NHH

INSTITUTT FOR FORETAKSØKONOMI

DEPARTMENT OF BUSINESS AND MANAGEMENT SCIENCE

FOR 4 2015

ISSN: 1500-4066

January 2015

Discussion paper

A new Lagrangean Approach for the Travelling Salesman Problem

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A new Lagrangean Approach for the Travelling Salesman Problem

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Abstract

In this paper, we use a reformulation of the symmetric and the asymmetric travelling salesman problem more suitable for Lagrangean relaxation and analyse the new approach on examples from TSP Lib. Furthermore the Lagrangean relaxed subproblems are travelling salesman alike which means that almost all that is known on the travelling salesman polytope can be used when the subproblems are to be solved.

Keywords: Travelling Salesman, Lagrangean Relaxation, Mathematical Programming

1. Introduction

The travelling Salesman problem is probably one of the most well-known and well-studied combinatorial optimization problems. Over the years, many authors, using various mathematical programming methods, have approached the problem. Early on, Dantzig et.al solved the 42-city problem using a cutting plane method. The polyhedral properties of the travelling salesman problem is also well known and has led to approaches such as the studies by Grötschel, Reinhart, Padberg and Rinaldi and more recent by Cook and his associates. There are several books written on the topic the most recent being " In pursuit of the travelling salesman" by W.J.Cook. According to the information on W.J Cooks website, the current world record in the solution of TSPs is a problem containing 85900 nodes and comes from an application in VLSI design.

In this paper, we present a new Lagrangean approach for the symmetric and asymmetric travelling salesman problem based on a reformulation of the problems that leads to a Lagrangean relaxation with only one single Lagrangean multiplier. Hence, the search for the optimal Lagrangean multiplier value consists of a simple one dimensional search procedure. If we only are interested in finding the optimal solution to the original problem, we can restrict the search for the multiplier value among the cost coefficients that exists in the original cost matrix. However if we also would like to find the optimal multiplier value, which has the economic interpretation of being the market price needed in order for all nodes to be visited, we need to extend the search to multiplier values not necessarily present in the original cost matrix. The subproblems that need to be solved in the iterative process are travelling salesman alike problems with a much smaller set of arcs/edges than the number that is present in the original problem.

The paper is organised as follows. In section 2 the standard formulation of the symmetric and the asymmetric travelling salesman problems are presented followed by the reformulation that is to be used in the rest of the article. Section 3 gives a short description of Lagrangean relaxation in general and its properties. We also describe the Lagrangean relaxation subproblems for the reformulated models. In section 4 we illustrate the procedure on two small examples taken from the literature and point out a slightly modification of our Lagrangean approach in which the optimal Lagrangean multiplier have an interesting economic interpretation. Section 5 is devoted to an economic interpretation of the optimal Lagrangean multiplier as the market price in a salesman market. Section 6 gives the computational results obtained on a few problem instances taken from TSPlib. Finally, in section 6 we give conclusions that can be made from our investigation.

2. The formulation of the Travelling Salesman problem and a reformulation more suitable for Lagrangean relaxation

The standard formulation of the asymmetric travelling salesman problem is as follows

$$Min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (1)$$

Subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \; \forall \; j \in J \qquad (2)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \; \forall \; i \in I \qquad (3)$$

$$no\ subcycles \qquad (4)$$

$$x_{ij} \in \{0,1\}, \qquad (5)$$

where

c_{ij} = the distance from customer i to customer j

x_{ij} = 1 if the arc ij is used in the Hamiltonian tour 0 otherwise

Equation (2) guarantees that all nodes have an incoming link. Constraint (3) is the requirement that all nodes have an outgoing link and (4) is the requirement that no subcycles are allowed and (5) are the integral requirements. In the sequel, we will use the following reformulation of the symmetric travelling salesman problem.

$$Min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (1)$$

Subject to

$$\sum_{i=1}^{n} x_{ij} \leq 1 \quad \forall j \in J \qquad (6)$$

$$\sum_{i=1}^{n} x_{ij} \leq 1 \quad \forall i \in I \qquad (7)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} = n \qquad (8)$$

$$no\ subcycles \qquad (4)$$

$$x_{ij} \in \{0,1\} \qquad (5)$$

The formulations are clearly equivalent and if the asymmetric travelling salesman problem were to be solved by any regular method, the reformulation makes no sense. However as we shall show if the solution approach is based on Lagrangean relaxation this formulation yields a procedure to identify the core of the problem.

The normal formulation of the symmetric travelling salesman problem is as follows

$$\text{Min}\sum_{e\in E} w(e)x(e) \qquad (9)$$

Subject to

$$\sum_{e\in\delta(v)} x(e) = 2 \;\; for\ all\ v \in V \qquad (10)$$

$$\sum_{e\in E} x(e) \leq |U| - 1 \quad for\ U\ being\ a\ subset\ of\ V, 3 \leq |U| \leq |V| - 1, \qquad (11)$$

where $\delta(v)$ are the edges connected to node v

$$x(e) \in \{0,1\} \qquad (12)$$

Also here we make the corresponding reformulation of the problem

$$\text{Min}\sum_{e\in E} w(e)x(e) \qquad (9)$$

Subject to

$$\sum_{e\in\delta(v)} x(e) \leq 2 \;\; for\ all\ v \in V \qquad (13)$$

$$\sum_{e\in E} x(e) = |V| \qquad (14)$$

$$\sum_{e\in E} x(e) \leq |U| - 1 \quad forU\ being\ a\ subset\ of\ V, |3| \leq |U| \leq |V| - 1 \qquad (11)$$

$$x(e) \in \{0,1\} \qquad (15)$$

In addition, in this case, the formulations are clearly equivalent and if the symmetric travelling salesman problem were to be solved by any regular method, the reformulation makes no sense. However as we shall show if the solution approach is based on Lagrangean relaxation this formulation yields anefficient relaxation procedure to identify the core problem.

3. Lagrangean Relaxation

The Lagrangean sub problems for the reformulated travelling salesmen problems are Max L(u) subject to u≥0 Where L(u) is defined by the following optimization problem

$$L(u) = Min \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - u(\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} - n) \qquad (16)$$

Subject to

$$\sum_{i=1}^{n} x_{ij} \leq 1 \quad \forall j \in J \qquad (6)$$

$$\sum_{i=1}^{n} x_{ij} \leq 1 \quad \forall i \in I \qquad (7)$$

$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \leq n$ (8)

$no\ subcycles$ (4)

$x_{ij} \ \in \{0,1\}$ (5)

For the asymmetric Lagrangean relaxed travelling salesman problem

and

L(u)=Min$\sum_{e \in E} w(e)x(e) - u(\sum_{e \in E} x(e) - |V|)$ (17)

Subject to

$\sum_{e \in \delta(v)} x(e) \leq 2 \ \ for\ all\ v \in V$ (13)

$\sum_{e \in E} x(e) \leq |V|$ (18)

$\sum_{e \in E} x(e) \leq |U| - 1 \quad for\ U\ being\ a\ subset\ of\ V, 3 \leq |U| \leq |V| - 1$ (11)

$x(e) \in \{0,1\}$ (15)

For the symmetric travelling salesman problem.

The Lagrangean relaxation approach presented above has the property that the duality gap is closed. To see this we just have to note the following.

Let P be defined as the polytope defined by the original constraint set for the respective problem i.e. either the polytope for the asymmetric problem defined by the constraints (2), (3), (4), and (5) or alternatively for the symmetric problem the polytope defined by the constraints (10), (11), and (12).

Let PU be the polytope of the feasible set for the Lagrangean relaxed asymmetric travelling salesman problem defined by the constraints (6), (7), (4), (5), and (8) or alternatively for the symmetric problem the polytope defined by the constraints (11), (13), (15), and (18) .

Now for the asymmetric problem we have that

$$Conv\{PU\} \cap Conv\left\{\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = n\right\}$$

which obviously is equal to $Conv\{P\}$ for the asymmetric problem.

Likewise for the symmetric problem we have that

$$Conv\{PU\} \cap \left\{\sum_{e \in E} x(e) = |V| \ \right\}$$

Which is equal to $Conv\{P\}$ for the symmetric problem. From these observation we know that the proposed Lagrangean relaxations has no duality gap.

Apart from this the optimal Lagrangean multiplier value u* has a meaningful economic interpretation. In fact the optimal multiplier value u^* corresponds to the market price that has to be payed in order for all customer demands to be fulfilled. Note that in both cases, we have only one Lagrangean multiplier to search for. This means that any one-dimensional search procedure can be used. It should also be noted that it is easy to get an initial estimate on u since we know that all nodes have to be visited. Hence a minimal u can in the asymmetric case be found by scanning over all outgoing and incoming links to all node to obtain obtained by $max\left(min_i(c_{ij}), min_j(c_{ij})\right)$ and in the symmetric case this conservative estimate of u is the maximum of the second least costly edge attached to any of the nodes.

Also note that, in the subproblems only arcs ij, or edges e with negative coefficients are of interest. Hence the subproblems will in most cases, have a much smaller number of edges, arc, links than that are present in the original problem.

In addition, the potential candidates for optimal u values are finite and limited to the values present in the distance matrices. This means that no complicated multiplier search procedure is needed.

In order to get an efficient Lagrangean implementation it is obviously not optimal to start with the conservative estimates of u presented above. A better alternative is to use a "guestimate" where the starting u is chosen to get a significant reduction in the number of edges, links, and arcs in the first studied subproblem. If the "guestimate" turned out to be too high. We have obtained an optimal solution to the problem.

If the" guestimate" turned out to be too low. We have obtained a lower bound on the optimal solution and since the optimal solution to the subproblems does not contain any subcycles and has less than| n| or |V| edges, arcs the solution must be a set of disconnected strings and some isolated nodes. In order to obtain an upper bound a simple assignment or matching heuristic can be used to connect the disconnected strings.

A further important property of the approach is that in case we have solved the subproblem for a u multiplier that is non-optimal the multiplier is increased to the next potential candidate level found in the distance matrix. This means that we get a new subproblem with more variables than in the subproblem recently solved. We have obtained at least one variable with a 0 cost in the objective function. Since the new subproblem is a relaxation of the former subproblem the old solution is still feasible and might be optimal for the new subproblem. If it is not optimal to the new subproblem, an optimal solution to the new subproblem must contain at least one of the newly introduced variables having a zero coefficient. This fact can be used when solving the new subproblem.

4. An illustrative Example

To illustrate how our semi-Lagrangean scheme works we use two small examples one for a symmetric problem, taken from Dantzig et al. (1959) and one for an asymmetric problem taken from Christofides book.

The cost matrix for the asymmetric example is

Variable costs c_{ij}

	1	2	3	4	5	6	7	8
1	-	76	43	38	51	42	19	80
2	42	-	49	26	78	52	39	87
3	48	28	-	36	53	44	68	61
4	72	31	29	-	42	49	50	38
5	30	52	38	47	-	64	75	82
6	66	51	83	51	22	-	37	71
7	77	62	93	54	69	38	-	26
8	42	58	66	76	41	52	83	-

In the example we have $max\left(min_i(c_{ij}), min_j(c_{ij})\right)$ = 41. Hence, the minimal possible semi-Lagrangean price is 41.

The subproblem for u=41 has value -95 giving a lower bound of 328-95=233.

The subproblem solution consists of two disconnected strings 65178 and 324. Searching for a feasible tour in the reduced graph is unsuccessful. For the next u=42 the solution to the subproblem solution remains the same and the bound is increased to 235. Now the search for a feasible tour in the reduced graph yields the tour 176532481 with value 251, and hence the upper bound is 251.

Continuing to increase the Lagrangean price to 43, 44, 47, 48, 50, 51, 52, 53, 54, and 58 yields increasing lower bounds. For instance for u=50 the new lower bound is 242 and for u=54 the lower bound is 250.

For u=58 the lower bound is 251 and hence optimality of the tour found in the first reduced subgraph is the optimal tour. Note that, since the subproblem solution obtained in iteration 1 for u=42 is feasible in all subsequent subproblems with increased u, we know that we need to increase u with at least 251-235=16 units in order for the subproblem sub string solution 65178 and 324 not to be the optimal solution to the subproblem. This is based on the fact that the string consist of 6 arcs and the lower bound will increase by 2 units for each increase in the Lagrangean multiplier value u. Hence after solving the subproblem for u=42 we can jump immediately to the value u=50. It is remarkable that although we already have found the optimal solution in the subgraph generated in the second iteration with u=42, the number of allowable arcs is 36% of the original number of arcs, hence a reduction of 64% in the number of arcs. In the subgraph with u=58 this reduction is only 28%. We will come back to this in a comment later.

The example with a symmetric travelling salesman problem has the following distance matrix

	1	2	3	4	5	6	7	8	9
2	28								
3	57	28							
4	72	45	20						
5	81	54	30	10					
6	85	57	28	20	22				
7	80	63	57	72	81	63			
8	113	85	57	45	41	28	80		
9	89	63	40	20	10	28	89	40	
10	80	63	57	45	41	63	113	80	40

The minimum possible value of u=63. The optimal solution to the subproblem has value -289 and consist of a simple string 12345968 leaving nodes 7 and 10 exposed. Hence, the lower bound is 630-289=341.

Continuing to increase the Lagrangean price gives continuously increasing lower bounds. For u=80 the new lower bound is 375 and the solution consist of a single string 1234685910. However searching for a complete tour in the reduced graph gives the tour 123451098671 with value 378.

For u=81 the lower bound is increase to 376.

For u=85 the bound is 378 and optimality of the tour 123451098671 is proven.

Note that also in this example we had to increase the Lagrangean price above the cost of the most expensive edge in the tour to prove optimality.

The reason for this is, in both examples that the subproblems have no constraints that require connectivity or requirements that all nodes should be visited.

In order to get a relaxation that has the property that the optimal Lagrangean multiplier has a value that is equal to the most expensive link/edge in the tour we need a stronger relaxation. Such relaxations can be obtained by requiring that we have equality in at least one of the inequality constraints requiring that a node has two edges connected to it (in the symmetric case), or that at least one arc is leaving or alternatively entering that node (in the asymmetric case).

To illustrate this lets again look at the symmetric example presented above.

For u=80 the smallest u value for which we could find a feasible tour, the subproblem solution consists of the isolated node 7 and the string with the two end nodes 1 and 10. Hence, a stronger relaxation can be obtained by requiring that one of the nodes 7, 1, or 10 shall have two edges connected to it. All of these three possible restrictions lead to an increase in the lower bound. The strongest restricted relaxation is the one where node 7 is selected as the node for which two edges have to be connected. Here, the restricted Lagrangean relaxation yields a lower bound of 378 and optimality is verified. Note that in order for this restricted relaxation to be true also the edges with positive costs connected to node 7 have to be included in the subproblem. This can be seen by adding this restriction for u=79. When the restricted subproblem is solved the lower bound will be larger than 378 if not also the edges with positive costs connected to node 7 are included in the restricted subproblem. An interesting observation is that when u=80 the restricted subproblem solution consists of edges which are non-positive . Hence, the restricted Lagrangean has an optimal multiplier value equal to the most costly edge in the optimal tour.

Likewise, for the asymmetric problem with u=41 the arcs which has caused the value to be chosen to be u=41 is of importance in this sense. In the example, this leads to four possibilities. These are one of the restriction requiring that exactly one arc leaves node 4, the requirement that exactly one arc leave node 8, the requirement that exactly one arc enters node 6 or the requirement that exactly one arc enters node 3. All the possible restrictions give a stronger relaxation and for most of them a slightly better lower bound. However, the one that requires that exactly one arc leaves node 8 leads to a relaxation which gives us a lower bound of 251 and the optimal solution. Hence, this stronger relaxation has the *property that the optimal Lagrangean multiplier is identical to the most expensive arc in the tour*. Note, that if we allow arcs with a positive objective function value to be present in the stronger relaxation optimality can be proven with a smaller u value.

5. An Economic interpretation of the optimal Semi-Lagrangean multiplier

The reformulated travelling salesman problem can be thought of as a problem in which we try to minimize the cost of serving a market with a total demand of $|V|$. The demand is distributed geographically with $|V|$ customers each having a demand of one unit. The market shall be served by one saleman making a sales tour to all customers.

$\text{Min} \sum_{e \in E} w(e)x(e)$ (9)

Subject to

$\sum_{e \in \delta(v)} x(e) \leq 2 \;\; for\ all\ v \in V$ (13)

$\sum_{e \in E} x(e) = |V|$ (14)

$\sum_{e \in E} x(e) \leq |U| - 1 \quad for\ U\ being\ a\ subset\ of\ V, |3| \leq |U| \leq |V| - 1$ (11)

$x(e) \in \{0,1\}$ (15)

When we Lagrangean relax the constraint stipulating the total demand we can look at the Lagrangean multiplier as a search for the market price in this market. *The optimal Lagrangean multiplier value has then the economic interpretation as the market price necessary in order to fulfill all customer demands on a single tour.*

L(u)=Min$\sum_{e\in E} w(e)x(e) - u(\sum_{e\in E} x(e) - |V|)$ (17)

Subject to

$\sum_{e\in\delta(v)} x(e) \le 2 \;\; for\ all\ v \in V$

$\sum_{e\in E} x(e) \le |V|$ (18)

$\sum_{e\in E} x(e) \le |U1 - 1 \quad for\ U\ being\ a\ subset\ of\ V, |3| \le |U| \le |V| - 1$

$x(e) \in \{0,1\}$

For a lower price than u*, the optimal Lagrangean price the salesman will not serve all customers on a single tour.

In the symmetric TSP used above for illustration the optimal Lagrangean price is u=85 and the profit obtained in the market is 850-378=472.

If we look at the problem in which node 7 is not included the optimal Lagrangean price will be 80 and the profit in this reduced market 720-295=425.

In the asymmetric illustrative example we get for the optimal multiplier value u*=58 we get the tour with cost 251 and the profit 464-251=113.

6. Computational results

In the computational tests we have performed we proceeded as follows. First, a heuristic is used to find a feasible solution with a good objective function value. We then select the Lagrangean multiplier to equal the most expensive arc in the feasible solution and solve the subproblem in the reduced graph obtained. Also, in the reduced graph we use a heuristic to find out if it contains a feasible tour with a lower objective function value than the value of the feasible tour we started off from. Based on the value of the feasible solution and the lower bound obtained from the subproblem, we can calculate the minimum increase necessary for the subproblem to give a bound necessary to either prove that our feasible solution is optimal or being able to generate a better feasible tour. This calculation is based on the fact that the feasible solution obtained when solving the subproblem remains feasible when we increase the value of the Lagrangean multiplier. Hence the increase in lower bound depends on the number of arcs contained in the subproblem solution. We have tested the procedure o a set of problems from TSPlib and the results are presented in the table below.

The calculations are used using CPLEX and we have not tried to develop an efficient code but used standard software. We have not used the fact that the optimal solution to the subproblem solved for a certain multiplier value u is feasible in the subsequent subproblems with larger multiplier values. The heuristic used to find feasible solution is Helsgauns LKH heuristic.

The computational results are presented in the table found below.

Instance	# Nodes	Total length of tour	longest link	time in sec	Total length of tour	longest link	Solution time in sec	u value	problem reduction	time in sec	Lower bound	no. of missing links
ftv55.tsp	55	1608	96	0,01	1608	121	4,79	96	70,0%	1,3	1544,95	2
								128	51,0%	6,8	1607,94	0
ftv44.tsp	44	1613	114	0,01	1613	114	2,7	114	61,0%	2,1	1601,96	1
								126	55,0%	9,8	1612,95	0
ftv35.tsp	35	1473	125	0,01	1473	125	1,9	125	54,0%	0,9	1447,96	1
								151	38,0%	20,2	1472,96	0
ft53.tsp	53	6905	978	0,02	6905	978	2	978	9,0%	143,7	6823,95	1
								1060	8,0%	153,1	6904,95	0
ft70.tsp	70	38673	1408	0,01	38673	1408	2,3	1408	14,0%	2,3	38388,9	1
								1694	9,0%	30,7	38672,9	0
ND4940.tsp	67	422508	7207	0,01	422508	7207	1,9	7207	73,0%	0,6	422314,0	4
								7256	54,0%	0,6	422462,0	1
								7302	35,0%	1,5	422508,0	0
ND4940m.tsp	51	336325	7231	0,01	336325	7207	1,4	7231	58,0%	0,2	336208,0	2
								7290	32,0%	0,6	336313,0	1
								7302	27,0%	0,3	336325,0	0
ND4941.tsp	70	452101	7275	0,02	452101	7302	5	7275	18,0%	5,0	452074,0	1
								7302	10,0%	7,5	452101,0	0
ND4942.tsp	80	517768	7188	0,01	517768	7188	9,6	7188	49,0%	3,7	517668,0	2
								7238	18,0%	6,2	517725,0	1
								7246	15,0%	12,2	517733,0	1
								7281	4,0%	9,0	517767,0	0

Note that in all problems the restricted relaxations obtained by requiring that an exposed node have to be visited or an end of a string shall have one more connection leads to the fact that optimality of the feasible solution found by the heuristic is proved.

7. Conclusions

In this paper, we have presented a Lagrangean relaxation method for the travelling salesman problem. We use a reformulation more suitable for Lagrangean relaxation. The optimal Lagrangean multiplier value has an interesting economic interpretation the market price needed in order to get all nodes visited. The number of potential Lagrangean multipliers is bounded and hence no subgradient search method needs to be used. In order for this approach to be computationally tractable, a special purpose code has to be developed for solving the subproblems and using the fact that the solution obtained in an earlier iteration of the procedure remains feasible when the multiplier value is increased. It should also be noted that once a subproblem has been solved based on a multiplier value obtained from a feasible solution one can easy construct a more restricted relaxation in which the optimal multiplier value is equal to the cost of the most costly edge in the tour.

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