Balanced scorecards: a relational contract approach

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Balanced scorecards: a relational contract approach*

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Abstract

Reward systems based on balanced scorecards typically connect pay to an index, i.e. a weighted sum of multiple performance measures. However, there is no formal incentive model that actually describe this kind of index contracts as an optimal solution. In this paper, we show that an index contract may indeed be optimal if performance measures are non-verifiable so that the contracting parties must rely on self-enforcement. Under standard assumptions, the optimal self-enforcing (relational) contract between a principal and a multitasking agent is an index contract where the agent gets a bonus if a weighted sum of performance outcomes on the various tasks (the index) exceeds a hurdle. For a parametric (multinormal) specification, the efficiency of the contract improves with higher precision of the index measure, since this strengthens incentives. Correlations between measurements may for this reason be beneficial. For a similar reason, the principal may also want to include verifiable performance measures in the relational index contract in order to improve incentives.

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1 Introduction

Very few jobs can be measured along one single dimension; employees usually multitask. This creates challenges for incentive providers: If the firm only rewards a subset of dimensions or tasks, agents will have incentives to exert efforts only on those tasks that are rewarded, and ignore others. A solution for the firm is to add more metrics to the compensation scheme, but this usually implies some form of measurement problem, leading either to more noise or distortions, or to the use of non-verifiable (subjective) performance measures.

The latter is often implemented by the use of a balanced scorecard (BSC). Kaplan and Norton's (1992, 1996) highly influential concept began with a premise that exclusive reliance on verifiable financial performance measures was not sufficient, as it could distort behavior and promote effort that is not compatible with long-term value creation. Their main ideas were indebted to the canonical multitasking models of Holmström and Milgrom (1991) and Baker (1992). However, their approach was more practical, guiding firms in how to design performance measurement systems that focus not only on short-term financial objectives, but also on long-term strategic goals (Kaplan and Norton, 2001).

While measuring performance is one issue, the question of how to reward performance is a different one. As noted by Budde (2007), there is a general understanding that efficient incentives must be based on multiple performance measures. Still, the implementation is a matter of controversy. Reward systems based on BSC typically connect pay to an index, i.e. a weighted sum of multiple performance measures. However, there is no formal incentive model that actually derive this kind of index contracts as an optimal solution. In fact, Kaplan and Norton (1996) were sceptical to compensation formulas that calculated incentive compensation directly via a sum of weighted metrics. Rather they proposed to establish different bonuses for a whole set of critical performance measures, more in line with the original ideas of Holmström and Milgrom (1991) and Feltham and Xie (1994).

Despite the large literature following the introduction of BSC (see Hoque, 2014, for a review), and the massive use of scorecards in practice, the index contracts that BSC-firms often prescribe, lacks a formal contract theoretic justification.[1] This paper aims to fill the gap. Our starting point is that the performance measures are non-verifiable. This means that the incentive contract cannot be enforced by a third party and thus needs to be self-enforcing - or what is commonly termed "relational". In the now large literature on self-enforcing relational contracts, relatively few papers have considered relational contracts with multitasking agents (prominent papers include Baker, Gibbons and Murphy, 2002; Budde, 2007, Schottner, 2008; Mukerjee and Vasconcelos, 2011; and Ishihara, 2016). We on the one hand generalize this literature in some dimensions (to an arbitrary number of tasks with stochastic measurements that are possibly correlated and/or distorted), and on the other hand invoke assumptions (normally distributed measurements) that make the model quite tractable.[2]

We first show that the optimal relational contract between a principal and a multitasking agent turns out to be an index contract, or what one may call a balanced scorecard. That is, the agent gets a bonus if a weighted sum of performance outcomes on the various tasks (an index) exceeds a hurdle. This in contrast to the optimal contract in e.g. Holmström and Milgrom (1991), where the agent gets a bonus on each task. The important difference from Holmström and Milgrom is that we consider a relational contracting setting where the size of the bonus is limited by the principal's temptation to renege (rather than risk considerations). In such a setting the marginal incentives to exert effort on each task is higher with index contracts than with bonuses awarded on each task.

The following example yields some intuition for the index result in a very simple setting. Consider an agent working on two tasks with outcomes that are, for each task, either a success or a failure for the principal. The agent controls the probability (a_i) of success on each task, and the outcomes are

[1] According to Hoque (2014), among the more than 100 papers published on BCS theory, only a handful have used principal agent theory to analyze BSC. See also Hesford et al (2009) for a review.

[2] Our paper is indebted to the seminal literature on relational contracts. The concept of relational contracts was first defined and explored by legal sholars (Macaulay, 1963, Macneil, 1978), while the formal literature started with Klein and Leffler (1981). MacLeod and Malcomson (1989) provides a general treatment of the symmetric information case, while Levin (2003) generalizes the case of asymmetric information. The relevance of the relational contract approach to management accounting and performance measurement is discussed in Glover (2012) and Baldenius et al. (2016).

(for algebraic simplicity here) stochastically independent. Suppose, and this is critical, that there is an upper limit (B) on total bonus payments, and compare two schemes: (i) a bonus with a hurdle (1 success) on each task, and (ii) a bonus based on an index that counts the number of successes.

Suppose the tasks are equally valuable for the principal, so she wants to treat them symmetrically. In Scheme 1, the bonus on each task can then at most be $\frac{1}{2}B$, yielding the agent expected revenue $\frac{1}{2}Ba_1 + \frac{1}{2}Ba_2$, and marginal revenue on each task $\frac{1}{2}B$. In Scheme 2, and with a hurdle set at 2 successes, the agent's expected income is Ba_1a_2, and his marginal revenue on task i is Ba_j. This exceeds the incentive in Scheme 1 if $a_j > \frac{1}{2}$. With a hurdle set at 1 success, we similarly find the agent's marginal revenue to be $B(1 - a_j)$, which exceeds the incentive in Scheme 1 if $a_j < \frac{1}{2}$. Scheme 2 can thus always be arranged so as to yield stronger incentives than Scheme 1. In other words, since there are upper bounds on the size of the bonuses that can be implemented in relational contracts, a bonus on each task puts more restrictions on the incentive problem than what is necessary. The index contract is more "flexible" and alleviates the problems caused by bonus limitations.

The performance measures within a scorecard may well be correlated. We point out that such correlations will affect the efficiency of the contract and we show, for a parametric (multinormal) specification, that the efficiency of the index contract depends on how correlations affect the precision of the overall scorecard measure. In particular, an index contract with non-negative weights on all relevant measures will work even better if the measures are negatively correlated. The reason is that negative correlation reduces the variance of the overall performance measure (the index) in such cases. This is beneficial in our setting not because a more precise measure reduces risk – since the agent is assumed to be risk neutral – but because it strengthens, for any given bonus level, the incentives for the agent to provide effort.[3]

We also consider the case where some measures are verifiable, and some are not. We show that the principal will include verifiable measures in the

[3]Similar effects are shown in Kvaløy and Olsen (2019), which analyzes relational contracts and correlated performances in a model with multiple agents, but single tasks.

relational index contract in order to strengthen incentives.[4] This resembles balanced scorecards seen in practice, which often include both verifiable measures such as sales or financial accounting data, and non-verifiable (subjective) measures, such as customer satisfaction, product quality, or other non-financial measures that are not subject to law enforcement (see e.g. Kaplan and Norton, 2001). By including a verifiable task in the relational contract, the variance of the performance index may be reduced, which again strengthens incentives. We also show that performance on the verifiable task is taken into the index as a benchmark, to which the other performances are compared. Moreover, the principal will still offer an explicit bonus contract on the verifiable task, but this bonus is generally affected by the optimal relational index contract.[5]

A paper that is closely related to ours is Budde (2007). It investigates the incentive effects of a balanced scorecard scheme under both formal (explicit) and relational contracts. First, in a setting with verifiable, but distorted, performance measures, it derives conditions under which a first-best allocation can be implemented by an explicit BSC-type of contract. The paper then extends the analysis to include non-verifiable measures and investigates when a relational contract can help to provide undistorted incentives. The paper is important, as it shows that BSC-types of contracts can provide undistorted incentives in settings with no noise and sufficient congruity/alignment between performance measures and the "true" value added.

In contrast to Budde, who takes the BSC-contract as given, we show that BSC-contracts can emerge as an optimal contract in a second-best world where noisy and potentially distorted measurements plus the limitations of self-enforcement preclude implementation of the first-best. The logic behind combining non-verifiable and verifiable measures in the relational contract

[4] Our analysis of this issue presumes short-term explicit (court enforced) contracts. Watson, Miller and Olsen (2020) presents a general theory for interactions between relatonal and court enforced contracts when the latter are long term and renegotiable, and show that optimal contracts are then non-stationary. Implications of this for the contracting problems considered in the current paper are left for future research.

[5] Our model thus complement the influential papers by Baker, Gibbons and Murphy (1994) and Schmidt and Schnitzer (1995) on the interaction between relational and explicit contracts. While their results are driven by differences in fallback options created by the explicit contracts, our results stem from correlation between the tasks and (or) misalignment between measurements and true values.

is also different in the model in this paper relative to Budde's, mainly since our model includes noise in the measurements. While Budde focuses on how non-verifiable measures can help remove distortions, we focus on how verifiable measures - used in relational contracts - can improve the precision of the BSC performance measure.

The rest of the paper is organized as follows: In section 2 we present the basic model and a preliminary result. In Section 3 we introduce distorted performance measures and present our main result, which shows that an optimal relational contract takes the form of a BSC (index) contract. The result relies on some assumptions, including validity of the "first-order approach"; and we discuss this assumption in two subsections. The discussion reveals that the approach is not valid if measurements are very precise, and a characterization of optimal contracts is thus lacking for such environments. We show that index contracts will nevertheless perform well under such conditions, and in fact become asymptotically optimal when measurement noise vanishes. In Section 4 we extend the model to include both verifiable and non-verfiable performance measures. Section 5 concludes.

2 Model

First we present the basic model between a principal and a multitasking agent. Consider an ongoing economic relationship between a risk neutral principal and a risk neutral agent. Each period the agent takes an n-dimensional action $a = (a_1, ..., a_n)'$, generating a gross value $v(a)$ for the principal, a private cost $c(a)$ for the agent, and a set of $m \leq n$ stochastic performance measurements $x = (x_1, ..., x_m)'$. These measurements are observable, but not verifiable, with joint density, conditional on action $f(x, a)$. Only the agent observes the action. We assume $v(a)$ to be increasing in each a_i and concave, and $c(a)$ to be increasing in a each a_i and strictly convex with $c(0) = 0$ and gradient vector (marginal costs) $\nabla c(0) = 0$. The total surplus (per period) in the relationship is $v(a) - c(a)$.

Given observable (but not verifiable) measurements, the agent is each period promised a bonus $\beta(x)$ from the principal. Specifically, the stage game proceeds as follows: 1. The principal offers the agent a contract consisting of a fixed payment w and a bonus $\beta(x)$. 2. If the agent accepts, he chooses some action a, generating performance measure x. If the agent declines, nothing happens until the next period. 3. The parties observe performance x, the principal pays w and chooses whether or not to honor the full contract and pay the specified bonus. 4. The agent chooses whether or not to accept the bonus he is offered. 5. The parties decide whether to continue or break off the relationship. Outside options are normalized to zero.

As shown by Levin (2002, 2003), we may assume trigger strategies and stationary contracts. The parties honor the contract only if both parties honored the contract in the previous period, and they break off the relationship and take their respective outside options otherwise. To prevent deviations, the self-enforced discretionary bonus payments must be bounded above and below. As is well known, the range of such self-enforceable payments is defined by the future value of the relationship, hence we have a dynamic enforceability condition given by

$$0 \leq \beta(x) \leq \frac{\delta}{1-\delta}(v(a) - c(a)), \qquad \text{all feasible } x. \tag{1}$$

The optimal relational contract maximizes the surplus $v(a)-c(a)$ subject to this constraint and the agent's incentive compatibility (IC) constraint. The latter is

$$a \in \arg\max_{a'} E(\beta(x)|\, a') - c(a'),$$

with first-order conditions (subscripts denote partials)

$$0 = \frac{\partial}{\partial a_i} E(\beta(x)|\, a) - c_i(a) = \int \beta(x) f_{a_i}(x,a) - c_i(a), \qquad i = 1, ...n.$$

A standard approach to solve this problem is to replace the global incentive constraint for the agent with the local first-order conditions. It is well known that this may or may not be valid, depending on the circumstances (see e.g. Hwang 2016 and Chi-Olsen 2018). We will in this paper mostly assume that it is valid, and subsequently state conditions for which this is true. So we invoke the following

Assumption A. *The first order approach (FOA) is valid.*

Unless explicitly noted otherwise, we will take this assumption for granted in the following. We then have an optimization problem that is linear in the bonuses $\beta(x)$. The optimal bonuses will then have a bang-bang structure, and hence be either maximal or minimal, depending on the outcome x. Introducing the likelihood ratios

$$l_{a_i}(x,a) = f_{a_i}(x,a)/f(x,a),$$

we obtain the following:

Lemma 1 *There is a vector of multipliers μ such that (at the optimal action $a = a^*$) the optimal bonus is maximal for those outcomes x where $\Sigma_i \mu_i l_{a_i}(x,a) > 0$, and it is zero otherwise, i.e.*

$$\beta(x) = \frac{\delta}{1-\delta}(v(a) - c(a)) \quad \textit{if} \quad \Sigma_i \mu_i l_{a_i}(x,a) > 0,$$

and $\beta(x) = 0$ if $\Sigma_i \mu_i l_{a_i}(x,a) < 0$.

The lemma says that there is an index $\tilde{y}(x) = \Sigma_i \mu_i l_{a_i}(x,a)$, with $a = a^*$ being the optimal action, such that the agent should be paid a bonus if and only if this index is positive, and the bonus should then be maximal. This index, which takes the form of a weighted sum of the likelihood ratios for the various tasks, is in this sense an optimal performance measure for the agent.

The index is basically a scorecard for the agent's performance, and since it is optimal, it is (more or less by definition) balanced. In the following we will introduce further assumptions to analyze its properties.

3 Scorecards and distorted measures

Following Baker (1992), Feltham-Xie (1994), and the often used modelling approach in the management accounting literature (e.g. Datar et al 2001, Huges et al 2005, Budde, 2007, 2009), we will in the remainder of the paper assume that the measurements x are potentially distorted and given by

$$x = Q'a + \varepsilon, \tag{2}$$

where Q' is an $m \times n$ matrix of rank $m \leq n$, and $\varepsilon \sim N(0, \Sigma)$ is multinormal with covariance matrix $\Sigma = [s_{ij}]$ (i.e. $x \sim N(Q'a, \Sigma)$). As is common in much of this literature, we assume multinormal noise for tractability The likelihood ratios for this distribution are linear in x, and this implies that the optimal performance index identified in the previous lemma is also linear in x. In particular, the vector of likelihood ratios is given by the gradient $\nabla_a \ln f(x; a) = Q\Sigma^{-1}(x - Q'a)$. Hence, defining vector τ by $\tau' = \mu' Q \Sigma^{-1}$, the index can be written as $\Sigma_i \mu_i l_{a_i}(x, a^*) = \tau'(x - Q'a^*)$; where the expression in accordance with Lemma 1 is evaluated at $a = a^*$. So we have:

Proposition 1 *In the multinormal case, there is a vector τ and a performance index $\tilde{y} = \Sigma_j \tau_j x_j$ such that the agent is optimally paid a bonus if and only if the index exceeds a hurdle ($\tilde{y}_0$). The hurdle is given by the agent's expected performance in this setting ($\tilde{y}_0 = \Sigma_j \tau_j E(x_j | a^*)$), and the bonus, when paid, is maximal: $\beta(x) = \frac{\delta}{1-\delta}(v(a^*) - c(a^*))$.*

This result parallels Levin's (2003) characterization of the single-task case, where the agent optimally gets a bonus if his performance on the single task exceeds a hurdle.. Here, in the multitask case, the principal offers an *index* $\tilde{y} = \Sigma_j \tau_j x_j$, i.e. a 'weighted sum' of performance outcomes on the various tasks, such that the agent gets a bonus if and only if this index exceeds a hurdle $\tilde{y}_0$. The optimal hurdle is given as the similar weighted sum of optimal expected performances. Hence, performance x_i is compared to expected performance, given (equilibrium) actions. If the weighted sum of performances exceeds what is expected, then the agent obtains the bonus.[6]

Figure 1 below illustrates the structure of the optimal bonus scheme. The index and its hurdle defines a hyperplane delineating outcomes "above" the plane from those "below", where the former are rewarded with full and maximal bonus while the latter yield no bonus at all. This is clearly different from a structure with separate bonuses and hurdles on each task. Such a structure is illustrated by the blue lines in the figure. In the two-dimensional

[6] The characterization given in the proposition relies on our maintained assumption that the first-order approach is valid. This is not innocous in the multinormal case. It is known that in such a setting with a single action ($n = 1$) the approach is not valid if measurements are very precise, i.e. if the variance of the performance measure is sufficiently small. On the other hand, it is valid in that setting if the variance is not too small; and as we will justify below, this is true also in the present multi-action setting.

case this structure defines four regions in the space of outcomes; where either zero, one or two bonuses are paid, respectively. The analysis shows that the structure defined by the index is better, and in fact optimal.

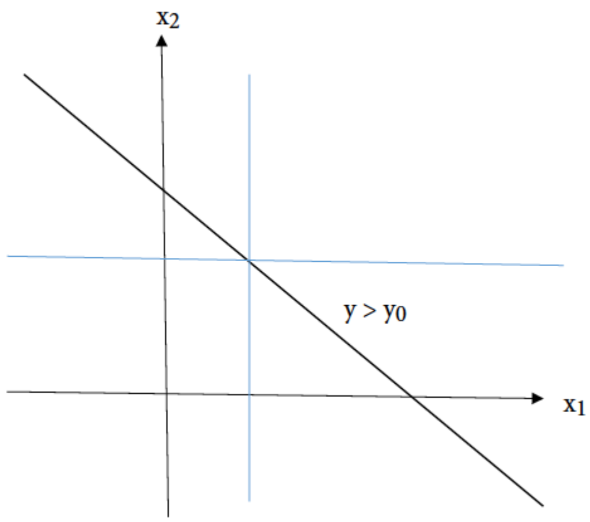

Figure 1. Structure of the optimal index contract.

Proposition 1 characterizes the type of bonus scheme that will be optimal. The next step is to characterize the parameters of the scheme, i.e. the weights τ and the hurdle $\tilde{y}_0$ that will generate optimal actions. To this we now turn.

Given the index $\tilde{y}$ with hurdle $\tilde{y}_0$, and the bonus $\beta = b$ being paid for $\tilde{y} > \tilde{y}_0$, the agent's performance related payoff is

$$b\,\Pr(\tilde{y} > \tilde{y}_0 |\, a) - c(a) = b\,\Pr(\,\tau' x > \tilde{y}_0 \big|\, a) - c(a)$$

Using the normal distribution we find that the agent's first order conditions for actions at their equilibrium levels ($a = a^*$), then satisfy

$$b\phi_0 \frac{1}{\tilde{\sigma}} Q\tau = \nabla c(a^*) \qquad (3)$$

where $\phi_0 = 1/\sqrt{2\pi}$ is a parameter of the distribution, and $\tilde{\sigma}$ is the standard deviation of the performance index:

$$\tilde{\sigma} = SD(\tilde{y}) = (\tau'\Sigma\tau)^{1/2}.$$

Note that incentives, given by the marginal revenues on the left hand side of (3), are inversely proportional to the standard deviation $\tilde{\sigma}$. All else equal, a more precise performance index (lower $\tilde{\sigma}$) will thus enhance the effectiveness of a given bonus in providing incentives to the agent. This indicates that more precise measurements will be beneficial in this setting, and that this will occur not because of reduced risk costs (there are none, by assumption) but because of enhanced incentives. The monetary bonus is constrained by self enforcement, and other factors that enhance its effectiveness will then be beneficial. We return to this below.

The optimal bonus paid for qualifying performance is the maximal one, so

$$b = \frac{\delta}{1-\delta}(v(a^*) - c(a^*))$$

For given actions a^* the elements b and τ of the optimal incentive scheme will be given by these relations.

On the other hand, optimal actions must maximize the surplus $v(a) - c(a)$ subject to these conditions. To characterize the associated optimization program for actions, it is convenient to introduce modified weights in the performance index, namely a weight vector θ given by

$$\theta = b\phi_0 \frac{1}{\tilde{\sigma}} \tau$$

Since θ is just a scaling of τ, i.e. $\theta = k\tau, k > 0$, the performance index can be expressed in terms of θ as $y = \theta' x$, and the agent is then given a bonus if this index exceeds its expected value $y_0 = \theta' E(x|\, a^*)$.

Note from the definitions of θ and $\tilde{\sigma}$ that $\theta'\Sigma\theta = (b\phi_0/\tilde{\sigma})^2 \tau'\Sigma\tau = \phi_0^2 b^2$, so we have:

$$\left(\theta'\Sigma\theta\right)^{1/2} / \phi_0 = b = \frac{\delta}{1-\delta}(v(a^*) - c(a^*)) \qquad (4)$$

Optimal actions a^* must thus satisfy (4) and the agent's first-order condition (3), which now takes the form $Q\theta = \nabla c(a^*)$. As noted, the optimal action vector must solve the problem of maximizing $v(a) - c(a)$ subject to these constraints. In fact, since the last equality in (4) reflects the dynamic enforcement constraint, we can replace it by weak inequality, and thus state the following result

Proposition 2 *In the multinormal case, optimal actions a^* are solutions to the following problem:*

$$\max_{a,\theta}(v(a) - c(a))$$

subject to $Q\theta = \nabla c(a)$ and

$$\frac{\delta}{1-\delta}(v(a) - c(a)) \geq \left(\theta'\Sigma\theta\right)^{1/2}/\phi_0 \tag{5}$$

The optimal solution yields actions a^* and associated weight parameters θ^* for the performance index. These weights are (from $Q\theta = \nabla c(a^*)$) given by·

$$\theta^* = (Q'Q)^{-1}Q'\nabla c(a^*).$$

As noted above, the optimal actions can be implemented by rewarding the agent with the largest dynamically enforceable bonus (as given in (4)) if and only if performance measured by the index $y = \theta^{*\prime}x$ exceeds its expected value $y_0 = \theta^{*\prime}E(x|\, a^*)$.

There are two sources for deviations from first-best actions in this setting, and they are reflected in the two constraints in the optimization problem. The first is due to distorted primary measures x, and will be relevant when the vector of marginal costs at the first-best actions (a^{FB}) cannot be written as $\nabla c(a^{FB}) = Q\theta$, for any θ; i.e. when this vector doesn't belong to the space spanned by (the column vectors of) Q.

Distorted measures have been studied extensively for the case when these measures are verifiable, see e.g. Feltham-Xie (1994), Baker (1992), Budde (2007); and particularly in settings where value- and cost-functions are linear and quadratic, respectively:

$$v(a) = p'a + v_0 \qquad \text{and} \qquad c(a) = \frac{1}{2}a'a. \tag{6}$$

Here $\nabla c(a) = a$ and first-best actions, characterized by marginal cost being equal to marginal value, are given by $a^{FB} = p$. If we now neglect the dynamic enforceability constraint (5) in the last proposition, we are lead to maximize the surplus $p'a - a'a/2$ subject to $a = Q\theta$. This maximization yields $\theta = (Q'Q)^{-1}Q'p$ and action, here denoted a_0^* given by $a_0^* = Q(Q'Q)^{-1}Q'p$. The best action, subject only to the agent's IC constraint $a = Q\theta$, is thus generally distorted relative to the first-best action.

It may be noted that the solution a_0^* just derived is also the optimal solution in a setting where the measurements x are verifiable and the agent is rewarded with a linear incentive scheme $\beta'x + \alpha$. This is the setting studied in several papers on distorted measures, and the literature has introduced indicators to measure the degree of distortion. One such indicator is the ratio of second-best to first-best surplus (as in Budde 2007), which for the the second-best solution just derived (and with $v_0 = 0$) amounts to

$$\frac{a_0^{*\prime}a_0^*}{p'p} = \frac{p'Q(Q'Q)^{-1}Q'p}{p'p}$$

In particular, when the measure x is one-dimensional, so Q is a vector, say $Q = q \in R^n$, the ratio is $(p'q/\,|p|\,|q|)^2$ and is thus a measure of the alignment between vectors p and q. Then the first-best can be attained only if the two vectors are perfectly aligned ($q = kp, k \neq 0$).

In the case of non-verfiable measurements x, which is the case analyzed in this paper, the solution must also respect the dynamic enforcement constraint, represented by (5) in the last proposition. When this constraint binds, the action a_0^* is generally no longer feasible. Moreover, since the stochastic properties of the measurements, represented by the covariance matrix Σ, affects the constraint, they will also affect the solution.

The expression $(\theta'\Sigma\theta)^{1/2}$ on the RHS of the constraint represents the standard deviation of the performance index $y = \theta'x$. It can be written as $(\Sigma_i\Sigma_j s_{ij}\theta_i\theta_j)^{1/2}$, where $s_{ij} = cov(x_i, x_j)$. It is clear that any variation in Σ that increases this expression will tighten the constraint, and hence reduce the total surplus. In particular, any increase of a variance in Σ will have this effect and, provided θ has no negative elements, any increase of a covariance in Σ will also have this effect. This substantiates the intuition discussed above about less precise measurements (larger variances) being detrimental in this setting. It is also noteworthy that positive correlations among elements in the measurement vector x will then be detrimental for the surplus, while negative correlations will be beneficial. This follows because, all else equal, the former increases and the latter reduces the variance of the performance index (when θ has no negative elements).

From the enforcement constraint (5) it may appear that any action a will satisfy this constraint if the standard deviation of the performance index on the RHS is sufficiently small; and hence that the constraint becomes irrelevant (non-binding) if measurements are sufficiently precise. The result in Proposition 2 builds, however, on the assumption that the frst-order approach is valid; and as we will demonstrate below, this is generally not the case for sufficiently precise measurements.

The approach replaces global IC constraints for the agent with a local one, and is only valid if the action (a^*) derived this way is in fact a global optimum for him under the given incentive scheme. Observe that, by choosing action a^* the agent gets a bonus if the index $y = \theta^{*\prime} x$ exceeds its expected value, an event which occurs with probability $\frac{1}{2}$. The agent's expected revenue is then $b/2$, with the bonus b given by (4), and this must strictly exceed the cost $c(a^*)$ in order for the agent to be willing to choose action a^*. This is so because by alternatively choosing action $a = 0$, the agent incurs zero costs but still obtains the bonus with some (small) positive probability. The following condition is thus necessary:

$$\frac{\delta}{1-\delta}(v(a^*) - c(a^*)) > 2c(a^*) \tag{7}$$

If a solution identified by the program in Proposition 2 doesn't satisfy this condition, it is not a valid solution. The reason is that the identified action is not a global optimum for the agent under the associated incentive scheme. A sufficient condition will be given below in Section 3.1.

We now present two examples to illustrate applications of Proposition 2.

Example 1. Suppose $n = 3$ and that we have $m = 2$ measurements, given by

$$x_1 = a_1 + \varepsilon_1, \qquad x_2 = k \cdot (a_2 + a_3) + \varepsilon_2, \quad k > 0,$$

Then Q' has rows $(1, 0, 0)$ and $(0, k, k)$, and we have $Q'Q = I$ (the identity matrix) if $k = 1/\sqrt{2}$. To simplify the algebra we will invoke this assumption regarding k. Assume also value- and cost-functions as in (6), with $v_0 = 0$.

Substituting from the agent's IC condition $a = Q\theta$ into the objective and the enforcement constraint in Proposition 2, we are lead to choose θ to maximize $p'Q\theta - \frac{1}{2}\theta'\theta$ subject to

$$\frac{\delta}{1-\delta}(p'Q\theta - \frac{1}{2}\theta'\theta) \geq (\theta'\Sigma\theta)^{1/2}/\phi_0$$

Given our assumptions about the measurements, we have $p'Q = (p_1, (p_2 + p_3)k)$. To simplify further, assume $p_1 = (p_2 + p_3)k$ and $var(\varepsilon_1) = var(\varepsilon_2) = s^2$, which implies that the objective and the constraint is entirely symmetric in θ_1 and θ_2. The optimal solution is then also symmetric, i.e. $\theta_1 = \theta_2$, and the (binding) enforcement constraint for the common value θ_1 takes the form

$$\frac{\delta}{1-\delta}(2p_1\theta_1 - \theta_1^2) = s\theta_1(2 + 2\rho)^{1/2}/\phi_0$$

where $\rho = corr(\varepsilon_1, \varepsilon_2)$. The optimal action is then $a^* = Q\theta = (1, k, k)'\theta_1$, and the associated surplus per period is $2p_1\theta_1 - \theta_1^2$. We see that a higher variance (s^2) or a higher correlation (ρ) for the observations will reduce θ_1 and reduce the surplus.

Given our assumptions about measurements in this example, we can promote action a_1 via incentives on x_1, and we can promote the sum $a_2 + a_3$ via incentives on x_2. As we have seen, the optimal incentive scheme rewards the agent with a fixed bonus (b) if performance measured by an index – a scorecard – $\theta_1 x_1 + \theta_2 x_2$ exceeds a hurdle. The agent will then clearly choose $a_2 = a_3$, since the marginal revenues on these two action elements are equal. This will entail a distortion from the first-best if the marginal values of these two elements for the principal are not equal ($p_2 \neq p_3$). The first best action is here $a^{FB} = (p_1, p_2, p_3)'$.

If this were the only distortion, the weight vector θ would be chosen to maximize the surplus, subject to the IC constraints, which would constrain actions such that $a_2 = a_3$. In our setting the enforcement constraint puts further bounds on these weights. We have in this example invoked an additional assumption ($p_1 = (p_2 + p_3)k$) that ensures equal weights $\theta_1 = \theta_2$ in the optimal index. The magnitude of this common weight, and therefore the strength of the agent's incentives, is bounded by the dynamic enforcement constraint. And as we have seen, the noise parameters s and ρ have negative influences in this respect.

Example 2. This example illustrates that distorted measurements may imply negative incentives on some measures, and that this has implications for comparative statics. Suppose again that there are $n = 3$ action elements and $n = 2$ measurements, but now given by

$$x_1 = a_1 + \varepsilon_1, \qquad x_2 = a_1 + \frac{1}{2}a_2 + \varepsilon_2.$$

Suppose further that $p = (1, 1, 1)'$, so the first-best action under quadratic costs is $a^{FB} = p = (1, 1, 1)'$ with surplus $3(1 - \frac{1}{2})$. For the given measurements we cannot provide incentives for a_3, and it follows that the second-best action that can be implemented via the IC constraint $a = Q\theta$ is $a_0^* = (1, 1, 0)$, with surplus $2(1 - \frac{1}{2}) = 1$. Geometrically this action is the projection of $a^{FB} = p$ on the plane spanned by Q, and it is achieved by setting $\theta_1 = -1, \theta_2 = 2$. Figure 2 below illustrates this. The positive incentive θ_2 on x_2 promotes a_1 and a_2, but with twice as strong incentives on a_1 as on a_2. The negative incentive θ_1 on x_1 dampens net incentives on a_1, and achieves in combination with θ_2 the desired balance between a_1 and a_2. As discussed above, this would be the optimal solution if measurements were verifiable.

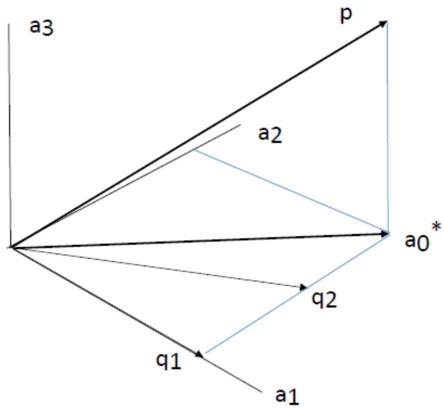

Figure 2. Illustration for Example 2.

When measures are non-verifiable, however, this solution would not be feasible if the enforcement constraint is violated, i.e. if

$$\phi_0 \frac{\delta}{1-\delta} 1 < (var(\theta_1 x_1 + \theta_2 x_2))^{1/2} = (s_{11} - 4s_{12} + 4s_{22})^{1/2},$$

where s_{ij} are the elements of Σ, and the last equality follows from $\theta_1 = -1, \theta_2 = 2$. The action a and the weights θ must then be modified to yield the highest surplus while satisfying both constraints. Observe that a larger covariance s_{12} will here reduce the variance of the performance index, and hence (at least locally) relax the enforcement constraint and thus allow for a larger surplus. In a case like this, where the weight elements have opposite signs, positive correlations between the measurements may thus be beneficial, and negative correlations detrimental.

Remark. It is of some interest to compare the result in Proposition 2 above to the Holmstrom-Milgrom (1991) and Feltham-Xie (1994) multitask models for verifiable measurements. In those models the agent is offered a linear incentive scheme $\beta' x + \alpha$, and for $E(x|\, a) = Q'a$ the IC constraint takes the form $Q\beta = \nabla c(a)$. With a risk averse (CARA) agent the total surplus (in certainty equivalents) is then $v(a) - c(a) - \frac{r}{2}\beta'\Sigma\beta$, where the last term captures risk costs, given by $\frac{r}{2}var(\beta' x)$. Letting $M = (Q'Q)^{-1}Q'$ we have $\beta = M\nabla c(a)$ and surplus $v(a) - c(a) - \frac{r}{2}(M\nabla c(a))'\Sigma(M\nabla c(a))$, which is to be maximized by choice of a. In the maximization problem in Proposition 2 we have similarly from IC that $\theta = M\nabla c(a)$, and the Lagrangian for the problem can then be written as $(v(a) - c(a))(1 + \lambda) - \lambda\frac{1-\delta}{\delta\phi_0}\left((M\nabla c(a))'\Sigma(M\nabla c(a))\right)^{1/2}$, where λ is the shadow price on the enforcement constraint. Hence the optimal solution maximizes $v(a) - c(a) - \psi\left((M\nabla c(a))'\Sigma(M\nabla c(a))\right)^{1/2}$, where $\psi = \frac{\lambda}{1+\lambda}\frac{1-\delta}{\delta\phi_0}$ can be seen as an (endogenous) cost factor.

There is thus a formal similarity between the models for the two contractual settings. But the mechanisms behind the trade-offs are different. When performance measures are verifiable, bonuses can in principle be arbitrarily large, but are optimally constrained due to the risk costs they generate for a risk averse agent. More precise measurements lowers the risk costs and consequently make bonuses in a sense more effective instruments to achieve higher surplus. With non-verifiable measures bonuses are constrained by self-enforcement at the outset, but are more effective in providing incentives if measurements are more precise. More precise measurements are thus beneficial in both settings, but for quite different reasons.

3.1 Validity of the first-order approach

We have throughout assumed FOA to be valid. Here we give sufficient conditions for this to be the case.

Let a^*, θ^* be a solution to the optimization problem in Proposition 2. The agent then gets a bonus (b) if the index $y = x'\theta^*$ exceeds the hurdle $y_0 = E(y|\, a^*) = a^{*\prime}Q\theta^*$. By construction, a^* satisfies the first-order conditions for the agent's optimization problem. These conditions are given by $Q\theta^* = \nabla c(a^*)$. We will find conditions guaranteeing that a^* is indeed an optimal choice for the agent. Observe that when the enforcement constraint binds, the necessary condition (7) implies a lower bound for the standard deviation of the performance index: $(\theta^{*\prime}\Sigma\theta^*)^{1/2} > 2c(a^*)\phi_0$.

If the agent chooses an action a, the index y has expectation $e = E(y|\, a) = a'Q\theta^*$ and variance $\sigma^2 = vary = \theta^{*\prime}\Sigma\theta^*$. Given our assumptions, the index y is $N(e, \sigma)$, and thus has a probability distribution that depends on action

a only via the (one-dimensional) expectation $e = E(y|\, a)$ The agent's expected revenue $(b\Pr(y > y_0|\, a))$ then also depends on a only via e. In light of this, it is natural to consider the action that induces e with minimal costs for the agent, i.e. action $\hat{a}(e)$ given by

$$\hat{a}(e) = \arg\min_a c(a) \quad \text{s.t.} \quad a'Q\theta^* = e,$$

and let $C(e) = c(\hat{a}(e))$ be the minimal cost.

We can then essentially write the agent's payoff as a function $u(e)$ (see the appendix for details), and seek conditions which guarantee that this function has a unique maximum. To this end, let $H(a) = [c_{ij}(a)]$ denote the Hessian for the cost function $c(\cdot)$, and define

$$h(a^*) = \sup_e \left\{ \frac{a^{*\prime}\nabla c(a)}{a'\nabla c(a^*)} \frac{\nabla c(a^*)' H(a)^{-1} \nabla c(a^*)}{a^{*\prime}\nabla c(a^*)} \,\middle|\, a = \hat{a}(e),\ 0 < e \leq a^{*\prime}\nabla c(a^*) \right\} \tag{8}$$

We may note that for a quadratic cost function[7] $c(a) = \frac{1}{2}a'Ka$ we have $h(a^*) = 1$. (In fact, the maximand here is the inverse of the elasticity of the marginal cost function $C'(e)$, see the appendix.) We then obtain the following result.

Proposition 3 *Let a^*, θ^* be a solution from Proposition 2 with the enforcement constraint binding. There is $\sigma_0^* > 0$ such that a^* is an optimal choice for the agent, and thus the first-order approach is valid, if and only if $\theta^{*\prime}\Sigma\theta^* \geq \sigma_0^{*2}$. A sufficient condition (for strict inequality, $\theta^{*\prime}\Sigma\theta^* > \sigma_0^{*2}$,) is $(\theta^{*\prime}\Sigma\theta^*)^{1/2} \geq a^{*\prime}\nabla c(a^*)\sqrt{h(a^*)}/2$, which is equivalent to*

$$\frac{\delta}{1-\delta}(v(a^*) - c(a^*)) \geq a^{*\prime}\nabla c(a^*)\sqrt{h(a^*)}/(2\phi_0). \tag{9}$$

Observe that for a quadratic cost function the expression on the right-hand side of (9) is $c(a^*)/\phi_0$ with $1/\phi_0 = \sqrt{2\pi} \approx 2.5$. A sufficient condition for the approach employed in Proposition 2 to be valid in this case is thus that the solution entails a cost for the agent that is no larger than 40% of the entire value of the future relationship.

It can be verified that for sufficiently imprecise measurements, a solution

[7] For $c(a) = (a'Ka)^r/2r$, $r \geq 1$, we find $h(a^*) = 2r - 1$.

from Proposition 2 will indeed, under some regularity conditions, satisfy condition (9). Specifically, assuming $\Sigma = s\Sigma_0$ and $\lim_{a \to 0} a' \nabla c(a) \sqrt{h(a)} = 0$ we can verify that if $s > 0$ is sufficiently large, a solution a^* will satisfy this condition when $v(0) > 0$.[8] This is so because a solution a^* will necessarily become "small" (approach zero) when measurements become very imprecise ($s \to \infty$), and then (9) will be satisfied under the given assumptions..

We conclude this section with an observation that can be helpful for characterizing properties of the solution in Proposition 2:

Corollary 1 *Let a^*, θ^* is a solution to the problem in Proposition 2 with the enforcement constraint binding, with surplus V^*, and which satisfies $\theta^{*\prime}\Sigma\theta^* > \sigma_0^{*2}$. Then a^*, θ^* solves*

$$\min_{\theta, a} \theta' \Sigma \theta \qquad st \qquad \nabla c(a) = Q\theta \qquad and \qquad v(a) - c(a) \geq V^*$$

Observe that the last constraint here must bind, since otherwise $a = 0$ and $\theta = 0$ would solve the minimization problem. Then, if the statement in the corollary were not true, there is a, θ satisfying the two constraints and $\theta' \Sigma \theta < \theta^{*\prime} \Sigma \theta^*$. Since the enforcement constraint in Proposition 2 would then be slack, a higher surplus than V^* would be feasible.

Applying this result to the linear-quadratic case (6), we find that a^* must satisfy $a^* = Q\theta^* = \lambda Q(2\Sigma + \lambda Q'Q)^{-1} Q' p$, where $\lambda > 0$ is a multiplier for the last constraint in the corollary. For the simple case of undistorted ($Q = I$) and uncorrelated measures, we then have $a_i^* = \lambda \frac{p_i}{2s_{ii} + \lambda}$, $i = 1...n$, where $s_{ii} = var(x_i)$. Comparing two action elements a_i^*, a_j^* with equal productivities ($p_i = p_j$), this reveals that the optimal solution entails less of the element that has the largest measurement variance.

3.2 Very precise measurements

We have seen that the first-order approach used to derive Proposition 2 may be invalid if measurements are noisy, but very precise. Specifically,

[8] This will also hold for $v(0) = 0$ if $(v(a) - c(a))/a' \nabla c(a) \sqrt{h(a)}$ is bounded away from zero when $a \to 0$.

the action a_0^* that maximizes surplus subject to the constraint $\nabla c(a) = Q\theta$ will be a solution to the program in Proposition 2 if measurements are sufficiently precise to make the index variance ($\theta'\Sigma\theta$) small enough to satisfy the enforcement constraint. This is true for any $\delta > 0$, but the action a_0^* will not satisfy the necessary condition (7) for a valid solution if δ is sufficiently small. Hence the first order approach is not valid in such a case.

We thus lack a characterization of optimal incentive schemes for settings with noisy but very precise measurements. On the other hand , the optimal scheme for an environment with no noise is known (Budde 2007). In this subsection we show that if V^{NF} is the optimal surplus in a setting with no noise, then any surplus value $V < V^{NF}$ can be implemented with an index contract if the measurements are sufficiently precise. Index contracts (scorecards) are in this sense at least approximately optimal for sufficiently precise measurements.

Measurements without noise. As a reference case we first consider measurements with no noise, i.e. of the form

$$x = Q'a.$$

We have then that an action a can be implemented by some bonus scheme $\beta(x)$ if and only if

$$\nabla c(a) = Q\gamma \tag{10}$$

for some $\gamma \in R^m$. The condition is necessary because, if a generating measurement $x = Q'a$ is optimal for the agent, then it must be cost-minimizing among all actions that generate the same x. So it must solve $\min_{\tilde{a}} c(\tilde{a})$ subject to $x = Q'\tilde{a}$, and hence satisfy the first-order condition (10) with Lagrange multiplier γ. Observe that γ is uniquely given by $\gamma = (Q'Q)^{-1}Q'\nabla c(a)$. On the other hand, if a satisfies (10), it is a cost-minimizing action generating measurement $x = Q'a$, and will be chosen by the agent under a bonus scheme with $\beta(x) \geq c(a)$ and $\beta(\tilde{x}) = 0, \tilde{x} \neq x$.

Being discretionary, bonuses must respect a dynamic enforcement constraint. Since the minimal bonus to implement an action a is its cost $c(a)$, the constraint here takes the form

$$c(a) \leq \frac{\delta}{1-\delta}(v(a) - c(a)) \tag{11}$$

The optimal contract in this setting thus maximizes the surplus $v(a) - c(a)$ subject to (10) and (11). Let a^{NF} denote the optimal action and V^{NF} the maximal surplus in this noise-free environment. In the following we will assume that the enforcement constraint binds and thus implies a surplus V^{NF} strictly less than the optimal surplus obtained without the constraint, thus $V^{NF} < V_0^* = \max\{v(a) - c(a)| \nabla c(a) = Q\theta, \theta \in R^m\}$

When the enforcement constraint here binds, we have $c(a^{NF}) = \delta v(a^{NF})$ We further have, from (10) that $\nabla c(a^{NF}) = Q\gamma$. In the linear-quadratic case as in (6) with $v_0 = 0$, this yields $a^{NF} = Q\gamma$ and (by optimization of the surplus with respect to γ) $\gamma = k(Q'Q)^{-1}Q'p$ with $k = 2\delta$ when the enforcement constraint binds, and $k = 1$ otherwise. The constraint binds for $\delta < \frac{1}{2}$. The optimal surplus is then $V^{NF} = (k - \frac{1}{2}k^2)p'Q(Q'Q)^{-1}Q'p$. This is a case considered in Budde (2007).

Measurements with noise. Consider again noisy measurements, and recall that the approach behind Proposition 2 is valid only if the solution (action a^*) satisfies condition (7). This condition is stricter than condition (11). This implies that, although noise-free measurements can be seen as a limiting case of noisy measurements when all variances go to zero, a valid solution from Proposition 2 can generally not converge to a^{NF}.

It may be noted that Chi and Olsen (2018) have found that for settings with a univariate action, an index contract derived form the likelihood ratio is still optimal even when the first-order approach is not valid. The only required modification is that the threshold for the index must be adjusted, taking into account not only a local IC constraint for the agent, but also non-local ones, which will be binding. It is an open question whether a similar property holds in settings with multivariate actions.

In the setting of this paper we can however show that for noisy but sufficiently precise measurements, any surplus $V < V^{NF}$ can be obtained by means of an index contract. This doesn't mean that such a contract is optimal, but it will at least be approximately optimal for such measurements.

Specifically, we will consider actions that satisfy

$$2c(a) \geq \frac{\delta}{1-\delta}(v(a) - c(a)) > c(a), \tag{12}$$

plus $\nabla c(a) = Q\theta$ for some $\theta \in R^n$. Such an action will be feasible for the optimization problem with noise free measurements, but not optimal in that problem, since the enforcement constraint (11) doesn't bind. Hence it generates a surplus $V < V^{NF}$, but the action a can be chosen such that V is arbitrarily close to V^{NF}.

The first inequality in (12) implies that the necessary condition (7) for FOA to be valid is violated, hence a cannot be implemented by the scheme applied in Proposition 2. Recall that this is a consequence of the scheme being designed such that, for the desired action the agent's expected revenue falls short of his costs. (The hurdle for the index is set to maximize marginal incentives, but this implies that the probability to obtain the bonus is 1/2, and the first inequality in (12) then implies a negative payoff for the agent, relative to choosing action $a = 0$.)

It seems intuitive that this problem can be alleviated by modifying the hurdle so as to make it less demanding for the agent to qualify for the bonus. On the other hand, such a modification will also negatively affect the agent's marginal incentives. It turns out that, if the measurements are sufficiently precise, a modification of the hurdle can achieve both goals: sufficiently strong incentives and a non-negative payoff for the agent, so that the desired action can be implemented. This is formally stated as follows.

Proposition 4 *Let action a satisfy $2c(a) \geq \frac{\delta}{1-\delta}(v(a) - c(a)) > c(a)$ and $\nabla c(a) = Q\theta$, for some $\theta \in R^m$. There is $\sigma_0 > 0$ with the following property: If Σ satisfies $\theta'\Sigma\theta \equiv \sigma^2 < \sigma_0^2$, then there is a hurdle $\kappa(\sigma) < E(x'\theta|\, a)$ such that the index $x'\theta$ with hurdle $\kappa(\sigma)$ implements a. Moreover, $\kappa(\sigma) \to E(x'\theta|\, a)$ as $\sigma \to 0$.*

The proposition implies that any surplus V smaller than, but close to V^{NF}, can be obtained by means of an index contract, provided measurements are sufficiently precise. It also implies that if such a contract is optimal in this class (of index contracts), then FOA must necessarily be violated, and hence some non-local incentive constraint must bind.

To make the last observation precise, let $V^M < V^{NF}$ be the surplus defined by

$$V^M = \max\left\{ v(a) - c(a) |\, 2c(a) \leq \frac{\delta}{1-\delta}(v(a) - c(a)) \text{ and } \nabla c(a) = Q\theta,\ \ \theta \in R^m \right\},$$

and observe that the first constraint in this problem must bind. (Otherwise we would have $V^M = V_0^*$ and $V^M \leq V^{NF}$, contradicting our basic assumption $V^{NF} < V_0^*$ here.)

Proposition 4 implies that any surplus $V \in (V^M, V^{NF})$ can be implemented with a linear index contract for some set of covariance matrices $\Sigma \in \Gamma(V)$. It follows that if an optimal such contract yields a surplus $V \in (V^M, V^{NF})$, it must be optimal for some Σ in the set $\Gamma(V)$. It must also be the case that the implemented optimal action, say a^*, satisfies $2c(a^*) > \frac{\delta}{1-\delta}(v(a^*) - c(a^*))$, since otherwise the surplus could not exceed V^M. It follows from this that the necessary condition (7) for FOA to be valid is violated, and we can state the following result

Corollary 2 *If an index contract that implements an action a^* with surplus $V \in (V^M, V^{NF})$ is optimal in the class of such contracts, for some Σ, then non-local incentive constraint(s) will be binding in the optimization program that defines the contract.*

This implies that characterizing the optimal (linear) index contract can be technically challenging in this setting. Of course this applies also for the overall optimal contract, since it must have non-local incentive constraints binding as well. (Otherwise it would be characterized by Proposition 2, and thus be an index contract with only a local constraint binding.) We leave these issues as topics for future research.

4 Non-verifiable and verifiable measurements

We have so far focused on non-verifiable measurements. But incentive schemes, at least for top management, will typically also include verifiable financial performance measures. Consider then a situation where there are both non-verifiable and verifiable measurements available. To simplify the exposition we will assume that there is one verifiable measure (x_0) in addition to the the non-verifiable measures (x) considered above. The latter depends stochastically on effort as in (2) and the former is assumed to have a similar representation:

$$x_0 = q_0' a + \varepsilon_0,$$

where $q_0 \in R^n$ and ε_0 is normally distributed noise generally correlated with the noise variables ε in x. (More precisely, the vector $(\varepsilon_0, \varepsilon)$ is multinormal.)

The agent can now be incentivized by a court enforced (explicit) bonus $b_0 x_0$ on the verifiable measure and a discretionary (relational) bonus $\beta(x_0, x)$ depending on the entire measurement vector (x_0, x). We consider a case where only short term explicit contracts are feasible, which allows us to confine attention to stationary contracts.[9].

In each period, the agent will now choose actions a to maximize $E(\, b_0 x_0 + \beta(x_0, x) |\, a) - c(a)$, yielding first-order conditions

$$\int (b_0 x_0 + \beta(x_0, x)) f_{a_i}(x_0, x, a) - c_i(a) = 0, \quad i = 1, ..., n.$$

(Here we use $f(x_0, x, a)$ to denote the joint density of all measurements, conditional on action.)

Returning to the assumption that FOA is valid, the principal then maximizes the total surplus $v(a) - c(a)$ subject to these constraints and the dynamic enforcement constraint. We assume as before that the parties separate if the relational contract is broken. The enforcement constraint is then the same as (1), just with x now replaced by the entire measurement vector (x_0, x).

From the same principles as before it follows that the agent should be

[9] Watson, Miller and Olsen (2020) analyse long term renegotiable court-enforced contracts, and show that it wil generally be optimal to renegotiate these contracts each period when in combination with relational contracts.

given the discretionary bonus if and only if an index exceeds a hurdle, and from the normal distribution it follows that this index is linear in the measurements; $y = \Sigma_{i=0}^{m}\tau_i x_i \equiv \tau_0 x_0 + \tau' x$, and moreover that the hurdle is $y_0 = E(\Sigma_{i=0}^{m}\tau_i x_i | \, a^*)$, where a^* is the equilibrium action. If the magnitude of the bonus is b, this leads to the following first-order conditions for the agent at the equilibrium action:

$$(b_0 + b\frac{\phi_0}{\sigma}\tau_0)q_0 + b\frac{\phi_0}{\sigma}Q\tau = \nabla c(a^*)$$

where now $\sigma^2 = var\Sigma_{i=0}^{m}\tau_i x_i = var(\tau_0 x_0 + \tau' x)$ is the variance of the performance index in this setting.

As before, it is convenient to introduce modified weights in the index:

$$\theta_0 = b\frac{\phi_0}{\sigma}\tau_0, \quad \theta = b\frac{\phi_0}{\sigma}\tau.$$

This yields $var(\Sigma_{i=0}^{m}\theta_i x_i)/\phi_0^2 = (b\frac{1}{\sigma})^2 var(\Sigma_{i=0}^{m}\tau_i x_i) = b^2$, and implies that the IC condition and the dynamic enforcement condition can be written as, respectively; the following relations:

$$(b_0 + \theta_0)q_0 + Q\theta = \nabla c(a)$$

$$\frac{\delta}{1-\delta}(v(a) - c(a)) \geq \frac{1}{\phi_0}(var(\theta_0 x_0 + \theta' x))^{1/2}$$

The principal maximizes the total surplus $v(a) - c(a)$ subject to these constraints.

Since the court-enforced bonus b_0 can be chosen freely, while the elements θ_0, θ of the discretionary bonus scheme are constrained by self-enforcement, we see that θ_0 should be chosen so as to minimize the variance appearing in the enforcement constraint. (If not, then for given θ we could modify b_0 and θ_0 so that the IC constraint holds and the enforcement constraint becomes slack.)

The variance is minimized for $\theta_0 = -cov(x_0, \theta' x)/s_0^2$, where $s_0^2 = var(x_0)$, and this implies in turn that the performance index takes the form

$$\theta_0 x_0 + \theta' x = \Sigma_{i=1}^{m}\theta_i(x_i - \frac{cov(x_0, x_i)}{s_0^2}x_0).$$

This shows that for correlated measurements ($cov(x_0, x_i) \neq 0$) performance on the verifiable measure is taken into the index as a benchmark, to which the other performances are compared.

The hurdle for the index is the expected value $\Sigma_{i=1}^m \theta_i (e_i^* - \frac{cov(x_0,x_i)}{s_0^2} e_0^*)$, where $e_i^* = E(x_i | a^*), i = 0, ..., m$. Since $e_i^* + \frac{cov(x_0,x_i)}{s_0^2}(x_0 - e_0^*)$ is the conditional expectation of x_i, given x_0 (and a^*), it follows that we can write the condition for the index to pass the hurdle as

$$\Sigma_{i=1}^m \theta_i (x_i - E(x_i | x_0, a^*)) > 0.$$

Performance x_i is thus compared to expected performance, given (equilibrium) actions and the outcome on the verifiable measure. If the performance exceeds what is expected, given this information, then it contributes positively to making the index exceed the hurdle, and thus for the agent to obtain the bonus.

Since the verifiable measure can be ignored (by setting $\theta_0 = b_0 = 0$, which is a feasible choice), the parties are here better off with this measure available than without it. They are certainly strictly better off when the optimal θ_0 is non-zero, which occurs when there is non-zero correlation between the verifiable and some non-verifiable measure. This enables the variance of the performance index to be reduced, and by that the dynamic enforcement constraint to be relaxed and the surplus to be increased. As we have seen, this is achieved by benchmarking the agent's performance on the non-verifiable measures to her performance on the verifiable one.

The minimized index variance is

$$\min_{\theta_0} var(\theta_0 x_0 + \theta' x) = var(\Sigma_{i=1}^m \theta_i \tilde{x}_i) = \theta' \tilde{\Sigma} \theta,$$

where $\tilde{x}_i = x_i - \frac{cov(x_0 x_i)}{s_0^2} x_0$, $i = 1, ..., m$, and $\tilde{\Sigma}$ is the covariance matrix for $\tilde{x}$. We have $cov(\tilde{x}_i, \tilde{x}_j) = s_{ij} - \rho_{0i}\rho_{0j} s_{ii} s_{jj}$, where $\rho_{0i} = corr(x_0, x_i)$, $i = 1, ..., m$ are the correlation coefficients between the verifiable and the non-verifiable measures. We see that if all of these have the same sign, then all elements in the new covariance matrix $\tilde{\Sigma}$ are reduced relative to the elements of matrix Σ. Moreover, the stronger are these correlations in such a case, the smaller are the elements of $\tilde{\Sigma}$, and the smaller is then the variance $\theta' \tilde{\Sigma} \theta$ if all elements of θ are non-negative. This will then relax the enforcement constraint and increase the surplus. Stronger correlations, either all positive or all negative, between the verifiable and each non-verifiable measure, will thus increase the surplus in such a case.

We finally outline an approach to solve for the optimal contract in the setting considered here, and apply this to the linear-quadratic case. First define $\tilde{b}_0 = b_0 + \theta_0$, so that the IC constraint takes the form $\tilde{b}_0 q_0 + Q\theta = \nabla c(a)$, and next define

$$S(\theta) = \max_{\tilde{b}_0, a} \{ v(a) - c(a) | \, \tilde{b}_0 q_0 + Q\theta = \nabla c(a)\}.$$

Then $S(0)$ would be the optimal surplus the parties could achieve if only the verifiable measure x_0 were available. The relational contract allows the parties to achieve

$$\max_{\theta} S(\theta) \quad \text{s.t.} \quad \frac{\delta}{1-\delta} S(\theta) \geq \theta' \tilde{\Sigma} \theta / \phi_0$$

In the linear-quadratic case ($v(a) = p'a$ and $c(a) = \frac{1}{2} a'a$), the IC constraint is $\tilde{b}_0 q_0 + Q\theta = a$, and using this to substitute for a, we find that the surplus to be maximized in the first step (with respect to $\tilde{b}_0$) is

$$\tilde{b}_0 p' q_0 - \frac{1}{2} \tilde{b}_0^2 q_0' q_0 - \tilde{b}_0 \theta' Q' q_0 + p' Q \theta - \frac{1}{2} \theta' Q' Q \theta$$

We see that, except if q_0 is orthogonal to all the columns of Q, i.e. $Q' q_0 = 0$, then the optimal bonus $\tilde{b}_0$ will depend on θ and hence be different from the optimal bonus for the verifiable measure alone.

The optimal value in this step is

$$S(\theta) = \frac{1}{2 q_0' q_0} (p' q_0 - \theta' Q' q_0)^2 + p' Q \theta - \frac{1}{2} \theta' Q' Q \theta$$

The formula illustrates that, relative to a situation with only non-verifiable measures, the verifiable one helps by (i) providing incentives that generate value (the first term in $S(\theta)$), and (ii) by relaxing the enforcement constraint; partly via the higher value, and partly by allowing for valuable benchmarking in the performance index.

5 Conclusion

Employees are often evaluated along many dimensions, and at least some of the performance measures will typically be non-verifiable to a third party. They may also be misaligned with (distorted from) the true values for the principal, and be stochastically dependent. The aim of this paper is to study this environment: Optimal incentives for multitasking agents whose performance measures are non-verifiable and potentially distorted and correlated. We extend and generalize the received literature in some important dimensions (to an arbitrary number of tasks with stochastic measurements that are possibly correlated and/or distorted), and we invoke assumptions (normally distributed measurements) that make the model quite tractable.

Our main result is that, under standard assumptions, the optimal relational contract is an index contract. That is, the agent gets a bonus if a weighted sum of performance outcomes on the various tasks (an index) exceeds a hurdle. The efficiency of this contract improves with higher precision of the index measure, since this strengthens incentives. Correlations between measurements may for this reason be beneficial. For a similar reason, the principal may also want to include verifiable performance measures in the relational index contract in order to improve incentives. These are then included as benchmarks, to which the other performances are compared.

The index contracts that turn out to be optimal in our model bear resemblance to key features of the performance measurement system known as balanced scorecards. Reward systems based on BSC typically connect pay to an index, but to the best of our knowledge there is no formal incentive model that actually describe this kind of index contracts as an optimal solution. In that sense, our paper provides at contract theoretic rationale for the way BSC schemes are implemented. However, while the scheme we present is a bonus contract with just one threshold (or 'hurdle), scorecards in practice often have several thresholds and bonus levels, where the *size* of the bonus depends on the score. Future research can extend the model we present to incorporate e.g. risk aversion or limited liability, in order to study under which broader conditions the index contract is optimal, and what kind of index contracts that are optimal under various model specifications.

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APPENDIX

Proof of Lemma 1. The lemma follows directly from the Lagrangian for the problem, which takes the form

$$L = v(a) - c(a) + \Sigma_i \mu_i (\int \beta(x) f_{a_i}(x,a) - c_i(a)) + \int \lambda(x) (\tfrac{\delta}{1-\delta}(v(a) - c(a)) - \beta(x))$$

At the optimal action $a = a^*$, the optimal bonus satisfies

$$\frac{\partial L}{\partial \beta(x)} = \Sigma_i \mu_i f_{a_i}(x,a) - \lambda(x) = 0 \text{ if } \beta(x) > 0, \qquad \leq 0 \text{ if } \beta(x) = 0$$

Hence we have

If $\Sigma_i \mu_i f_{a_i}(x,a) > 0$ then $\lambda(x) > 0$ and hence $\beta(x) = \frac{\delta}{1-\delta}(v(a) - c(a))$.

If $\Sigma_i \mu_i f_{a_i}(x,a) < 0$ then $\frac{\partial L}{\partial \beta(x)} < 0$ and hence $\beta(x) = 0$ (implying $\lambda(x) = 0$).

Verification of (3). Given that $\tilde{y} = \tau' x$ is nomal with expectation $E(\tilde{y}|\, a)$ and variance $\tilde{\sigma}^2 = \tau' \Sigma \tau$, we have

$$\Pr(\tilde{y} > \tilde{y}_0|\, a) = \Pr(\frac{\tilde{y} - E(\tilde{y}|\, a)}{\tilde{\sigma}} > \frac{\tilde{y}_0 - E(\tilde{y}|\, a)}{\tilde{\sigma}}\bigg|\, a) = 1 - \Phi(\frac{\tilde{y}_0 - E(\tilde{y}|\, a)}{\tilde{\sigma}}) \tag{13}$$

where $\Phi(\cdot)$ is the standard normal CDF. Since $E(\tilde{y}|\, a) = \tau' Q' a$ has gradient $\nabla_a E(\tilde{y}|\, a) = Q\tau$, we then obtain

$$\nabla_a \Pr(\tilde{y} > \tilde{y}_0|\, a) = \phi(\frac{\tilde{y}_0 - E(\tilde{y}|\, a)}{\tilde{\sigma}})\frac{1}{\tilde{\sigma}} Q\tau$$

where $\phi = \Phi'$ is the standard normal density. This verifies (3), since $\tilde{y}_0 = E(\tilde{y}|\, a^*)$.

Proof of Proposition 3. For an action a the index $y = x'\theta^*$ has variance $\sigma^2 = \theta^{*\prime}\Sigma\theta^*$ and expected value $e = E(y|\, a) = a' Q\theta^*$. For given e, let $C(e)$ be the minimal cost for the agent to achieve this expected value, i.e.

$$C(e) = \min_a c(a) \quad \text{s.t.} \quad a' Q\theta^* = e. \tag{14}$$

From a formula corresponding to (13) we see that the agent's expected revenue depends on a only via $e = E(y|\, a)$, hence consider the payoff

$$u(e) = b(1 - \Phi(\frac{y_0 - e}{\sigma})) - C(e) = \frac{\sigma}{\phi_0}(1 - \Phi(\frac{e^* - e}{\sigma})) - C(e),$$

where we have used $b = \sigma/\phi_0$ and defined $e^* = a^{*\prime} Q\theta^* = y_0$. Note that for $e = e^*$ we have $C(e^*) = c(a^*)$, since a^* satisfies the first-order condition in the convex cost-minimization problem. Hence the agent's payoff from a^* is $u(e^*)$, which equals $b\frac{1}{2} - c(a^*)$.

It is clear that if $u(e) \leq u(e^*)$ for all feasible e, then action a^* is an optimal choice for the agent. (If not, there exists an action $\tilde{a}$ yielding a higher payoff. This payoff is $u(\tilde{e})$, where $\tilde{e} = \tilde{a}' Q\theta^*$, and thus $u(\tilde{e}) > u(e^*)$, a contradiction.) Observe that

$$u'(e) = \phi(\frac{e^* - e}{\sigma})/\phi_0 - C'(e),$$

where $\phi = \Phi'$ is the standard normal density.

Since $Q\theta^* = \nabla c(a^*)$, the first-order conditions for the cost minimization problem defining $C(e)$ are

$$\nabla c(\hat{a}) = \gamma \nabla c(a^*) \qquad \text{and} \qquad e = \hat{a}' \nabla c(a^*), \tag{15}$$

where γ is a Lagrange multiplier. Differentiation wrt e yields

$$H(\hat{a}) d\hat{a} = d\gamma \nabla c(a^*) \qquad \text{and} \qquad \nabla c(a^*)' d\hat{a} = de,$$

hence $d\hat{a} = H(\hat{a})^{-1} \nabla c(a^*) d\gamma$ and so

$$\frac{d\gamma}{de} = (\nabla c(a^*)' H(\hat{a})^{-1} \nabla c(a^*))^{-1} > 0,$$

where the inequality follows from H being positive definite. From the envelope property we have $C'(e) = \gamma$ and so $C''(e) = \frac{d\gamma}{de} > 0$.

Observe for later use that from conditions (15) we have $e = \hat{a}' \nabla c(a^*)$ and $\gamma = a^{*\prime} \nabla c(\hat{a}) / (a^{*\prime} \nabla c(a^*))$, and hence

$$\eta(e) \equiv e \frac{C''(e)}{C'(e)} = \hat{a}' \nabla c(a^*) \frac{a^{*\prime} \nabla c(a^*)}{a^{*\prime} \nabla c(\hat{a})} \frac{1}{\nabla c(a^*)' H(\hat{a})^{-1} \nabla c(a^*)}. \tag{16}$$

Now consider $u(e)$ for $e > e^*$. Here we have $u'(e) < u'(e^*) = 0$ since $\phi(\frac{e^* - e}{\sigma})$ is decreasing and $C'(e)$ is increasing in e, where the latter property follows from $C''(e) = \frac{d\gamma}{de} \geq 0$. This verifies $u(e) < u(e^*)$ for $e > e^*$.

Next consider $u(e)$ for $e < e^*$. We will show that $u(e) \leq u(e^*)$ for all $e \leq e^*$ iff $\sigma \geq \sigma_0^*$. To this end we first state and prove the following claim.

Claim. For $\sigma \geq e^* \sqrt{h(a^*)}/2 \equiv \sigma_m$ we have $u'(e) \geq 0$ for all $e < e^*$.

The statement obviously implies $u(e) \leq u(e^*)$ for all $e < e^*$. To prove the claim, observe first that $u'(0) > 0 = u'(e^*)$ (since $C'(0) = 0$ due to $\hat{a}(0) = 0$ and therefore $\gamma = 0$ for $e = 0$). If $u'(e)$ has no local minimum in $(0, e^*)$, then

$u'(e)$ is non-negative on this interval. So consider a local minimum, where then $u''(e) = 0$. Using $\phi'(z) = -z\phi(z)$ we have

$$0 = u''(e) = -\phi'(\frac{e^* - e}{\sigma})\frac{1}{\phi_0 \sigma} - C''(e) = \frac{e^* - e}{\sigma}\phi(\frac{e^* - e}{\sigma})\frac{1}{\phi_0 \sigma} - C''(e).$$

This yields $\phi(\frac{e^*-e}{\sigma})/\phi_0 = \frac{\sigma^2}{e^*-e}C''(e)$ and thus, from the definition of the elasticity $\eta(e)$ above:

$$u'(e) = \phi(\frac{e^* - e}{\sigma})/\phi_0 - C'(e) = C''(e)(\frac{\sigma^2}{e^* - e} - \frac{e}{\eta(e)})$$

By the expression for $\eta(e)$ in (16) and the definition of $h(a^*)$ in (8) we have $h(a^*) \geq 1/\eta(e)$ and hence

$$\frac{\sigma^2}{e^* - e} - \frac{e}{\eta(e)} \geq \frac{\sigma^2}{e^* - e} - eh(a^*).$$

The last expression is non-negative if $\sigma^2/h(a^*) \geq \max_e e(e^* - e) = (e^*/2)^2$, i.e. if $\sigma \geq e^*\sqrt{h(a^*)}/2 \equiv \sigma_m$. This verifies that $u'(e) \geq 0$ for all $e \leq e^*$ if $\sigma \geq \sigma_m$, and thus proves the claim.

So we have $u'(e) \geq 0$ for all $e < e^*$ when $\sigma \geq \sigma_m$. Let σ_l be the smallest σ for which $u'(e) \geq 0$ for all $e < e^*$. (We must have $\sigma_l > 0$ since otherwise the necessary condition (7) would be violated.) So for $\sigma < \sigma_l$ there is $e < e^*$ such that $u'(e) < 0$. Then, since $u'(0) > 0$ as noted above, $u(e)$ must have a local maximum at some $e^0 \in (0, e^*)$. Since both e^0 and e^* are local maxima, we have then, for $\sigma < \sigma_l$

$$\frac{d}{d\sigma}(u(e^*) - u(e^0))\phi_0 = \Phi(\frac{e^* - e^0}{\sigma}) - \Phi(0) - \sigma\Phi'(\frac{e^* - e^0}{\sigma})\frac{e^* - e^0}{\sigma^2} > 0,$$

where the inequality follows from $\Phi(z)$ being strictly concave for $z > 0$, and thus $\Phi(z) - \Phi(0) - \Phi'(z)z > 0$.

Hence, the smaller is σ, the smaller is the payoff difference $u(e^*) - u(e^0)$. Let σ_0^* be the smallest σ for which $u(e^*) - u(e^0) \geq 0$. By the monotonicity of $u(e^*) - u(e^0)$.we have $u(e^*) \geq u(e^0)$ iff $\sigma \geq \sigma_0^*$. This proves the first statement in the proposition.

The second statement follows from the sufficient condition $\sigma \geq e^*\sqrt{h(a^*)}/2$

in the stated Claim above, taking account of $e^* = a^{*\prime} Q\theta^* = a^{*\prime}\nabla c(a^*)$ and the binding enforcement constraint in Proposition 2. This completes the proof.

Remark. The proof uses only two properties of a^* and θ^*; namely that they satisfy $\nabla c(a^*) = Q\theta^*$ and the binding enforcement constraint (5). Its conclusions regarding a^* being implementable (an optimal choice for the agent) with index $x'\theta^*$ are therefore valid for any a^* and θ^* that satisfy these conditions

Proof of Proposition 4. To take advantage of the notation developed in the previous proofs, we will in this proof denote the given a and θ by a^* and θ^*, respectively. We thus consider a^* and θ^* that satisfy $2c(a^*) \geq \frac{\delta}{1-\delta}(v(a^*) - c(a^*)) > c(a^*)$ and $\nabla c(a^*) = Q\theta^*$.

We will consider the index $y = x'\theta^*$ with a hurdle $\kappa < E(y|\, a^*)$, and with bonus b paid for qualifying performance ($y > \kappa$). The bonus is

$$b = \frac{\delta}{1-\delta}(v(a^*) - c(a^*)).$$

The proof will show that the hurdle κ can be chosen such that this index scheme implements a^*, provided the index has a sufficiently low variance.

By assumption we have $c(a^*) < b$. Choose $\xi_0 > 0$ and σ_0 such that

$$c(a^*) = (\Phi(\xi_0) - \Phi(-\xi_0))b \quad \text{ and } \quad \sigma_0 = b\phi(-\xi_0)$$

The index $y = x'\theta^*$ has variance $\sigma^2 = \theta^{*\prime}\Sigma\theta^*$, and assume now $\sigma < \sigma_0$. Define $\xi > \xi_0$ by

$$\sigma = b\phi(-\xi),$$

and let the hurdle for the index be $\kappa = E(y|\, a^*) - \xi\sigma = \theta^{*\prime}Q'a^* - \xi\sigma$.

The agent's payoff from an action a is then $b(1 - \Phi(\frac{\kappa - E(y|a)}{\sigma})) - c(a)$ with gradient $b\frac{1}{\sigma}\phi(\frac{\kappa-\theta^{*\prime}Q'a}{\sigma})Q\theta^* - \nabla c(a)$. It follows that action a^* satisfies the first-order condition for an optimum, since we have $\kappa - \theta^{*\prime}Q'a^* = -\xi\sigma$, $b\frac{1}{\sigma}\phi(-\xi) = 1$ and $Q\theta^* = \nabla c(a)$ Since $\xi > 0$, we can also verify that the Hessian at a^* is positive definite, hence action a^* is a local optimum for the agent under the given incentive scheme.

It remains to show that a^* is a global optimum. As in the proof of Proposition 3, it suffices to consider the payoff

$$u(e) = b(1 - \Phi(\frac{\kappa - e}{\sigma})) - C(e).$$

where e is the expected index value ($e = E(y|\, a)$), $C(e)$ is the minimal cost to obtain a given expected value e, see (14); and κ is here the hurdle for the index. For action a^* this payoff is $u(e^*)$, where $e^* = E(y|\, a^*) = \theta^{*\prime} Q' a^*$. The proof is complete if we show $u(e) \leq u(e^*)$ for all feasible e.

First note that by the definition of κ we have $\frac{\kappa - e^*}{\sigma} = -\xi$ and so

$$u(e^*) = b(1 - \Phi(-\xi)) - c(a^*),$$

where we have used the fact that $C(e^*) = c(a^*)$, by virtue of a^* being the cost-minimizing action to generate expectation $e^* = \theta^{*\prime} Q' a^*$.

Next consider $e < e^*$. Since $u'(0) > 0$ (by virtue of $C'(0) = 0$, see the previous proof), we have $u(e) \leq u(e^*)$ for all $e \in [0, e^*]$ if $u(\cdot)$ has no local maximum in the interior of the interval. So suppose $u(\cdot)$ has a local maximum at some $e^0 \in (0, e^*)$. Then $u'(e^0) = 0$ and so $b\frac{1}{\sigma}\phi(\frac{\kappa - e^0}{\sigma}) = C'(e^0)$. Since $C'(e^0) < C'(e^*)$, and e^* is also a local maximum, we then have $\phi(\frac{\kappa - e^0}{\sigma}) < \phi(\frac{\kappa - e^*}{\sigma})$. Since $\phi(\cdot)$ is symmetric around zero, this implies $\kappa - e^0 > e^* - \kappa$ and hence, by definition of $\kappa = e^* - \xi\sigma$, that $\kappa - e^0 > \xi\sigma$. This yields

$$u(e^0) = b(1 - \Phi(\frac{\kappa - e^0}{\sigma})) - C(e^0) \leq b(1 - \Phi(\xi)),$$

and hence

$$u(e^*) - u(e^0) \geq b(1 - \Phi(-\xi)) - c(a^*) - b(1 - \Phi(\xi)).$$

The last expression is increasing in ξ and is (by definition of ξ_0) zero for $\xi = \xi_0$. Hence $u(e^*) - u(e^0) \geq 0$, since $\xi > \xi_0$. This verifies $u(e) \leq u(e^*)$ for all feasible $e < e^*$.

Now consider $e > e^*$. As in the proof of Proposition 3, we have $u'(e) < u'(e^*) = 0$ when $e > e^*$ This follows because $C'(e)$ is increasing (as shown in the proof of Proposition 3), and because $\phi(\frac{\kappa - e}{\sigma})$ is decreasing in e when

$e > e^*$, since $e^* > \kappa$ and thus $\kappa - e < 0$. This verifies $u(e) < u(e^*)$ for $e > e^*$.

We finally verify that $\kappa \to E(y|\, a^*)$ when $\sigma \to 0$. From the definition of κ and ξ we have $E(y|\, a^*) - \kappa = \xi\sigma = \xi\phi(-\xi)b$, where $\xi \to \infty$ when $\sigma \to 0$. The density $\phi(\cdot)$ has the property that $\xi\phi(-\xi) \to 0$ when $\xi \to \infty$, and this completes the proof.

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