Contingent Claims Evaluation when the Convenience Yield is Stochastic: Analytical Results

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Abstract

This paper considers contingent claims on a commodity when both the spot price and the convenience yield are generated by diffusion processes. By adopting the Gibson and Schwartz (1990) assumptions on the economy, we derive analytical solutions to both the futures price and the European call option.[1]

1 Introduction

Brennan and Schwartz (1985) interpret a copper mine as a contingent claim on future production of output. The essence is to obtain optimal management strategies and the corresponding project value. The spot price of copper is assumed to follow a geometrical Brownian motion, and the convenience yield rate is assumed constant. Similar assumptions have been adopted in articles related to other commodities.[2]

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[1]After I wrote the first draft of this paper, I became aware of Jamshidian and Fein (1990), who use a somewhat different approach. The results stated as Propositions 1 and 2 in their paper are equivalent to Theorems 1 and 2 below.

[2]See, e.g., Bjerksund and Ekern (1990), Kemna (1987), Lund (1987), Mac-Kie Mason (1988), Paddock, Siegel, and Smith (1988), and Pindyck (1988 b).

Empirical evidence suggests that modeling the commodity spot price process as a geometrical Brownian motion is not obviously unreasonable, whereas assuming a non-stochastic and even constant convenience yield rate is more questionable.[3]

In a recent article analyzing long-term oil-related assets, Gibson and Schwartz (1990) take an important step to a more realistic model of the economy by introducing a stochastic convenience yield rate. The convenience yield rate is described by an Ornstein-Uhlenbeck process. The authors estimate the necessary input parameters for evaluation, and use a numerical method to approximate the value of long-term oil-related assets (futures contracts).

This paper adopts the Gibson-Schwartz assumptions on the economy. By using the equivalent martingale-approach, we derive analytical solutions to both futures contracts on the commodity and European call options written on the commodity.

2 Assumptions

The assumptions on the economy in this paper are similar to Gibson and Schwartz (1990), and will only be stated briefly here. The first assumption is that the spot price $S(t)$ of the commodity is described by a geometrical Brownian motion, i.e.,

$$dS(t) = \mu S(t)dt + \eta S(t)dW(t). \tag{1}$$

In Eq. (1), μ is the drift term, η is the volatility term, and $dW(t)$ is the increment of a standard Brownian motion, $W(t)$.[4]

The second assumption is that the instantaneous net marginal convenience yield rate $\delta(t)$ on the commodity is described by an Ornstein-Uhlenbeck

[3] See, e.g., Brennan (1991), Gibson and Schwartz (1990), Kemna (1987), and Pindyck (1988 a).

[4] The assumption may equivalently be stated as

$$S(T) = S(t)\exp\left\{(\mu - \tfrac{1}{2}\eta^2)(T-t) + \eta\int_t^T dW(s)\right\}$$

where t and $T > t$ are different dates.

process, i.e.,

$$d\delta(t) = k(\alpha - \delta(t))dt + \sigma dZ(t). \tag{2}$$

In Eq. (2), α is the long range mean to which $\delta(t)$ tends to revert, k is the speed of adjustment, σ is the volatility term, and $dZ(t)$ is the increment of a standard Brownian motion, $Z(t)$.[5] We define

$$X(t) \equiv \int_0^t \delta(s)ds \tag{3}$$

where $X(t)$ represents the cumulative convenience yield rate from date 0 to date t.[6]

The third assumption is that the risk free interest rate r is constant through time, and the the fourth assumption is that there is a constant market price λ per unit of convenience yield risk.

By invoking the usual assumptions of perfect frictionless markets and absence of risk-free arbitrage opportunities, the market value of any contingent claim $B(S, \delta, t)$ must satisfy the partial differential equation

$$\tfrac{1}{2}\eta^2 S^2 B_{SS} + \rho\eta\sigma S B_{S\delta} + \tfrac{1}{2}\sigma^2 B_{\delta\delta} + (r-\delta)SB_S + (k(\alpha-\delta)-\lambda\sigma)B_\delta + B_t - rB = 0 \tag{4}$$

---

[5]The assumption may alternatively be written as

$$\delta(T) = \theta\delta(t) + (1-\theta)\alpha + \sigma e^{-kT}\int_t^T e^{ks}dZ(s)$$

where we define $\theta \equiv e^{-k(T-t)}$, c.f. Merton (1990) Eqs. (5.108) and (5.109). $\delta(T)$ is normally distributed with mean and variance

$$\begin{aligned} E_t[\delta(T)] &= \theta\delta(t) + (1-\theta)\alpha \\ Var_t[\delta(T)] &= \frac{\sigma^2}{2k}(1-\theta^2) \end{aligned}$$

respectively.

[6]It is shown in Appendix A that

$$\begin{aligned} X(T) &= X(t) + (1-\theta)\tfrac{1}{k}(\delta(t)-\alpha) + \alpha(T-t) \\ &+ \tfrac{1}{k}\sigma\int_t^T dZ(s) - \tfrac{1}{k}\sigma e^{-kT}\int_t^T e^{ks}dZ(s) \end{aligned}$$

where t and $T > t$ are alternative dates, and θ is defined in footnote 5. $X(T)$ is normally distributed.

c.f. Eq. (4) in Gibson and Schwartz (1990). In Eq. (4), ρ is the correlation coefficient between the increments $dW(t)$ and $dZ(t)$ of the two Brownian motions.

3 Risk Adjustment and Evaluation

Let

$$Y(T) \equiv Y(T, S(T), X(T)) \tag{5}$$

be the pay-off from a contingent claim at the maturity date T. The main result from the equivalent martingale theory[7] is that the current value of this contingent claim may be expressed by

$$V_t[Y(T)] = e^{-r(T-t)} E_t^*[Y(T)] \tag{6}$$

where $E_t^*[\cdot]$ represents the expectation taken under the equivalent martingale probability measure. Note that by inserting $Y(T) = 1$ into the evaluation formula just above, we obtain the current value of a one-dollar discount bond with time to maturity $T - t$.

In the economy outlined in the section just above, the relation between the true probability measure and the martingale probability measure is[8]

$$dW(t) = dW^*(t) - \left(\tfrac{\mu+\delta(t)-r}{\eta}\right) dt, \tag{7}$$

$$dZ(t) = dZ^*(t) - \lambda dt. \tag{8}$$

The fraction in Eq. (7), $(\mu + \delta(t) - r)/\eta$, may be interpreted as the market price per unit of oil risk, and corresponds to the definition of λ' in Gibson and Schwartz (1990) footnote 7. In Eq. (8) above, λ represents the market

[7]See, e.g., Aase (1988), Cox and Ross (1976), Harrison and Kreps (1979), and Harrison and Pliska (1981).

[8]Eq. (7), combined with Eq. (3), implies

$$\int_t^T dW(s) = \int_t^T dW^*(s) - \tfrac{\mu-r}{\eta}(T-t) - \tfrac{1}{\eta}(X(T) - X(t)).$$

price per unit of convenience yield risk.[9]

The current value of the contingent claim is found as follows: First, insert Eqs. (7) and (8) into the arguments $S(T)$ and $X(T)$ of Y in Eq. (5) by using footnotes 4 and 5. Second, express the future pay-off $Y(T)$ as a stochastic integral over the processes $W^*(t)$ and $Z^*(t)$. Third, apply the evaluation formula in Eq. (6) by taking the expectation, and discounting back at the risk-free rate, r.[10]

4 A Self-Financing Portfolio

Consider a portfolio with initial value $P(0) = S(0)$ at date 0, and where the convenience yield is continuously reinvested. The value of this self-financing portfolio at the future date T is

$$P(T) = \exp\left\{\int_0^T \delta(s)ds\right\} S(T). \tag{9}$$

[9]By inserting Eq. (7) into Eq. (1), and Eq. (8) into Eq. (2), and some rearranging, we obtain the two transformed "risk-neutral" processes

$$\begin{aligned}\frac{dS(t)}{S(t)} &= (r - \delta(t))dt + \eta dW^*(t)\\ d\delta(t) &= k\left(\left(\alpha - \frac{\sigma\lambda}{k}\right) - \delta(t)\right)dt + \sigma dZ^*(t)\end{aligned}$$

stated in Jamshidian and Fein (1990).

[10]The transformation in Eqs. (7) and (8) will not be proved directly, but are direct consequences of the Girsanov theorem, see, e.g., Cox and Huang (1989) p. 276. However, we have already noted that the general evaluation formula in Eq. (6) provides consistent market prices on riskless dicount bonds. Furthermore, we show in the two following sections that the evaluation procedure also provides consistent market prices for two other self-financing portfolios, i.e., a commodity portfolio where the convenience yield is continuously reinvested, and a claim on future delivery of the commodity.

In our economy, with two dimensions of uncertainty (in addition to the time dimension), we claim that every future pay-off $Y \equiv Y(T, S(T), X(T))$ can be replicated by a dynamic self-financing portfolio consisting of the three basic portfolios described just above. Furthermore, we claim that our price system, by which we obtain consistent market prices for the three basic self-financing portfolios, can be extended to evaluate every contingent claim.

To prevent risk-free arbitrage opportunities, the value at the current date t of receiving this self-financing portfolio at the future date T must be

$$V_t[P(T)] = P(t). \tag{10}$$

Both as an illustration and as a partial verification of Eqs.(7) and (8), we evaluate this simple contingent claim by the procedure outlined above. First, rewrite the future pay-off in Eq. (9) by using the defition of $X(t)$ in Eq. (3),

$$P(T) = \exp\{X(T)\} S(T).$$

Next, use the expression of $S(T)$ stated in footnote 4

$$P(T) = \exp\left\{(\mu - \tfrac{1}{2}\eta^2)(T-t) + \eta \int_t^T dW(s) + X(T)\right\} S(t).$$

Insert Eq. (7), use footnote 8 and the definition of $P(t)$, and obtain

$$P(T) = \exp\left\{(r - \tfrac{1}{2}\eta^2)(T-t) + \eta \int_t^T dW^*(s)\right\} P(t).$$

By applying the evaluation rule in Eq. (6) above, i.e., by taking the expectation of the expression just above, and discounting back at the risk-free rate of return, we get

$$\begin{aligned} V_t[P(T)] &= e^{-r(T-t)} E_t^*[P(T)] \\ &= e^{-r(T-t)} E_t^*\left[\exp\left\{(r - \tfrac{1}{2}\eta^2)(T-t) + \eta \int_t^T dW^*(s)\right\}\right] P(t) \\ &= P(t) \end{aligned}$$

which is the desired result.[11]

In the typical case, the evaluation of a contingent claim is far more complicated. The derivations of the main results, presented in the following sections, are thus relegated to the appendixes.

[11] Inserting the partial derivatives of P

$$\begin{aligned} P_S &= \tfrac{P}{S} \\ P_{SS} &= P_\delta = P_{S\delta} = P_{\delta\delta} = 0 \\ P_t &= \delta P \end{aligned}$$

into Eq.(4) verifies that $P(t)$ satisfies the PDE.

5 The Value of a future delivery

Theorem 1 *The current value (at date t) of a claim on a future delivery of the commodity on the future date T is*

$$\begin{aligned} V_t[S(T)] &= S(t)\exp\Big\{\Big[-\alpha + \tfrac{1}{k}(\sigma\lambda - \sigma\eta\rho) + \tfrac{1}{2}(\tfrac{1}{k})^2\sigma^2\Big](T-t) \\ &- \tfrac{1}{k}\Big[\delta(t) - \alpha + \tfrac{1}{k}(\sigma\lambda - \sigma\eta\rho) + (\tfrac{1}{k})^2\sigma^2\Big](1-\theta) \\ &+ \tfrac{1}{2}(\tfrac{1}{k})^2\tfrac{\sigma^2}{2k}(1-\theta^2)\Big\} \end{aligned}$$

where $\theta \equiv e^{-k(T-t)}$.

Proof: See Appendix B and C.

The absence of risk-free arbitrage opportunities implies that the futures price F is determined by the relation

$$V_t[S(T) - F] = 0. \tag{11}$$

We may thus conclude that the futures price on a contract on the commodity with time to maturity $(T-t)$ is

$$F(T-t, S, \delta) = e^{r(T-t)}V_t[S(T)] \tag{12}$$

where the argument $T-t$ is time to maturity, and $V_t[S(T)]$ is given by Theorem 1 above.

Consider the hypothetical case where the speed of adjustment k is large. The interpretation is that the force pushing the convenience yield rate $\delta(t)$ back to its mean α is strong, c.f. Eq. (2). With k sufficiently large, this means that the stochastic property of $\delta(t)$ will vanish. From Theorem 1, it is straightforward to verify that

$$\lim_{k\to\infty} V_t[S(T)] = e^{-\alpha(T-t)}S(t) \tag{13}$$

which is the evaluation formula in the case of a constant proportional convenience yield rate $\delta(t) = \alpha$. Eq. (13) translates into the futures price being $F = e^{(r-\alpha)(T-t)}$.

6 The European Call Option

Theorem 2 *The current value (at date t) of an European call option with exercise price K and maturity date T is*

$$V_t[(S(T)-K)^+] = V_t[S(T)]N[d_1] - e^{-r(T-t)}KN[d_2]$$

where $N[\cdot]$ is the standard cumulative normal distribution, and we define

$$\begin{aligned}
d_1 &\equiv \frac{\ln\left(V_t[S(T)]/K\right) + r(T-t) + \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}} \\
d_2 &\equiv \frac{\ln\left(V_t[S(T)]/K\right) + r(T-t) - \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}} \\
\hat{\sigma}^2 &\equiv \left(\eta^2 - 2\tfrac{1}{k}\sigma\eta\rho + (\tfrac{1}{k})^2\sigma^2\right)(T-t) \\
&+ 2\left((\tfrac{1}{k})^2\sigma\eta\rho - (\tfrac{1}{k})^3\sigma^2\right)(1-\theta) \\
&+ (\tfrac{1}{k})^2\tfrac{\sigma^2}{2k}(1-\theta^2)
\end{aligned}$$

Proof: See Appendix D.

By examining Theorem 2, we see that the expressions have a structure similar to the famous Black-Scholes option pricing formula. As a special case of Theorem 2, we have

$$\begin{aligned}
&\lim_{k\to\infty} V_t[(S(T)-K)^+] \\
= \;& e^{-\alpha(T-t)}S(t)N\left[\frac{\ln(S(t)/K) + (r-\alpha+\frac{1}{2}\eta^2)(T-t)}{\eta\sqrt{T-t}}\right] \\
- \;& e^{-r(T-t)}KN\left[\frac{\ln(S(t)/K) + (r-\alpha-\frac{1}{2}\eta^2)(T-t)}{\eta\sqrt{T-t}}\right] \qquad (14)
\end{aligned}$$

which corresponds to the option value when the convenience yield rate is constant, $\delta(t) = \alpha$. With $\alpha = 0$, we have the standard Black-Scholes option pricing formula.

By applying the evaluation formula in Eq. (6) to the identity

$$(S(T)-K)^+ - (K-S(T))^+ = S(T) - K,$$

and inserting the result in Theorem 2, we obtain

$$V_t[(K-S(T))^+] = -V_t[S(T)]N[-d_1] + e^{-r(T-t)}KN[-d_2].$$

The expression just above represents the current value of an European put option with time to maturity $T-t$ and exercise price K.

7 Conclusion

This paper presents the formula for the futures price and the European call option when the commodity price follows a geometrical Brownian motion, and the convenience yield rate follows an Ornstein-Uhlenbeck process. The results are natural generalizations of the standard Black-Scholes economy where the underlying asset pays a constant proportional dividend.

The formulas provide useful benchmarks when considering some more complex contingent claims for which no closed form solutions are known. Moreover, by knowing the equivalent martingale measure, under which contingent claims prices may be represented as the discounted expected future pay-off, simulation techniques may be easily implemented to approximate the current market value, particularly for European style derivative assets.[12]

[12]See, e.g., Boyle (1977).

A The derivation of $X(T) - X(t)$

Eq. (2) implies

$$\begin{aligned} \int_t^T d\delta(s) &= \int_t^T k(\alpha - \delta(s))ds + \int_t^T \sigma dZ(s) \\ &= k\alpha(T-t) - k\int_t^T \delta(s)ds + \sigma \int_t^T dZ(s). \end{aligned} \tag{15}$$

It follows that

$$\delta(T) - \delta(t) = \int_t^T d\delta(s). \tag{16}$$

Insert Eq. (16) and the implication of the definition in Eq. (3),

$$X(T) - X(t) \equiv \int_t^T \delta(s)ds, \tag{17}$$

into Eq. (15), and obtain

$$\delta(T) - \delta(t) = k\alpha(T-t) - k\left(X(T) - X(t)\right) + \sigma \int_t^T dZ(s). \tag{18}$$

Inserting the equation from footnote 5 into the lefthand side of Eq. (18), and rearranging, leads to the result stated in footnote 6.

B The derivation of $V_t[S(T)]$

In this appendix, we evaluate a claim on a future delivery of the commodity, i.e.

$$V_t[S(T)] = e^{-r(T-t)}E_t^*[S(T)]. \tag{19}$$

From footnote 4, it follows that the discounted future pay-off may be expressed as

$$e^{-r(T-t)}S(T) = S(t)\exp\left\{(\mu - r - \tfrac{1}{2}\eta^2) + \int_t^T dW(s).\right\} \tag{20}$$

We want to express $\eta \int_t^T dW(s)$ by the two processes $W^*(t)$ and $Z^*(t)$. Multiply through the expression in footnote 8 by η, and use the equation in footnote 6 to substitute for $X(T) - X(t)$. Some rearranging leads to

$$\begin{aligned} \eta \int_t^T dW(s) &= -(\mu - r + \alpha - \tfrac{1}{k}\sigma\lambda)(T-t) \\ &+ \tfrac{1}{k}(\alpha - \delta(t) - \tfrac{1}{k}\sigma\lambda)(1-\theta) \\ &+ \eta \int_t^T dW^*(s) - \tfrac{1}{k}\sigma \int_t^T dZ^*(s) \\ &+ \tfrac{1}{k}\sigma e^{-kT} \int_t^T e^{ks} dZ^*(s). \end{aligned} \tag{21}$$

By inserting Eq. (21) into Eq. (20), we may express the discounted future pay-off as

$$\begin{aligned} e^{-r(T-t)}S(T) &= S(t)\exp\Big\{-\left(\tfrac{1}{2}\eta^2 + \alpha - \tfrac{1}{k}\sigma\lambda\right)(T-t) \\ &\quad +\tfrac{1}{k}\left(\alpha - \delta(t) - \tfrac{1}{k}\sigma\lambda\right)(1-\theta) \\ &\quad +\eta \int_t^T dW^*(s) - \tfrac{1}{k}\sigma \int_t^T dZ^*(s) \\ &\quad +\tfrac{1}{k}\sigma e^{-kT} \int_t^T e^{ks} dZ^*(s)\Big\} \\ &\equiv S(t)\exp\{z^*\} \end{aligned} \tag{22}$$

where we for notational convenience denote the exponent by z^*. It follows from stochastic calculus that

$$\begin{aligned} \hat{\mu} \equiv\ & E_t^*[z^*] \\ & -\left(\tfrac{1}{2}\eta^2 + \alpha - \tfrac{1}{k}\sigma\lambda\right)(T-t) \\ & +\tfrac{1}{k}\left(\alpha - \delta(t) - \tfrac{1}{k}\sigma\lambda\right)(1-\theta). \end{aligned} \tag{23}$$

Furthermore, we have

$$E_t^*\left[\left(\eta \int_t^T dW^*(s)\right)^2\right] = \eta^2(T-t) \tag{24}$$

$$E_t^*\left[\left(\tfrac{1}{k}\sigma \int_t^T dZ^*(s)\right)^2\right] = (\tfrac{1}{k})^2\sigma^2(T-t) \tag{25}$$

$$E_t^*\left[\left(\tfrac{1}{k}\sigma e^{-kT} \int_t^T e^{ks} dZ^*(s)\right)^2\right] = (\tfrac{1}{k})^2 \tfrac{\sigma^2}{2k}(1-\theta^2) \tag{26}$$

$$E_t^*\left[\left(\eta\int_t^T dW^*(s)\right)\left(\tfrac{1}{k}\sigma\int_t^T dZ^*(s)\right)\right] = \tfrac{1}{k}\sigma\eta\rho(T-t) \quad (27)$$

$$E_t^*\left[\left(\eta\int_t^T dW^*(s)\right)\left(\tfrac{1}{k}\sigma e^{-kT}\int_t^T e^{ks}dZ^*(s)\right)\right] = (\tfrac{1}{k})^2\sigma\eta\rho(1-\theta) \quad (28)$$

$$E_t^*\left[\left(\tfrac{1}{k}\sigma\int_t^T dZ^*(s)\right)\left(\tfrac{1}{k}\sigma e^{-kT}\int_t^T e^{ks}dZ^*(s)\right)\right] = (\tfrac{1}{k})^3\sigma^2(1-\theta) \quad (29)$$

Eqs. (23) - (29) implies that the variance of the exponent z^* in Eq. (22) above is

$$\begin{aligned}\hat{\sigma}^2 &\equiv E_t^*[(z^*)^2] - (E_t^*[z^*])^2 \\ & \left(\eta^2 - 2\tfrac{1}{k}\sigma\eta\rho + (\tfrac{1}{k})^2\sigma^2\right)(T-t) \\ &+ 2\left((\tfrac{1}{k})^2\sigma\eta\rho - (\tfrac{1}{k})^3\sigma^2\right)(1-\theta) \\ &+ (\tfrac{1}{k})^2\tfrac{\sigma^2}{2k}(1-\theta^2).\end{aligned} \quad (30)$$

The exponent z^* is normally distributed, and the right hand side of Eq. (22) is hence log-normally distributed. The value of the contingent claim is thus

$$\begin{aligned}V_t^*[S(T)] &= E_t^*\left[e^{-r(T-t)}S(T)\right] \\ &= E_t^*\left[S(t)\exp\{z^*\}\right] \\ &= S(t)\exp\left\{\hat{\mu} + \tfrac{1}{2}\hat{\sigma}^2\right\}\end{aligned} \quad (31)$$

where we have applied the standard formula for the expected value of a log-normal random variable. Inserting $\hat{\mu}$ and $\hat{\sigma}^2$ into Eq. (31) leads to the result stated as Theorem 1.

C $V_t[S(T)]$ satisfies the PDE

This appendix shows that the the current value of a claim on future delivery of the commodity, stated as Theorem 1 above, satisfies the partial differential equation in Eq. (4). For notational convenience, define for the moment

$$\Psi(t, S, \delta) \equiv V_t[S(T)].$$

The partial derivatives of Ψ are as follows:

$$\Psi_S = \tfrac{\Psi}{S} \quad (32)$$

$$
\begin{aligned}
\Psi_{SS} &= 0 && (33)\\
\Psi_{\delta} &= -(1-\theta)\tfrac{1}{k}\Psi && (34)\\
\Psi_{\delta\delta} &= (1-\theta)^2(\tfrac{1}{k})^2\Psi && (35)\\
\Psi_{S\delta} &= -(1-\theta)\tfrac{1}{k}\tfrac{\Psi}{S} && (36)\\
\Psi_{t} &= \alpha-\tfrac{1}{k}\sigma\lambda+\tfrac{1}{k}\sigma\eta\rho-\tfrac{1}{2}(\tfrac{1}{k})^2\sigma^2 && (37)\\
&+ \delta\theta-\alpha\theta+\tfrac{1}{k}\sigma\lambda\theta-\tfrac{1}{k}\sigma\eta\rho\theta+(\tfrac{1}{k})^2\sigma^2\theta-\tfrac{1}{2}(\tfrac{1}{k})^2\sigma^2\theta^2
\end{aligned}
$$

By inserting Eqs. (32) - (37) into Eq. (4), it is straightforward to verify that $\Psi(t,S,\delta)\equiv V_t[S(T)]$ satisfies the partial differential equation.

D The derivation of $V_t[(S(T)-K)^+]$

In this appendix, we evaluate an European call option written on the commodity with time to maturity $T-t$ and exercise price K. Apply the evaluation formula in Eq. (6)

$$V_t\left[(S(T)-K)^+\right]=e^{-r(T-t)}E_t^*\left[(S(T)-K)^+\right]. \qquad (38)$$

By rearranging, and using the definition in Eq. (22), we may express the right hand side of Eq. (38) as

$$V_t\left[(S(T)-K)^+\right]=E_t^*\left[\left(S(t)e^{z^*}-e^{-r(T-t)}K\right)^+\right]. \qquad (39)$$

Recall from Appendix B that the exponent z^* is normal with expected value $\hat{\mu}$ and variance $\hat{\sigma}^2$, c.f. Eqs. (23) and (30).

By using standard results used for evaluating the classical Black-Scholes European call option, we get

$$
\begin{aligned}
& V_t\left[(S(T)-K)^+\right]\\
=\ & \exp\left\{\hat{\mu}+\tfrac{1}{2}\hat{\sigma}^2\right\}S(t)N\left[\frac{\ln(S(t)/K)+r(T-t)+\hat{\mu}+\hat{\sigma}^2}{\hat{\sigma}}\right]\\
-\ & \exp\left\{-r(T-t)\right\}KN\left[\frac{\ln(S(t)/K)+r(T-t)+\hat{\mu}}{\hat{\sigma}}\right]. \qquad (40)
\end{aligned}
$$

Recall from Eq. (31) in Appendix B that

$$V_t[S(T)]=\exp\left\{\hat{\mu}+\tfrac{1}{2}\hat{\sigma}^2\right\}S(t). \qquad (41)$$

By inserting Eq. (41) into Eq. (40), and rearranging, we obtain the result stated as Theorem 2.

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