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Discussion paper

Recursive utility and jump-diffusions

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Abstract

We derive the equilibrium interest rate and risk premiums using recursive utility for jump-diffusions. Compared to the continuous version, including jumps allows for a separate risk aversion related to jump size risk in addition to risk aversion related to the continuous part. The jump part also introduces moments of higher orders that may matter in many circumstances. We consider the version of recursive utility which gives the most unambiguous separation of risk preference from time substitution, and use the stochastic maximum principle to analyze the model. This method uses forward/backward stochastic differential equations. We demonstrate how the stochastic process for the market portfolio is determined in terms the corresponding processes for future utility and aggregate consumption. It is indicated that this model has the potential to give reasonable explanations of empirical puzzles.

KEYWORDS: recursive utility, jump dynamics, the stochastic maximum principle

JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

Rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time. Questions like the following are sometimes asked: Are asset prices too volatile relative to the information arriving in the market? Is the mean risk premium on equities over the riskless

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rate too large? Is the real interest rate too low? Is the market's risk aversion too high?

The results of Mehra and Prescott (1985) gave rise to some of these questions in their well-known paper, using a variation of Lucas's (1978) pure exchange economy with a Kydland and Prescott (1982) "calibration" exercise. They chose the parameters of the endowment process to match the sample mean, variance and the annual growth rate of per capita consumption in the years 1889-1978. The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

The puzzle has been verified by many others, e.g., Hansen and Singleton (1983), Ferson (1983), Grossman, Melino, and Shiller (1987). Many theories have been suggested during the years to explain the puzzle, but to date there does not seem to be any consensus that the puzzles have been fully resolved by any single of the proposed explanations ¹.

In the present paper we reconsider recursive utility in a continuous-time model including jump dynamics along the lines of Øksendal and Sulem (2013). This is an extension of the model developed by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994) which elaborates the foundational work by Kreps and Porteus (1978) and Epstein and Zin (1989) of recursive utility in dynamic models. The data set we consider is the same as that used by Mehra and Prescott (1985) in their seminal paper on this subject².

While jump dynamics has been introduced in the conventional model, among other things to throw some light on the puzzles (see Aase (1993a-b)), in the recursive models that we analyze in this paper, jump dynamics may play an even more interesting role. The reason for this are several:

¹Constantinides (1990) introduced habit persistence in the preferences of the agents. Also Campbell and Cochrane (1999) used habit formation. Rietz (1988) introduced financial catastrophes, Barro (2005) developed this further, Weil (1992) introduced non-diversifiable background risk, and Heaton and Lucas (1996) introduce transaction costs. There is a rather long list of other approaches aimed to solve the puzzles, among them are borrowing constraints (Constantinides et al. (2001)), taxes (Mc Grattan and Prescott (2003)), loss aversion (Benartzi and Thaler (1995)), survivorship bias (Brown, Goetzmann and Ross (1995)), and heavy tails and parameter uncertainty (Weitzmann (2007)).

²There is by now a long standing literature that has been utilizing recursive preferences. We mention Avramov and Hore (2007), Avramov et al. (2010), Eraker and Shaliastovich (2009), Hansen, Heaton, Lee, Roussanov (2007), Hansen and Scheinkman (2009), Wachter (2012), Bansal and Yaron (2004), Campbell (1996), Bansal and Yaron (2004), Kocherlakota (1990 b), and Ai (2012) to name some important contributions. Related work is also in Browning et. al. (1999), and on consumption see Attanasio (1999). Bansal and Yaron (2004) study a richer economic environment than we employ.

One is that the recursive model has already changed matters so much in the right direction, that second, and higher order effects may be enough to get satisfactory results. Another reason is that jump dynamics in the recursive model allow for one new preference parameter related to relative risk aversion for jump size risk, which gives the model added flexibility. This addition may also throw some light on the behavioral puzzle of 'loss aversion'. A third reason is that jump sizes are governed by an entire probability distribution, not just a few moments. This can be utilized to move out of the local mean/variance type analyses offered by the continuous model, and combine the best properties of the discrete time with continuous time analysis.

It has been a goal in the modern theory of asset pricing to internalize probability distributions of financial assets. To a large extent this has been achieved in our approach. As with a Lucas-style model, aggregate consumption is a given jump/diffusion process. The solution of a backward stochastic differential equations (BSDE) provides the main characteristics in the probability distributions of future utility. With existence of a solution to the BSDE granted, market clearing finally determines the characteristics in the market portfolio from the corresponding characteristics of the utility and aggregate consumption processes.

The paper is organized as follows: In Section 2 we explain the problems with the conventional, time additive model including jump dynamics. Here we illustrate some effects of deviation from the standard mean/variance analysis in financial economics. Section 3 contains a preview of results. Section 4 starts with a brief introduction to recursive utility in continuous time including jump dynamics, Section 5 derives the first order conditions, Section 6 details the financial market, and Section 7 presents the analysis relevant for recursive utility with jumps. In Section 8 we summarize our results. In Section 9 we present some calibrations, and Section 10 concludes.

2 The problems with the conventional model

The conventional asset pricing model in financial economics, the consumption-based capital asset pricing model (CCAPM) of Lucas (1978) and Breeden (1979), assumes a representative agent with a utility function of consumption that is the expectation of a sum, or a time integral, of future discounted utility functions. The model has been criticized for several reasons. First, it does not perform well empirically. Second, the usual specification of utility can not separate the risk aversion from the elasticity of intertemporal substitution, while it would clearly be advantageous to disentangle these two conceptually different aspects of preference. Third, while this representation

seems to function well in deterministic settings, and for *timeless* situations, it is not well founded for *temporal* problems (derived preferences do not in general satisfy the substitution axiom, e.g., Mossin (1969)).

In the conventional model the utility $U(c)$ of a consumption stream c_t is given by $U(c) = E\{\int_0^T u(c_t, t) dt\}$, where the felicity index u has the separable form $u(c, t) = \frac{1}{1-\gamma} c^{1-\gamma} e^{-\delta t}$. The parameter γ is the representative agent's relative risk aversion and δ is the utility discount rate, or the impatience rate, and T is the time horizon. These parameters are assumed to satisfy $\gamma > 0$, $\delta \geq 0$, and $T < \infty$.

When jumps are included the risk premium $(\mu_R - r)$ of any risky security labeled R (for "risky") is given by

$$\mu_R(t) - r_t = \gamma \sigma_{Rc}(t) - \int_{\mathcal{Z}} ((1 + \gamma_c(t, \zeta))^{-\gamma} - 1) \gamma_R(t, \zeta) \nu(d\zeta). \quad (1)$$

Here r_t is the equilibrium real interest rate at time t , and the term $\sigma_{Rc}(t) = \sum_{i=1}^d \sigma_{R,i}(t) \sigma_{c,i}(t)$ is the covariance rate between returns of the risky asset and the growth rate of aggregate consumption at time t , a measurable and adaptive process satisfying standard conditions. The dimension of the Brownian motion is $d > 1$. Underlying the jump dynamics we have $\{N_j\}$, $j = 1, 2, \dots, l$ independent Poisson random measures with Levy measures ν_j coming from l independent (1-dimensional) Levy processes. The possible time inhomogeneity in the jump processes is expressed through the terms denoted $\gamma_{R,j}(t, \zeta_j)$ for the risky asset under consideration, and $\gamma_{c,j}(t, \zeta_j)$ for the aggregate consumption process, both measuring the jump sizes. Here also jump frequencies at time t are embedded. The "mark space" $\mathcal{Z} = \mathbb{R}^l$ in this paper, where $\mathbb{R} = (-\infty, \infty)$. Thus the above term in (1) is short-hand notation for the following

$$\begin{aligned} \int_{\mathcal{Z}} ((1 + \gamma_c(t, \zeta))^{-\gamma} - 1) \gamma_R(t, \zeta) \nu(d\zeta) \\ = \sum_{j=1}^l \int_{\mathbb{R}} ((1 + \gamma_{c,j}(t, \zeta_j))^{-\gamma} - 1) \gamma_{R,j}(t, \zeta_j) \nu(d\zeta_j). \end{aligned}$$

This is a continuous-time version of the consumption-based CAPM, allowing for jumps at random time points. Similarly the expression for the risk-free,

real interest rate is

$$r_t = \delta + \gamma \mu_c(t) - \frac{1}{2} \gamma (\gamma + 1) \sigma'_c(t) \sigma_c(t) - \left(\gamma \int_{\mathcal{Z}} \gamma_c(t, \zeta) \nu(d\zeta) + \int_{\mathcal{Z}} ((1 + \gamma_c(t, \zeta))^{-\gamma} - 1) \nu(d\zeta) \right). \quad (2)$$

In the risk premium (1) the last term stems from the jump dynamics of the risky asset and aggregate consumption, while in (2) the last two terms have this origin. These results follow from Aase (1993a,b).

If the consumption process were as volatile as the stock market index, the jump dynamics could potentially contribute to giving a better explanation of empirical regularities than the continuous model can alone. However, because of the relatively small sizes of the potential jumps in the consumption process, it is unlikely that the last terms in these two relationships move these quantities enough in the right direction. As with the continuous model, the problem stems from the low covariance rate between consumption and the market index.

The process $\mu_c(t)$ is the annual growth rate of aggregate consumption and $(\sigma'_c(t)\sigma_c(t))$ is the annual variance rate of the consumption growth rate, both at time t , again dictated by the Ito-isometry. Both these quantities are measurable and adaptive stochastic processes, satisfying usual conditions. The return processes as well as the consumption growth rate process in this paper are also assumed to be ergodic processes, implying that statistical estimation makes sense.

Notice that in the model is the instantaneous correlation coefficient between returns and the consumption growth rate given by

$$\kappa_{Rc}(t) = \frac{\sigma_{Rc}(t)}{\|\sigma_R(t)\| \cdot \|\sigma_c(t)\|} = \frac{\sum_{i=1}^d \sigma_{R,i}(t) \sigma_{c,i}(t)}{\sqrt{\sum_{i=1}^d \sigma_{R,i}(t)^2} \sqrt{\sum_{i=1}^d \sigma_{c,i}(t)^2}},$$

and similarly for other correlations given in this model. Here $-1 \leq \kappa_{Rc}(t) \leq 1$ for all t . With this convention we can equally well write $\sigma'_R(t)\sigma_c(t)$ for $\sigma_{Rc}(t)$, and the former does *not* imply that the instantaneous correlation coefficient between returns and the consumption growth rate is equal to one. Prime means transpose.

Similarly the term $\sum_{j=1}^l \int_{\mathbb{R}} \gamma_{R,j}(t, \zeta_j) \gamma_{c,j}(t, \zeta_j) \nu(d\zeta_j)$ is the covariance rate at time t between returns of the risky asset and the growth rate of aggregate consumption stemming from the discontinuous dynamics. We use the shorthand notation $\int_{\mathcal{Z}} \gamma_R(t, \zeta) \gamma_c(t, \zeta) \nu(d\zeta)$ for this term as well.

Using a Taylor series expansion, the risk premium is approximately

$$\begin{aligned} \mu_R(t) - r_t = & \gamma \left(\sigma_{Rc}(t) + \int_{\mathcal{Z}} \gamma_R(t, \zeta) \gamma_c(t, \zeta) \nu(d\zeta) \right) \\ & - \frac{1}{2} \gamma(\gamma + 1) \int_{\mathcal{Z}} \gamma_R(t, \zeta) \gamma_c^2(t, \zeta) \nu(d\zeta) + \dots \quad (3) \end{aligned}$$

and an approximation for the interest rate is

$$\begin{aligned} r_t = & \delta + \gamma \mu_c(t) - \frac{1}{2} \gamma(1 + \gamma) \left(\sigma'_c(t) \sigma_c(t) + \int_{\mathcal{Z}} \gamma_c^2(t, \zeta) \nu(d\zeta) \right) \\ & + \frac{1}{6} \gamma(\gamma + 1)(\gamma + 2) \int_{\mathcal{Z}} \gamma_c^3(t, \zeta) \nu(d\zeta) - \dots \quad (4) \end{aligned}$$

Here the term $\int_{\mathcal{Z}} \gamma_c^2(t, \zeta) \nu(d\zeta)$ is the variance rate of the consumption growth rate at time t , stemming from the discontinuous dynamics, so that the total consumption variance rate is $(\sigma'_c(t) \sigma_c(t) + \int_{\mathcal{Z}} \gamma_c^2(t, \zeta) \nu(d\zeta))$ at time t . Similarly the total covariance rate between returns of the risky asset and the consumption growth rate is $(\sigma_{Rc}(t) + \int_{\mathcal{Z}} \gamma_R(t, \zeta) \gamma_c(t, \zeta) \nu(d\zeta))$.

In Table 1 we present the key summary statistics of the data in Mehra and Prescott (1985), of the real annual return data related to the S&P-500, denoted by M , as well as for the annualized consumption data, denoted c , and the government bills, denoted b ³.

| | Expectat. | Standard dev. | Covariances |
|--------------------|-----------|---------------|-------------------------------|
| Consumption growth | 1.83% | 3.57% | $\text{cov}(M, c) = .002226$ |
| Return S&P-500 | 6.98% | 16.54% | $\text{cov}(M, b) = .001401$ |
| Government bills | 0.80% | 5.67% | $\text{cov}(c, b) = -.000158$ |
| Equity premium | 6.18% | 16.67% | |

Table 1: Key US-data for the time period 1889-1978. Discrete-time compounding.

Here we have, for example, estimated the covariance between aggregate consumption and the stock index directly from the data set to be .00223. This gives the estimate .3770 for the correlation coefficient⁴.

Since our development is in continuous time, we have carried out standard adjustments for continuous-time compounding, from discrete-time compounding. The results of these operations are presented in Table 2. This

³There are of course newer data by now, but these retain the same basic features. If our model can explain the data in Table 1, it can explain any of the newer sets as well.

⁴The full data set was provided by Professor Rajnish Mehra.

gives, e.g., the estimate $\hat{\kappa}_{Mc} = .4033$ for the instantaneous correlation coefficient $\kappa(t)$. The overall changes are in principle small, and do not influence our comparisons to any significant degree, but are still important.

| | Expectation | Standard dev. | Covariances |
|--------------------|-------------|---------------|--------------------------------|
| Consumption growth | 1.81% | 3.55% | $\hat{\sigma}_{Mc} = .002268$ |
| Return S&P-500 | 6.78% | 15.84% | $\hat{\sigma}_{Mb} = .001477$ |
| Government bills | 0.80% | 5.74% | $\hat{\sigma}_{cb} = -.000149$ |
| Equity premium | 5.98% | 15.95% | |

Table 2: Key US-data for the time period 1889-1978. Continuous-time compounding.

Interpreting the risky asset R as the value weighted market portfolio M corresponding to the S&P-500 index, equations (3) and (4) are two equations in two unknowns that can provide estimates of the two preference parameters by the "method of moments". Ignoring the higher order terms in each of these equations, the result is $\gamma = 26.3$ and $\delta = -.015$, i.e., a relative risk aversion of about 26 and an impatience rate of minus 1.5%.

The jump terms might mitigate these numbers somewhat, since the jump model can, under certain assumptions, produce a larger equity premium than the continuous model can alone. As an example, suppose the cross-moment term $\int_{\mathcal{Z}} \gamma_R(t, \zeta) \gamma_c^2(t, \zeta) \nu(d\zeta)$ is of the order $-1.3 \cdot 10^{-3}$ and the third moment term $\int_{\mathcal{Z}} \gamma_c^3(t, \zeta) \nu(d\zeta)$ is of the order $-1.6 \cdot 10^{-3}$. Then the model produces results of the order $\delta = .08$ and $\gamma = 7.7$. By taking all the higher moments into account, these numbers could potentially be further improved. It is an empirical question to estimate these quantities (e.g., Ait Sahalia and Jacod (2009-11)), but see below.

2.1 Deviations from normality

In the conventional model we may use jump dynamics to study the effects of deviations from normality. This we have done by using the pure jump model alone to fit the data summarized in Table 1, and its logarithmic version (Table 4). In doing so we have fixed the frequency of "jumps" to one per year on the average. The advantage with this approach is that we do not have to separate the jump dynamics from the continuous part in the data. We have modeled the simultaneous jumps in the Levy-measure $\nu(d\zeta_1, d\zeta_2)$ by a joint Normal Inverse Gaussian (NIG)-distribution. This distribution measures heavy tails, kurtosis, skewness, etc, often found in financial stock market data. It fits fat-tailed and skewed data very well and is analyti-

cally tractable. This distribution was brought to the attention of workers in empirical finance by Barndorff-Nielsen (1997).

The result of this analysis weakened the puzzle using the above model when calibrated to the data (for details, see Aase and Lillestøl (2015)).

Although there is no canonical definition of a bivariate Normal inverse Gaussian distribution, the most common one is obtained by a mean-variance mixture of a multivariate Normal distribution with respect to the Inverse Gaussian (IG) distribution. This is convenient, since it leads to a relatively simple expression for its moment generating function, which may be taken as the definition of the distribution itself. By maximum likelihood estimators for the NIG-parameters, we obtain the same estimates of the moments as given in Table 1 (and Table 4), from which we obtain the following calibrated values: $(\gamma, \delta) = (22.2, 0.0083)$. Moreover, by varying the NIG-estimates, one by one, within the bounds given by sampling errors, and using resampling techniques, the puzzle was further weakened to $(17.7, 0.058)$.

As a comparison, under joint normality instead we get $(\gamma, \delta) = (24.3, -.044)$. Jumps alone move the risk premium down somewhat relative to the diffusion model, deviations from normality accounts for the rest.

The result of this is encouraging for the task we now set out to do, namely to include jumps in the recursive model.

3 Preview of results

Turning to recursive utility, one more parameter occurs in its most basic form. It is the time preference denoted by ρ . In the form we consider, the parameter $\psi = 1/\rho$ is the elasticity of intertemporal substitution in consumption (EIS), which we refer to as the EIS-parameter. In the conventional Eu-model $\gamma = \rho$, but relative risk tolerance $(1/\gamma)$ is something quite different from EIS.

We show that the standard recursive model extended to include jump dynamics takes the following form: For $\rho \neq 1$ and with the same notation as above

$$\begin{aligned} \mu_R(t) - r_t = & \rho \sigma_c(t)' \sigma_R(t) + (\gamma - \rho) \sigma_V(t)' \sigma_R(t) + \\ & \int_{\mathcal{Z}} \left\{ \gamma_0 K_V(t, \zeta) - \left(\left(\frac{1 + K_V(t, \zeta)}{1 + \gamma_c(t, \zeta)} \right)^\rho - 1 \right) (1 - \gamma_0 K_V(t, \zeta)) \right\} \gamma_R(t, \zeta) \nu(d\zeta). \end{aligned} \tag{5}$$

Here the term $K_V(t, \cdot)$ signify the jump sizes in the future utility process V , and $\gamma_c(t, \zeta)$ is the corresponding quantity for the growth rate of aggregate consumption, both parts of the primitives of the economic model.

The jump size function of the market portfolio is then determined in equilibrium as

$$1 + \gamma_M(t, \zeta) = \frac{1 + K_V(t, \zeta)(1 - \gamma_0 - \gamma_0 K_V(t, \zeta))}{1 - \gamma_0 K_V(t, \zeta) + \left(\left(\frac{1 + K_V(t, \zeta)}{1 + \gamma_c(t, \zeta)}\right)^\rho - 1\right)(1 - \gamma_0 K_V(t, \zeta))} \quad (6)$$

where the equality holds $\nu(\cdot)$ a.e.

Also the volatility of the market portfolio, $\sigma_M(t)$, is determined as a linear combination of corresponding utilities of future utility and the growth rate of aggregate consumption, $\sigma_V(t)$ and $\sigma_c(t)$ respectively, as follows

$$\sigma_M(t) = (1 - \rho)\sigma_V(t) + \rho\sigma_c(t), \quad (7)$$

i.e., also from primitives of the model. The jump term in (5) reduces to the jump term in (1) when $K_V(t, \cdot) = 0$ a.e., so K_V has strictly to do with recursive utility. Similarly if $\sigma_V(t) = 0$ a.e., we obtain the risk premium of the conventional model for the continuous part, so this term has also to do with recursive utility. The equation (6) is seen to be linear in $\gamma_M(t, \cdot)$, and can be seen to reduce to simpler forms in special cases.

The short term real interest rate is given by

$$\begin{aligned} r_t = & \delta + \rho\mu_c(t) - \frac{1}{2}\rho(\rho + 1)\sigma'_c(t)\sigma_c(t) \\ & - \rho(\gamma - \rho)\sigma_c(t)'\sigma_V(t) - \frac{1}{2}(\gamma - \rho)(1 - \rho)\sigma'_V(t)\sigma_V(t) \\ & - \int_{\mathcal{Z}} \left\{ \frac{1}{2}(1 + \rho)\gamma_0 K'_V(t, \zeta)K_V(t, \zeta) + \left(\left(\frac{1 + K_V(t, \zeta)}{1 + \gamma_c(t, \zeta)}\right)^\rho - 1\right)(1 - \gamma_0 K(t, \zeta)) \right. \\ & \left. + \rho\gamma_c(t, \zeta) - \rho K_V(t, \zeta) \right\} \nu(d\zeta). \quad (8) \end{aligned}$$

In the model the covariances are assumed to be measurable, adaptive, ergodic stochastic processes satisfying standard conditions. The parameter γ_0 we interpret as the agent's relative risk aversion related to jump size risk. When there are no jumps, we obtain the standard recursive model. When $\rho = \gamma$ the latter model reduces to the conventional, additive Eu-model. When $K_V = 0$ and $\sigma_V = 0$ the standard model with jumps, presented in the previous section, results.

3.1 The pure jump part

In order to study the effects from the nonlinearities caused by the jump dynamics, we may remodel the jump part slightly by letting $y := \gamma_M(t, \zeta)$, $c =$

$\gamma_c(t, \zeta)$, $v = K_V(t, \zeta)$ and $x = \gamma_R(t, \zeta)$. Assuming a stationary distribution for the jumps, for $l = 1$ the equation (6) can be written

$$1 + y = \frac{1 + v(1 - \gamma_0 - \gamma_0 v)}{1 - \gamma_0 v + \left(\left(\frac{1+v}{1+c}\right)^\rho - 1\right)(1 - \gamma_0 v)}. \quad (9)$$

Similarly $\nu(d\zeta) = \nu(d\zeta_1, d\zeta_2, d\zeta_3)$ can be remodeled as $\lambda_t dH_t(c, v, x)$ for a jump frequency λ_t and a cumulative distribution function $H_t(c, v, x)$ for the jump parts of the consumption growth rate, utility growth rate, and the return rate on the risky security R . The transformation (9) gives the following connection in terms of the variables y , c and x . The jump contribution to the risk premium can be written

$$\int_{-1}^{\infty} \int_{-1}^{\infty} \int_{-1}^{\infty} \left\{ \gamma_0 v - \left(\left(\frac{1+v}{1+c} \right)^\rho - 1 \right) (1 - \gamma_0 v) \right\} x \lambda_t dF_t(c, y, x) = \quad (10)$$

$$\iiint_{\mathcal{Z}'} \left\{ \gamma_0 v - \left(\left(\frac{1+v}{1+c} \right)^\rho - 1 \right) (1 - \gamma_0 v) \right\} x \lambda_t dH_t(c, v, x)$$

where $F(c, y, x)$ is the joint probability distribution of the jump parts of the consumption growth rate, the return rate on the market (wealth) portfolio, and the return rate on the risky security R . Assuming F has a probability density function $f(c, y, x)$, the connection to the given H , with density h , is that $h(c, v, x) = J(c, v, x) f(c, y(c, v), x)$. Here $y = y(c, v)$ is given in (9), and $J(c, v, x)$ is the Jacobian associated with the change of variables from (c, v, x) to (c, y, x) , given by

$$J(c, v, x) = \text{mod} \left| \begin{array}{ccc} 1 & 0 & 0 \\ \frac{\partial y}{\partial c} & \frac{\partial y}{\partial v} & 0 \\ 0 & 0 & 1 \end{array} \right| = \left| \frac{\partial y}{\partial v} \right|$$

where "mod" means the absolute value of the expression following it. Here $\frac{\partial y}{\partial v}$ can be written

$$\frac{\partial y}{\partial v} = \left((1 - \gamma_0 - 2\gamma_0 v)(1 - \gamma_0 v + \left(\frac{1+v}{1+c}\right)^\rho - 1)(1 - \gamma_0 v) - \right.$$

$$\left. (1 + v(1 - \gamma_0 - \gamma_0 v)) \left(\left(\frac{1+v}{1+c} \right)^\rho \left(\frac{(\rho - \gamma_0) - \gamma_0(1 + \rho)v}{1+v} \right) \right) \right) /$$

$$\left(1 - \gamma_0 v + \left(\left(\frac{1+v}{1+c} \right)^\rho - 1 \right) (1 - \gamma_0 v) \right)^2 \quad (11)$$

For this to be well defined, the Jacobian must be different from zero in the relevant domain, where the set \mathcal{Z}' is the image of $(-1, +\infty) \times (-1, +\infty) \times (-1, +\infty)$ under the change of variables.

This version contains the higher order terms in addition to the extra parameter γ_0 for the jump size risk. As for the conventional model, one can also consider deviations from normality in this framework. A similar rewriting can be formulated for the jump part of the interest rate r_t .

Notice the logic of the the equation (10): From the probability distribution H governing the 'primitives' of the model, which include (the jump parts of) consumption and utility, the probability distribution F is determined in equilibrium by the transformation (9). Turning this around, by the same relationship we also connect the mostly 'unobservable' H to the partly 'observable' F .

3.2 The CAPM++: $\rho = 0$

When $\rho = 0$ the equation (6) takes on the simple form

$$\gamma_M(t, \zeta) = K(t, \zeta) \quad \text{for all } t \text{ and } \zeta \in \mathcal{Z},$$

and the relationship (7) reduces to $\sigma_M(t) = \sigma_V(t)$, in which case we have perfect substitutability of consumption across time. This corresponds to a dynamic version of the classical one-period CAPM:

$$\mu_R(t) - r_t = \gamma \sigma'_M(t) \sigma_R(t) + \gamma_0 \int_{\mathcal{Z}} \gamma'_M(t, \zeta) \gamma_R(t, \zeta) \nu(d\zeta) \quad (12)$$

and

$$r_t = \delta - \frac{1}{2} \gamma \sigma'_M(t) \sigma_M(t) - \frac{1}{2} \gamma_0 \int_{\mathcal{Z}} \gamma'_M(t, \zeta) \gamma_M(t, \zeta) \nu(d\zeta). \quad (13)$$

Notice that these results are exact. We denote the dynamic version of the CAPM model based on recursive utility by CAPM++.

3.3 The second order approximation

If we disregard moments of order three and higher, the expressions for the risk premiums and the real rate can be simplified for any non-negative value

of $\rho \neq 1$ to the following:

$$\begin{aligned} \mu_R(t) - r_t &= \frac{\rho(1-\gamma)}{1-\rho} \sigma_c(t)' \sigma_R(t) + \frac{\gamma-\rho}{1-\rho} \sigma_M(t)' \sigma_R(t) + \\ &\frac{\rho(1-\gamma_0)}{1-\rho} \int_{\mathcal{Z}} \gamma_c(t, \zeta) \gamma_R(t, \zeta) \nu(d\zeta) + \frac{\gamma_0-\rho}{1-\rho} \int_{\mathcal{Z}} \gamma_M(t, \zeta) \gamma_R(t, \zeta) \nu(d\zeta) + \dots \end{aligned} \quad (14)$$

and

$$\begin{aligned} r_t &= \delta + \rho \mu_c(t) - \frac{1}{2} \frac{\rho(1-\rho\gamma)}{1-\rho} \sigma_c'(t) \sigma_c(t) + \frac{1}{2} \frac{\rho-\gamma}{1-\rho} \sigma_M(t)' \sigma_M(t) \\ &- \frac{1}{2} \frac{\rho(1-\rho\gamma_0)}{1-\rho} \int_{\mathcal{Z}} \gamma_c(t, \zeta) \gamma_c(t, \zeta) \nu(d\zeta) + \frac{1}{2} \frac{\rho-\gamma_0}{1-\rho} \int_{\mathcal{Z}} \gamma_M(t, \zeta) \gamma_M(t, \zeta) \nu(d\zeta) \\ &+ \dots \end{aligned} \quad (15)$$

The possibility that γ_0 is different from γ gives the recursive model an extra degree of freedom in these relationships.

In the above the jump sizes in the market portfolio is approximately internalized as follows

$$\gamma_M(t, \zeta) = (1-\rho) K_V(t, \zeta) + \rho \gamma_c(t, \zeta) + \dots$$

This is an approximation derived from (6) disregarding higher order terms. In the above we have used (7) as it stands.

These results show that our jump/diffusion version (5)-(8) is a natural extension of the continuous recursive model, just as (3) and (4) show that (1) and (2) is a natural extension to jump/diffusions of the conventional Eu-model with continuous dynamics only.

4 Recursive Stochastic Differentiable Utility

In this section we give a brief introduction to recursive, stochastic differential utility in the continuous-time model including jumps, along the lines of Øksendal and Sulem (2013). The starting point for this theory for the continuous model is Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994). Our approach based on Øksendal and Sulem (2013) includes jump dynamics, and is a more general.

We are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0, T], P)$ satisfying the 'usual' conditions, and a standard model for the stock market with Levy-

process driven uncertainty, N risky securities and one riskless asset (Section 6 provides more details). Consumption processes are chosen from the space L of square integrable progressively measurable processes with values in R_+ . The agent has utility function U , to be specified below, and an endowment process $e \in L$.

The stochastic differential utility $U : L \rightarrow R$ is defined as follows by three primitive functions: $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $A : \mathbb{R} \rightarrow \mathbb{R}$ and $A_0 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

The function $f(t, c_t, V_t, \omega)$ corresponds to a felicity index at time t , A is associated with a measure of absolute risk aversion related to the continuous dynamics, while A_0 is connected to a similar measure related to jump size risk. Both the latter two terms may also depend on t . In addition to current consumption c_t , the function f also depends on utility V_t .

The utility process V for a given consumption process c that we consider, satisfying $V_T = 0$, is given by the representation

$$V_t = E_t \left\{ \int_t^T \left(f(s, c_s, V_s) - \frac{1}{2} A(V_s) Z(s)' Z(s) - \frac{1}{2} \int_{\mathcal{Z}} A_0(V_s, \zeta) K'(s, \zeta) K(s, \zeta) \nu(d\zeta) \right) ds \right\}, \quad t \in [0, T] \quad (16)$$

where $E_t(\cdot)$ denotes conditional expectation given \mathcal{F}_t , and $Z(t)$ as well as $K(t, \cdot)$ are square-integrable progressively measurable processes, to be determined in our analysis. Here d is the dimension of the Brownian motion B_t , and $K(t, \cdot)$ is an l dimensional vector. We think of V_t as the utility for c at time t , conditional on current information \mathcal{F}_t . The term $A(V_t)$ is penalizing for risk in the continuous model, while the term $A_0(V_t, \cdot)$ penalizes for jump size risk.

Recall the *timeless* situation with a mean zero risk X having variance σ^2 , where the certainty equivalent m is defined by $Eu(w + X) := u(w - m)$ for a constant wealth w . Then the Arrow-Pratt approximation to m , valid for "small" risks, is given by $m \approx \frac{1}{2} A(w) \sigma^2$, where $A(\cdot)$ is the absolute risk aversion associated with u . This approximation is often good also when risks are not necessarily "small". The financial risks in this paper we consider small enough.

If, for each consumption process c_t , there is a well-defined utility process V , the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The triplet (f, A, A_0) generating V is called an aggregator.

Since $V_T = 0$ and $\int Z(t) dB_t$ and $\int \int_{\mathcal{Z}} K(t, \zeta) \tilde{N}(dt, d\zeta)$ are assumed to be

martingales, (16) has the stochastic differential equation representation

$$dV_t = \left(-f(t, c_t, V_t) + \frac{1}{2}A(V_t) Z(t)'Z(t) + \frac{1}{2} \int_{\mathcal{Z}} A_0(V_t, \zeta) K'(t, \zeta) K(s, \zeta) \nu(d\zeta) \right) dt + Z(t) dB_t + \int_{\mathcal{Z}} K(t, \zeta) \tilde{N}(dt, d\zeta). \quad (17)$$

Here $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$ is an l -dimensional compensated Poisson random measure of the underlying l -dimensional Levy process, and $B(t)$ is an independent d dimensional, standard Brownian motion.

If terminal utility different from zero is of interest, like for applications to life insurance, then V_T may be different from zero. We may think of A and A_0 as associated with functions $h, h_0 : \mathbb{R} \rightarrow \mathbb{R}$ such that $A(v) = -\frac{h''(v)}{h'(v)}$, where h is two times continuously differentiable, and similarly for h_0 . U is monotonic and risk averse if $A(\cdot) \geq 0$, $A_0(\cdot, \cdot) \geq 0$ and f is jointly concave and increasing in consumption.

The preference ordering represented by recursive utility is usually assumed to satisfy A1: Dynamic consistency (in the sense of Johnsen and Donaldson (1985)), A2: Independence of past consumption, and A3: State independence of time preference (see Skiadas (2009a)).

The version we consider has the Kreps-Porteus CES utility representation in discrete time, which here corresponds to the aggregator with the specification

$$f(c, v) = \frac{\delta}{1 - \rho} \frac{c^{1-\rho} - v^{1-\rho}}{v^{-\rho}}, \quad A(v) = \frac{\gamma}{v} \text{ and } A_0(v, \zeta) = \frac{\gamma_0}{v}, \quad \forall \zeta \in R \quad (18)$$

If, for example, $A(v) = A_0(v) = 0$ for all v , this means that the recursive utility agent is risk neutral.

Here $\rho \geq 0, \rho \neq 1, \delta \geq 0, \gamma \geq 0, \gamma_0 \geq 0$. The elasticity of intertemporal substitution in consumption is $\psi := 1/\rho$. The parameter ρ we call the time preference parameter. When $\rho \neq \gamma$ or $\rho \neq \gamma_0$ the desired disentangling of risk aversion from consumption substitution results.

For the model with continuous dynamics only, an ordinally equivalent specification can be derived as follows. When an aggregator (f_1, A_1) is given corresponding to the utility function U_1 , there exists a strictly increasing and smooth function $\varphi(\cdot)$ such that the ordinally equivalent $U_2 = \varphi \circ U_1$ has the aggregator (f_2, A_2) where

$$f_2(c, v) = ((1 - \gamma)v)^{-\frac{\gamma}{1-\gamma}} f_1(c, ((1 - \gamma)v)^{\frac{1}{1-\gamma}}), \quad A_2 = 0.$$

The function φ is given by

$$U_2 = \frac{1}{1-\gamma} U_1^{1-\gamma}, \quad (19)$$

for the Kreps-Porteus specification. It has the aggregator

$$f_2(c, v) = \frac{\delta}{1-\rho} \frac{c^{1-\rho} - ((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \quad A_2(v) = 0. \quad (20)$$

The normalized version is used to prove existence and uniqueness of the solution to the BSDE with semimartingale dynamics, see Theorem 1 in Duffie and Epstein (1992b). For the aggregator of the Kreps and Porteus type, the Lipschitz condition on utility in the above reference is not satisfied, but existence and uniqueness has then been proven in Duffie and Lions (1992).

The reduction to a normalized aggregator $(f_2, 0)$ does not mean that intertemporal utility is risk neutral, or that the representation has lost the ability to separate risk aversion from substitution (see Duffie and Epstein(1992a)). This version will not be used in this paper.

In Aase (2014a) it is shown that these two versions have the same risk premiums and the same short term interest rate in recursive model with continuous dynamics only.

It is instructive to recall the that the conventional additive and separable utility has aggregator

$$f(c, v) = u(c) - \delta v, \quad A = 0. \quad (21)$$

in the present framework (an ordinally equivalent one). As can be seen, even if $A = 0$, the agent of the conventional model is not risk neutral.

4.1 Homogeneity

The following result will be made use of in sections 7.3-4. For a given consumption process c_t we let $(V_t^{(c)}, Z_t^{(c)}, K_t(\zeta)^{(c)})$ be the solution of the BSDE

$$\begin{cases} dV_t^{(c)} = \left(-f(t, c_t, V_t^{(c)}) + \frac{1}{2}A(V_t^{(c)}) Z(t)^{(c)} Z(t)^{(c)} + \right. \\ \left. \frac{1}{2} \int_{\mathcal{Z}} A_0(V_t^{(c)}, \zeta) K'(t, \zeta)^{(c)} K(s, \zeta)^{(c)} \nu(d\zeta) \right) dt + Z(t)^{(c)} dB_t \\ \left. + \int_{\mathcal{Z}} K(t, \zeta)^{(c)} \tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \right. \\ \left. V_T^{(c)} = 0 \right. \end{cases} \quad (22)$$

Theorem 1 *Assume that, for all $\lambda > 0$,*

- (i) $\lambda f(t, c, v) = f(t, \lambda c, \lambda v); \forall t, c, v, \omega$
(ii) $A(\lambda v) = \frac{1}{\lambda} A(v); \forall v$
(iii) $A_0(\lambda v) = \frac{1}{\lambda} A_0(v); \forall v$
Then

$$V_t^{(\lambda c)} = \lambda V_t^{(c)}, Z_t^{(\lambda c)} = \lambda Z_t^{(c)} \text{ and } K_t^{(\lambda c)}(\zeta) = \lambda K_t^{(c)}(\zeta); \forall \zeta, t \in [0, T]. \quad (23)$$

Proof By uniqueness of the solution of the BSDEs of the type (22), all we need to do is to verify that the triple $(\lambda V_t^{(c)}, \lambda Z_t^{(c)}, \lambda K_t(\cdot)^{(c)})$ is a solution of the BSDE (22) with c_t replaced by λc_t , i.e. that

$$\begin{cases} d(\lambda V_t^{(c)}) = \left(-f(t, \lambda c_t, \lambda V_t^{(c)}) + \frac{1}{2} A(\lambda V_t^{(c)}) \lambda Z(t)^{(c)} \lambda Z(t)^{(c)} + \right. \\ \left. \frac{1}{2} \int_{\mathcal{Z}} A_0(\lambda V_t^{(c)}, \zeta) \lambda K'(t, \zeta)^{(c)} \lambda K(s, \zeta)^{(c)} \nu(d\zeta) \right) dt + \lambda Z(t)^{(c)} dB_t \\ + \int_{\mathcal{Z}} \lambda K(t, \zeta)^{(c)} \tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\ \lambda V_T^{(c)} = 0 \end{cases} \quad (24)$$

By (i), (ii) and (iii) the BSDE (24) can be written

$$\begin{cases} \lambda dV_t^{(c)} = \left(-\lambda f(t, c_t, V_t^{(c)}) + \frac{1}{2} \frac{1}{\lambda} A(V_t^{(c)}) \lambda^2 Z(t)^{(c)} Z(t)^{(c)} + \right. \\ \left. \frac{1}{2} \int_{\mathcal{Z}} \frac{1}{\lambda} A_0(V_t^{(c)}, \zeta) \lambda^2 K'(t, \zeta)^{(c)} K(s, \zeta)^{(c)} \nu(d\zeta) \right) dt + \lambda Z(t)^{(c)} dB_t \\ + \lambda \int_{\mathcal{Z}} K(t, \zeta)^{(c)} \tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\ \lambda V_T^{(c)} = 0 \end{cases} \quad (25)$$

But this is exactly the equation (22) multiplied by the constant λ . Hence (25) holds and the proof is complete. \square

Remarks 1) Note that the system need not be Markovian in general, since we allow

$$f(t, c, v, \omega); (t, \omega) \in [0, T] \times \Omega$$

to be an adapted process, for each fixed c, v .

2) Similarly, we can allow A_0 and A to depend on t as well⁵.

Corollary 1 Define $U(c) = V_0^{(c)}$. Then $U(\lambda c) = \lambda U(c)$ for all $\lambda > 0$.

Notice that the aggregator in (18) satisfies the assumptions of the theorem.

⁵not common in economics

5 The First Order Conditions

In the following we solve the consumer's optimization problem using the stochastic maximum principle and forward/backward stochastic differential equations. We have the specification in (17) and (18) in mind, formulated in the previous section, where the \tilde{f} to appear below is the drift term in (17). However, in principle the analysis is valid for any f , A and A_0 satisfying the stated conditions. The representative agent's problem is to solve

$$\sup_{c \in L} U(c)$$

subject to

$$E \left\{ \int_0^T c_t \pi_t dt \right\} \leq E \left\{ \int_0^T e_t \pi_t dt \right\},$$

where e is the endowment process of the agent. Here $V_t = V_t^c$, and $(V_t, Z(t), K(t, \cdot))$ is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dV_t = -\tilde{f}(t, \tilde{c}_t, V_t, Z(t), K(t, \zeta)) dt + Z(t) dB_t + \int_{\mathcal{Z}} K(t, \zeta) \tilde{N}(dt, d\zeta) \\ V_T = 0. \end{cases} \quad (26)$$

For $\alpha > 0$ we define the Lagrangian

$$\mathcal{L}(c; \lambda) = U(c) - \alpha E \left(\int_0^T \pi_t (c_t - e_t) dt \right).$$

The volatility $Z(t)$ as well as the jump size quantity $K(t, \zeta)$ are both part of the solution, together with the dynamics of utility V . Market clearing combined with properties of recursive utility in Theorem 1 will be used to internalize the corresponding quantities for "prices", by connecting these to Z and K .

In order to set down the first order condition for the representative consumer's problem, we use Kuhn-Tucker and either directional derivatives in function space, or the stochastic maximum principle. Both these methods are rather robust. The problem is well posed since U is increasing and concave and the constraint is convex.

Below we utilize the stochastic maximum principle (see Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Øksendal and Sulem (2013), Hu and Peng (1995), or Peng (1990)): We then have a forward backward stochastic differential equation (FBSDE) system consisting of the simple FSDE $dX(t) = 0; X(0) = 0$ and the BSDE (26). The Hamiltonian for

this problem is

$$H(t, c, v, z, k, y) = y_t \tilde{f}(t, c_t, v_t, z_t, k_t) - \alpha \pi_t(c_t - e_t), \quad (27)$$

where

$$\tilde{f}(t, c, v, z, k) = f(c, v) - \frac{1}{2} A(v) z' z - \frac{1}{2} \int_{\mathcal{Z}} A_0(v, \zeta) k'(t, \zeta) k(t, \zeta) \nu(d\zeta) \quad (28)$$

with A and A_0 given in (18). Here y_t refers to the adjoint variable to be defined shortly. Let $\nabla_k \tilde{f}$ denote the Frechet derivative of \tilde{f} with respect to k , and $\frac{d\nabla_k \tilde{f}}{d\nu}(\zeta)$ denote its Radon-Nikodym derivative with respect to ν . From the general theory, the adjoint equation is then

$$\begin{cases} dY_t = Y(t-) \left\{ \left(\frac{\partial f}{\partial v}(t, c_t) - \frac{1}{2} \left(\frac{\partial}{\partial v} A(V_t) \right) Z'(t) Z(t) \right. \right. \\ \left. \left. - \frac{1}{2} \int_{\mathcal{Z}} \left(\frac{\partial}{\partial v} A_0(V_t, \zeta) \right) K'(t, \zeta) K(t, \zeta) \nu(d\zeta) \right) dt \right. \\ \left. - \frac{1}{2} \frac{\partial}{\partial z} \left(A(V_t) Z'_t Z_t \right) dB_t + \int_{\mathcal{Z}} \frac{d\nabla_k \tilde{f}}{d\nu}(t, c_t, V_t, Z_t, K(t, \cdot))(\zeta) \tilde{N}(dt, d\zeta) \right\}, \\ Y_0 = 1. \end{cases}$$

With a general form of $A_0(v, \zeta)$ as in (16), we see that the Frechet derivative, $\nabla_k \tilde{f}$, is the linear operator

$$h \rightarrow (\nabla_k \tilde{f})(h) = - \int_{\mathcal{Z}} A_0(v, \zeta) k'(\zeta) h(\zeta) \nu(d\zeta); \quad h \in L^2(\nu).$$

Therefore, as a random measure we have that $\nabla_k \tilde{f} \ll \nu$, with Radon-Nikodym derivative

$$\frac{d\nabla_k \tilde{f}}{d\nu}(\zeta) = -A_0(v, \zeta) k(\zeta).$$

Based on this, the adjoint equation can be written

$$\begin{cases} dY_t = Y(t-) \left\{ \left(\frac{\partial f}{\partial v}(t, c_t) + \frac{1}{2} \frac{\gamma}{V_t^2} Z'(t) Z(t) \right. \right. \\ \left. \left. + \frac{1}{2} \int_{\mathcal{Z}} \frac{\gamma_0}{V_t^2} K'(t, \zeta) K(t, \zeta) \nu(d\zeta) \right) dt \right. \\ \left. - \frac{\gamma}{V_t} Z(t) dB_t - \int_{\mathcal{Z}} \frac{\gamma_0}{V_t} K(t, \zeta) \tilde{N}(dt, d\zeta) \right\}, \\ Y(0) = 1, \end{cases} \quad (29)$$

which has the solution

$$\begin{aligned}
Y_t = \exp & \left(\int_0^t \left(\frac{\partial f}{\partial v}(s, c_s) + \frac{1}{2} \frac{\gamma(1-\gamma)}{V_s^2} Z'(s) Z(s) \right. \right. \\
& + \frac{1}{2} \int_{\mathcal{Z}} \frac{\gamma_0}{V_{s-}^2} K'(s, \zeta) K(s, \zeta) \nu(d\zeta) ds - \int_0^t \frac{\gamma}{V_s} Z(s) dB_s \\
& + \int_0^t \int_{\mathcal{Z}} \left\{ \ln \left(1 - \frac{\gamma_0}{V_s} K(s, \zeta) \right) + \frac{\gamma_0}{V_s} K(s, \zeta) \right\} \nu(d\zeta) ds \\
& \left. \left. + \int_0^t \int_{\mathcal{Z}} \ln \left(1 - \frac{\gamma_0}{V_s} K(s, \zeta) \right) \tilde{N}(ds, d\zeta) \right). \right.
\end{aligned} \tag{30}$$

The adjoint equation is now reduced to depend on primitives of the economy only. The interpretation of Y_t is a shadow price; the marginal value as of time zero of an additional unit of utility at time t .

Sufficient conditions for a unique, optimal solution using the stochastic maximum principle are the same as the corresponding conditions for the existence and uniqueness of a solution to the BSDE (26).

Maximizing the Hamiltonian with respect to c gives the first order equation

$$y \frac{\partial \tilde{f}}{\partial c}(t, c^*, v, z, k) - \alpha \pi = 0$$

or

$$\alpha \pi_t = Y(t) \frac{\partial \tilde{f}}{\partial c}(t, c_t^*, V(t), Z(t), K(t, \cdot)) \quad \text{a.s. for all } t \in [0, T]. \tag{31}$$

where c^* is optimal. The state price deflator π_t at time t depends, through the adjoint variable Y_t , on the entire optimal paths $(c_s^*, V_s, Z(s), K(s, \cdot))$ for $0 \leq s \leq t$, which means that marginal value at time t depends on the consumption history.

When $\gamma = \gamma_0 = \rho$ then $Y_t = e^{-\delta t}$ for the aggregator (21) of the conventional model, so the state price deflator is a Markov process, the utility is additive in which case dynamic programming is known to work well.

For the representative agent equilibrium the optimal consumption process is the given aggregate consumption c in society, and for this consumption process the utility V_t at time t is optimal.

We now have the first order conditions for recursive utility. Before we proceed to a solution of the problem, we need to specify the financial market model.

6 The financial market

Having established the general recursive utility form of interest, in this section we specify our model for the financial market. The model is much like the one used by Duffie and Epstein (1992a), except that we do not assume any unspecified factors in our model.

Let $\nu_R(t) \in R^N$ denote the vector of expected rates of return of the N given risky securities in excess of the riskless instantaneous return r_t , and let $\sigma(t)$ denote the $N \times d$ -matrix of diffusion coefficients of the risky asset prices, normalized by the asset prices, so that $\sigma(t)\sigma(t)'$ is the instantaneous covariance matrix for the continuous part of asset returns. The jumps in the various assets are captured by the $N \times l$ -matrix $\gamma(t, \zeta)$ and a vector valued, compensated random measure

$$\begin{aligned} \tilde{N}(dt, d\zeta)' &= (\tilde{N}_1(dt, d\zeta_1), \dots, \tilde{N}_l(dt, d\zeta_l)) = \\ &= (N_1(dt, d\zeta_1) - \nu_1(d\zeta_1)dt, \dots, N_l(dt, d\zeta_l) - \nu_l(d\zeta_l)dt), \end{aligned}$$

where $\{N_j\}$ are independent Poisson random measures with Levy measures ν_j coming from l independent (1-dimensional) Levy processes.

The representative consumer's problem is, for each initial level w of wealth to solve

$$\sup_{(c, \varphi)} U(c) \tag{32}$$

subject to the intertemporal budget constraint

$$\begin{aligned} dW_t &= (W_t(\varphi_t' \cdot \nu_R(t) + r_t) - c_t)dt + W_t \varphi_t' \cdot \sigma(t)dB_t \\ &\quad + W_t \varphi_t' \cdot \int_{R^l} \gamma(t, \zeta) \tilde{N}(dt, d\zeta). \end{aligned} \tag{33}$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \dots, \varphi_t^{(N)})$ are the fractions of total wealth W_t held in the risky securities. The processes $\nu_R(t)$, $\sigma(t)$ and $\gamma(t)$ are progressively measurable, ergodic processes.

Market clearing requires that $\varphi_t' \sigma(t) = (\delta_t^M)' \sigma(t) = \sigma_M(t)$ and $\varphi_t' \gamma(t, \cdot) = (\delta_t^M)' \gamma(t, \cdot) = \gamma_M(t, \cdot)$ in equilibrium, where $\sigma_M(t)$ is the volatility of the return on the market (wealth) portfolio, $\gamma_M(t, \cdot)$ is the corresponding jump size function, and δ_t^M are the fractions of the different securities, $j = 1, \dots, N$ held in the value-weighted market portfolio. That is, the representative agent must hold the market portfolio in equilibrium, by construction.

The model is a pure exchange economy where the aggregate consumption process c_t in society is exogenously given, and the single agent optimally

consumes $c_t = e_t$ in every period. The main issue is then the determination of prices, including risk premiums and the interest rate, consistent with this behavior.

7 The development of the recursive model

We now turn our attention to pricing restrictions relative to the given optimal consumption plan. Recall the first order conditions are given in (31).

It is convenient to use the notation $Z(t)/V_t := \sigma_V(t)$ and $K(t, \cdot)/V(t-) := K_V(t, \cdot)$, where V_{t-} means the value of V just before a possible jump at time t , assuming $V \neq 0$. By Theorem 1, $\sigma_V(t)$ and $K_V(t, \cdot)$ are both homogeneous of degree zero in c . With this convention the utility process V_t satisfies the following backward equation

$$\begin{aligned} \frac{dV_t}{V_{t-}} = & \left(-\frac{\delta}{1-\rho} \frac{c_t^{1-\rho} - V_t^{1-\rho}}{V_t^{-\rho+1}} + \frac{1}{2} \gamma \sigma'_V(t) \sigma_V(t) \right. \\ & \left. + \frac{1}{2} \int_{\mathcal{Z}} \gamma_0 K'_V(t, \zeta) K_V(t, \zeta) \nu(d\zeta) \right) dt \\ & + \sigma_V(t) dB_t + \int_{\mathcal{Z}} K_V(t, \zeta) \tilde{N}(dt, d\zeta), \quad (34) \end{aligned}$$

where $V(T) = 0$. The short-hand notation for the integrals with jump dynamics is as explained in Section 2. Since the jump times have Lebesgue measure zero, $V_t = V_{t-}$ a.e. on $[0, T]$.

Aggregate consumption is exogenous, with dynamics on of the form

$$\frac{dc_t}{c_{t-}} = \mu_c(t) dt + \sigma_c(t) dB_t + \int_{\mathcal{Z}} \gamma_c(t, \zeta) \tilde{N}(dt, d\zeta), \quad (35)$$

where $\mu_c(t)$, $\sigma_c(t)$ and $\gamma_c(t, \cdot)$ are measurable, \mathcal{F}_t adapted stochastic processes, satisfying appropriate integrability conditions. We assume these processes to be ergodic, so that they can be estimated.

Under these conditions the adjoint variable Y has dynamics given in (29). From the FOC in equation (31) we derive the dynamics of the state price deflator. We then seek the connection between V_t , $\sigma_V(t)$ and $K_V(t, \cdot)$ and the rest of the economy. Towards this end, by Ito's generalized lemma, normalizing to $\alpha = 1$, we get

$$d\pi_t = f_c(c_t, V_t) dY_t + Y_t df_c(c_t, V_t) + d[Y, f_c(c, V)](t), \quad (36)$$

since $\tilde{f}_c = f_c$, where $[X, Y](t)$ is the quadratic covariation of the processes X

and Y given by

$$[X, Y](t) = \int_0^t (\sigma_X(s)\sigma_Y(s) + \int_{\mathcal{Z}} \gamma_X(s, \zeta)\gamma_Y(s, \zeta)\nu(d\zeta)) ds \\ + \int_0^t \int_{\mathcal{Z}} \gamma_X(s, \zeta)\gamma_Y(s, \zeta)\tilde{N}(ds, d\zeta).$$

By the dynamics of the adjoint and the backward equations, this can be written, using Ito's multi-dimensional formula

$$\begin{aligned} d\pi_t &= Y_t f_c(c_t, V_t) \left(\{f_v(c_t, V_t) + \frac{1}{2}\gamma\sigma'_V(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 K'_V K_V \nu(d\zeta)\} dt \right. \\ &- \gamma\sigma_V(t)dB_t - \left. \int_{\mathcal{Z}} \gamma_0 K_V(t, \zeta)\tilde{N}(dt, d\zeta) \right) + Y_t \frac{\partial f_c}{\partial c}(c_t, V_t)(c_t \mu_c(t)dt + c_t \sigma_c(t)dB_t) \\ &+ Y_t \frac{\partial f_c}{\partial v}(c_t, V_t) \left(\{-f(c_t, V_t) + \frac{1}{2}\gamma V_t \sigma'_V(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} V_{t-} \gamma_0 K'_V K_V \nu(d\zeta)\} dt \right. \\ &+ \left. V_t \sigma_V(t)dB_t \right) + Y_t \left(\frac{1}{2} \frac{\partial^2 f_c}{\partial c^2}(c_t, V_t) c_t^2 \sigma'_c(t)\sigma_c(t) + \frac{\partial^2 f_c}{\partial c \partial v}(c_t, V_t) \sigma'_c(t)\sigma_V(t) \right. \\ &+ \left. \frac{1}{2} \frac{\partial^2 f_c}{\partial v^2}(c_t, V_t) V_t^2 \sigma'_V(t)\sigma_V(t) \right) dt + Y_t \left(\int_{\mathcal{Z}} \{f_c(c_{t-}(1+\gamma_c(t, \zeta)), V_{t-}(1+K_V(t, \zeta))) \right. \\ &- f_c(c_{t-}, V_{t-}) - \gamma_c(t, \zeta)c_{t-} \frac{\partial f_c}{\partial c}(c_t, V_t) - K_V(t, \zeta)V_{t-} \frac{\partial f_c}{\partial v}(c_t, V_t)\} \nu(d\zeta) dt \\ &+ \left. \int_{\mathcal{Z}} \{f_c(c_{t-}(1+\gamma_c(t, \zeta)), V_{t-}(1+K_V(t, \zeta))) - f_c(c_{t-}, V_{t-})\} \tilde{N}(dt, d\zeta) \right) \\ &- \gamma\sigma_V(t)Y_t \{c_t \sigma_c(t) \frac{\partial f_c}{\partial c}(c_t, V_t) + V_t \sigma_V(t) \frac{\partial f_c}{\partial v}(c_t, V_t)\} dt \\ &- Y_t \int_{\mathcal{Z}} \gamma_0 K_V(t, \zeta) \{f_c(c_{t-}(1+\gamma_c(t, \zeta)), V_{t-}(1+K_V(t, \zeta))) - f_c(c_{t-}, V_{t-})\} \nu(d\zeta) dt \\ &- Y_t \int_{\mathcal{Z}} \gamma_0 K_V(t, \zeta) \{f_c(c_{t-}(1+\gamma_c(t, \zeta)), V_{t-}(1+K_V(t, \zeta))) - f_c(c_{t-}, V_{t-})\} \tilde{N}(dt, d\zeta). \end{aligned} \tag{37}$$

Here

$$f_c(c, v) := \frac{\partial f(c, v)}{\partial c} = \delta c^{-\rho} v^\rho, \quad f_v(c, v) := \frac{\partial f(c, v)}{\partial v} = -\frac{\delta}{1-\rho} (1 - \rho c^{1-\rho} v^{\rho-1}), \\ \frac{\partial f_c(c, v)}{\partial c} = -\delta \rho c^{-(1+\rho)} v^\rho, \quad \frac{\partial f_c(c, v)}{\partial v} = \delta \rho v^{\rho-1} c^{-\rho},$$

$$\frac{\partial^2 f_c}{\partial c^2}(c, v) = \delta\rho(\rho + 1)v^\rho c^{-(\rho+2)}, \quad \frac{\partial^2 f_c}{\partial c \partial v}(c, v) = -\delta\rho^2 v^{\rho-1} c^{-(\rho+1)},$$

and

$$\frac{\partial^2 f_c}{\partial v^2}(c, v) = \delta\rho(\rho - 1)v^{\rho-2} c^{-\rho}.$$

From the canonical representation of the state price deflator

$$d\pi_t = \mu_\pi(t)dt + \sigma_\pi(t)dB_t + \int_{\mathcal{Z}} \gamma_\pi(t, \zeta) \tilde{N}(dt, d\zeta),$$

from (37) we find the key characteristics of π . They are

$$\begin{aligned} \mu_\pi(t) = & Y_t \left(f_c(c_t, V_t) (f_v(c_t, V_t) + \frac{1}{2} \gamma \sigma'_V(t) \sigma_V(t) \right. \\ & \left. + \frac{1}{2} \int_{\mathcal{Z}} \gamma_0 K'_V K_V \nu(d\zeta)) + \frac{\partial f_c}{\partial c}(c_t, V_t) c_t \mu_c(t) \right. \\ & \left. + \frac{\partial f_c}{\partial v}(c_t, V_t) \left\{ -f(c_t, V_t) + \frac{1}{2} \gamma V_t \sigma'_V(t) \sigma_V(t) + \frac{1}{2} \int_{\mathcal{Z}} V_{t-} \gamma_0 K'_V K_V \nu(d\zeta) \right\} \right. \\ & \left. + \frac{1}{2} \frac{\partial^2 f_c}{\partial c^2}(c_t, V_t) c_t^2 \sigma'_c(t) \sigma_c(t) + \frac{\partial^2 f_c}{\partial c \partial v}(c_t, V_t) \sigma'_c(t) \sigma_V(t) \right. \\ & \left. + \frac{1}{2} \frac{\partial^2 f_c}{\partial v^2}(c_t, V_t) V_t^2 \sigma'_V(t) \sigma_V(t) + \int_{\mathcal{Z}} \left\{ f_c(c_{t-}(1 + \gamma_c(t, \zeta)), V_{t-}(1 + K_V(t, \zeta))) \right. \right. \\ & \left. \left. - f_c(c_{t-}, V_{t-}) - \gamma_c(t, \zeta) c_{t-} \frac{\partial f_c}{\partial c}(c_t, V_t) - K_V(t, \zeta) V_{t-} \frac{\partial f_c}{\partial v}(c_t, V_t) \right\} \nu(d\zeta) \right. \\ & \left. - \gamma \sigma_V(t) \left\{ c_t \sigma_c(t) \frac{\partial f_c}{\partial c}(c_t, V_t) + V_t \sigma_V(t) \frac{\partial f_c}{\partial v}(c_t, V_t) \right\} \right. \\ & \left. - \int_{\mathcal{Z}} \gamma_0 K_V(t, \zeta) \left\{ f_c(c_{t-}(1 + \gamma_c(t, \zeta)), V_{t-}(1 + K_V(t, \zeta))) - f_c(c_{t-}, V_{t-}) \right\} \nu(d\zeta) \right), \end{aligned} \quad (38)$$

$$\sigma_\pi(t) = Y_t \left(-f_c(c_t, V_t) \gamma \sigma_V(t) + \frac{\partial f_c}{\partial c}(c_t, V_t) c_t \sigma_c(t) + V_t \sigma_V(t) \frac{\partial f_c}{\partial v}(c_t, V_t) \right) \quad (39)$$

and

$$\begin{aligned} \gamma_\pi(t, \zeta) = & Y_t \left(f_c(c_t, V_t) (-\gamma_0 K_V(t, \zeta)) \right. \\ & \left. + \left\{ f_c(c_{t-}(1 + \gamma_c(t, \zeta)), V_{t-}(1 + K_V(t, \zeta))) - f_c(c_{t-}, V_{t-}) \right\} \right. \\ & \left. - \gamma_0 K_V(t, \zeta) \left\{ f_c(c_{t-}(1 + \gamma_c(t, \zeta)), V_{t-}(1 + K_V(t, \zeta))) - f_c(c_{t-}, V_{t-}) \right\} \right). \end{aligned} \quad (40)$$

7.1 The risk premiums

The risk premium of any risky security with return process R is given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t} \sigma'_\pi(t) \sigma_R(t) - \frac{1}{\pi_t} \int_{\mathcal{Z}} \gamma_\pi(t, \zeta) \gamma_R(t, \zeta) \nu(d\zeta) \quad (41)$$

where the last term follows from Aase (1993a,b). Since $\pi_t = Y_t f_c(c_t, V_t)$, it is a consequence of the expressions in (39) and (41) that the risk premium of any risky security is given by

$$\begin{aligned} \mu_R(t) - r_t = & \left(-\frac{\frac{\partial f_c}{\partial c}(c_t, V_t)}{f_c(c_t, V_t)} c_t \sigma'_c(t) \sigma_R(t) + \left(\gamma - \frac{\frac{\partial f_c}{\partial v}(c_t, V_t)}{f_c(c_t, V_t)} V_t \right) \sigma'_V(t) \sigma_R(t) \right) \\ & + \int_{\mathcal{Z}} \left(\gamma_0 K_V(t, \zeta) - \frac{1}{f_c(c_t, V_t)} (f_c(c_{t-}(1+\gamma_c(t, \zeta)), V_{t-}(1+K_V(t, \zeta))) - f_c(c_{t-}, V_{t-})) \right. \\ & \left. (1 - \gamma_0 K_V(t, \zeta)) \right) \gamma_R(t, \zeta) \nu(d\zeta). \quad (42) \end{aligned}$$

This is our basic result for risk premiums. We now substitute in for f given in (18) and the various partial derivatives derived above. This gives

$$\begin{aligned} \mu_R(t) - r_t = & \rho \sigma_c(t)' \sigma_R(t) + (\gamma - \rho) \sigma_V(t)' \sigma_R(t) \\ & + \int_{\mathcal{Z}} \left(\gamma_0 K_V(t, \zeta) - \left(\left(\frac{1 + K_V(t, \zeta)}{1 + \gamma_c(t, \zeta)} \right)^\rho - 1 \right) (1 - \gamma_0 K_V(t, \zeta)) \right) \gamma_R(t, \zeta) \nu(d\zeta). \quad (43) \end{aligned}$$

It remains to connect the characteristics of the market portfolio to the fundamentals σ_V and K_V of the economy, which we do below. Before that we turn to the interest rate.

7.2 The equilibrium interest rate

The equilibrium short-term, real interest rate r_t is given by the formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \quad (44)$$

The real interest rate at time t can be thought of as the expected exponential rate of decline of the representative agent's marginal utility, which is π_t in equilibrium.

In order to find an expression for r_t in terms of the primitives of the model, we use (38). Using the expression for f and its various partial derivatives,

we obtain the expression

$$\begin{aligned}
r_t = & \delta + \rho\mu_c(t) - \frac{1}{2}\rho(\rho+1)\sigma'_c(t)\sigma_c(t) \\
& - \rho(\gamma-\rho)\sigma_c(t)\sigma'_V(t) - \frac{1}{2}(\gamma-\rho)(1-\rho)\sigma'_V(t)\sigma_V(t) \\
- \int_{\mathcal{Z}} & \left\{ \frac{1}{2}(1+\rho)\gamma_0 K'_V(t, \zeta) K_V(t, \zeta) + \left(\frac{1+K_V(t, \zeta)}{1+\gamma_c(t, \zeta)} \right)^\rho - 1 \right\} (1-\gamma_0 K(t, \zeta)) \\
& + \rho\gamma_c(t, \zeta) - \rho K_V(t, \zeta) \Big\} \nu(d\zeta), \quad (45)
\end{aligned}$$

first presented in (8) of Section 2, which is our basic result for the equilibrium short rate.

7.3 The determination of the volatility and jump characteristics of the market portfolio

In order to determine $\sigma_M(t)$, and $\gamma_M(t, \cdot)$ from the primitives of the model, which in this case involve $\sigma_V(t)$, $K_V(t, \cdot)$, $\sigma_c(t)$ and $\gamma_c(t, \cdot)$, first notice that the wealth at any time t is given by

$$W_t = \frac{1}{\pi_t} E_t \left(\int_t^T \pi_s c_s ds \right), \quad (46)$$

where c is optimal. By the definition of directional derivatives (the Frechet derivative) we have that

$$\begin{aligned}
\nabla U(c; c) &= \lim_{\alpha \downarrow 0} \frac{U(c + \alpha c) - U(c)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{U(c(1 + \alpha)) - U(c)}{\alpha} \\
&= \lim_{\alpha \downarrow 0} \frac{(1 + \alpha)U(c) - U(c)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{\alpha U(c)}{\alpha} = U(c),
\end{aligned}$$

where the third equality uses that U is homogeneous of degree one as shown in Theorem 1. By the Riesz representation theorem and dominated convergence theorem it follows from the linearity and continuity of the directional derivative that

$$\nabla U(c; c) = E \left(\int_0^T \pi_t c_t dt \right) = W_0 \pi_0 \quad (47)$$

where W_0 is the wealth of the representative agent at time zero, and the last equality follows from (46) for $t = 0$. Thus $U(c) = \pi_0 W_0$.

Let $V_t(c)$ denote future utility at the optimal consumption for our representation. Since this function is also homogeneous of degree one and is

continuously differentiable, by Riesz' representation theorem and the dominated convergence theorem, the same type of basic linear relationship holds here for the associated directional derivatives at any time t , i.e.,

$$\nabla V_t(c; c) = E_t \left(\int_t^T \pi_s^{(t)} c_s ds \right) = V_t(c)$$

where the Riesz representation $\pi_s^{(t)}$ for $s \geq t$ is the state price deflator at time $s \geq t$, as of time t . As for the discrete time model, it follows by results in Skiadas (2009a) that with assumption A2, implying that this quantity is independent of past consumption, the consumption history in the adjoint variable Y_t is 'removed' from the state price deflator π_t , so that $\pi_s^{(t)} = \pi_s/Y_t$ for all $t \leq s \leq T$. By this it follows that

$$V_t = \frac{1}{Y_t} \pi_t W_t. \quad (48)$$

This gives us the dynamics of V in terms of the other variables, which determines the relationship between primitives and the endogenous variables. By the product rule,

$$dV_t = d(Y_t^{-1})(\pi_t W_t) + Y_t^{-1} d(\pi_t W_t) + d[Y_t^{-1}, (\pi_t W_t)](t) \quad (49)$$

where

$$d(\pi_t W_t) = W_t d\pi_t + \pi_t dW_t + d[\pi_t, W_t](t). \quad (50)$$

Ito's lemma gives

$$\begin{aligned} d\left(\frac{1}{Y_t}\right) &= -\frac{1}{Y_{t-}} \left(f_v(c_t, V_t) + \frac{1}{2} \gamma \sigma'_V(t) \sigma_V(t) + \frac{1}{2} \int_{\mathcal{Z}} \gamma_0 K'_V(t, \zeta) K_V(t, \zeta) \nu(d\zeta) \right) dt \\ &\quad + \frac{\gamma^2}{Y_{t-}} \sigma'_V(t) \sigma_V(t) dt + \frac{1}{Y_{t-}} \gamma \sigma_V(t) dB_t \\ &+ \int_{\mathcal{Z}} \left\{ \frac{1}{Y_{t-}(1 - A_0(t, \zeta)K(t, \zeta))} - \frac{1}{Y(t-)} - \frac{1}{Y(t-)} A_0(t, \zeta) K(t, \zeta) \right\} \nu(d\zeta) dt \\ &\quad + \int_{\mathcal{Z}} \left\{ \frac{1}{Y_{t-}(1 - A_0(t, \zeta)K(t, \zeta))} - \frac{1}{Y(t-)} \right\} \tilde{N}(dt, d\zeta). \quad (51) \end{aligned}$$

From the equations (49)-(51) it follows by the market clearing condition $\varphi'_t \cdot \sigma(t) = \sigma_M(t)$ that

$$V_t \sigma_V(t) = \frac{1}{Y_t} \left(\pi_t W_t \gamma \sigma_V + \pi_t W_t \sigma_M(t) - \pi_t W_t (\rho \sigma_c(t) + (\gamma - \rho) \sigma_V(t)) \right) \quad (52)$$

From the expression (48) for V_t we obtain the following equation for σ_V

$$\sigma_V(t) = \gamma\sigma_V(t) + \sigma_M(t) - (\rho\sigma_c(t) + (\gamma - \rho)\sigma_V(t))$$

from which it follows that

$$\sigma_M(t) = (1 - \rho)\sigma_V(t) + \rho\sigma_c(t). \quad (53)$$

This relationship internalizes this important quantity in equilibrium. The relationship can also be written

$$\sigma_V(t) = \frac{1}{1 - \rho}(\sigma_M(t) - \rho\sigma_c(t)), \quad (54)$$

which, when inserted into (43) and (45) gives the model of Section 3 for the continuous part of the dynamics.

The version treated by Duffie and Epstein (1992a) is the ordinally equivalent one based on (20), which was claimed to be better suited for dynamic programming, the solution method used by them. One assumption which must be made in order to solve the associated Bellman equation is then that the volatilities involved are constants. In the conventional model the result of this is that the volatility of the consumption growth rate must equal the volatility of the wealth portfolio. In the recursive model in continuous time, this happens only if $\rho = 1$. In any case, this assumption does not seem well supported by data.

Under the assumptions of this paper and without any jump dynamics, the two ordinally equivalent versions give the same expressions for the risk premiums and the real interest rate (see Aase (2014a)).

We turn to the equilibrium determination of $\gamma_M(t, \cdot)$. From equation (51) we define

$$\gamma_{Y^{-1}}(t, \zeta) := \frac{\gamma_0 K_V(t, \zeta)}{1 - \gamma_0 K_V(t, \zeta)}.$$

From the equations (48)-(51), using the market clearing condition $\varphi'_t \gamma(t, \cdot) = \gamma_M(t, \cdot)$, it follows that

$$\begin{aligned} K_V(t, \zeta) &= \gamma_{Y^{-1}}(t, \zeta) + (\gamma_\pi(t, \zeta)/\pi_t + \gamma_M(t, \zeta) + (\gamma_\pi(t, \zeta)/\pi_t)\gamma_M(t, \zeta)) \\ &\quad + \gamma_{Y^{-1}}(t, \zeta)(\gamma_\pi(t, \zeta)/\pi_t + \gamma_M(t, \zeta) + (\gamma_\pi(t, \zeta)/\pi_t)\gamma_M(t, \zeta)), \end{aligned}$$

or

$$K_V(t, \zeta) = \gamma_{Y^{-1}}(t, \zeta) + \left(\frac{\gamma_\pi(t, \zeta)}{\pi_t} + \gamma_M(t, \zeta) + \frac{\gamma_\pi(t, \zeta)}{\pi_t} \gamma_M(t, \zeta) \right) (1 + \gamma_{Y^{-1}}(t, \zeta))$$

We now use the expression for $\gamma_\pi(t, \cdot)$ found in (40). It can be simplified to

$$\gamma_\pi(t, \zeta) = \pi_t \left(-\gamma_0 K_V(t, \zeta) + \left(\frac{1 + K_V(t, \zeta)}{1 + \gamma_c(t, \zeta)} \right)^\rho - 1 \right) (1 - \gamma_0 K_V(t, \zeta)).$$

This immediately gives the following relationship between $\gamma_M(t, \cdot)$, $K_V(t, \cdot)$ and $\gamma_c(t, \cdot)$:

$$\begin{aligned} K_V(t, \zeta) \left(1 - \gamma_0 - \gamma_0 K_V(t, \zeta) \right) &= \gamma_M(t, \zeta) + \\ \left(1 + \gamma_M(t, \zeta) \right) \left(-\gamma_0 K_V(t, \zeta) + \left(\frac{1 + K_V(t, \zeta)}{1 + \gamma_c(t, \zeta)} \right)^\rho - 1 \right) &(1 - \gamma_0 K_V(t, \zeta)) \end{aligned} \quad (55)$$

ν a.e.⁶. This relationship is seen to be linear in $\gamma_M(t, \zeta)$, and the solution is

$$1 + \gamma_M(t, \zeta) = \frac{1 + K_V(t, \zeta) (1 - \gamma_0 - \gamma_0 K_V(t, \zeta))}{1 - \gamma_0 K_V(t, \zeta) + \left(\frac{1 + K_V(t, \zeta)}{1 + \gamma_c(t, \zeta)} \right)^\rho - 1} (1 - \gamma_0 K_V(t, \zeta)) \quad (56)$$

where the equality holds $\nu(\cdot)$ a.e.

This proves the results of Section 3.1, which we formulate in the next section.

8 The results

We can now summarize our results so far in the following

Theorem 2 *For the standard recursive model with jump dynamics included,*

⁶By starting with the identity $Y_t V_t = \pi_t W_t$ instead of using (48), these computations can be made somewhat easier.

in equilibrium the risk premium of any risky asset R is given by

$$\begin{aligned} \mu_R(t) - r_t &= \frac{\rho(1-\gamma)}{1-\rho} \sigma_c(t)' \sigma_R(t) + \frac{\gamma-\rho}{1-\rho} \sigma_M(t)' \sigma_R(t) + \\ &\int_{\mathcal{Z}} \left(\gamma_0 K_V(t, \zeta) - \left(\left(\frac{1+K_V(t, \zeta)}{1+\gamma_c(t, \zeta)} \right)^\rho - 1 \right) (1 - \gamma_0 K_V(t, \zeta)) \right) \gamma_R(t, \zeta) \nu(d\zeta), \end{aligned}$$

and the real interest rate by

$$\begin{aligned} r_t &= \delta + \rho \mu_c(t) - \frac{1}{2} \frac{\rho(1-\rho\gamma)}{1-\rho} \sigma_c'(t) \sigma_c(t) + \frac{1}{2} \frac{\rho-\gamma}{1-\rho} \sigma_M(t)' \sigma_M(t) \\ &- \int_{\mathcal{Z}} \left\{ \frac{1}{2} (1+\rho) \gamma_0 K_V'(t, \zeta) K_V(t, \zeta) + \left(\left(\frac{1+K_V(t, \zeta)}{1+\gamma_c(t, \zeta)} \right)^\rho - 1 \right) (1 - \gamma_0 K(t, \zeta)) \right. \\ &\quad \left. + \rho \gamma_c(t, \zeta) - \rho K_V(t, \zeta) \right\} \nu(d\zeta). \end{aligned}$$

The volatility of the market portfolio $\sigma_M(t)$ is given by

$$\sigma_M(t) = (1-\rho) \sigma_V(t) - \rho \sigma_c(t),$$

and $\gamma_M(t, \zeta)$ is given in terms of $K_V(t, \zeta)$ and $\gamma_c(t, \zeta)$ by the equation (56).

Under the reformulation in Section 3 regarding the jump part, the relationship (56) can be integrated in the formula for the risk premium and the real rate, by a change of variables. Let $y := \gamma_M(t, \zeta)$, $c = \gamma_c(t, \zeta)$, $v = K_V(t, \zeta)$ and $x = \gamma_R(t, \zeta)$. As explained in Section 3, for $l = 1$ the relationship (56) with this notation is

$$1 + y = \frac{1 + v(1 - \gamma_0 - \gamma_0 v)}{1 - \gamma_0 v + \left(\left(\frac{1+v}{1+c} \right)^\rho - 1 \right) (1 - \gamma_0 v)}. \quad (57)$$

For the Levy-measure $\nu(d\zeta, d\zeta_2, d\zeta_3)$ on the form $\lambda_t dH_t(y, c, x)$ the risk premium and the real rate can be written respectively

$$\begin{aligned} \mu_R(t) - r_t &= \frac{\rho(1-\gamma)}{1-\rho} \sigma_c(t)' \sigma_R(t) + \frac{\gamma-\rho}{1-\rho} \sigma_M(t)' \sigma_R(t) + \\ &\iiint_{\mathcal{Z}'} \left\{ \gamma_0 v - \left(\left(\frac{1+v}{1+c} \right)^\rho - 1 \right) (1 - \gamma_0 v) \right\} x \lambda_t dH_t(c, v, x) \quad (58) \end{aligned}$$

and

$$\begin{aligned}
r_t &= \delta + \rho\mu_c(t) - \frac{1}{2} \frac{\rho(1-\rho\gamma)}{1-\rho} \sigma'_c(t)\sigma_c(t) + \frac{1}{2} \frac{\rho-\gamma}{1-\rho} \sigma_M(t)\sigma'_M(t) \\
&- \iiint_{\mathcal{Z}'} \left(\frac{1}{2}\gamma_0(1+\rho)v^2 + \left(\left(\frac{1+v}{1+c} \right)^\rho - 1 \right) (1-\gamma_0v) + \rho c - \rho v \right) \lambda_t dH_t(c, v, x).
\end{aligned} \tag{59}$$

where $h_t(x, v, x) = J(c, v, x) f_t(c, y(c, v), x)$ is the pdf of H_t . As pointed out in Section 3, $F_t(c, y, x)$, the joint probability distribution function of the jump sizes in consumption, the market (wealth) portfolio and the risky asset under consideration, is internalized this way, assuming F_t has a probability density function $f_t(c, y, x)$.

Further transformations of variables suitable for computations are sometimes needed, as we show below.

8.1 The pure jump part when $\rho = 0$

Let us consider the pure jump part of the risk premium and the interest rate in Theorem 2. The expression (56) becomes particularly simple when $\rho = 0$. In this situation the agent is neutral to consumption transfers across time, where

$$\gamma_M(t, \zeta) = K_V(t, \zeta) \quad \text{for all } t \text{ and for all } \zeta \in \mathcal{Z}, \tag{60}$$

which means that the expression for the risk premium and the real rate are

$$\mu_R(t) - r_t = \gamma_0 \int_{\mathcal{Z}} \gamma'_M(t, \zeta) \gamma_R(t, \zeta) \nu(d\zeta),$$

and

$$r_t = \delta - \frac{1}{2} \gamma_0 \int_{\mathcal{Z}} \gamma'_M(t, \zeta) \gamma_M(t, \zeta) \nu(d\zeta)$$

respectively. No approximations were used in deriving this result.

Notice that this model can be considered as a dynamic version of the classical CAPM of Mossin (1968). The CAPM is derived in a time-less setting, where there is consumption only on the last time point, so that the interest rate has no real meaning. In contrast, our model is valid in a dynamic setting with recursive utility, and has an associated real, equilibrium interest rate as given above.

As an illustration of the pure jump model, it can be seen to fit the data summarized in in Table 1 by modeling the discrete data by a marked point process of frequency one per year, on the average. The result of this calibra-

tion is: $\gamma = 2.38$, and $\delta = .038$. As we shall see in Section 8, this is the same result as obtained using the continuous model with no jumps, and also using the combined model when $\gamma_0 = \gamma$.

8.2 The approximation for the pure jump part for general $\rho \neq 1$

Let us consider the relationship (56) and expand the power function in the denominator in a Taylor series, retaining only first order terms. This gives

$$1 + \gamma_M(t, \zeta) \approx \frac{1 + K_V(t, \zeta)(1 - \gamma_0 - \gamma_0 K_V(t, \zeta))}{1 - \gamma_0 K_V(t, \zeta) + \rho(K_V(t, \zeta) - \gamma_c(t, \zeta) - \rho \gamma_c(t, \zeta) K_V(t, \zeta))(1 - \gamma_0 K_V(t, \zeta))}.$$

Disregarding terms of higher order in the denominator, this can be written

$$1 + \gamma_M(t, \zeta) \approx (1 + K_V(t, \zeta)(1 - \gamma_0 - \gamma_0 K_V(t, \zeta)))(1 + \gamma_0 K_V(t, \zeta) + \rho \gamma_c(t, \zeta) - \rho K_V(t, \zeta)) + \dots$$

or

$$\gamma_M(t, \zeta) \approx K_V(t, \zeta)(1 - \rho) + \rho \gamma_c(t, \zeta) + \dots \quad \text{for all } t \text{ and for all } \zeta \in \mathcal{Z}, \quad (61)$$

an approximate internalization of $\gamma_M(t, \cdot)$. This reduces to (60) when $\rho = 0$. Inverting this we also have

$$K_V(t, \zeta) \approx \frac{1}{1 - \rho} (\gamma_M(t, \zeta) - \rho \gamma_c(t, \zeta)) + \dots \quad (62)$$

These relationship can be compared to the corresponding between the volatilities $\sigma_M(t)$, $\sigma_V(t)$ and $\sigma_c(t)$ given in Theorem 2.

The jump part of the expression for the risk premium we approximate as follows

$$\int_{\mathcal{Z}} \left(\gamma_0 K_V(t, \zeta) - ((1 + \rho K_V(t, \zeta))(1 - \rho \gamma_c(t, \zeta)) - 1)(1 - \gamma_0 K_V(t, \zeta)) \right) \gamma_R(t, \zeta) \nu(d\zeta).$$

Retaining second order moments only, this expression can be written

$$\int_{\mathcal{Z}} \left((\gamma_0 - \rho) K_V(t, \zeta) + \rho \gamma_c(t, \zeta) \right) \gamma_R(t, \zeta) \nu(d\zeta).$$

Inserting the expression for $K_V(t, \zeta)$ from (62), we obtain an approximation

to the risk premium for the pure jump model as follows

$$\begin{aligned} \mu_R(t) - r_t = & \frac{\rho(1 - \gamma_0)}{1 - \rho} \int_{\mathcal{Z}} \gamma_c(t, \zeta) \gamma_R(t, \zeta) \nu(d\zeta) + \\ & \frac{\gamma_0 - \rho}{1 - \rho} \int_{\mathcal{Z}} \gamma_M(t, \zeta) \gamma_R(t, \zeta) \nu(d\zeta) + \dots \end{aligned} \quad (63)$$

We notice that up to the first two moments this expression has the same structure as the corresponding formula for the continuous part, except that γ_0 now replaces γ .

Turning to the interest rate, we proceed as follows. The jump part of the interest rate can be written

$$\begin{aligned} - \int_{\mathcal{Z}} \left\{ \frac{1}{2} (1 + \rho) \gamma_0 K'_V(t, \zeta) K_V(t, \zeta) + \left(\left(\frac{1 + K_V(t, \zeta)}{1 + \gamma_c(t, \zeta)} \right)^\rho - 1 \right) \left(1 - \gamma_0 K_V(t, \zeta) \right) \right. \\ \left. + \rho \gamma_c(t, \zeta) - \rho K_V(t, \zeta) \right\} \nu(d\zeta) \end{aligned}$$

To obtain the same order of approximation as above, we must include one more term in the Taylor series expansion for the power term. This gives

$$\begin{aligned} - \int_{\mathcal{Z}} \left\{ \frac{1}{2} (1 + \rho) \gamma_0 K'_V(t, \zeta) K_V(t, \zeta) + \left(1 - \gamma_0 K_V(t, \zeta) \right) \cdot \right. \\ \left. \left((1 + \rho K_V(t, \zeta) + \frac{1}{2} \rho(\rho - 1) K_V^2(t, \zeta)) (1 - \rho \gamma_c(t, \zeta) + \right. \right. \\ \left. \left. \frac{1}{2} \rho(\rho + 1) \gamma_c^2(t, \zeta)) - 1 \right) + \rho \gamma_c(t, \zeta) - \rho K_V(t, \zeta) \right\} \nu(d\zeta). \end{aligned} \quad (64)$$

First we focus on the term $K'_V K_V$ which leads to $\gamma'_M \gamma_M / (1 - \rho)^2$ using (62). Examining (64), we see that three terms contribute to the coefficient of $\gamma'_M(t, \zeta) \gamma_M(t, \zeta)$: They are

$$- \left(\frac{1}{2} \frac{\gamma_0(1 + \rho)}{(1 - \rho)^2} - \frac{\gamma_0 \rho}{(1 - \rho)^2} + \frac{1}{2} \frac{\rho(\rho - 1)}{(1 - \rho)^2} \right) = \frac{1}{2} \frac{\rho - \gamma_0}{1 - \rho}. \quad (65)$$

From Theorem 2, or (59), we see that this is the coefficient multiplying the corresponding $\sigma'_M(t) \sigma_M(t)$ -term in the continuous part of r_t .

Next we focus on the term that gives $\gamma'_M(t, \zeta) \gamma_c(t, \zeta)$, which we obtain from $K'_V(t, \zeta) \gamma_c(t, \zeta)$. In addition to the previous component, the following contributes directly from (64) to this product:

$$- (\gamma_0 \rho - \rho^2) \frac{1}{1 - \rho} (\gamma'_M(t, \zeta) \gamma_c(t, \zeta) - \rho \gamma'_c(t, \zeta) \gamma_c(t, \zeta)) \quad (66)$$

where we have used (62). Recall that (65) is obtained as the first part of

$$K'_V(t, \zeta)K_V(t, \zeta) = \frac{1}{(1 - \rho)^2} (\gamma'_M(t, \zeta)\gamma_M(t, \zeta) - 2\rho\gamma'_M(t, \zeta)\gamma_c(t, \zeta) + \rho^2\gamma'_c(t, \zeta)\gamma_c(t, \zeta)) \quad (67)$$

Taking these two terms into account, the coefficient in question is is:

$$\left(\frac{1}{2} \frac{\rho - \gamma_0}{1 - \rho} (-2\rho) - (\gamma_0\rho - \rho^2) \frac{1}{1 - \rho} \right) \gamma'_M(t, \zeta)\gamma_c(t, \zeta) = 0,$$

in agreement with Theorem 2, or (59): No such term appears in the interest rate. Finally we turn to the term $\gamma'_c(t, \zeta)\gamma_c(t, \zeta)$. We obtain from (66) the contribution

$$-\frac{\gamma_0\rho - \rho^2}{1 - \rho} (-\rho)\gamma'_c(t, \zeta)\gamma_c(t, \zeta).$$

Directly from (64) we have

$$-\frac{1}{2}\rho(\rho + 1)\gamma'_c(t, \zeta)\gamma_c(t, \zeta).$$

From (65) and (67) we get

$$\frac{1}{2} \frac{\rho - \gamma_0}{1 - \rho} \rho^2 \gamma'_c(t, \zeta)\gamma_c(t, \zeta).$$

Adding these three terms gives the following result

$$-\frac{1}{2} \frac{\rho(1 - \rho\gamma_0)}{1 - \rho} \gamma'_c(t, \zeta)\gamma_c(t, \zeta).$$

This proves that the pure jump part of the real interest rate is given by

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2} \frac{\rho(1 - \rho\gamma_0)}{1 - \rho} \int_{\mathcal{Z}} \gamma'_c(t, \zeta)\gamma_c(t, \zeta) \nu(d\zeta) + \frac{1}{2} \frac{\rho - \gamma_0}{1 - \rho} \int_{\mathcal{Z}} \gamma'_M(t, \zeta)\gamma_M(t, \zeta) \nu(d\zeta) + \dots$$

to the required order of approximation. Thus we have shown (14) and (15) in Section 3.2.

8.3 Summary of the approximative model

Combining the results in Section 3.1 and 8.2 with the formulation of the jump model Section 8, the risk premiums and the real interest rate can be written

$$\begin{aligned} \mu_R(t) - r_t &= \frac{\rho(1-\gamma)}{1-\rho} \sigma_c(t)' \sigma_R(t) + \frac{\gamma-\rho}{1-\rho} \sigma_M(t)' \sigma_R(t) + \\ &\frac{\rho(1-\gamma_0)}{1-\rho} \int_{-1}^{\infty} \int_{-1}^{\infty} c x \lambda_t dF_t^{(1)}(c, x) + \frac{\gamma_0-\rho}{1-\rho} \int_{-1}^{\infty} \int_{-1}^{\infty} y x \lambda_t dF_t^{(2)}(y, x) \end{aligned} \quad (68)$$

and

$$\begin{aligned} r_t &= \delta + \rho \mu_c(t) - \frac{1}{2} \frac{\rho(1-\rho\gamma)}{1-\rho} \sigma'_c(t) \sigma_c(t) + \frac{1}{2} \frac{\rho-\gamma}{1-\rho} \sigma_M(t)' \sigma_M(t) \\ &- \frac{1}{2} \frac{\rho(1-\rho\gamma_0)}{1-\rho} \int_{-1}^{\infty} \int_{-1}^{\infty} c^2 \lambda_t dF_t^{(1)}(c, x) + \frac{1}{2} \frac{\rho-\gamma_0}{1-\rho} \int_{-1}^{\infty} \int_{-1}^{\infty} y^2 \lambda_t dF_t^{(2)}(y, x) \end{aligned} \quad (69)$$

respectively. Here $F_t^{(1)}(c, x)$ and $F_t^{(2)}(y, x)$ are the marginal distributions of $F_t(c, y, x)$. We assume $F_t^{(1)}(c, x)$ has density $f_t^{(1)}(c, x)$ and $F_t^{(2)}(y, x)$ has density $f_t^{(2)}(y, x)$.

In the following it will be an advantage to consider the model in exponential, rather than in the stochastic exponential form. We therefore make the substitution $1+y = e^{z_W}$, $1+c = e^{z_C}$ and $1+x = e^{z_M}$ which leads to the following expressions

$$\begin{aligned} \mu_R(t) - r_t &= \frac{\rho(1-\gamma)}{1-\rho} \sigma_c(t)' \sigma_R(t) + \frac{\gamma-\rho}{1-\rho} \sigma_M(t)' \sigma_R(t) + \\ &\frac{\rho(1-\gamma_0)}{1-\rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{z_C} - 1)(e^{z_M} - 1) \lambda_t dG_t^{(1)}(z_C, z_M) + \\ &\frac{\gamma_0-\rho}{1-\rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{z_W} - 1)(e^{z_M} - 1) \lambda_t dG_t^{(2)}(z_W, z_M) \end{aligned} \quad (70)$$

and

$$\begin{aligned}
r_t = & \delta + \rho\mu_c(t) - \frac{1}{2} \frac{\rho(1-\rho\gamma)}{1-\rho} \sigma'_c(t)\sigma_c(t) + \frac{1}{2} \frac{\rho-\gamma}{1-\rho} \sigma_M(t)\sigma_M(t) \\
& - \frac{1}{2} \frac{\rho(1-\rho\gamma_0)}{1-\rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{z_c} - 1)^2 \lambda_t dG_t^{(1)}(z_c, z_M) \\
& + \frac{1}{2} \frac{\rho-\gamma_0}{1-\rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{z_W} - 1)^2 \lambda_t dG_t^{(2)}(z_W, z_M) \quad (71)
\end{aligned}$$

where $G^{(1)}$ has density function $g^{(1)}$ given by

$$g^{(1)}(z_c, z_M) = f^{(1)}(c(z_c), x(z_M))J^{(1)}(z_c, z_M)$$

and where the $J^{(1)}$ is the Jacobian

$$J^{(1)}(z_c, z_M) = \text{mod} \begin{vmatrix} e^{z_c} & 0 \\ 0 & e^{z_M} \end{vmatrix} = e^{z_c+z_M}$$

and similarly for $G^{(2)}(z_W, z_M)$. This form we will make use of in the calibrations of the next section.

9 Some calibrations

In this section we calibrate the recursive model to the data summarized in Table 1. Some calibrations of the recursive model with only continuous diffusion dynamics are shown in Table 2. This model is based on the aggregator (20) in Section 4, and the risk premium was first derived by Duffie and Epstein (1992a)⁷. The interest rate was first derived in Aase (2014a), and also follows from our approach in the present paper. In the calibration we have fixed the time impatience rate δ and solved the two equations (5) and (8) in the two remaining unknowns γ and ρ , for values of δ between 1.5 and 3.8 per cent.

The values obtained for γ and ρ seem reasonable, in particular the ones corresponding to $\delta \geq 0.022$. For comparisons, the results for the conventional additive model is given in the first line of Table 2, and for the pure jump model with joint NIG-distributed jump sizes in the second line. In applied economics values of the impatience rate between 1 and 4 per cent seem common. One reason for this is of course that the conventional, additive Eu-model is often taken for granted, and from the expression for the

⁷The coefficients were all constants, since dynamic programming was used. This is not necessary in our approach

| | γ | ρ | EIS |
|--|----------|--------|-----------|
| Conventional Eu-Model | | | |
| $\delta = -.015$ | 26.37 | 26.37 | .037 |
| Conventional Eu-model with jumps only (NIG) | | | |
| $\delta = .0083$ | 22.2 | 22.2 | .045 |
| Continuous recursive model | | | |
| $\delta = .015$ | .46 | 1.34 | .74 |
| $\delta = .020$ | .90 | 1.06 | .94 |
| $\delta = .025$ | 1.33 | .78 | 1.28 |
| $\delta = .030$ | 1.74 | .48 | 2.08 |
| $\delta = .035$ | 2.14 | .18 | 5.56 |
| $\delta = .038$ CAPM+ | 2.38 | .00 | $+\infty$ |

Table 3: Calibrations of the continuous recursive model

interest rate in (2) (disregarding the jump terms) one simply does not obtain reasonable values for the short rate unless δ is in this range, or smaller.

With the jump terms included, we may expect some changes. The above continuous model gives interesting results in itself. One might conjecture that only minor adjustments may be required, which the discontinuous part could provide.

To investigate this, we employ the model on the form summarized in Section 8.3. Since this requires a transformation to log returns, the relevant statistics is summarized in Table 4. Notice that this table is not a mere transformation of Table 1, but developed from the the original data set used in the Mehra and Prescott (1985)-study, by taking logarithms of the relevant yearly quantities, and basing the statistical analysis on these transformed data points. As an illustration, of the total annual variation of .02509 in

| | Expectat. | Standard dev. | Covariances |
|--------------------|-----------|---------------|-------------------------------|
| Consumption growth | 1.75% | 3.55% | $\text{cov}(M, c) = .002268$ |
| Return S&P-500 | 5.53% | 15.84% | $\text{cov}(M, b) = .001477$ |
| Government bills | 0.64% | 5.74% | $\text{cov}(c, b) = -.000149$ |
| Equity premium | 4.89% | 15.95% | |

Table 4: Key US-data for the time period 1889-1978 in terms of log returns of discrete-time compounding.

the stock market, measured as variance, suppose we allocate .0126 to jumps. Similarly, the total annual variance of the consumption growth rate of .00126

is divided in two so that the jump part retains .00063. The expected growth rate of the jump part of the consumption variable is set to .01 and the jump parts contribution to the annual return on the market portfolio is set to .028. Higher order terms are ignored, so it is the new aspect of $\gamma_0 \neq \gamma$, as well as the existence of a full, joint probability distribution for the jump sizes that is investigated here. The latter distribution is assumed to be joint lognormal, and as a consequence both $G^{(1)}$ and $G^{(2)}$ are both joint normal probability distribution functions. Some results are presented in Table 5. As can be

| | γ | ρ | γ_0 | EIS |
|-------------------------|----------|--------|------------|------|
| Recursive model | | | | |
| including jump dynamics | | | | |
| $\delta = .005$ | 1.06 | 1.02 | .95 | .98 |
| $\delta = .010$ | 1.04 | 1.02 | .95 | .98 |
| $\delta = .015$ | 1.04 | 1.03 | .90 | .97 |
| $\delta = .020$ | .98 | .96 | 1.10 | 1.04 |
| $\delta = .025$ | 1.04 | .87 | 1.30 | 1.15 |
| $\delta = .030$ | 1.14 | .55 | 2.00 | 1.82 |
| $\delta = .035$ | 1.71 | .69 | 1.50 | 1.45 |
| $\delta = .040$ | 2.60 | .57 | 1.30 | 1.75 |
| $\delta = .045$ | 2.97 | .62 | 1.10 | 1.61 |

Table 5: Calibrations of the model including jump dynamics

seen from this table, with the new elements added, the model explains the data also for small values of the impatience rate δ , (as well as for large ones). When the impatience rate becomes small, the continuous model does not fit the data at all. With jumps included, and the risk aversion on jump size risk γ_0 can differ from γ , we obtain better results, although values of the risk aversion around one is considered a bit low. It may be interesting to notice that log utility in the conventional model, the so-called 'Kelly-criterion', is known to have certain advantages as an objective in the long run (see e.g., Breiman (1960)).

9.1 CAPM++: $\rho = 0$

When the time preference parameter $\rho = 0$ no approximations are involved. This model corresponds to the dynamic version of the classical one-period CAPM, which we denote by CAPM++. Some results are presented in Table 6. For the upper six rows in the table the assumptions are as in the last section. For the last three rows we have set the return rate to minus one

percent annually for the jump part. This is to check the casual observation that jumps in the stock market often seem associated with slumps, or even market crashes.

| CAPM++ | γ | ρ | γ_0 |
|--|----------|--------|------------|
| Recursive model with $\rho = 0$ including jump dynamics | | | |
| $\delta = .038$ | 2.38 | .00 | 2.38 |
| $\delta = .038$ | 1.25 | .00 | 3.00 |
| $\delta = .038$ | 2.42 | .00 | 2.00 |
| $\delta = .038$ | 4.18 | .00 | .50 |
| $\delta = .038$ | 3.59 | .00 | 1.00 |
| $\delta = .038$ | 3.01 | .00 | 1.50 |
| $\delta = .038$ | 4.77 | .00 | .00 |
| $\delta = .038$ | 5.27 | .00 | -.50 |
| $\delta = .038$ | 5.76 | .00 | -1.00 |

Table 6: Calibrations of the model including jump dynamics

The parameters γ and γ_0 are seen to supplement each other; when one is large the other is small, and vice versa, and both the impatience rate and the risk aversions calibrate to plausible values.

For the last three rows of the table jumps are associated with negative shocks in the stock market, so we may check if *loss aversion* is supported by the model by choosing a low, or even a negative value of γ_0 (since loss aversion is associated with risk proclivity for losses). The ordinary risk aversion γ is then a between four and six, and the impatience rate is still reasonable at 3.8 per cent. This indicates a utility based connection to loss aversion (see e.g., Kahneman and Tversky (1979)).

9.2 The market portfolio is not a proxy for the wealth portfolio

In the above calibrations we have assumed that all income is investment income. This may be justified in the present paper, since we compare different models. If we can view exogenous income streams as dividends of some shadow asset, our model is valid if the market portfolio is expanded to include the new asset. However, if the latter is not traded, then the return to the wealth portfolio is not readily observable or estimable from available data. Still we should be able to get a fair impression of what can be expected in a more elaborate model. This is analyzed in Aase (2014a,b) for the pure

continuous recursive and the discrete time model respectively, and the results are promising. Using a pure jump model, the results are even more promising as we shall see.

Below we take the pure jump model to represent the data, with frequency one per year on the average, as explained in Section 2.1. The advantage of this approach is that we do not need to separate jumps from the continuous part of the data paths. This way we may study the deviations from the local mean square analysis in isolation, and take advantage of the joint probability distribution of jump sizes. The model of Section 8.3 then takes the form

$$\begin{aligned} \mu_R(t) - r_t = & \frac{\rho(1 - \gamma_0)}{1 - \rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{z_c} - 1)(e^{z_M} - 1) \lambda dG^{(1)}(z_c, z_M) + \\ & \frac{\gamma_0 - \rho}{1 - \rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{z_W} - 1)(e^{z_M} - 1) \lambda dG^{(2)}(z_W, z_M) \quad (72) \end{aligned}$$

and

$$\begin{aligned} r_t = & \delta + \rho\mu_c(t) - \frac{1}{2} \frac{\rho(1 - \rho\gamma_0)}{1 - \rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{z_c} - 1)^2 \lambda dG^{(1)}(z_c, z_M) \\ & + \frac{1}{2} \frac{\rho - \gamma_0}{1 - \rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{z_W} - 1)^2 \lambda dG^{(2)}(z_W, z_M) \quad (73) \end{aligned}$$

Now W signify the wealth portfolio. Below we set $\mu_W(t) = .02$, $\sigma_W(t) = .10$, $\kappa_{c,W} = .40$ as before, and set $\kappa_{W,R} = .80$. We assume a joint lognormal distribution F for the the various variables c , W and M , in which case the two marginal distributions $G^{(1)}$ and $G^{(2)}$ are both joint normal distribution functions.

The results of the calibrations are given in Table 7. As can be seen, they correspond to plausible values of the various parameters.

In the above we have set the growth rate of the wealth portfolio to two per cent, which is captured by the joint probability distribution function $G^{(2)}$ (or $H^{(2)}$). The result is a fairly patient agent. Since the majority (9/10) did not invest in the stock market for the period the data covers, this value for the growth growth rate is likely to better reflect the average return that this majority received. This group also dominates in aggregate consumption because of the sheer number of people it represents.

For the agent in the stock market receiving about seven per cent annual return, this higher return may not change the results all that much in a two agent model, because this 'agent' consumes only a smaller fraction of the aggregate consumption (because of the small size of the population that this agent represents). Typically, one would expect that the latter agent will

| | ρ | γ_0 | EIS |
|--------------------|--------|------------|-------|
| Recursive model. | | | |
| Pure jump dynamics | | | |
| $\delta = .015$ | .97 | 1.11 | 1.03 |
| $\delta = .017$ | .86 | 1.53 | 1.16 |
| $\delta = .019$ | .74 | 1.93 | 1.35 |
| $\delta = .020$ | .68 | 2.51 | 1.47 |
| $\delta = .022$ | .55 | 2.13 | 1.81 |
| $\delta = .025$ | .35 | 3.06 | 2.86 |
| $\delta = .027$ | .21 | 3.42 | 4.76 |
| $\delta = .028$ | .14 | 3.59 | 7.14 |
| $\delta = .029$ | .07 | 3.75 | 14.28 |

Table 7: Calibrations when the market portfolio is not a proxy for the wealth portfolio.

display more impatience than the former.

9.3 Some related issues

In the above we have ignored the higher order moments than the two first ones, specific for the jump model. In Section 2 we noticed that including all the features of the jump model can make some difference. A numerical analysis can be based on the results of Section 3.3, where the pure jump model may, as above, be taken to represent the data alone with frequency one per year on the average. This way we may study the deviations from the local mean square analysis in isolation. Some of this deviation was indeed captured in the previous section, but not all. This approach would also allow for deviations from normality. The associated numerical analysis we do not consider here.

The topic of separating the jump part from the continuous part of a data set is dealt with in e.g., Ait Sahalia and Jacod (2009-11).

More numerical work is of course desired, possibly combined with statistical estimation theory, aimed at separating the discontinuous dynamics from the continuous part. However, the general picture seems to be that jumps may be of particular interest in the recursive model.

10 Conclusions

We have addressed the well-known empirical deficiencies of the conventional asset pricing model in financial- and macro economics. We have considered the recursive model in continuous time jump/diffusion setting with the Kreps-Porteus specification where we have derived both the equilibrium real interest rate, and risk premiums.

We use a general method of optimization, the stochastic maximum principle, together with the theory of forward/backward stochastic differential equations, which allow for an extension to jump dynamics.

The recursive model has several interesting features when jumps are allowed in the dynamics of the aggregate consumption process as well as in the future volatility process. In addition to the nonlinear terms that are introduced, it also gives a new parameter for the risk aversion related to jump size risk. Both together, and in isolation, these new features may improve many results for the continuous recursive model in relation to explaining real data.

We have demonstrated that for the US-data in the Mehra and Prescott (1985)-study, our extended model may calibrate, with a few simplifications regarding the jump dynamics, to reasonable values of the preference parameters. We also consider the more realistic situation where the market portfolio is not a proxy for the wealth portfolio.

From a theoretical point of view, a most important step in our derivation is the internalization of the probability distribution, or more precisely, the stochastic process, for the market and the wealth portfolios. They are determined in equilibrium, by the first order conditions and market clearing, from the primitives of the underlying economic model, which are the stochastic process for future utility (preferences) and the process determining the dynamics of the growth rate of aggregate consumption (the given endowment process).

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