \Pi^r = \pi_1^r(r_1) + \phi_2(r_1) \times \underbrace{\Big[\frac{\alpha_2}{\alpha_1}\overline{\pi}_2^r(r_2^*) + \cdots + \frac{\alpha_n}{\alpha_k}\overline{\pi}_n^r(r_n^*)\prod_{i=3}^{n}\phi_i(r_{i-1}^*)\Big]}_{\mathcal{F}_1^r \text{ future expected profit, given (obtained from pre-observation analysis)}} \tag{51}$$

Therefore the vector of optimal decision variables for the retailer after observing $\hat{\epsilon}_1$ is $[\hat{r}_1, r_2^*, \cdots, r_n^*]$ where $\hat{r}_1$ is obtained from (51) and r_k^*s ($k = 2\cdots n$) are equal to the ones obtained from the pre-observation optimization problems.

Now we proceed to the manufacturer's part of the equilibrium (50), considering the effect of the new retail pricing scheme on future (time interval $\{2, \cdots, n\}$) demand.

$$\max_{\{(b_2,w_2),\cdots,(b_n,w_n)\}} J_2^m = \max_{\{(b_2,w_2),\cdots,(b_n,w_n)\}} \Big[\alpha_2 \Phi_2(\hat{r}_1)\, \overline{\pi}_2^m(b_2, w_2) + \cdots$$
$$+\alpha_n \Phi_n(\hat{r}_1, r_2^*, \cdots, r_{n-1}^*)\, \overline{\pi}_n^m(b_n, w_n)\Big] \tag{52}$$
$$\text{where } \hat{r}_1 = \hat{r}_1(b_1^*, w_1^*), r_k^* = r_k^*(b_k, w_k) \quad k \in \{2, \cdots, n\}$$

where each $\overline{\pi}_k^m$ is calculated from (27) and (28).

Analogously, observing that each of the terms b_n and w_n appear only in the profit expression for the final period $\overline{\pi}_n^m$, we start the backward induction process from the nth period.

$$\max_{b_n,w_n} J_2^m \equiv \max_{b_n,w_n} \overline{\pi}_n^m \tag{53}$$

But this problem has already been solved in the open-loop analysis and it will yield the same optimal decision variable as before, i.e. b_n^* and w_n^*. Going backward in time, in general at each period $j \in \{2, \cdots, n\}$ the manufacturer faces the optimization problem (54). Note that for this arbitrary period j we have $\max_{b_j,w_j} J_2^m \equiv \max_{b_j,w_j} J_j^m$. This is due to the result of the Proposition 4.5 about the subgame perfection of the equilibrium aimed at maximization of the expected profits on time interval between 2 and n.

$$\max_{w_j} J_2^m \equiv \max_{b_j,w_j} J_j^m = \max_{b_j,w_j} \alpha_k \Phi_j(\tilde{\mathbf{r}}_{j-1}) \overbrace{\Big[\overline{\pi}_k^m(b_j, w_j) + \phi_{j+1}\big(r_k^*(b_j, w_j)\big)\mathcal{F}_j^m\Big]}^{\mathfrak{J}_k^m(b_j,w_j)}$$
$$\text{where } \tilde{\mathbf{r}}_{j-1} = \{\hat{r}_1(b_1^*, w_1^*), r_2^*(b_2, w_2), \cdots, r_{j-1}^*(b_{j-1}, w_{j-1})\} \tag{54}$$
$$\Phi_j(\tilde{\mathbf{r}}_{j-1}) = \phi_2(\hat{r}_1) \prod_{i=3}^{j} \phi_i(r_{i-1}^*)$$

$$\mathcal{F}_j^m = \frac{\alpha_{j+1}}{\alpha_j}\, \overline{\pi}_{j+1}^m(b_{j+1}, w_{j+1}) + \cdots + \frac{\alpha_n}{\alpha_j}\, \overline{\pi}_n^m(b_n, w_n) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*) \tag{55}$$
$$\mathcal{F}_n^m = 0$$

Again we have $\max_{b_j,w_j} J_j^m \equiv \max_{b_j,w_j} \mathfrak{J}_j^m$. Plus, when solving $\max_{b_j,w_j} \mathfrak{J}_j^m$ we observe that the choice of $\hat{r}_1$ does not affect $\mathcal{F}_j^m$. Therefore the results of $\max_{b_j,w_j} \mathfrak{J}_j^m$ will be exactly as equal to ones obtained by the open-loop solutions.

However, $\phi_2(\hat{r}_1)$ in (55) will scale $\mathfrak{J}_j^m$ differently from $\phi_2(r_1^*)$ in the corresponding open-loop equilibrium. Hence while the same b_j^*s and w_j^*s will come out of the two equilibrium problems, the expected values of the total profits will be different due to different memory elements.

After analyzing the two-step solution for the players in the period 1, we try to find a general solution procedure at a period k. The players arrive at period k with the memory function containing the already declared $\hat{\mathbf{r}}_{k-1}$. In the first step they have to solve the following bilevel optimization (Stackelberg equilibrium) problem.

$$\max_{r_k,\cdots,r_n} J_k^r \tag{56}$$

$$\max_{(b_k,w_k),\cdots,(b_n,w_n)} J_k^m \tag{57}$$

From Proposition 4.5 we know that the equilibrium aimed at maximization of the expected profits is subgame perfect. Hence, in the first step, each decision maker obtains a subset of her original open-loop equilibrium results; i.e. $[r_k^*,\cdots,r_n^*]$, $[b_k^*,\cdots,b_n^*]$, and $[w_k^*,\cdots,w_n^*]$. Thus, at the first step in period k, the manufacturer declares w^* and the retailer orders $\hat{q}_k = \Phi_k(\hat{\mathbf{r}}_{k-1})\left[\mu_k(r_k^*) + \sigma_k(r_k^*)\, F_{\epsilon_k}^{-1}\left(\frac{r_k^*-w_k^*-c_{r_k}}{r_k^*-b_k^*-s_k}\right)\right]$.

At the second step, after the retailer observes $\hat{\epsilon}_k$ the following bilevel equation has to be solved.

$$\max_{r_k,\cdots,r_n} \alpha_k \Phi_k(\hat{\mathbf{r}}_{k-1})\pi_k^r(r_k) + J_{k+1}^r \qquad \text{over } k,\cdots,n \tag{58}$$

$$\max_{(b_{k+1},w_{k+1}),\cdots,(b_n,w_n)} J_{k+1}^m = \max_{w_{k+1},\cdots,w_n} \sum_{i=k+1}^{n} \alpha_i \Phi_i(\tilde{\mathbf{r}}_{i-1})\overline{\pi}_i^m(b_i,w_i) \qquad \text{over } k+1,\cdots,n$$

$$\text{where } \tilde{\mathbf{r}}_{i-1} = [\hat{\mathbf{r}}_{k-1}, r_k,\cdots,r_{i-1}] \tag{59}$$

Similarly, starting the backward induction from the final period, it is evident that from period n to $k+1$ the retailer will face the exact same optimization problems as the ones in the pre-observation analysis. The only term in the entire objective function which is different from its corresponding term in (34) is π_k^r (the real profit at k which has replaced its own expected value, $\overline{\pi}_k^r$). Therefore the retailer's optimization problem boils down to the following.

$$J_k^r = \alpha_k \underbrace{\Phi_k(\hat{\mathbf{r}}_{k-1})}_{\text{price history}} \overbrace{\left(\pi_k^r(r_k) + \phi_{k+1}(r_k)\underbrace{\left[\frac{\alpha_{k+1}}{\alpha_k}\overline{\pi}_{k+1}^r(r_{k+1}) + \cdots + \frac{\alpha_n}{\alpha_k}\overline{\pi}_n^r(r_n)\prod_{i=k+2}^{n}\phi_i(r_{i-1})\right]}_{\mathcal{F}_k^r=\text{expected (future) values, given (obtained from pre-observation problem)}}\right)}^{\mathfrak{J}_k^r(r_k)}$$

$$\max_{r_k} J_k^r \equiv \max_{r_k} \mathfrak{J}_k^r \tag{60}$$

Note that by the time the backward induction process reaches the kth period $\mathcal{F}_k^r$ in (60), i.e. the future expected profit, is already calculated and is treated as a constant. Solving the single-variable optimization problem in (60) yields $\hat{r}_k$ while the rest of the optimal retail prices remain equal to those obtained in the pre-observation (open-loop) optimization problem. Thus at the second step in the kth period, the retailer obtains her optimal decision variables $[\hat{r}_k(b_k^*, w_k^*), r_{k+1}^*(b_{k+1}, w_{k+1}), \cdots, r_n^*(b_n, w_n)]$ as functions of the corresponding buy-back and manufacturing prices.

In order to obtain numerical values for $\hat{r}_k(b_k, w_k)$ and the rest of the optimal retail prices, the retailer has to solve the manufacturer's problem of finding optimal responses for the next periods.

$$\max_{(b_{k+1},w_{k+1}),\cdots,(b_n,w_n)} J_{k+1}^m$$

$$\begin{aligned} J_{k+1}^m &= \alpha_{k+1}\Phi_{k+1}(\hat{\mathbf{r}}_k)\overline{\pi}_{k+1}^m + \cdots + \alpha_n\Phi_k(\hat{\mathbf{r}}_k)\prod_{i=k+2}^{n}\phi_i(r_{i-1}^*)\overline{\pi}_n^m \\ &= \alpha_{k+1}\Phi_{k+1}(\hat{\mathbf{r}}_k)\Big[\overline{\pi}_{k+1}^m + \cdots + \frac{\alpha_n}{\alpha_{k+1}}\prod_{i=k+2}^{n}\phi_i(r_{i-1}^*)\overline{\pi}_n^m\Big] \end{aligned} \tag{61}$$

The numerical results for optimal buy-back and wholesale prices are obtained using the recursive solution procedure delineated below.

$$\begin{aligned} J_j^m &= \alpha_j\Phi_j(\tilde{\mathbf{r}}_{j-1})\Big[\overline{\pi}_j^m(b_j, w_j) + \phi_{j+1}(r_j^*)\mathcal{F}_j^m\Big] \quad k+1 \le j \le n \\ \tilde{\mathbf{r}}_{j-1} &= (\hat{\mathbf{r}}_k, r_{k+1}^* \cdots, r_{j-1}^*) \Rightarrow \Phi_j(\tilde{\mathbf{r}}_{j-1}) = \prod_{i=1}^{k+1}\phi_i(\hat{r}_{i-1})\prod_{i=k+2}^{j}\phi_i(r_{i-1}^*) \end{aligned} \tag{62}$$

$$\begin{aligned} \mathcal{F}_n^m &= 0 \\ \mathcal{F}_j^m &= \frac{\alpha_{j+1}}{\alpha_j}\overline{\pi}_{j+1}^m(b_{j+1}, w_{j+1}) + \cdots + \frac{\alpha_n}{\alpha_j}\overline{\pi}_n^m(b_n, w_n)\prod_{i=k+2}^{n}\phi_i(r_{i-1}^*) \end{aligned} \tag{63}$$

Using the result of Proposition 4.5, it is straightforward to see that after declaration of $\hat{r}_k$ at period k, and when solving manufacturer's optimization problem for the time interval $\{k+1, \cdots, n\}$, the backward induction process will yield the same $\{(b_{k+1}^*, w_{k+1}^*), \cdots, (b_n^*, w_n^*)\}$ as those obtained in the open-loop equilibrium problem. However, the manufacturer's expected profit will be different from the results of the open-loop solutions. This is due to the scaling factor $\Phi_k(\hat{\mathbf{r}}_{k-1})$. The results of this section are expressed in the following theorem.

Theorem 5.1.

In a retail price postponement scenario where the retailer and the manufacturer face the uncertain demand described in Theorem 4.4, the retailer at each period k postpones the declaration of her price until after observing demand uncertainty ϵ_k.

Assuming that there exists an equilibrium state $[\boldsymbol{r}_k^*, \boldsymbol{w}_k^*]$ for the open-loop problem described *in Theorem 4.4, if the following objective function has a global maximum, $\hat{r}_k$,*

$$\mathfrak{J}^r_k(r_k) = \pi^r_k(r_k) + \phi_{k+1}(r_k)\mathcal{F}^r_k$$
$$\textit{where } \mathcal{F}^r_k = \frac{\alpha_{k+1}}{\alpha_k}\,\overline{\pi}^r_{k+1}(r^*_{k+1}) + \cdots + \frac{\alpha_n}{\alpha_k}\,\overline{\pi}^r_n(r^*_n)\prod_{i=k+2}^{n}\phi_i(r^*_{i-1}) \quad \textit{given value}$$

then the closed-loop problem of price postponement has an equilibrium with the following optimal decision variables.

$$\begin{aligned}
\hat{\boldsymbol{r}}_n &= [\hat{r}_1, \cdots, \hat{r}_n] \\
\hat{\boldsymbol{b}}_n &= [\hat{b}_1, \cdots, \hat{b}_n] \\
\boldsymbol{w}^*_n &= [w^*_1, \cdots, w^*_n] \\
\hat{\boldsymbol{q}}_n &= [\hat{q}_1, \cdots, \hat{q}_n] \\
\textit{where } \hat{q}_k &= \Phi_k(\hat{\boldsymbol{r}}_{k-1})\left[\mu_k(r^*_k) + \sigma_k(r^*_k)\,F^{-1}_{\epsilon_k}\left(\frac{r^*_k - w^*_k - c_{r_k}}{r^*_k - b^*_k - s_k}\right)\right] \\
&= \frac{\Phi_k(\hat{\boldsymbol{r}}_{k-1})}{\Phi_k(\boldsymbol{r}^*_{k-1})} q^*_k
\end{aligned}$$

5.2 Price Postponement and Retailer's Profit

At the end of period n, the set of post-observation optimal retail prices, $[\hat{\mathbf{r}}_n]$ is the result of the optimization problem $\max_{\mathbf{r}_n} \Pi^r$ considering the real values of π^r_ks. Whereas the set of pre-observation optimal retail prices, $[\mathbf{r}^*_n]$ is the result of optimization $\max_{\mathbf{r}_n} \overline{\Pi}^r$ considering the expected values of the profits at each period $\overline{\pi}^r_k$s. Thus it is trivial that in a hypothetical n-period scenario where two retailers face the same ϵ_k at each period k, the one that postpones the declaration of her prices ($\hat{r}_k$s) until after observation of each ϵ_k gains higher profit compared to the retailer who adheres to sub-optimal r^*_ks. In other words, in a price-postponement scenario, because $\hat{r}_k$s are the results of the real profit optimizations, any other set of decision variables (including the set of r^*_ks) will be sub-optimal. Therefore we have $\Pi^r_{CL} \geq \Pi^r_{OL}$ where Π^r is the total discounted real profit gained through n periods.

5.3 Price Postponement and Manufacturer's Profit

In general, in the closed-loop optimization scenario, at each period k the retailer enjoys the strategic means to find an optimal $\hat{r}_k$ maximizing the sum of her current profit and expected future profits. Whereas the manufacturer always faces the structurally identical (though differently scaled) expected profit optimization.

At each period k after observing $\hat{\epsilon}_k$, the retailer deviates from the previously obtained equilibrium price r^*_k by declaring $\hat{r}_k$ instead. Due to the structure of the memory functions, this new pricing scheme will affect the future demand and thereby the future earnings for both the retailer and the manufacturer. The retailer's optimization problem as generalized in Theorem 5.1 is tailored such that an optimal $\hat{r}_k$ will maximize the sum of the current real profit and expected future profits. Thus after declaring each $\hat{r}_k$, it is the manufacturer's turn to modify her own optimal pricing scheme for the future considering the effects of the retail prices on future demand and expected earnings.

Comparing to the non-postponement solutions, the retailer always benefits from postponing her retail price. Whereas the manufacturer's may either benefit or lose potential profit compared to the non-postponement case, depending on the structure of demand mean and variance, and different realizations of the uncertain demand.

6 Numerical Implementation of the Model

In this section, we illustrate the theoretical results and implement the solution algorithms discussed in Sections 4 and 5. In the examples analyzed in this section, we use Cobb-Douglas demand functions.

Note that when deriving the results in those sections, we did not made any assumption on the underlying distribution of ϵ except that its inverse commutative distribution function, F_ϵ^{-1} must exist. In the examples we use a truncated and re-normalized normal distribution function for ϵs to ensure that the negative noise terms do not cause the entire demand to become negative.

It should be noted that the main purpose of this section is merely to offer practical advice on how our theory can be implemented in some special cases. The examples we present are particularly simple, and hardly reflect the potential of this framework. To take full advantage of the model, one should try to vary scaling factors and functional forms in a systematic way. This makes it is possible to model a wide range of economic contexts. A full discussion of the model and all the variations it can cover, is, however, beyond the scope of this paper.

6.1 The Open-loop Equilibria Solutions

Following the order by which the scenarios were presented, we begin by providing examples of the open-loop equilibria wherein optimization takes place based on the expected values of discounted profits within a (possibly infinite) time horizon.

Multiplicative memory functions scale the future demand such that an increase in the current retail price decreases the future demand. Thus the memory function at period $k+1$ which will scale the future demand D_{k+1} is monotonically decreasing with respect to the retail price at all previous periods.

$$\forall k \in \{1, \cdots n\} \quad \frac{\partial \Phi_{k+1}(r_1, \cdots, r_n)}{\partial r_k} = \frac{\partial \prod_{k=1}^{n} \phi_{k+1}(r_k)}{\partial r_k} < 0 \tag{64}$$

This means that the memory element at $k+1$ must be monotonically decreasing with respect to r_k.

$$\frac{\partial \phi_{k+1}(r_k)}{\partial r_k} < 0 \tag{65}$$

Here, for illustration purpose, we use the following functional structure for memory elements

$$\phi_{k+1}(r_k) = 1 + \gamma_k(\kappa_k - r_k) \tag{66}$$

where $\gamma_k \geq 0$, the memory strength factor at period k, is a given parameter. The given parameter $\kappa_k \geq 0$ can be interpreted as a price cap; i.e., any initial price above κ_k reduces demand, whereas demand is more likely to increase if $r_k < \kappa_k$. If the scaling factor is negative, maxima are turned into minima. Hence, if $\phi_{k+1}(r_k) \leq 0$, the optimal order q_{k+1} is zero. To avoid this problem, we consider

$$\phi_{k+1}(r_k) = [1 + \gamma_k(\kappa_k - r_k)]^+. \tag{67}$$

Example 3

For this example, we consider the following scaled demand function.

$$\begin{aligned} d_k(r_k) &= \mu_k(r,k) + \sigma_k(r,k)\epsilon_k \\ \mu_k(r,k) &= \frac{10000e^{-0.05(k-1)}}{r_k^{4+0.1(k-1)}} \quad \sigma_k(r,k) = 0.2\mu_k(r,k) + \frac{10}{r_k^4} \end{aligned} \tag{68}$$

Note that in the expression for demand, the absolute value of price elasticity of demand increases as k increases. Thus, as time goes by, the market gradually becomes more sensitive to price increases. Moreover, due to competition, we have considered demand to be monotonically decreasing with respect to time.

For simplicity, we set $\gamma_k = 0.02$, $\kappa_k = 5$, $\alpha_k = 0.95$, $c_{m_k} = 5$, $c_{r_k} = 0$, and $s_k = 4$ for all ks. The number of periods, n, is set to be 25.

The three pricing decision variables (r^*, b^*, w^*) in equilibrium state at each period are illustrated in Figure 3. The corresponding expected profits for the whole duration of time between period 1 and n are $\overline{\Pi}^m = 9.78$ and $\overline{\Pi}^r = 11.04$. We observe that the optimal buy-back price for the manufacturer is non-zero at all times. The lowest buy-back price is 0.05 offered at $k = n = 25$ and the highest buy-back price is 1.39 offered at $k = 1$.

Now, we solve the closed-loop coordination problem in which the retailer is allowed to postpone her retail price at each period. That is, we simulate scenarios in which a channel who has already solved the open-loop solutions, faces different values of price-dependent

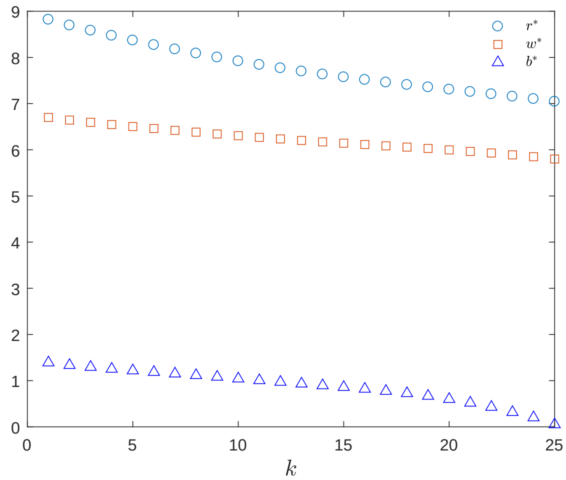

Figure 3: Three pricing decision variables in equilibrium state at each period (open-loop solutions)

stochastic demand. In doing so, we gerenerate a set of ϵ_ks representing demand uncertainty at each period: $\mathcal{E} = \{\epsilon_1, \cdots, \epsilon_n\}$ using the given f_{ϵ_k} and $F^{-1}_{\epsilon_k}$. The retailer, after observing each stochastic ϵ_k sometime within the period k, has to re-solve the coordination problem and find her $\hat{r}_k$.

In a postponement scenario, in order to find out the effect of postponement on the channel members' profits, we compare two sets of results for each channel member and for the whole channel. The first set, which we denote by the subscript NP (standing for no postponement) are the profits obtained in the face $\mathcal{E}$ if the channel members adhere to the decision variables obtained by open-loop solution $\{\mathbf{r}^*_n, \mathbf{q}^*_n, \mathbf{w}^*_n, \mathbf{b}^*_n\}$. The second group of the results, denoted by the subscript P are the profits of the channel in the face of the same $\mathcal{E}$ if the retailer postpones her retail price. Thus the second group represent the profits obtained by the set of optimal variables $\{\hat{\mathbf{r}}_n, \hat{\mathbf{q}}_n, \mathbf{w}^*_n, \mathbf{b}^*_n\}$. The superscripts r, m, c denote retailer, manufacturer, and channel, respectively.

$$\Pi^r_{NP} = 11.87 \qquad \Pi^m_{NP} = 10.11 \qquad \Pi^c_{NP} = 21.99$$
$$\Pi^r_{P} = 13.27 \qquad \Pi^m_{P} = 10.66 \qquad \Pi^c_{P} = 23.94$$

We observe that price postponement has been beneficial to each channel member and thus to the whole channel. Figure 4 illustrates the sets of $\hat{r}_k$s corresponding to the specific $\mathcal{E}$ generated for this example and r^*_ks at different periods.

Example 4

In this example, we consider a slightly different channel facing a different demand struc-

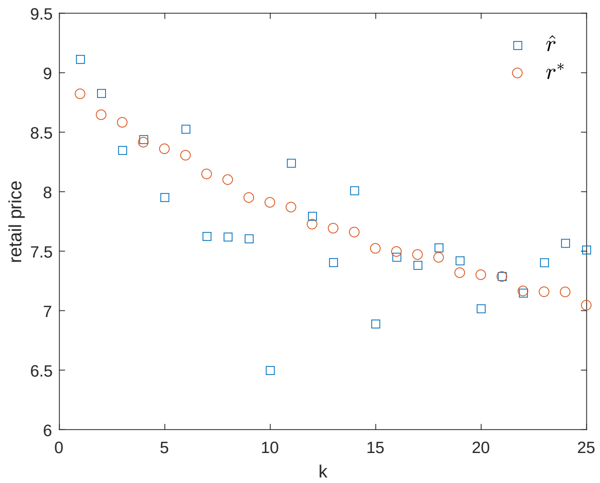

Figure 4: Open-loop and closed-loop optimal retail prices

ture.

$$d_k(r_k) = \mu_k(r,k) + \sigma_k(r,k)\epsilon_k$$
$$\mu_k(r,k) = \frac{10000e^{-0.1(k-1)}}{r_k^{4+0.1(k-1)}} \quad \sigma_k(r,k) = \frac{1}{r_k^3} \tag{69}$$

In this example, too, the absolute value of price elasticity of demand increases with time. The model parameters differing from those in Example 3 are $c_{m_k} = 2$ and $s_k = 1$ for all ks. The number of periods, n, is set to be 25.

The three pricing decision variables (r^*, b^*, w^*) in equilibrium state at each period are illustrated in Figure 5. The corresponding expected profits for the whole duration of time between period 1 and n are $\overline{\Pi}^m = 233.57$ and $\overline{\Pi}^r = 274.36$. We observe that all buy-back prices are found to be zero at equilibrium states.

Next, we generate a vector $\mathcal{E}$. Figure 6 illustrates the two sets of retail prices $\mathbf{r}_n^*$ and $\hat{\mathbf{r}}_n$, where the latter is the retailer's optimal response to demand realization due to $\mathcal{E}$. Analogous to the comparison in Example 3, we compare the results of the open-loop (no postponement) and closed-loop coordination solutions.

$$\Pi_{NP}^r = 273.98 \qquad \Pi_{NP}^m = 233.57 \qquad \Pi_{NP}^c = 507.56$$
$$\Pi_P^r = 279.82 \qquad \Pi_P^m = 233.48 \qquad \Pi_P^c = 513.30$$

In both Examples 3 and 4, the retailer benefits from postponement. In Example 3, the manufacturer also benefits from retail price postponement, while in Example 4, the manufacturer is slightly worse off due to postponement. All of these observations are consistent with the result of the analysis in Sections 5.2 and 5.3. The aggregate channel profit in both examples has increased due to price postponement.

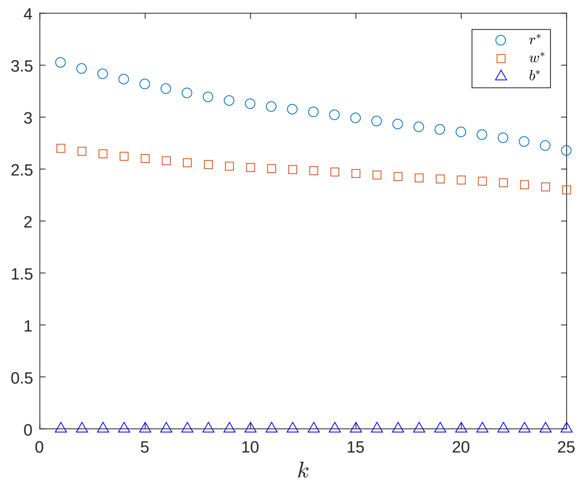

Figure 5: Three pricing decision variables in equilibrium state at each period (open-loop solutions)

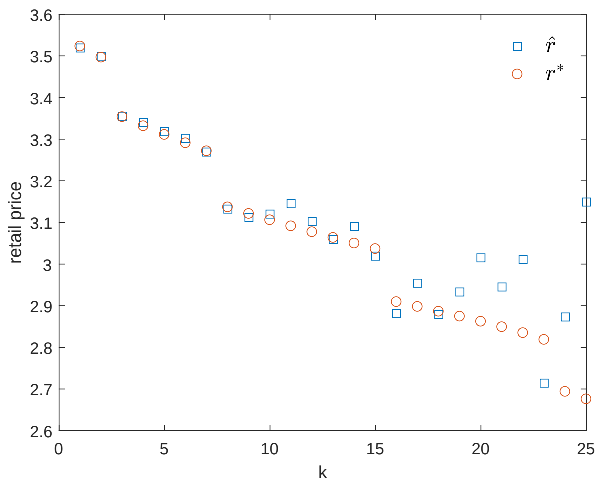

Figure 6: Open-loop and closed-loop optimal retail prices

7 Concluding Remarks

The main objective of this paper has been to offer an analytic tool to solve the general problem of channel coordination with optimal buy-back contracts at different times in a multi-period setting when addressing a time-varying and uncertain demand. We have also embedded an important feature in the model which allows the downstream supplier to postpone (and enhance) her pricing decision until demand uncertainty is observed.

Through these two approaches, first, the upstream supplier also takes part of risk caused by demand uncertainty. Moreover, with price postponement, the downstream supplier reduces the level of the risk stemming from uncertainty at each period (she still has to order before uncertainty is resolved, so the uncertainty risk is not entirely eliminated).

In addition, in the theoretical results presented in theorems 4.4 and 5.1, all model parameters and variables are considered as time-dependent. The model also allows for functional forms such as ϕ_ks, f_{ϵ_k}s, μ_ks, and σ_ks to vary with time. This feature also adds up to the level of non-autonomy our solution method can cover.

An interesting next step would be to extend our construction to include inventory management. Such a model can cover channel coordination in cases where the channel is to address demand for a non-perishable good which can be stored at the end of the current period and supplied in the next. Also, multiple downstream suppliers can be added to the channel.

Quantity discount contracts in which the upstream supplier offers the downstream channel partner(s) with a discount proportional to their order quantities will also increase the level of the non-linearity of the original problem. The nestedness of the ensuing equilibrium problems may need memory structures different from those offered in Section 6.1 to decouple.

8 Appendix 1: Proof of Proposition 4.1

In order to obtain the expected value of the retailer's profit, we need to calculate $\mathrm{E}\big[\min(D,q)\big]$. Given f_ϵ, F_ϵ, and $\underline{\epsilon}$ we define and calculate the expected sales, $\mathcal{S}$, as follows.

$$\begin{aligned}\mathcal{S}(q) &:= \mathrm{E}\big[\min(D,q)\big] = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \min(\mu+\sigma t, q)\, f_\epsilon(t)\, dt \\ &= \int_{\underline{\epsilon}}^{\frac{q-\mu}{\sigma}} (\mu+\sigma t)\, f_\epsilon(t)\, dt + \int_{\frac{q-\mu}{\sigma}}^{\overline{\epsilon}} q\, f_\epsilon(t)\, dt \\ &= q - (q-\mu) F_\epsilon\Big(\frac{q-\mu}{\sigma}\Big) + \sigma \int_{\underline{\epsilon}}^{\frac{q-\mu}{\sigma}} t f_\epsilon(t) dt\end{aligned} \tag{70}$$

$$\frac{\partial\, \mathcal{S}(q)}{\partial q} = 1 - F_\epsilon\Big(\frac{q-\mu}{\sigma}\Big) \tag{71}$$

From (8) and (70), we obtain the expected value of the retailer's profit $\overline{\pi}^r$.

$$\overline{\pi}^r(r,w,q) := \mathrm{E}[\pi^r(r,w,q)] = (r-s-b)\,\mathcal{S}(q) + (b+s-c_r-w)\, q \tag{72}$$

Following the outline in (10), now the retailer can calculate her optimal order quantity, q^* as a function of r and w.

$$\frac{\partial \overline{\pi}^r}{\partial q} = (r-s-b)\left(1 - F_\epsilon\Big(\frac{q-\mu}{\sigma}\Big)\right) + (b+s-c_r-w) = 0 \tag{73}$$

From the expressions in (71) and (72) it is readily observable that $\mathrm{E}[\pi^r(r, w, q)]$ is convex with respect to q; therefore, solving (73) yields $q^*(r, w)$ as the argmax of the retailer's expected profit.

$$q^*(r, w) = \mu(r) + \sigma(r)\, F_\epsilon^{-1}\left(\frac{r - w - c_r}{r - s - b}\right) \tag{74}$$

Substituting (74) in (70) and the result in (72), we obtain the following.

$$\begin{aligned} \overline{\pi}^r(r, w) &= (r - w - c_r)\, \mu(r) + (r - s - b)\, \sigma(r) \int_{\underline{\epsilon}}^{z} t f_\epsilon(t) dt \\ \text{where} \quad z(r, w) &= F_\epsilon^{-1}\left(\frac{r - w - c_r}{r - s - b}\right) \end{aligned} \tag{75}$$

According to the procedure outlined in (10) a numerical solution to $\max_r \overline{\pi}^r(r, b, w)$ in (75) yields $r^*(b, w)$ which is in turn substituted in the expression for the manufacturer's expected profit (76).

Using the expression for π^m given in (9), we have:

$$\begin{aligned} \overline{\pi}^m(w) &= \mu\left(r^*(w)\right)\left(w - c_m\right) + \sigma\left(r^*(w)\right)\Big[(z^*(w)\Big(w - c_m - \frac{r^* - w - c_r}{r^* - s - b}\Big) \\ &+ b \int_{\underline{\epsilon}}^{z^*} t f_\epsilon(t) dt\Big] \\ &\text{where} \quad z^*(w) = F_\epsilon^{-1}\left(\frac{r^* - w - c_r}{r^* - s - b}\right) \end{aligned} \tag{76}$$

References

[1] Cachon G. P., 2003. Supply chain coordination with contracts. In: *The Handbook of Operations Research and Management Science: Supply Chain Management: Design, Coordination and Operation.* Chapter 6. A. G. de Kok and S. C. Graves (eds.), Elsevier, Amsterdam, pp. 229-340.

[2] Cheng, T.E., Li, J., Wan, C.J. and Wang, S., 2010. *Postponement strategies in supply chain management* (Vol. 143). New York: Springer.

[3] Granot, D. and Yin, S., 2008. Price and order postponement in a decentralized newsvendor model with multiplicative and price-dependent demand. *Operations Research*, 56(1), pp.121-139.

[4] Gümüş, M., Ray, S. and Yin, S., 2013. Returns policies between channel partners for durable products. *Marketing Science*, 32(4), pp.622-643.

[5] Kim, G., Wu, K. and Huang, E., 2015. Optimal inventory control in a multi-period newsvendor problem with non-stationary demand. *Advanced Engineering Informatics*, 29(1), pp.139-145.

[6] Li, Y., Wei, C. and Cai, X., 2012. Optimal pricing and order policies with B2B product returns for fashion products. *International Journal of Production Economics*, 135(2), pp.637-646.

[7] Nan, Z. and Fang, Z.Z., 2016. A Literature Review on the Supply Chain Coordination of the Buyback Contract. *3rd International Conference on Education, Management, Arts, Economics and Social Science (ICEMAESS 2015)*

[8] Pasternack, B.A., 1985. Optimal pricing and return policies for perishable commodities. *Marketing science*, 4(2), pp.166-176.

[9] Van Mieghem, J.A. and Dada, M., 1999. Price versus production postponement: Capacity and competition. *Management Science*, 45(12), pp.1639-1649.

[10] Wei, J. and Tang, J., 2013. Analysis on the Stackelberg game model and risk sharing based on buy back contract. *Journal of Theoretical & Applied Information Technology*, 48(2).

[11] Xu, Y. and Bisi, A., 2011. Price postponement in a newsvendor model with wholesale price-only contracts. *Purdue CIBER Working Paper*, Purdue University

[12] Yao, Z., Leung, S.C. and Lai, K.K., 2008. Analysis of the impact of price-sensitivity factors on the returns policy in coordinating supply chain. *European Journal of Operational Research*, 187(1), pp.275-282.

[13] Young, L. 1978 . Price, inventory and the structure of uncertain demand. *New Zealand Operations Research* 6, No.2, 157-177.

NHH

NORGES HANDELSHØYSKOLE
Norwegian School of Economics

Helleveien 30
NO-5045 Bergen
Norway

T +47 55 95 90 00
E nhh.postmottak@nhh.no
W www.nhh.no