Elements of economics of uncertainty and time with recursive utility

BY Knut K. Aase

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Institutt for foretaksøkonomi
Department of Business and Management Science

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Knut K. Aase *

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Abstract

We address how recursive utility affects important results in the theory of economics of uncertainty and time, as compared to the standard model, where the focus is on dynamic models in discrete time. Several puzzles associated with the standard theory are less puzzling with recursive utility, even if this type of preference representation seems close to the standard one at first sight. An inconsistency with the axioms behind the standard, separable and additive expected utility representation is pointed out and extended to also be relevant for recursive utility. The basic difference from the standard model is that recursive utility allows a form of separation of consumption substitution from risk aversion. This also means that the timing of resolution of uncertainty matters. In dynamic models, however, this turns out to be a rather crucial step.

KEYWORDS: Recursive utility, axioms, scale invariance, utility gradients, the equity premium puzzle, precautionary savings

JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

In this paper we discuss some elements of the economics of uncertainty and time in a discrete time setting, when individuals have preferences represented by recursive utility.

We start by a description of recursive utility of the Kreps-Porteus type, where certainty equivalents are determined by expected utility.

*The Norwegian School of Economics, 5045 Bergen Norway Telephone: (+47) 55959249. E-mail: Knut.Aase@NHH.NO.

We briefly discuss the axioms behind the both the standard, and the recursive preference representations. Some of the shortcomings of the standard model are pointed out, one of which related to a counterexample, showing a violation of the substitution axiom. This counterexample is also relevant for recursive utility, pointing to a deeper result in axiomatization theory and logic.

We restrict attention to a standard, scale invariant version of recursive utility, belonging to the Kreps-Porteus class. Here we present equilibrium risk premiums and the equilibrium short term, real interest rate, and illustrate by some calibrations to market data. We compare with the corresponding results of the standard model, and more than indicate that the "equity premium puzzle" is less puzzling with recursive utility.

We end by considering precautionary savings and related issues, and compare with the standard model. In these applications it also turns out to be important to be able to separate risk aversion from consumption substitution, a property of recursive utility that the standard model lacks.

The paper is organized as follows: Section 1 is an introduction, where we point out some weaknesses of the standard separable and additive expected utility representation in settings where consumption takes place in more than one period. In Section 2 we present the basic elements of scale invariant, recursive utility. It is emphasized precisely where recursive utility (RU) departs from the standard additive representation (EU). In some ways one can view this version of RU as the closest, non-trivial extension from a standard form of EU. In Section 4 we discuss the axioms; both those behind EU in the one-period and the dynamic version, and the axioms behind RU. In Section 4.1 we discuss the counterexample. In Section 5 we consider the implications of RU in a market economy, and in Section 6 we end the paper with a self-contained discussion of precautionary savings and related issues. In the appendix we attempt to explain the issue of early/late resolution of uncertainty.

2 The discrete time development

2.1 Introduction

The conventional asset pricing model in financial economics, the consumption-based capital asset pricing model (CCAPM) of Lucas (1978) and Breeden (1979), assumes a representative agent with a utility function of consumption that is the expectation of a sum of future discounted utility functions. The model has been criticized for several reasons. First, the conventional specification of utility can not separate the risk aversion from the elasticity of intertemporal substitution, while it would clearly be advantageous to disentangle these two conceptually different aspects of an individual's preferences. Second, the representation appears not "well founded" for *temporal* problems including uncertainty, in that derived preferences do not satisfy the substitution axiom, although this axiom is behind the very representation (e.g., Mossin (1969)). Third, it does not perform well empirically. Fourth, the agent has a myopic perspective, and treats every period as if it were the last one.

Nevertheless, this representation satisfies three of the most basic axioms of dynamic utilities, and the additive structure provides certain advantages and is simple to work with in many applied problems, in particular in deterministic settings.

The basic problem, however, seems to be that two agents having identical preferences over deterministic consumption plans must also have the same preferences. This fact leads to strange situations, where expected discounted utility is the same for different random consumption streams that obviously are very different in terms of risk, indicating that the additive nature of utility may be too limiting.

Example 1. The following simple example illustrates: Consider two random consumption sequences, $a_0 = 0, a_1, a_2, \cdots a_T$ and $b_0, b_1, b_2, \cdots b_T$. The random variables a_i, $i \geq 1$, are independent and identically distributed, where each a_i takes the values 0 or 1 with equal probability. The sequence b is determined in terms of a as $b_0 = a_0, b_1 = b_2 = \cdots = b_T = a_1$. In other words, for the consumption stream a consumption in each period t is is determined by the toss of a fair coin at the beginning of the period, and the tosses are independent, while for the consumption stream b everything depends on what happens on the first toss of the coin. If the results in 0, consumption will be 0 ever afterwards, while if this toss results in 1, consumption will be 1 in all the consecutive periods.

With expected, additive utility $U_0(a) = U_0(b)$ regardless of the felicity index u_t in the representation $U_0(x) = E\big(\sum_{t=0}^{T} \beta^t u_t(x_t)\big)$ (here β is the "patience" factor). So the individual is indifferent between these two rather different consumption sequences. Since u_t is supposed to determine risk aversion, this is a rather odd, and far from intuitive result, since obviously plan b will be considered significantly more risky than plan a by most people.

To illustrate this latter claim, suppose we concentrate future consumption to time 1. Then time-1 consumption of the a-plan, $\sum_{t=0}^{T} a_t$, is binomially distributed $B(T, \frac{1}{2})$, while time-1 consumption of plan b, $\sum_{t=0}^{T} b_t$, takes the value 0 or T with probability 1/2 each. In this case the random variable

$\sum_{t=0}^{T} b_t$ happens to be a mean preserving spread of $\sum_{t=0}^{T} a_t$ in the sense of Rothschild and Stiglitz (1970)[1], meaning that all risk averters prefer $\sum_{t=0}^{T} a_t$ to $\sum_{t=0}^{T} b_t$. $\square$

While expected utility seems to work well in the one period setting, with consumption taking place at the end of the period only, with consumption in several periods, or consumption at more than one point in time, the additive and separable utility representation is faced with some serious problems.

The above example, along with backward recursion, can be used to show that with recursive utility of the Kreps-Porteus class and felicity index $u(\cdot)$ related to the certainty equivalent, as long as this function is strictly concave, the agent will strictly prefer the sequence a to the sequence b.

Another issue is the timing of the resolution of uncertainty. For example, suppose for plan a that all the independent coin tosses were performed at the beginning. While the agent of the standard model would be indifferent to this modification and plan a, this will not be so with recursive utility. In the models we consider, there will be "early resolvers" and "late resolvers", where the former would prefer this modification to plan a, and the latter would prefer a to the modification.

We focus on uncertainty in this paper, but recursive utility has been used for deterministic models in macro economics as well. For example, for the *Ramsey optimal growth problem* the standard model leads to problems, notably among them is the *impatience problem*. Here a form of recursive utility, the Epstein-Hymes utility, can be shown to solve this puzzle (see e.g., Becker and Boyd (1997), Koopmans (1960)).

3 Recursive utility

The basic notions are roughly summarized as follows: First consider a riskless economy, where preferences over consumption sequences $(c_0, c_1, \cdots, c_T)$ are characterized by Koopmans' (1960) time aggregator f, which takes into account both the present (t) and the future. This framework is then generalized to evaluate uncertain consumption sequences essentially by replacing the second argument in f by the period t certainty equivalent of the probability distribution over all possible consumption continuations. The resultant class of recursive preferences may be characterized as

$$U_t(c_t, c_{t+1}, \cdots, c_T) = f_{y_t}(c_t, m_t(U_{t+1}(c_{t+1}, c_{t+2} \cdots, c_T))),$$

[1] or Blackwell (1951).

where $m_t(\cdot)$ describes the certainty equivalent function based on the conditional probability distribution over consumption sequences beginning in period $t+1$, and $y_t = (c_0, c_1, \cdots, c_{t-1})$ represents the past. We use the notation $U_t = U_t(c_t, c_{t+1}, \cdots, c_T)$ from now on.

Recursive preferences have an axiomatic underpinning in the basic work in the field by Kreps and Porteus (1978). With reference to that article, we assume Axioms 2.1 (preference relation), 2.2 (continuity), 2.3 (the substitution axiom) and 3.1 (temporal consistency). This gives preference for early or late resolution of uncertainty depending on the convexity or concavity of the aggregator, properly defined, in its second argument (see below). In addition we will assume Axiom 6.1 (Payoff history independence), which removes y_t as an argument in f.

Such preferences are dynamically consistent, Axiom 3.1 in Kreps and Porteus (1978))[2].

3.1 The aggregator

The general form of the aggregator is the following

$$U_t = f(c_t, m_t) = v^{-1}((1-\beta)v(c_t) + \beta v(m_t)),\ t < T,\ U_T = c_T, \tag{1}$$

where v is a felicity index with inverse function v^{-1}, m_t is a conditional certainty equivalent as of time t, and β is a parameter linked to patience satisfying $0 < \beta < 1$, with impatience rate δ defined via $\beta = 1/(1+\delta)$. When the parameter β is large, the agent is patient in that she puts more weight on the future utility and less weight on the present. Also the larger the impatience rate δ, the more impatient is the agent, and the smaller is β.

So, where does such an aggregator come from? The standard separable and additive expected utility representation has an ordinally equivalent version which, when normalized, can be expressed in recursive form. For example is the representation

$$U_t = E_t\Big[\sum_{s=t}^{T-1} \beta^{s-t} v(c_s) + \frac{\beta^{T-t}}{1-\beta} v(c_T)\Big] \tag{2}$$

ordinally equivalent to the recursive version in (1), provided the conditional certainty equivalent $m_t = v^{-1}(E_t(v(U_{t+1})))$ is the one of expected utility with utility (felicity) index v.

[2]In the infinite horizon case the Axiom "Recurssivity" in Chew and Epstein (1991) is essentially identical to the notion of dynamic consistency, as outlined in Johnsen and Donaldson (1985).

Thus, in order to deviate, in a non-trivial way, from the standard, additive representation of preferences, it is assumed that the conditional certainty equivalent can be represented as above, but with a different felicity index u: $m_t = u^{-1}(E_t(u(U_{t+1})))$, $u \neq v$. This turns out to be an important step, since consumption substitution in a deterministic world is something very different from risk aversion, where the latter only makes sense under uncertainty.

On the one hand this approach stays close enough to the standard, additive representation of preferences to still benefit from many of its useful properties and interpretations, on the other this step is significant enough to avoid some of its unrealistic and negative features. However, this generalization comes at a price of added complexity, as is naturally the case with most generalizations.

In this paper we employ the two standard functions v and u, defined up to affine transformations as $v(w) = \frac{1}{1-\rho}(w^{1-\rho} - 1)$ and $u(w) = \frac{1}{1-\gamma}(w^{1-\gamma} - 1)$, with inverse functions $v^{-1}(y) = ((1-\rho)y+1)^{\frac{1}{\rho-1}}$ and $u^{-1}(y) = ((1-\gamma)y+1)^{\frac{1}{\gamma-1}}$ respectively, the following scale invariant aggregator results from (1)

$$U_t = f(c_t, m_t) = ((1-\beta)c_t^{1-\rho} + \beta m_t^{1-\rho})^{\frac{1}{1-\rho}}, \tag{3}$$

where the conditional certainty equivalent m is given by

$$m_t = (E_t[U_{t+1}^{1-\gamma}])^{\frac{1}{1-\gamma}}.$$

The parameter $\gamma \geq 0$ corresponds to the agent's relative risk aversion in the standard one-period model (the time-less model), and has the same interpretation here. Similarly, in a deterministic setting the parameter $\rho \geq 0$, and $\frac{1}{\rho}$ is the elasticity of intertemporal substitution in consumption (EIS). These parameters correspond to different properties of the individual's preferences, and should be measured independently. In the standard, additive expected utility model $\gamma = \rho$, which turns out to be rather restrictive.

When $\rho = 1$, the felicity index $v(x) = \ln(x)$, and $U_t = m_t^{\beta} c_t^{1-\beta}$, and when $\gamma = 1$, then we have $u(x) = \ln(x)$, and $m_t = \exp(E_t[\ln(U_{t+1})])$.

The parameter β is the 'patience' factor, where $0 \leq \beta \leq 1$ as explained above. The impatience rate $\delta = -\ln(\beta)$ is typically used in continuous-time models, and is approximately equal to δ defined as $\delta = 1/\beta - 1$.

While preferences over deterministic consumption plans are solely determined by the function v, the limitation of expected additive, discounted utility in the presence of uncertainty rests on the fact that the function determining risk aversion also governs the purely deterministic development.

Recursive utility overcomes this latter problem, and many of the other problems mentioned as well, by simply separating v from u.

The version in (3) is known as the Epstein-Zin aggregator.

4 Back to the Axioms

In the above we have referred to the various axioms behind the preference relations, and here we return to this issue. As this topic can be rather complex in its full mathematical description, we shall limit ourselves an informal discussion, and refer to the literature for precise definitions of the underlying mathematical structure.

Starting with the axioms behind expected utility in a one-period model - the timeless case - with no consumption at the initial time, they can briefly be described as follows. Let $\mathcal{P}_{\mathcal{S}}$ be the set of all finite lotteries. The symbol $\oplus$ means lottery composition (mixing of probability distributions). The three fundamental axioms behind Eu-theory are the following:

Axiom 1. $\succeq$ is a preference relation on $\mathcal{P}_{\mathcal{S}}$.

Axiom 2. The Substitution Axiom. Given any three lotteries $p, q, r \in \mathcal{P}_{\mathcal{S}}$ where $p \succ q$ and $a \in (0, 1]$. Then $ap \oplus (1-a)r \succ aq \oplus (1-a)r$.

Axiom 3. The Archimedean axiom. Consider any $p, q, r \in \mathcal{P}_{\mathcal{S}}$ such that $p \succ q \succ r$. Then there exist numbers $a, b \in (0, 1)$ such that $ap \oplus (1-a)r \succ q \succ bp \oplus (1-b)r$.

Let C be the consumption space. Here we may think of it as the real line or a metric space. We then have the following:

Theorem 1 *Let $\succeq$ satisfy axioms 1, 2 and 3. Then there exists a function $u : C \rightarrow R$ such that*

$$p \succeq q \Leftrightarrow \sum_{x \in supp(p)} u(x)p(x) \geq \sum_{x \in supp(q)} u(x)q(x).$$

Moreover, if u represents $\succeq$ in this sense, then a function $\tilde{u} : C \rightarrow R$ also represents $\succeq$ in this sense if and only if there exist real numbers $c > 0$ and d such that $\tilde{u}(x) = cu(x) + d$ for all $x \in C$.

It is the Substitution Axiom that is instrumental in obtaining the additive form of the Eu-representation in probabilities. Consider the relationship $U(ap \oplus (1-a)q) = aU(p) + (1-a)U(q)$, where $U(p) = \sum_{x \in \mathrm{supp}(p)} u(x)p(x)$. This is what we mean when we say that a utility function U is additive in probabilities, and this is what this axiom provides. To see that Eu satisfies this, just consider

$$U(ap \oplus (1-a)q) = E_{ap\oplus(1-a)q}\, u = \sum_{x \in supp(p)\cup supp(q)} u(x)(ap \oplus (1-a)q)(x) =$$
$$\sum_{x \in supp(p)\cup supp(q)} u(x)(ap(x) \oplus (1-a)q(x)) =$$
$$a \sum_{x \in supp(p)} u(x)p(x) + (1-a) \sum_{x \in supp(q)} u(x)q(x) =$$
$$aE_p\, u + (1-a)E_q\, u = aU(p) + (1-a)U(q). \quad (4)$$

4.1 The RU-axioms

Moving to dynamics, properly reformulated, the three first axioms are still fundamental, where the Archimedean Axiom is strengthened so that the dynamic preference relation $\succcurlyeq_t$ is continuous (see Kreps and Porteus (1978) for the full formulation of this theory).

These three axioms, properly modified to a dynamic context, must be supplemented by *dynamic consistency*: Given a dynamic utility process $U_t(c)$, if for any t and any $c, c+x \in C$, $U_t(c+x) > U_t(c)$, then $U_0(c+x) > U_0(c)$. An agent with dynamically consistent preferences that prefers to add x to a consumption plan c at time t also prefers the plan $c+x$ to c at time zero.

With such a set of axioms the dynamic preference relation can be represented by a recursive utility of the type we have seen in the above. As mentioned, the consumer may have preference for early resolution of uncertainty, or the opposite. To see what is involved, we consider an ordinally equivalent version of the recursive utility function in (3). Recall that this utility function can be written

$$U_t = f(c_t, m_t) = \left((1-\beta)c_t^{1-\rho} + \beta(E_t[U_{t+1}^{1-\gamma}])^{\frac{1-\rho}{1-\gamma}}\right)^{\frac{1}{1-\rho}},$$

where the conditional certainty equivalent m is given by

$$m_t = (E_t[U_{t+1}^{1-\gamma}])^{\frac{1}{1-\gamma}}.$$

The utility functions U and V are ordinally equivalent if and only if there exists a unique increasing continuous function g such that $V = g(U)$. Two ordinally equivalent utility functions represent the same preferences. Consider the following ordinally equivalent version of the above U: $V = U^{1-\gamma}/(1-\gamma)$.

It can be written

$$V_t(c_t,\xi_t) = \frac{1}{1-\gamma}\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)E_t(V_{t+1})\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{1-\gamma}{1-\rho}}.$$

The connection between m and ξ is given by $m^{1-\gamma} = (1-\gamma)\xi_t$. From the simple conditional expctation $E_t(V_{t+1})$ in the second argument of V_t, one may be lead to think that this corresponds to risk-neutrality, however, this is incorrect. Recall, U and V are ordinally equivalent, and U is risk averse, hence V is as well. This particular ordinally equivalent version we refer to as the *non-normalized* one.

With recursive utility there is a well-defined notion of the time at which uncertainty is revealed, and although for compound lotteries there is also an implicit axiom perceiving them as equivalent to the one-shot lottery they reduce to at a single time, there is no axiom which says that uncertainties at two different times are equivalent.

Given any consumption plans c' and c'' in the domain of V and any $\alpha \in (0,1)$, let $c^\alpha = \alpha c' \oplus (1-\alpha)c''$. The RU-agent is supposed to choose form the space of random, temporal consumption plans.

This means that, in our notation, the following sum

$$\alpha V_t\big(c_t, E(V_{t+1}(c'))\big) + (1-\alpha)V_t\big(c_t, E(V_{t+1}(c''))\big) \tag{5}$$

can be strictly larger than, or strictly smaller than

$$V_t\big(c_t, E(V_{t+1}(c^\alpha))\big), \tag{6}$$

while for expected additive utility we have seen that these two representations must be equal. If the sum in (5) is larger than the expression in (6), we say that the agent has preference for early resolution of uncertainty, if the sum is smaller, the agent has preference for late resolution of uncertainty.

There is a more general definition of early/late resolution of uncertainty in Skiadas (2009), which we return to in Appendix 1.

Let $\xi_t := E_t(V_{t+1})$. Then the result in Kreps and Porteus (1978), Theorem 3, applied to our version, is that if the function $V_t(c_t,\xi_t)$ is convex (respctively concave) in its second argument for every $t < T$, then $V_0(\cdot)$ represents preference for early (respectively late) resolution of uncertainty. If the function is affine in ξ the individual is indifferent, and we have an ordinal equivalent to separable and additive expected utility.

We use this result and demonstrate (in Appendix 1) that for our version of recursive utility, the scale invariant one, when $\gamma > \rho$, the agent has preference for early resolution of uncertainty, when $\rho > \gamma$ the agent has preference for late resolution, and when $\gamma = \rho$ she is indifferent. In this appendix we attempt a proof this theorem using the more general definition of early/late resolution of uncertainty mentioned above.

In the general theory the aggregator depends also on the history represented by y_t at time t. We have assumed history independence in the above. Formally, we have added an axiom about history independence, leaving us with four basic axioms behind our representation.

Here we come to an important part of our story. Provided we add a last axiom to the above four stating that the individual is indifferent between early and late resolution of uncertainty, then the preference relation can be represented by separable and additive expected utility of the form given in the representation (2).

Accordingly, this dynamic utility has an axiomatic underpinning as well. Notice that one of the above three axioms is the substitution axiom, adjusted to a dynamic setting.

In the next section we present an example, which may be hard to comprehend in the light of the above.

4.2 The additive expected utility: An example.

In preparation for the example, consider the following temporal decision problem: We are given a two period model, with consumption at times 0 and 1, denoted c_0 and c_1. Income in the two periods are denoted y_1 and y_2. We suppose that after c_0 has been chosen, all uncertainty is revealed. The problem is to solve

$$\max_{c_0 \geq 0, c_1 \geq 0} \{Eu(c_0) + \beta Eu(c_1)\} \quad \text{subject to} \quad c_0 + c_1 \leq y_0 + y_1. \tag{7}$$

Here y_0 is observed before the decision is taken, y_1 is a random variable and so is c_1. We suppose that the five axioms of the previous section hold, so that this additive form of expected utility represents the preferences of the agent. In other words, this dynamic utility representation has a sound axiomatic underpinning.

Since there is uncertainty only in the last period, the above problem can be written

$$\max_{c_0 \geq 0} \int_y \big(u(c_0) + \beta u(y + y_0 - c_0)\big) dF_1(y) := U(p), \tag{8}$$

s.t. $0 \leq c_1 \leq y_0 + y_1 - c_0$. Here $F_1(y)$ is the cumulative probability distribution function of the random variable y_1, and p denotes the associated "lottery" (i.e., the probability distribution of y). Note that the max operation is out-

x	1	2
$p_1(x)$	0.5	0.5

Table 1: The consumption lottery p_1.

x	0.6	6.702
$p_2(x)$	0.5	0.5

Table 2: The consumption lottery p_2.

side the integral; the optimal c_0 has to be determined before the uncertainty about y_1 is revealed.

Example 2. Consider the above problem, and let us here refer to the five axioms behind the representation as the von Neumann-Morgenstern (vN-M) axioms.

With only two time points, every future action can be planned ahead at date 0, with no possibility to change one's mind.

Let $u_0(c_0) = \ln(c_0)$ and $u_1(c_1) = \ln(c_1)$, and the impatience factor $\beta = 1.0$. Let $y_0 = 0$, and consider the two time 1 lotteries in tables 1 and 2.

Here $U(p_1) = \max_{c_0} g_1(c_0)$ where the expected additive utility is given by the function $g_1(c_0)$ as follows:

$$g_1(c_0) := \ln(c_0) + \frac{1}{2}\ln(1 - c_0) + \frac{1}{2}\ln(2 - c_0).$$

This problem has solution $c_{01} = 0.6096$, which gives $U(p_1) = -0.8004$. Similarly $U(p_2) = \max_{c_0} g_2(c_0)$ where

$$g_2(c_0) := \ln(c_0) + \frac{1}{2}\ln(0.6 - c_0) + \frac{1}{2}\ln(6.702 - c_0).$$

This problem has solution $c_{02} = 0.3957$, which gives $U(p_2) = -0.8004$ as well. Thus the decision maker is indifferent between p_1 and p_2. Since her preferences obey the vN-M axioms, she is indifferent between each of these and any of their convex combinations $p := ap_1 \oplus (1-a)p_2$, for any $a \in [0, 1]$. The relationship in (4) holds in this two period problem as well. Accordingly the substitution axiom requires "additivity" in probability (mixtures) of the following kind

$$U(p) = U(ap_1 \oplus (1-a)p_2) = aU(p_1) + (1-a)U(p_2) = -0.8004. \quad (9)$$

But it is easy to show that the decision maker strictly prefers both p_1 and p_2 to p. There is an implicit axiom that the decision maker is indifferent between the compound lottery p for $a = \frac{1}{2}$ and the following 'one shot' lottery that it reduces to (see Raiffa (1968))

x	0.6	1	2	6.702
$p(x)$	0.25	0.25	0.25	0.25

Using this distribution, we can again compute $U(p) = \max_{c_0} g_p(c_0)$ where

$$g_p(c_0) := \ln(c_0) + \frac{1}{4}\ln(0.6 - c_0) + \frac{1}{4}\ln(1 - c_0) + \frac{1}{4}\ln(2 - c_0) + \frac{1}{4}\ln(6.702 - c_0).$$

This problem has solution $c_{03} = 0.4440$, which gives $U(p) = -0.8542$. All solutions are interior, so the various constraints are not binding. Hence the vN-M preferences do not satisfy (9), an underlying assumption, which is a contradiction. $\square$

The above problem was discovered by Mossin (1969) using a different example, and is also discussed in the textbook by Kreps (1988) using essentially the above example. It is further discussed and extended in Aase (2017-19). The papers that originally treated temporal resolution of uncertainty are Dreze and Modigliani (1972), Mossin (1969), and Spence and Zechhauser (1972).

Mossin (1969) points out ..."for temporal prospects, unlike time-less ones, representation of preferences in terms of an expected utility function may be inappropriate - and for reasons that are obvious once you think of it. On further reflection it is also apparent that in the real world *temporal* prospects, not time-less ones, are the rule rather than the exception".

According to Kreps (1988) ..."when uncertainty resolves at dates in the future after important decisions must be taken, then the use of the standard expected utility model is suspect and often quite wrong. We have seen this for the von Neumann-Morgenstern model. Similar conclusions can be obtained in settings appropriate for the Savage model. Since this includes many important economic/social decision making contexts that one is likely to encounter, there is obviously a need for care in applications, as well as in theory".

In models with no uncertainty, a temporal independence axiom is often recalled, stating that preferences over a consumption pair (c_0, c_1) do not depend upon the remaining consumption path over the $T - 2$ periods. Then some authors claim that this implies that utility $U(c)$ must be additive and separable, $U(c) = \sum u(c_t, t)$, where $u(\cdot, t)$ is the utility of consumption at time t. As demonstrated by Scott and Suppes (1958), even in the two-period case it may be necessary to go beyond this type of temporary independence axiom to obtain an additive and separable representation of preferences. In this case a sufficient condition is, for example, given in Fishburn (1970), Theorem 4.1 for the case with no uncertainty. As the author remarks, (p 46): "It can not be emphasized too strongly that additive utilities might not exist in some situations where their use seems attractive for ease in analysis. Possibly the best way to test the conditions of Theorem 4.1, it to try deliberately to find T-tuples that violate the condition."

Fishburn (1970) also covers the situation with uncertainty e.g., Theorem 11.4, and sufficient conditions for an additive representation are rather involved and strong. With this in mind, it is, perhaps, not surprising that examples can be found where these conditions are violated.

Let us see what results are obtained using recursive utility in the context of Example 2.

Example 3. Consider the same two consumption "lotteries" as in Example 2, but with recursive utility, in the same two period model. The utility is given by

$$U_0 = \Big((1-\beta)c_0^{1-\rho} + \beta\big(\exp(E(\ln U_1))\big)^{1-\rho}\Big)^{\frac{1}{1-\rho}},$$

where $U_1 = c_1$. Here we have set the relative risk aversion $\gamma = 1$ as well, for comparison. Furthermore, $\beta = 0.50$ corresponds to equal weighting of the utilities at the two points in time, as in Example 2. The non-normalized version is the ordinally equivalent $V = \ln(U)$, with aggregator

$$V_0 = \ln\Big((1-\beta)c_0^{1-\rho} + \beta\big(\exp(E(V_1))\big)^{1-\rho}\Big)^{\frac{1}{1-\rho}},$$

where $V_1 = \ln(c_1)$. With lottery p_1 we solve the maximization problem corresponding to (7), but with recursive utility V_0, for the two other preference parameters $\beta = 0.50$ and $\rho = 0.70$. We obtain the utility $V_{01} = -0.3984$ for the optimal consumption at time zero $c_{01} = 0.5892$. For the lottery p_2 the result is $V_{02} = -0.3476$ for $c_{02} = 0.3372$, while for the one-shot lottery p the result is $V_p = -0.3996$ for $c_p = 0.4010$. All the solutions are interior. Here we check that

$$\frac{1}{2}V_{01} + \frac{1}{2}V_{02} = -0.3730 > V_p = -0.3996,$$

in accordance with the above theory, since here $\gamma > \rho$, in which case the above inequality prescribes preference for early resolution of uncertainty.

Consider the same problem for the preference parameters $\rho = 1.2 > \gamma = 1$ and $\beta = 0.50$. The optimal consumption levels at time zero are, respectively $c_{01} = 0.6181$, $c_{02} = 0.4204$ and $c_p = 0.4622$, all interior solutions. The corresponding optimal utilities are given by $V_{01} = -0.4010$, $V_{02} = -0.4246$

and $V_p = -0.4403$, respectively. This means that $\frac{1}{2}V_{01} + \frac{1}{2}V_{02} = -0.4128 > V_p = -0.4403$. The example indicates preference for early resolution of uncertainty, despite of the fact that the theory predicts preference for late resolution of uncertainty. The same situation repeats itself for $\rho = 2 > \gamma = 1$ and $\beta = 0.50$.

Lastly we investigate the case when $\rho = \gamma = 1$ and $\beta = 0.50$ in this example, which corresponds exactly to Example 2. We then know from the above that the recursive (normalized) utility can here be written as $U_t = m_t^{\beta} c_t^{1-\beta}$, where the certainty equivalent $m_t = \exp(E_t(\ln U_{t+1}))$, which means that $V_t = \beta(E_t(V_{t+1})) + (1-\beta)\ln c_t$, and $V_1 = \ln(c_1)$.

The optimal consumption levels at time zero are, respectively $c_{01} = 0.6096$, $c_{02} = 0.3957$ and $c_p = 0.444$, all interior solutions, and the exact same optimal consumptions as in Example 2. The corresponding optimal utilities are given by $V_{01} = 0.6702$, $V_{02} = 0.6702$ and $V_p = 0.6524$ respectively. This gives that $\frac{1}{2}V_{01} + \frac{1}{2}V_{02} = 0.6702$, not equal to V_p. Instead the example indicates preference for early resolution of uncertainty, but should display indifference. $\square$

Hence contradiction in Example 2 is seen to also carry over to recursive utility when $\gamma = \rho$, i.e., when the two dynamic utilities are ordinally equivalent. This is not surprising, but noteworthy. The contradictary result when $\rho > \gamma$ is, on the other hand, surprising.

The contradictions revealed in examples 2 and 3 may call for some deeper thought into axiomatization theory and logic. A famous theorem of Kurt Gődel (1931) comes to mind. He demonstrated, to the shock of the few who could read his proof (including John von Neumann), that the ordinary arithmetics of integers could not be axiomatized in the way people normally thought it could: That theorems of mathematics could all be deduced from a set of axioms with the sole help of principles of logic. As Nagel and Newman (1960) formulate this: "In the light of his conclusions, no final systematization of many important areas of mathematics is attainable, and no absolutely impeccable guarantee can be given that many significant branches of mathematical thought are entirely free from internal contradictions".

This concern may be seen to also carry over to economics (and e.g., physics), since mathematics is used in many of its central parts.

5 Market consequences

We give a short sketch of dynamic equilibrium, and discuss briefly what the assumption about recursive utility adds to the standard theory. There is a rich literature on these topics, both in discrete time and in continuous-time modelling. In discrete time it of great importance to establish a closed form of the stochastic discount factor. In equilibrium the ratio of the Arrow-Debreu state prices p_{t+1}/p_t must equal the intertemporal marginal rate of substitution, which depends on the agent's preferences and equilibrium consumption plan. This was characterized in the fundamental papers of Epstein and Zin (1989-91). In continuous time the analogous references are Duffie and Epstein (1992a,b). In these developments dynamic programming were the basic tools behind the optimizations.

In Aase (2016a) the stochastic maximum principle was used to establish closed form expressions for risk premiums of risky assets and an expression for the equilibrium, short term real interest rate, in a continuous time model with continuous dynamics. These expressions were calibrated to various market data, and the results were promising, compared to the corresponding calibrations for the standard model.

Also optimal consumption and portfolio selection problems have been dealt with, see e.g., Schroder and Skiadas (1999), for continuous time, and Skiadas (2009) for discrete time problems. Similar problems related to insurance has been dealt with in Aase (2016b), also in continuous time.

5.1 The first order conditions

In order to determine an equilibrium we must solve the first order conditions of agent optimalization, and then determine prices such that markets clear.

The agent is characterized by a utility function U and an endowment process $e \in L$. The agent's problem is

$$\sup_{c \in L_+} U(c) \text{ subject to } E\big(\sum_{s=0}^{T} p_s c_s\big) \leq E\big(\sum_{s=0}^{T} p_s e_s\big)$$

where L is the space of adapted consumption processes, L_+ its positive cone, and p is the state price deflator (the Arrow-Debreu state price in units of probability).

The Lagrangian of the problem is

$$\mathcal{L}(c, \lambda) = U(c) - \lambda\, E\big(\sum_{s=0}^{T} p_s (c_s - e_s)\big)$$

where $\lambda > 0$ is the Lagrangian multiplier. Assuming U to be continuously differentiable, the gradient of U at c in the direction x is denoted by $\bigtriangledown U(c; x)$. This directional derivative is a linear functional, and by the Riesz Representation Theorem and e.g., dominated convergence, it is given by

$$\bigtriangledown U(c;x) = E\big(\sum_{s=0}^{T} \pi_s x_s\big).$$

Here π is the Riesz representation of $\bigtriangledown U(c;\cdot)$. The first order condition is

$$\bigtriangledown \mathcal{L}(c,\lambda;x) = 0 \text{ for all } x \in L.$$

This is equivalent to

$$E\big\{\sum_{s=0}^{t}(\pi_s - \lambda p_s)x_s\big\} = 0 \text{ for all } x \in L.$$

This implies that $\pi_t = \lambda p_t$ for all $t \leq T$.

Our next task is to characterize the Riesz representation π of U. When this is done, by the above result we have the state price in the economy modulo a constant.

5.2 The state prices in the economy

In order to characterize the state price in this economy, we need to find the Riesz representation π of the utility function U as explained in the last section.

When c is an equilibrium allocation, or the aggregate endowment in a representative agent economy, π_t has the interpretation of being the state price deflator at time t as demonstrated above.

Using directional derivatives and backward induction, we can show that the utility gradient is given by the following expression

$$\bigtriangledown U(c;x) = \bigtriangledown U_0(c;x) = E\Big\{\sum_{t=0}^{T} x_t\ f_c(c_t, m_{t+1}) \prod_{s=0}^{t-1} \frac{f_m(c_s, m_{s+1})}{h'(m_{s+1})} h'(U_{s+1})\Big\}, \quad (10)$$

from which it follows that the state price deflator is given as

$$\pi_t = f_c(c_t, m_{t+1}) \prod_{s=0}^{t-1} \frac{f_m(c_s, m_{s+1})}{h'(m_{s+1})} h'(U_{s+1}) \quad (11)$$

for $t = 0, 1, \cdots, T$. In (11) c is assumed optimal from now on. Notice how the agent lifts her perspective to take into account both the future via the term m_{t+1}, and the past via the product term, while the expected utility maximizer is just myopic ($\pi_t = u_c(c_t, t)$ when u is the felicity index).

The intertemporal marginal rate of substitution, or the stochastic discount factor, $\mathcal{M}_{t+1} = \pi_{t+1}/\pi_t$ in equilibrium, and is given by the formula

$$\mathcal{M}_{t+1} = \frac{f_c(c_{t+1}, m_{t+2})}{f_c(c_t, m_{t+1})} f_m(c_t, m_{t+1}) \frac{h'(U_{t+1})}{h'(m_{t+1})}. \tag{12}$$

Along the optimal consumption path $\mathcal{M}_{t+1} = \pi_{t+1}/\pi_t = p_{t+1}/p_t$, i.e., the ratio between the state prices at times $t+1$ and t.

5.3 The Stochastic Discount Factor

In order to find the stochastic discount factor we must compute the quantities in (12). These are

$$\frac{\partial}{\partial c} f(c_t, m_t) = (1-\beta) U_t^{\rho} c_t^{-\rho}, \quad \frac{\partial}{\partial m} f(c_t, m_t) = \beta U_t^{\rho} m_t^{-\rho}$$

and

$$\frac{h'(U_{t+1})}{h'(m_t)} = \frac{U_{t+1}^{-\gamma}}{m_t^{-\gamma}}.$$

This means that the stochastic discount factor takes the form

$$\mathcal{M}_{t+1} = \frac{\pi_{t+1}}{\pi_t} = \beta \Big(\frac{c_{t+1}}{c_t}\Big)^{-\rho} \Big(\frac{U_{t+1}}{m_t}\Big)^{\rho-\gamma}. \tag{13}$$

Let c signify optimal consumption, and W_t is the agent's wealth at time t, given by

$$W_t = \frac{1}{\pi_t} E_t\Big(\sum_{s=t}^{T} \pi_s c_s\Big). \tag{14}$$

Our definition of wealth W_t includes current consumption (dividend), so the gross real rate of return on the wealth portfolio over the period $(t, t+1)$ is

$$R_{t+1}^{W} := \frac{W_{t+1}}{W_t - c_t}. \tag{15}$$

By the definition in (15), it now follows by a string of manipulations that

$$\mathcal{M}_{t+1} = \beta^{\frac{1-\gamma}{1-\rho}} \Big(\frac{c_{t+1}}{c_t}\Big)^{-\rho\frac{1-\gamma}{1-\rho}} \big(R_{t+1}^{W}\big)^{\frac{\rho-\gamma}{1-\rho}}. \tag{16}$$

This expression has been the starting point for much of the literature on recursive utility in discrete time models, see e.g., Mehra and Donaldson (2008) and Cochrane (2008), among many others. This is the stochastic discount factor, first derived by Epstein and Zin (1989-91) in their seminal papers based on dynamic programming techniques.

5.4 The financial market

Having established the general, homogeneous recursive utility of interest, in this section we turn our attention to pricing restrictions relative to the given optimal consumption plan.

Suppose S_t is the price process (possibly adjusted for dividends) of any risky asset in this economy, with corresponding gross return $R^R_{t+1} := \frac{S_{t+1}}{S_t}$. Since we have a state price deflator π, there is no arbitrage in this economy if and only if $S_t \pi_t$ is a martingale. The martingale property implies the following pricing relation

$$S_t = \frac{1}{\pi_t} E_t\{\pi_{t+1} S_{t+1}\}$$

for any $t \in [0, T-1]$. This implies the pricing restriction

$$E_t\{\mathcal{M}_{t+1} R^R_{t+1}\} = 1. \tag{17}$$

From this it follows by the defining property of covariance that

$$-\frac{\mathrm{cov}_t(\mathcal{M}_{t+1}, R^R_{t+1})}{E_t(\mathcal{M}_{t+1})} = E_t(R^R_{t+1}) - R^f_{t+1}, \tag{18}$$

provided that we interpret the reciprocal of $E_t(\mathcal{M}_{t+1})$ as the gross rate of return on the riskless asset over the period $(t, t+1)$, i.e.,

$$R^f_{t+1} := \frac{1}{E_t(\mathcal{M}_{t+1})}. \tag{19}$$

This interpretation is seen from (18) to be correct, by replacing R^R_{t+1} by R^f_{t+1}, in which case

$$\mathrm{cov}_t(\mathcal{M}_{t+1}, R^f_{t+1}) = 0,$$

the defining property of the risk-less asset. The right-hand side of (18) is of course the *risk premium* of the risky asset.

The main question of interest is then the determination of prices, including risk premiums and the interest rate that makes the agent's behavior optimal.

We adopt the assumption that one can view exogenous income streams as dividends of some shadow asset. Then our model is valid if the market portfolio is expanded to include the new asset. While this is the most important addition, a few more portfolios must be included in order to be a reasonable proxy for a nation's wealth portfolio. We assume the latter is marketed, in which case W_t is the time-t wealth required to finance the consumption plan c from time t on; in other words (c, W) is considered to be a traded contract.[3]

5.5 Risk premiums and the interest rate

Based on the above, we can derive expressions for the equilibrium risk premiums and the equilibrium, real interest rate. To this end we use the pricing restriction $E_t\{\mathcal{M}_{t+1} R^R_{t+1}) = 1$, valid for any risky security R in the market, together with the relationship $\ln R^f_{t+1} = -\ln(E_t(\mathcal{M}_{t+1}))$.

By making the assumption that the the random variables of interest are jointly log-normally distributed, the analysis becomes particularly simple. Since the multinormal distribution has moments of all orders, and the conditional joint probability distribution is fully characterized by the conditional mean vector and conditional covariance matrix, our expressions depend on the first two moments of the various random variables involved. In Aase and Lillestøl (2015) it is demonstrated that deviations from normality may matter to some degree, but can in no way explain the equity premium puzzle. The results of this approach are as follows: The risk premium of any risky asset, denoted R, is given at any time t by the formula

$$E_t(\ln R^R_{t+1}) - \ln R^f_{t+1} = \rho \frac{1-\gamma}{1-\rho} \operatorname{cov}_t(\ln \frac{c_{t+1}}{c_t}, \ln R^R_{t+1}) + \frac{\gamma - \rho}{1-\rho} \operatorname{cov}_t(\ln R^W_{t+1}, \ln R^R_{t+1}) - \frac{1}{2} \operatorname{var}_t(\ln R^R_{t+1}). \quad (20)$$

[3]In reality the (c, W) is not traded, so the return to the wealth portfolio is not readily estimated from available data. However, see below.

The log-return on the risk-free asset takes the form

$$\ln R^f_{t+1} = \frac{1-\gamma}{1-\rho}\ln\big(\frac{1}{\beta}\big) + \frac{\rho(1-\gamma)}{1-\rho}\, E_t \ln\big(\frac{c_{t+1}}{c_t}\big) - \frac{1}{2}\frac{\rho^2(1-\gamma)^2}{(1-\rho)^2}\mathrm{var}_t(\ln \frac{c_{t+1}}{c_t})$$
$$+ \frac{\gamma-\rho}{1-\rho}E_t \ln R^W_{t+1} - \frac{1}{2}\frac{(\rho-\gamma)^2}{(1-\rho)^2}\mathrm{var}_t(\ln R^W_{t+1})$$
$$+ \rho\frac{1-\gamma}{1-\rho}\frac{\rho-\gamma}{1-\rho}\mathrm{cov}_t\big(\ln\big(\frac{c_{t+1}}{c_t}\big), \ln R^W_{t+1}\big). \quad (21)$$

The expressions in (20) and (21) are not relying on any approximations under the joint normality assumption.

In the expression for the risk premium the first term on the right-hand side corresponds to the consumption capital asset pricing mode (CCAPM) of Breeden (1979), while the next term corresponds to the market based CAPM of Mossin (1966). The first model was originally developed in continuous time with continuous dynamics, while the second was developed, independently by several people at about the same time, in the time-less setting of one period.

For comparisons, in the conventional, expected utility model these relationships are

$$E_t(\ln R^R_{t+1}) - \ln R^f_{t+1} = \gamma\, \mathrm{cov}_t(\ln \frac{c_{t+1}}{c_t}, \ln R^R_{t+1}) - \frac{1}{2}\mathrm{var}_t(\ln R^R_{t+1}) \qquad (22)$$

and

$$\ln R^f_{t+1} = \ln\big(\frac{1}{\beta}\big) + \gamma\, E_t(\ln \frac{c_{t+1}}{c_t}) - \frac{1}{2}\gamma^2\, \mathrm{var}_t(\ln \frac{c_{t+1}}{c_t}), \qquad (23)$$

which can be obtained from the recursive formulas by simply setting $\rho = \gamma$. This is the discrete time version of Breeden's CCAPM.[4]

Similar, but not identical expressions can be derived using Taylor series approximations, ignoring moments of order three and higher. The small discrepancy occurs in the expression for the risk premiums only:

$$E_t(\tilde{r}^R_{t+1}) - \ln R^f_{t+1} \approx \rho\, \frac{1-\gamma}{1-\rho}\mathrm{cov}_t(\ln \frac{c_{t+1}}{c_t}, \ln R^R_{t+1})$$
$$+ \frac{\gamma-\rho}{1-\rho}\mathrm{cov}_t(\ln R^W_{t+1}, \ln R^R_{t+1}) + \frac{1}{2}(E_t(\tilde{r}^R_{t+1}))^2 \quad (24)$$

for any asset R in the economy. Here $\tilde{r}^R_{t+1}$ is the simple return on the risky

[4] Weil (1989) do not develop expressions like (20) and (21), but rather analyze (17) and (19) directly directly using a stationary two state Markov process and numerical methods.

asset R. That this expression is approximately equal to the one given in (20), follows since

$$E_t(\ln R^R_{t+1}) + \frac{1}{2}\text{var}_t(\ln R^R_{t+1}) \approx E_t(\tilde{r}^R_{t+1}) - \frac{1}{2}\big(E_t(\tilde{r}^R_{t+1})\big)^2,$$

where this approximation holds precisely when we ignore moments of order three and higher.

A more detailed discussion of the theoretical topics of this section, including proofs, can be found in the Aase (2020), where the formulas for the risk premiums (20) and the real short rate (21), as well as (24) and the corresponding formula for short rate are derived (by both methods).

5.6 Calibrations

In Table 3 we provide the key summary statistics of the data in Mehra and Prescott (1985) of the real annual return data related to the S&P-500, denoted by M, as well as for the annualized consumption data, denoted c, and the Government bills, denoted b [5].

	Expectat.	Standard dev.	Covariances
Consumption growth	1.83%	3.57%	$\text{cov}(M, c) = .002226$
Return S&P-500	6.98%	16.54%	$\text{cov}(M, b) = .001401$
Government bills	0.80%	5.67%	$\text{cov}(c, b) = -.000158$
Equity premium	6.18%	16.67%	

Table 3: Key US-data for the time period 1889-1978. Discrete-time compounding.

In our calibrations the equations (20)-(21) tell us that we must consider a log transformation and use log returns. The relevant summary statistics are given in Table 4. Notice that this table is not a mere transformation of Table 3, but developed from the the original data set used in the Mehra and Prescott (1985)-study, by taking logarithms of the relevant yearly quantities, and basing the statistical analysis on these transformed data points.[6]

[5]There are of course newer data by now, but these retain the same basic features. If we can explain the data in Table 1, we can most likely explain any of the newer sets as well.

[6]We have obtained the original data set from Professor R. Mehra. For example, a log return is not obtained simply adjusted as $\mu - (1/2)\sigma^2$ from Table 3, which would be (almost) true if returns and growth rates of consumption were normally distributed. We observe some deviations from normality in the data, albeit not significant ones.

	Expectat.	Standard dev.	Covariances
Consumption growth	1.75%	3.55%	$\text{cov}(M, c) = .002268$
Return S&P-500	5.53%	15.84%	$\text{cov}(M, b) = .001477$
Government bills	0.64%	5.74%	$\text{cov}(c, b) = -.000149$
Equity premium	4.89%	15.95%	

Table 4: Key US-data for the time period 1889-1978 in terms of log returns of discrete-time compounding.

Assuming for the moment that the market portfolio can be used as a proxy for the wealth portfolio, we then interpret the risky asset as the value weighted market portfolio M corresponding to the S&P-500 index. The result is two equation in two unknowns to provide estimates for the preference parameters by the "method of moments". The impatience rate $\delta = 1/\beta - 1$ in Table 5. We denote the elasticity of intertemporal substitution in consumption by $\psi := 1/\rho$, and refer to it as the EIS-parameter. Under this assumption we calibrate our model (20) and (21) for various values of β. The results are given in Table 5 when the market portfolio is assumed a proxy for the wealth portfolio.

Parameters	γ	ρ	EIS	δ
The expected utility model :				
$\beta = 1.01$	27.07	27.07	.037	-0.01
The recursive model:				
$\beta = .965$	2.32	.10	10.35	.036
$\beta = .968$	2.06	.29	3.48	.033
$\beta = .970$	1.88	.41	2.43	.030
$\beta = .975$	1.44	.71	1.39	.020
$\beta = .980$	.99	1.01	0.97	.025
$\beta = .985$	.52	1.29	0.77	.015

Table 5: Various Calibrations Consistent with Table 4.

When β is between .965 and .975, the other two parameters take on rather reasonable values, giving preference for early resolution of uncertainty. In this range the EIS-parameter is larger than 1. When β increases beyond this range, the parameters are not unreasonable, but now indicate preference for late resolution of uncertainty, where the EIS-parameter is smaller than 1. In this region the relative risk aversion is too low to be considered reasonable.

These results are in agreement with those of Aase (2016a) based on the continuous-time model with continuous dynamics.

5.7 Discussion

The above results should be contrasted with the comments made above, where some researchers express disappointing results for the Epstein-Zin-model's ability to explain empirical observations. When the agent is as patient as commonly assumed in applied work, with β around .96 -.99, the results for γ may become too low when the patience rate is close to 1, but still fairly reasonable when β is not too high in this range, in particular when compared to the results from the additive and separable EU-model.

Weil (1989) do not develop expressions like (20) and (21), but rather analyze (17) and (19) directly using a stationary two state Markov process and numerical methods. A similar remark can be made for the paper by Kocherlakota (1990). From these expressions one one can readily infer, with a minimal amount of calculations, that both higher risk premiums, and lower real rates can be obtained from this model, compared to the standard one, for reasonable values of the preference parameters. This we illustrate below.

Weil (1989), for example, obtained the values $\beta = .95, \gamma = 45, \psi = 1/\rho = .10$ when the net risk premium is .0572 and the risk-free rate is .0085. According to Table 1 this risk premium is a bit smaller than the one estimated, and the risk-free rate is a bit larger, but this may be close enough for the present purposes. Using these values for the moment, our model produces two possible solutions for $\beta = .96$: $(\gamma = 1.8, \rho = 0.30)$ and $(\gamma = 30.8, \rho = 19.0)$. It is this last one that is closest to what Weil (1989) obtained, using the two-state Makrov model directly on (18) and (19).

In other words, sometimes more than one solution is obtained, but Weil (1989) only reports one solution. The more interesting one was not discovered by his approach. In contrast, he launched a new puzzle, the "risk-free rate puzzle". By our calibrations, there is no such puzzle.

In Aase (2020) two more data sets are considered, one of the US-economy for the period of (1960-2015) where we also have data related to national wealth. The other, a data set for the Norwegian economy is also discussed, in Aase (2016a) for the continuous model, and in Aase (2020) for the discrete, scale invariant recursive utility model. For these we have data for the national wealth as well. These data sets calibrate reasonably well to the recursive model, while the standard model does considerably worse in calibrations to these newer sets of market and consumption data.

Figure 1 illustrates the feasible region in (γ, ρ)-space. For the conventional model it is the 45°-line shown $(\gamma = \rho)$. For the recursive utility it is all of the first quadrant, including the axes. The points above the 45°-line represent late resolution of uncertainty, the points below correspond to early resolution.

Notice the distance between the typical calibrated point "Calibr" in Fig-

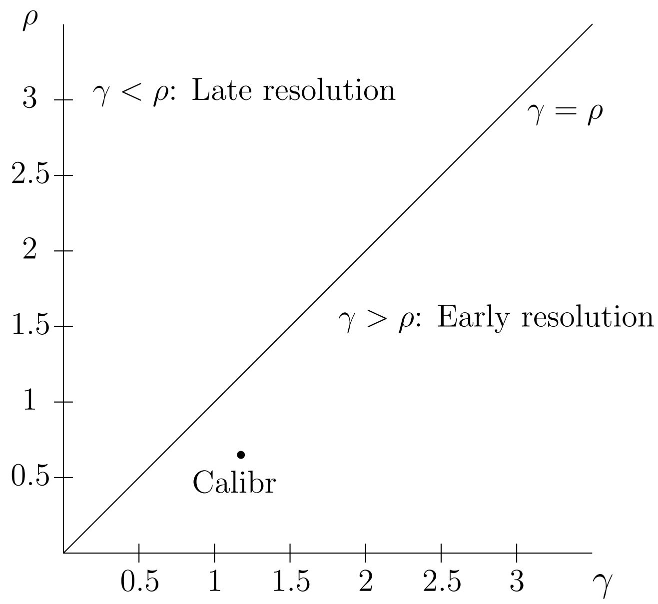

Figure 1: Calibration points in the (γ, ρ)-space

ure 1 and the corresponding uniquely determined point for the expected additive model: It is located on the diagonal, far outside the boundaries of Figure 1. A relative risk aversion of the order of 27 is considered implausible.

The larger region for the (γ, ρ)-combinations permitted by recursive utility is not a frivolous generalization of the conventional, additive model. That the richer structure of the recursive model is not a modest extension is demonstrated by the interpretations and plausible results yielded in the simple expressions (20) and (21). The model is based on fundamental assumptions and axioms of rational behaviour (Kreps and Porteus (1978), or Chew and Epstein (1991) for the infinite time horizon). But remember Kurt Gődel, no guarantees issued.

6 Precautionary savings

From our discussion in Section 3.1 we noticed from the basic relations in (1) and (3) that the essential difference between the standard model and the recursive one is contained in the separation of the function v from the function u, i.e., the ability the recursive model has to separate risk aversion from consumption substitution. In this last section we discuss some of the implications of this separation for optimal saving decisions of individuals. This treatment does not require an excessive preparation, yield some interesting results, by throwing some more light on the subjects of this paper.

We consider $T = 1$, i.e., a two date economy (which is dynamic). To start, we first assume a sure income z_0 at date 0, and uncertain income $\tilde{z}_1$ at date 1. We assume that this risk is exogenous. For example, the consumer might plan for the future knowing that his future labor income is subject to changes that might be higher or lower than anticipated. Note that this simple set-up is dynamic because consumption takes place both at the initial time and at time 1, while In the one-period model no consumption takes place at the initial time.

Consumers select how much to save (s) at date 0 in order to maximize their lifetime utility, giving $(z_0 - s)$ to consume at the initial time.

In the standard model the utility function is

$$U_{EU}(s) = u_0(z_0 - s) + \beta E(u_1((1 + r)s + \tilde{z}_1),$$

for some strictly increasing and concave felicity indices $u_i(\cdot)$, $i = 0, 1$. Let us denote the solution to the maximization problem $\max_s U_{EU}(s)$ by s^*. Here r is the interest rate, assumed deterministic for the time being.

The uncertainty affecting future consumption introduces a new motive for saving. The intuition is that it induces consumers to increase their wealth accumulation in order to prepare themselves to face future risk. This is the precautionary motive for saving, and it relies on the technical concept on prudence, to be defined shortly. The result can be derived by comparing s^* with optimal saving s^0 when the uncertain future income $\tilde{z}_1$ is replaced by its expectation $E(\tilde{z}_1)$. The answer is that $s^* \geq s^0$ whenever u_1' is convex, or equivalently, whenever u_1''' is positive, which is referred to as *prudence*.

Aside from the technicalities, this seems like an intuitive and reasonable result, and our first question is if this is also true for recursive utility. But what do we mean by recursive utility here? In general, the recursive utility function is defined by the two functions u and v satisfying

$$U(s) = v^{-1}\big((1 - \beta)v(z_0 - s) + \beta u(m)\big),$$

where $m = u^{-1}\big(E(u(U_1))\big)$, $u \neq v$, so that m is the certainty equivalent of time 1 utility, and u and v are two different felicity indices.

Below we shall be interested in finding out more about risk aversion and consumption substitution, in which case we choose to work with the scale invariant version of recursive utility with the Epstein-Zin parameterization. This means that the objective with recursive utility is the following:

$$U(s) = \Big((1-\beta)(z_0 - s)^{1-\rho} + \beta\big(E((1+r)s + \tilde{z}_1)^{1-\gamma}\big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{1}{1-\rho}} \tag{25}$$

For the problems of this section, we could alternatively drop the factor $(1-\beta)$ at time zero, but here we choose to keep this factor, consistent with our treatment in Sections 3 and 4.

The optimal saving s^* is the solution to $\max_s U(s)$, and the first order condition is

$$\frac{1-\beta}{\beta(1+r)}(z_0 - s)^{-\rho} = \big(E((1+r)s + \tilde{z}_1)^{1-\gamma}\big)^{\frac{\gamma-\rho}{1-\gamma}} E\big((1+r)s + \tilde{z}_1)\big)^{-\gamma}. \tag{26}$$

Let us denote by $E(\tilde{z}_1) = z_1$, and first consider the case of certainty. That is, we replace $\tilde{z}_1$ by its expectation z_1 in the first order condition. The optimal saving s under certainty we call s^0. Let $\alpha = \frac{\beta}{1-\beta}$. It is easy to see from the above that

$$s^0 = \frac{z_0 - kz_1}{1 + (1+r)k}, \quad \text{where} \quad k = (\alpha(1+r))^{-\frac{1}{\rho}}, \tag{27}$$

valid if $\rho > 0$. [7] When $z_0 \geq kz_1$ positive savings take place under certainty. If $z_0 = z_1$, this happens if $\alpha(1+r) \geq 1$, i.e., when the gross interest $1+r$ more than cancels the effect of the factor $\beta/(1-\beta)$. If we define the impatience rate δ in this two-period model by $\alpha = \frac{1}{1+\delta}$, this means that $r \geq \delta$.

Now we turn to uncertainty, and we ask when is $s^* \geq s^0$. This is when we have *precautionary savings* in the recursive utility model: Faced with future income uncertainty, the 'prudent' consumer saves more than in a world of certainty.

The analysis become rather simple once we observe that we may replace the two expectations in (26) by expressions containing certainty equivalents, or even simpler, by inserting the certainty equivalent m right away in the expression for $U(s)$ given in (25): We can alternatively write

$$U(s) = \Big((1-\beta)(z_0 - s)^{1-\rho} + \beta\big(E((1+r)s + \tilde{z}_1)^{1-\gamma}\big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{1}{1-\rho}} =$$

[7] In the special case that $\alpha(1+r) = 1$, s^0 does not depend upon ρ. Equation (27) is then still valid in the limiting case when $\rho \to 0$.

$$\Big((1-\beta)(z_0-s)^{1-\rho}+\beta\big((1+r)s+m\big)^{1-\rho}\Big)^{\frac{1}{1-\rho}}.$$

The first order condition in s is, using this latter expression

$$\frac{1-\beta}{\beta(1+r)}(z_0-s^*)^{-\rho}=\big((1+r)s^*+m\big)^{-\rho}. \tag{28}$$

This gives the optimal amount of saving

$$s^*=\frac{z_0-km}{1+(1+r)k}. \tag{29}$$

From equation (28) it may seem as if the optimal value of s^* depends mainly upon ρ, but this is not the case. As the result (29) tells us, it depends on the marginal utility flexibility parameter ρ through k, while the certainty equivalent m depends upon the relative risk aversion γ. When $k=1$ we notice that s^* does not depend upon ρ. [8]

By replacing the certainty equivalent m in (28) with the expected value $z_1=E(\tilde{z}_1)$, we again obtain that the optimal saving s^0 under certainty is the same as above, and given in (27). Thus precautionary savings is obtained, since $s^*\geq s^0$ then follows from $m\leq z_1$ (Jensen's inequality) and $k>0$.

In the above we have used the CRRA-version for the felicity index u. For expected utility, when the time one felicity index u_1 is of this type, it is prudent in the above definition, so here we have consistency between these two different preference representations as far as precautionary savings is concerned. But notice that for recursive utility the particular form of the function v also matters, in other words, it depends on the parameter ρ, while for the additive, expected utility representation the result is independent of the particular form of the initial felicity index u_0, so long as it is increasing and concave.

6.1 Risky savings and precautionary demand

In the previous section we considered only labor-income risk. The individual had a risk-free savings alternative but was unsure about how much income would be earned at date 1. We now look at a model where labor income is known, but the rate on return on savings is risky. Here we abstract from the portfolio selection problem, which is treated in some detail in Section 5.9. First we assume there is only one fund for risky savings paying a random return $1+\tilde{r}$, where $E\tilde{r}=r_0>0$.

[8] In this particular case it is seen that s^* is stable as $\rho\to 0$.

Saving at the stochastic rate is then compared to saving at the deterministic rate r_0, and the question is in which case does the agent saves the most.

6.1.1 The standard model

We consider a consumer with an investment horizon of two periods. Since lifetime income is known with certainty, we assume without loss of generality that all income is paid at date $t = 0$. Letting z_0 denote this wealth, the consumer's objective is to maximize the function $U(s)$ given by

$$U(s) = u(z_0 - s) + \beta E(u((1 + \tilde{r})s).$$

The first order condition is given by

$$u'(z_0 - s^*) = \beta E[(1 + \tilde{r})u'((1 + \tilde{r})s^*)],$$

where s^* is the optimal saving under uncertain interest rate. Since the objective function $U(s)$ is concave, the second order condition holds for this problem.

The first order condition for the corresponding problem where saving takes place at the deterministic savings rate $r_0 = E(\tilde{r})$ is given by

$$u'(z_0 - s^0) = \beta(1 + r_0)u'((1 + r_0)s^0),$$

where s^0 is the optimal saving with the deterministic interest rate r_0. In this case we denote the objective function by $\hat{U}(s)$.

We will now compare saving under these two conditions, and ask when is $s^* > s^0$. Since the objective $\hat{U}(s)$ is concave in s, the answer to this question is that saving to an uncertain interest rate will increase the amount of saving provided $\hat{U}'(s^*) < 0$.

It follows that the uncertainty in the rate of return will cause the optimal level of savings to increase whenever

$$E[(1 + \tilde{r})u'((1 + \tilde{r})s^*)] > (1 + r_0)u'((1 + r_0)s^*),$$

This inequality will hold, by Jensen's inequality, if the function $f(x) := xu'(xs^*)$ is strictly convex ($x = 1 + r$), which happens if and only if relative prudence, $-wu'''(w)/u''(w)$, is larger than 2 (see e.g., Eeckhoudt, Gollier and Schlesinger (2005), Ch. 6).

For the expected utility model there are two competing influences at work: On the one hand, the riskiness of returns makes savings less attractive than saving at a risk-free rate with the same average return. On the other hand, the date-1 risk will induce a precautionary motive to the prudent consumer. It turns out that when relative prudence exceeds 2 the precautionary motive dominates.

With CRRA utility relative prudence is given by $1 + \gamma$, and this result translates to

$$s^* \text{ will } \begin{cases} \text{increase} & \text{if relative risk aversion } \gamma > 1, \\ \text{remain the same} & \text{if relative risk aversion } \gamma = 1, \\ \text{decrease} & \text{if relative risk aversion } \gamma < 1. \end{cases} \tag{30}$$

Thus the result (30) says that when $\gamma > 1$, then $s^* > s^0$, when $\gamma < 1$, then $s^* < s^0$, and when $\gamma = 1$, then $s^* = s^0$.

We now extend the analysis to include a risk-free asset with return $1+r_{rf}$, where $r_{rf} < r_0$, both constants. This inequality follows, since the return on the risk-free asset must be lower that the expected return $E(\tilde{r})$ on a risky asset, a direct consequence of risk aversion.[9]

We then compare savings at these two deterministic rates. At first sight this may appear as a rather simple question, as one would, perhaps, think that the rational agent would save more at the highest interest rate. Denoting the optimal saving s_{rf} at the risk free interest rate r_{rf}, proceeding as above, saving will be largest with the highest certain rate of return provided

$$\beta(1 + r_0)u'((1 + r_0)s^0)] > \beta(1 + r_{rf})u'((1 + r_{rf})s^0).$$

This will take place when the function $g(x) = xu'(xs^0)$ is strictly increasing in x, which holds if and only if $-wu''(w)/u'(w) < 1$. With the utility function $u(c) = c^{1-\rho}/(1 - \rho)$, where ρ represent the 'resistance' to consumption substitution across time (there is no uncertainty here), this holds if and only if $\rho < 1$. The result of this comparison can be summarized as

$$s_{rf} \text{ will } \begin{cases} \text{increase} & \text{if } \rho > 1, \\ \text{remain the same} & \text{if } \rho = 1, \\ \text{decrease} & \text{if } \rho < 1. \end{cases} \tag{31}$$

Thus $s_{rf} > s^0$ if $\rho > 1$, so the smallest interest rate leads to the highest saving when $\rho > 1$. Also $s_{rf} < s^0$ if $\rho < 1$, and $s_{rf} = s^0$ if $\rho = 1$. When $\rho < 1$, the consumer will save more when the certain interest rate increases, which is, perhaps, the natural savings result one would expect.

[9]This is introduced for comparative statics only, so arbitrage is not a subject here.

To understand the result (31), let us first digress as to what the elasticity of substitution in consumption, $\psi = 1/\rho$, for an individual describes. Let $R_{c_1 c_0}$ denote the marginal rate of substitution between consumption at the two dates 0 and 1, and let $El_x f(x)$ be the elasticity of the function $f(x)$ with respect to x. Then

$$\psi_{c_1 c_0} = El_{R_{c_1 c_0}}(\frac{c_1}{c_0}).$$

The elasticity of substitution in consumption tells us, approximately, how many percent the ratio c_1/c_0 changes, when the consumer moves along her indifference curve $U(c_0, c_1) = c$, a constant, in such a manner that $R_{c_1 c_0}$ increases by 1%.

If $\psi > 1$ this indicates that the consumer has lower resistance to consumption substitution than if $\psi < 1$. For the utility of this section, $\psi = 1/\rho$, so this property translates to higher resistance to consumption substitution when $\rho > 1$ then when $\rho < 1$.

When $\rho > 1$, the consumer's resistance against consumption substitution dominates the temptation to save more when the interest rate is higher, as this would lead to too much transfer of consumption between the two dates. When $\rho < 1$ the consumer has less resistance to consumption substitution, and as a result she will save more when the interest rate increases.

It is clear that with saving under certainty in (31) it is the resistance to intertemporal substitution of consumption interpretation of the parameter ρ that is relevant. In the saving under uncertainty given in (30) this is at best unclear.

It is striking why the explanations for the results change so much in these two situations. In the standard model $\gamma = \rho$, so we can only guess what is the correct interpretation under uncertainty; is it really the risk aversion property that drives the result, or the consumption substitution property?

In order to answer this question, we move to recursive utility, where these two properties are separated.

6.1.2 Recursive utility

The objective function can now be written

$$U(s) = \Big((1-\beta)(z_0 - s)^{1-\rho} + \beta\big(E((1+\tilde{r})s)^{1-\gamma}\big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{1}{1-\rho}} \tag{32}$$

where $\tilde{r}$ is the uncertain interest rate, a random variable. The optimal saving s^* with saving at the interest rate $\tilde{r}$ is a solution to $\max_s U(s)$. We first find the certainty equivalent interest rate, call it r, to the stochastic rate $\tilde{r}$. It satisfies

$$E\big((1+\tilde{r})s)^{1-\gamma}\big) := ((1+r)s)^{1-\gamma}.$$

By Jensen's inequality

$$\frac{1}{1-\gamma}E\big((1+\tilde{r})s)^{1-\gamma}\big) < \frac{1}{1-\gamma}((1+E(\tilde{r})s)^{1-\gamma}.$$

Since the CRRA-utility function is a strictly increasing function, this implies that $r < r_0$, where $r_0 = E(\tilde{r})$. With the certainty equivalent interest rate r inserted in the objective, it becomes

$$U(s) = \Big((1-\beta)(z_0 - s)^{1-\rho} + \beta\big((1+r)s)^{1-\rho}\big)\Big)^{\frac{1}{1-\rho}},$$

and the first order condition is s is

$$\frac{(1-\beta)}{\beta(1+r)}(z_0 - s)^{-\rho} = \big((1+r)s\big)^{-\rho}.$$

Because the objective is strictly concave, the first order condition is both necessary and sufficient for optimality. The solution is

$$s^* = \frac{z_0}{1 + \alpha^{-\frac{1}{\rho}}(1+r)^{\frac{\rho-1}{\rho}}}, \tag{33}$$

where $\alpha = \frac{\beta}{1-\beta}$. In other words, the recursive agent is indifferent between saving at the uncertain interest rate $\tilde{r}$ and saving at the deterministic rate r, the certainty equivalent interest rate, where $r < r_0 = E(\tilde{r})$. The inequality follows by Jensen's inequality, since the agent is risk averse. If we denote optimal saving at the certainty equivalent interest rate by s^r, we here have that $s^r = s^*$, which gives this indifference conclusion from (32).

First we compare this level of saving to the corresponding optimal amount of saving s^0 when the stochastic interest rate $\tilde{r}$ is replaced by its expectation $E(\tilde{r}) = r_0$, so that saving takes place under certainty. Under this condition the optimal amount of saving is

$$s^0 = \frac{z_0}{1 + \alpha^{-\frac{1}{\rho}}(1+r_0)^{\frac{\rho-1}{\rho}}}.$$

Since $r < r_0$, this gives the following conclusion

$$s^* \text{ will} \begin{cases} \text{increase} & \text{if } \rho > 1, \\ \text{remain the same} & \text{if } \rho = 1, \\ \text{decrease} & \text{if } \rho < 1. \end{cases} \tag{34}$$

In other words, $s^* > s_0$ when $\rho > 1$, $s^* < s_0$ when $\rho < 1$, while $s^* = s^0$ when $\rho = 1$.

Comparing the result in (34) to the corresponding result (30) for the standard model, we see that it is the elasticity of substitution in consumption that is behind the result, also with saving at an uncertain rate $\tilde{r}$, not risk aversion. Under uncertainty, we are not able to separate these properties in the standard model, since there $\gamma = \rho$.

In the present model, however, where these different properties of an individual has been separated under uncertainty, we are finally able to recognize what property is behind the result (30) (at least in the CRRA case). When resistance to consumption substitution across time is low, i.e., when $\rho < 1$, the amount of saving s^* under uncertainty ($\tilde{r}$) will go down compared to saving under certainty (r_0), when $E(\tilde{r}) = r_0$. The role played by risk aversion here is to secure that $r < r_0$, i.e., the certainty equivalent interest rate r is smaller than $E(\tilde{r})$. This holds as long as $\gamma > 0$. It is this fact that makes saving less attractive than saving at the certain rate r_0, when $\rho < 1$, not because the relative risk aversion $\gamma < 1$ as suggested by (30). The larger the relative risk aversion, the larger is the difference $(r_0 - r)$, and the larger the difference between s_0 and s^*. Thus the explanation for this phenomenon is the same as for the situation where $\rho < 1$ in (31): The consumer will save more in this situation when the certain interest is the highest, because of low resistance to consumption fluctuation across the two time periods.

It is true that there is a precautionary savings motive with recursive utility as well, shown in the last section, and here this motive dominates only when $\rho > 1$, i.e., when the resistance to consumption substitution is high. In this situation the highest certain interest rate gives the lowest saving, for the same reason as in (31) when $\rho > 1$.

Here we remind the reader that with the constant elasticity of substitution (CES)-utility function which is formally the aggregator of the recursive utility version we consider, the scale invariant one, it is also the case that

$$\psi := \psi_{c_1 c_0} = El_{R_{c_1 c_0}}(\frac{c_1}{c_0}) = \frac{1}{\rho},$$

so that the parameter ρ has the same interpretation as for the standard expected utility model, and the CES-function indeed lives up to its name.

Finally, let us compare saving at the two deterministic rates r_0 and r_{rf} where $r_{rf} < r_0$, as we did for the standard model. By analogy with the result just shown for recursive utility, we obtain

$$s_{rf} \text{ will } \begin{cases} \text{increase} & \text{if } \rho > 1, \\ \text{remain the same} & \text{if } \rho = 1, \\ \text{decrease} & \text{if } \rho < 1. \end{cases} \tag{35}$$

which is the same result as for the standard model, provided we interpret the parameter in (31) as $\rho = 1/\psi$ and not as risk aversion γ (there is no risk in this situation and hence risk aversion is irrelevant).

The example of this section demonstrates the advantages of having one parameter for relative risk aversion, another for consumption substitution, where one is not merely the reciprocal of the other.

7 Appendix 1.

7.1 Preference for early/late resolution of uncertainty.

With recursive utility there is a well-defined notion of the time at which uncertainty is revealed, and although for compound lotteries there is also an implicit axiom perceiving them as equivalent to the one-shot lottery they reduce to at a single time, there is no axiom which says that uncertainties at two different times are equivalent.

Given any consumption plans c' and c'' in the domain of V and any $\alpha \in (0,1)$, let $c^\alpha = \alpha c' \oplus (1-\alpha)c''$. The RU-agent is supposed to choose form the space of random temporal consumption plans.

This non-indifference means, in our notation, that the expressions in (5) and (6) in Section 4.1 will, in general, not be equal.

Let us now try to extend this concept by using the definition due to Skiadas (2009). Towards this end, let C be the consumption set and $\mathcal{F}$ the set of all filtrations $\{\mathcal{F}_t : t = 0, 1, \cdots, T\}$ satisfying $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_T = 2^\Omega$.

Consider the non-normalized utility function $V_0(\cdot)$ over the set of pairs $(c_t, \{\mathcal{F}_t\}) \in C \times \mathcal{F}$ such that $c_t^\alpha = c_t$ is $\mathcal{F}_t$ adapted.

Definition 1 *The utility function $V_0(\cdot)$ represents preferences for early resolution of uncertainty if for any $(c, \{\mathcal{F}_t^1\})$ and $(c, \{\mathcal{F}_t^2\})$ in its domain.*

$$\mathcal{F}_t^1 \subset \mathcal{F}_t^2 \textit{ for all } t \quad \textit{implies } V_0(c, \{\mathcal{F}_t^1\}) \le V_0(c, \{\mathcal{F}_t^2\}).$$

Preferences for late resolution of uncertainty are defined likewise, with the last inequality reversed.

Denote the aggregator of V_t by f_t, a concave function. Let $\xi_t := E(V_{t+1}|\mathcal{F}_t)$. The result is then:

Theorem 2 *For fixed $t < T$ the recursive utility function $V_0(\cdot)$ represents early (resp. late) resolution of uncertainty if and only the function f is convex (resp. concave) in the ξ-argument. If f is affine in ξ_t the individual is indifferent.*

Proof: Consider the following set of inequalities:

$$V_t\big(c_t^\alpha, E(V_{t+1}(c^\alpha)|\mathcal{F}_t^2)\big) \geq V_t\big(c_t, E(V_{t+1}(c^\alpha)|\mathcal{F}_t^1)\big) \quad \text{if and only if}$$

$$\alpha V_t\big(c_t, E(V_{t+1}(c')|\mathcal{F}_t^2)\big) + (1-\alpha)V_t\big(c_t, E(V_{t+1}(c'')|\mathcal{F}_t^2)\big)$$
$$\geq V_t\big(c_t, E(V_{t+1}(c^\alpha)|\mathcal{F}_t^1)\big) \quad \text{if and only if}$$
$$\alpha E\big\{V_t\big(c_t, E(V_{t+1}(c')|\mathcal{F}_t^2)\big)|\mathcal{F}_t^1\big\} + (1-\alpha)E\big\{V_t\big(c_t, E(V_{t+1}(c'')|\mathcal{F}_t^2)\big)|\mathcal{F}_t^1\big\}$$
$$\geq V_t\big(c_t, E(V_{t+1}(c^\alpha)|\mathcal{F}_t^1)\big) \quad \text{if and only if}$$
$$\alpha V_t\big(c_t, E(E(V_{t+1}(c')|\mathcal{F}_t^2)|\mathcal{F}_t^1)\big) + (1-\alpha)V_t\big(c_t, E(E(V_{t+1}(c'')|\mathcal{F}_t^2))|\mathcal{F}_t^1)\big)$$
$$\geq V_t\big(c_t, \alpha E(V_{t+1}(c')|\mathcal{F}_t^1) + (1-\alpha)E(V_{t+1}(c'')|\mathcal{F}_t^1)\big) \quad \text{if and only if}$$
$$\alpha f_t\big(c_t, \xi_t'\big) + (1-\alpha)f_t\big(c_t, \xi_t''\big)\big) \geq f_t\big(c_t, \alpha\xi_t' + (1-\alpha)\xi_t''\big),$$

where $\xi_t' = E(V_{t+1}(c')|\mathcal{F}_t^1)$ and $\xi_t'' = E(V_{t+1}(c'')|\mathcal{F}_t^1)$.

The first equivalence follows from the development in Kreps and Porteus (1978) related to (5) and (6), which is what they mean by early resolution of uncertainty in the present setting, the second follows by taking conditional expectation across the inequality, given $\mathcal{F}_t^1$, observing that $f\big(c_t, E(V_{t+1}(c^\alpha)|\mathcal{F}_t^1)\big)$ is $\mathcal{F}_t^1$-measurable, the third follows from Jensen's inequality on conditional form, and the fourth follows from the rule of iterated expectations. □

By this result, we have to investigate under what conditions the function $V_t(c_t, \xi_t)$ is convex/concave/affine in the variable ξ_t, where

$$V(c_t, \xi_t) = \frac{1}{1-\gamma}\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi_t\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{1-\gamma}{1-\rho}}. \tag{36}$$

The two first partial derivatives are

$$\frac{\partial V(c,\xi)}{\partial \xi} = \beta\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{\rho-\gamma}{1-\rho}}\big((1-\gamma)\xi\big)^{\frac{\gamma-\rho}{1-\gamma}},$$

and

$$\frac{\partial^2 V(c,\xi)}{\partial \xi^2} = \beta^2(\rho-\gamma)\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{\rho-\gamma}{1-\rho}-1}\big((1-\gamma)\xi\big)^{\frac{2(\gamma-\rho)}{1-\gamma}} +$$

$$\beta(\gamma-\rho)\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{\rho-\gamma}{1-\rho}}\big((1-\gamma)\xi\big)^{\frac{(\gamma-\rho)}{1-\gamma}-1}.$$

The second partial derivative we rearrange as follows

$$\frac{\partial^2 V(c,\xi)}{\partial \xi^2} = \beta(\rho-\gamma)\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{\rho-\gamma}{1-\rho}}\big((1-\gamma)\xi\big)^{\frac{(\gamma-\rho)}{1-\gamma}} \cdot$$

$$\Bigg(\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{-1}\beta\big((1-\gamma)\xi\big)^{\frac{(\gamma-\rho)}{1-\gamma}} - \big((1-\gamma)\xi\big)^{-1}\Bigg).$$

By polynomial division, the term

$$\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{-1}\beta\big((1-\gamma)\xi\big)^{\frac{(\gamma-\rho)}{1-\gamma}} =$$

$$\big((1-\gamma)\xi\big)^{-1} - \frac{(1-\beta)c_t^{1-\rho}\big((1-\gamma)\xi\big)^{-1}}{(1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}}.$$

Accordingly

$$\frac{\partial^2 V(c,\xi)}{\partial \xi^2} = \beta(\rho-\gamma)\Big((1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{\rho-\gamma}{1-\rho}}\big((1-\gamma)\xi\big)^{\frac{(\gamma-\rho)}{1-\gamma}} \cdot$$

$$\Bigg(-\frac{(1-\beta)c_t^{1-\rho}\big((1-\gamma)\xi\big)^{-1}}{(1-\beta)c_t^{1-\rho} + \beta\Big((1-\gamma)\xi\Big)^{\frac{1-\rho}{1-\gamma}}}\Bigg). \quad (37)$$

From this expression we observe that
1) If $\gamma > \rho$, then $\frac{\partial^2 V(c,\xi)}{\partial \xi^2} > 0$ implying that $V(c,\xi)$ is convex in the second variable. According to Theorem 3 in Kreps and Porteus (1978), the consumer prefers early resolution of uncertainty to late.
2) If $\gamma < \rho$, then $\frac{\partial^2 V(c,\xi)}{\partial \xi^2} < 0$ implying that $V(c,\xi)$ is concave in the second variable. Accordingly, the consumer prefers late resolution of uncertainty to early.
3) If $\gamma = \rho$, then $\frac{\partial^2 V(c,\xi)}{\partial \xi^2} = 0$ implying that $V(c,\xi)$ is affine in the second variable. Accordingly, the consumer is indifferent to the timing of the resolution of uncertainty.

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NHH

NORGES HANDELSHØYSKOLE
Norwegian School of Economics

Helleveien 30
NO-5045 Bergen
Norway

T +47 55 95 90 00
E nhh.postmottak@nhh.no
W www.nhh.no