An age-structured model for the effect of interest rate changes on consumption

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Abstract

A model for the effect of an interest rate change on household consumption is developed. The approach is age-structured: households reconsider their consumption patterns at the moment of the interest rate change and the changes of the consumption patterns are age dependent. These changes for different age groups contribute to the modification of aggregate consumption. Numerical simulation shows that a decrease of the interest rate leads to a consumption boost (a substantial increase of consumption in the short run), which diminishes as time passes and consumption gets fully adjusted to the new interest rate value. The consumption boost is achieved by an increase of the debt load.

Key words:
interest rate change,
consumption,
aggregate consumption,
debt load

1 Introduction

Households spending is often financed by borrowing. It is particularly relevant for housing and durables. Consumption of services and nondurables can also rely on credit. Figure 1 shows the increasing role of the consumer credit

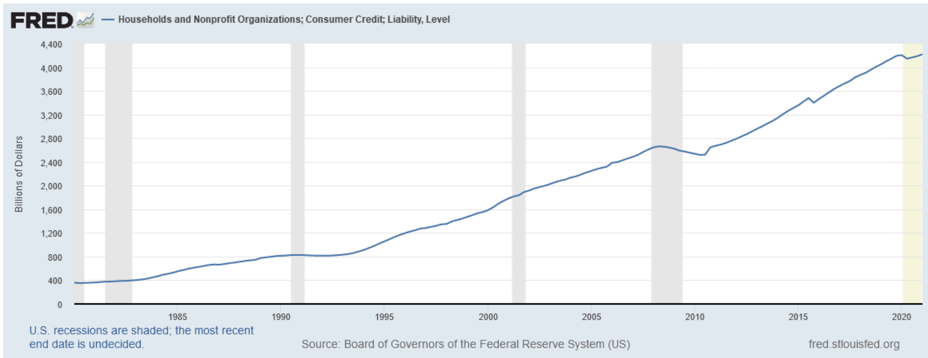


Figure 1: US Households and Nonprofit Organizations; Consumer Credit; 1980–2021. Source: FRED

in the USA. At the same time a significant fraction of households saves little or no liquid wealth (see [1] and references therein). Under such conditions consumption depends on the interest rates, which affect borrowing against future earnings.

Over the past few decades the interest rates gradually declined. In the case of the USA it can be easily seen in the interest rates data in Fig. 2 and in the US government bonds data in Fig. 3. The general trend for the interest rates during past decades, let us say from 1980s, was the decline from high (double digit) values to values close to zero. Substantial decreases of the interest rates are usually implemented to stimulate consumption and business activities in times of economic difficulties. For example, the last substantial decrease took place because of the financial crisis 2007-2009. The last but one was related to dot-com bubble of 2000. Inevitably such changes had effect on the consumption pattern.

The intertemporal relationship between the interest rates and consumption (saving) attracted many economists [2, 3, 4, 5]. The impact of interest rate changes on consumption needs to be understood, in particular, for efficient implementation of monetary policy. There were comparatively few empirical attempts to study the effect of the interest rate changes on consumption and especially few attempts to approach this topic at a disaggregated level. Most of the authors argue in favor of the inverse relation between changes of consumption and the interest rates. They also argue that the real (not nominal) rate of interest is the appropriate value to consider. In [2, 3] it was discussed that some results of the empirical studies are inconsistent.

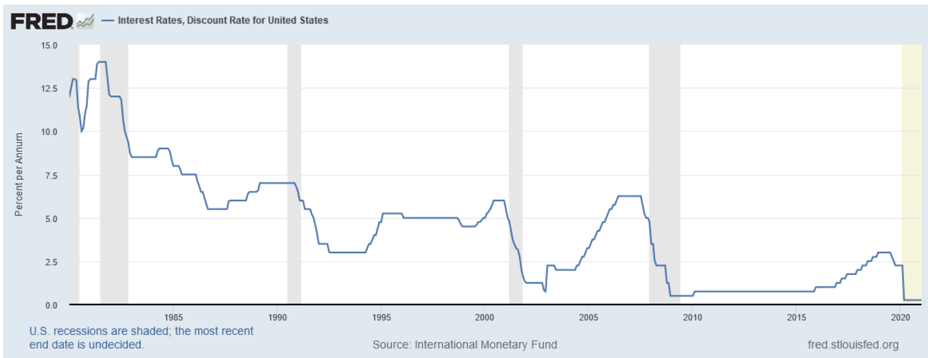


Figure 2: Discount Rate for United States 1980–2021. Source: FRED

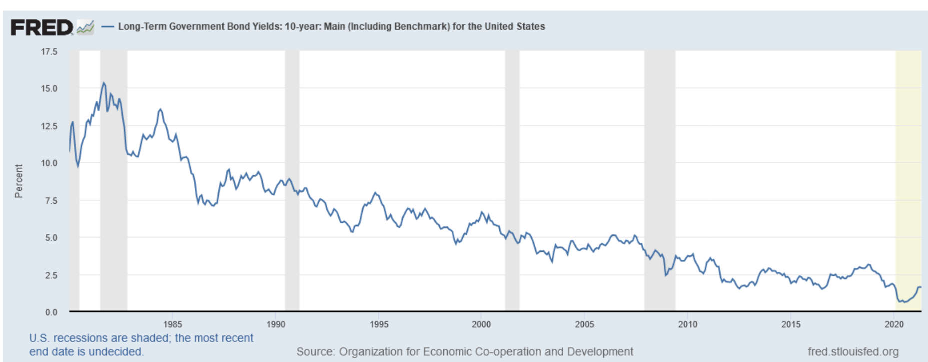


Figure 3: 10-year US Government Bond Yields 1980–2021. Source: FRED

It should be noted that most of the known results were established before low interest rates became the new reality that happened during the past two decades.

Falling mortgage rates can be used for refinancing that frees up disposable income for additional consumption [6]. In [7] there was considered the effect of lower mortgage rates on consumption and mortgage repayment. Car purchases were used as the main measure of consumption. It was found that a substantial decline in mortgage payments (up to 50 %) led to a noticeable increase in car purchases (up to 35 %).

In this paper we suggest a model for the effect of an interest rate change on consumption. We consider households which rely on income and loans (student loans, mortgage loans, car loans, credit cards, etc.), assuming that a household behaves as a unit. Households can borrow against future earnings to optimize the consumption over their life cycles. The borrowing strategies depend on the interest rates. For this reason the interest rate decrease can be an efficient tool for consumption stimulation.

Of course, the analysis of the impact of the interest rate changes needs to be based on comprehensive economic development models, which include all relevant factors such as firms, government spending, taxation, public debt, economic growth, technological development, etc. All these factors make the analysis very complicated. Especially because it is difficult to make future predictions for some of these factors. We substitute this complicated problem by a simpler problem: Which effect can an interest rate change have on consumption provided that the other conditions (employment, labor income, the value of assets, etc.) remain the same? The only other parameter which can get changed is the subjective discount rate, which is induced by the interest rate value. The analysis does not take into account reasons of the interest rage change.

For further simplification we consider only households in labor force because for them the consumption changes are expected to be the most significant (both consumption and a flexibility to change consumption patterns drop at retirement). These households are divided into age groups. The results on consumption obtained for particular age groups are used to derive aggregate consumption.

A number of simplifying assumptions help to isolate the interest rate effect and to provide a high degree of analytical tractability. We assume that the interest rate was constant for a long time. Then, it gets changed to another value, which will be held for a long time. This change is *unexpected* for households and makes them to reconsider their consumption patterns. Of course, this scenario is a simplification of the reality, in real life the interest rate changes happen relatively often and the interest rate cannot be assumed to be constant for a long time. However, the considered approach can reveal some important features of the interest rate effect on consumption.

All representative households for particular age groups stay the same number of years in labor force. They start with some initial debt and finish working life with some savings. To simplify further we assume no demographic changes: the same number of households joins and quits labor force every year.

The focus of the paper is devoted to the effect of the interest rate changes on consumption. There are several practical points in such analysis. Probably, the most important is to understand the stimulating effect of an interest rate decrease on consumption. On the other hand, there are concerns related to the reverse changes, namely possible increases of low interest rates. Households debt increased during last decades relative to the disposable income and bank deposits [8]. A possible increase of the interest rate will reduce disposable income more than previously. It will lead to a follow up decrease of consumption.

The paper is organized as follows. In Section 2 we describe a basic model for a constant value of the interest rate. Section 3 introduces an interest rate change as a parameter change for the basic model. Numerical simulations are provided in Section 4. Finally, Section 5 presets discussion and concluding remarks. Some technical results are separated into the Appendix.

2 The basic model

In this section we present a basic model for a constant interest rate. The model describes a representative household for a particular age group. This representative household can be considered as average for the age group. Changes of the interest rate will be introduced in the forthcoming section.

2.1 Model formulation

All households are assumed to stay in labor force for T years (time will be measured in years). The starting year for a particular age group is denoted as t_0. Such households will be in labor force during the time interval $[t_0, t_0 +$

$T]$. The representative household chooses a consumption plan $C(t) > 0$ to maximize the functional

$$\max_{C(t)>0} \int_{t_0}^{t_0+T} \alpha(t) U(C(t)) dt, \tag{2.1}$$

where $\alpha(t)$ is a discount function and $U(C)$ is a utility function.

The standard choice is the exponential discount function

$$\alpha(t) = e^{-\delta t}, \tag{2.2}$$

where $\delta > 0$ is the subjective rate of discount. It should be noted that for borrowing scenario we expect $\delta > r$, where r is the constant interest rate.

The utility function is taken in the exponential form

$$U(C) = U_1(C) = 1 - e^{-C}. \tag{2.3}$$

Other popular choices are the logarithmic

$$U_2(C) = \ln C \tag{2.4}$$

and power

$$U_3(C) = \frac{1}{1-\gamma} C^{1-\gamma}, \quad 0 < \gamma < 1 \tag{2.5}$$

utility functions. All three utility functions are compared in Appendix A. The utility function (2.3) gives linear $C(t)$ for the considered optimal control problem that simplifies analysis as well as visualization of the results.

Remark 2.1 *Instead of (2.3) it is more appropriate to consider the utility function*

$$U_1(C) = 1 - e^{-C/w_0},$$

where w_0 is a parameter. The value $w_0 = 1$ suits well to the other parameters, which will be used for numerical simulation. That's why we omit w_0 and continue with (2.3).

The consumption rate, the loan value and the income rate are related by the ordinary differential equation

$$K'(t) = rK(t) + C(t) - w. \tag{2.6}$$

This equation states that the consumption is provided by earned income, given by the constant income rate w, and borrowing against future earnings. We denote the loan value as $K(t)$ assuming that it describes borrowing if $K(t) > 0$ and saving if $K(t) < 0$. For households starting in labor force at year t_0 there are boundary conditions

$$K(t_0) = K^0, \qquad K(t_0 + T) = K^T. \tag{2.7}$$

Here K^0 and K^T are constant. They denote the loan values at the initial time t_0 and at the final time $t_0 + T$. As it was mentioned before negative values can also be considered, they are interpreted as savings. For example, a positive value K^0 can stand for a student loan and a negative value K^T can be saving for retirement.

Remark 2.2 *For simplicity the suggested framework assumes that the income rate, the initial loan and the final loan are constant. In a more general case they can be time dependent functions*

$$w(t, t_0), \qquad K^0(t_0) \qquad \text{and} \qquad K^T(t_0).$$

We summarize the basic model as the following consumption–borrowing/saving problem: The households optimize borrowing/saving during T years expressed by the maximization of the functional (2.1) subject to the dynamic relation (2.6) and the boundary conditions (2.7).

2.2 Solution of the control problem

There are many textbooks on control problems and their applications [9, 10, 11, 12]. The equations for the optimal control problem (2.1), (2.6) are provided by the Hamiltonian function

$$H = e^{-\delta t} U(C(t)) + \lambda(t)(rK(t) + C(t) - w). \tag{2.8}$$

Here we assume that there is no need to care about the constraint $C(t) > 0$. Otherwise, this constraint should be added to the Hamiltonian function.

The Hamiltonian leads to the system of equations

$$K'(t) = H_\lambda, \tag{2.9a}$$

$$\lambda'(t) = -H_K, \tag{2.9b}$$

$$H_C = 0, \tag{2.9c}$$

which takes the form

$$K'(t) = rK(t) + C(t) - w, \tag{2.10a}$$

$$\lambda'(t) = -r\lambda(t), \tag{2.10b}$$

$$e^{-\delta t}U'(C(t)) + \lambda(t) = 0. \tag{2.10c}$$

We denote the consumption rate and the loan value for the system of equations (2.10) with the boundary conditions (2.7) as

$$C(t, t_0) \qquad \text{and} \qquad K(t, t_0). \tag{2.11}$$

Here time t is the independent variable and t_0 stands as a parameter. For the utility function U_1, given in (2.3), the consumption rate and the loan value for the starting year t_0 are

$$C(t, t_0) = w + (r - \delta)T\left(\frac{t - t_0}{T} + \frac{1}{e^{rT} - 1} - \frac{1}{rT}\right) + r\frac{K^T - e^{rT}K^0}{e^{rT} - 1} \tag{2.12}$$

and

$$K(t, t_0) = \frac{\delta - r}{r}T\left(\frac{t - t_0}{T} - \frac{e^{r(t-t_0)} - 1}{e^{rT} - 1}\right) + K^0\frac{e^{rT} - e^{r(t-t_0)}}{e^{rT} - 1} + K^T\frac{e^{r(t-t_0)} - 1}{e^{rT} - 1}. \tag{2.13}$$

Note that the convenient choice of the utility function makes $C(t, t_0)$ linear in time t.

2.3 Aggregation for age groups

The solution of the control problem leads to several important averaged quantities for the whole time in labor force, i.e. over the interval $[t_0, t_0 + T]$:

1. the average consumption rate

$$\bar{C}([t_0, t_0 + T]) = \frac{1}{T}\int_{t_0}^{t_0+T} C(t, t_0)dt, \tag{2.14}$$

2. the average loan value

$$\bar{K}([t_0, t_0 + T]) = \frac{1}{T}\int_{t_0}^{t_0+T} K(t, t_0)dt, \tag{2.15}$$

3. the payment for the loan (this short name, which is relevant only for the case $K^T = K^0$, is used for convenience)

$$P([t_0, t_0+T]) = \int_{t_0}^{t_0+T} w dt - \int_{t_0}^{t_0+T} C(t, t_0) dt \\ = T(w - \bar{C}([t_0, t_0+T])). \quad (2.16)$$

For the solution (2.12),(2.13) these average values are

$$\bar{C}([t_0, t_0+T]) = w + (r-\delta)T\left(\frac{1}{2} + \frac{1}{e^{rT}-1} - \frac{1}{rT}\right) + r\frac{K^T - e^{rT}K^0}{e^{rT}-1}, \quad (2.17)$$

$$\bar{K}([t_0, t_0+T]) = \frac{\delta - r}{r}T\left(\frac{1}{2} + \frac{1}{e^{rT}-1} - \frac{1}{rT}\right) \\ + \frac{e^{rT}K^0 - K^T}{e^{rT}-1} + \frac{K^T - K^0}{rT}, \quad (2.18)$$

$$P([t_0, t_0+T]) = (\delta - r)T^2\left(\frac{1}{2} + \frac{1}{e^{rT}-1} - \frac{1}{rT}\right) + rT\frac{e^{rT}K^0 - K^T}{e^{rT}-1}, \quad (2.19)$$

respectively. Note that for $K^0 = K^T = 0$ we have

$$P([t_0, t_0+T]) = rT\bar{K}([t_0, t_0+T]).$$

Another set of average values corresponds to some particular time t and averaging over all age groups. We recall that there are no demographic changes. The average consumption rate and the average loan value at some particular time t are contributed by the age group with starting time t_0 satisfying $t - T < t_0 < t$:

$$\bar{C}(t) = \frac{1}{T}\int_{t-T}^{t} C(t, t_0) dt_0, \quad (2.20)$$

$$\bar{K}(t) = \frac{1}{T}\int_{t-T}^{t} K(t, t_0) dt_0. \quad (2.21)$$

It is also possible to consider the aggregate consumption rate, which is proportional to the average consumption rate in the considered framework.

We obtain

$$\bar{C}(t) = w + (r-\delta)T\left(\frac{1}{2} + \frac{1}{e^{rT}-1} - \frac{1}{rT}\right) + r\frac{K^T - e^{rT}K^0}{e^{rT}-1}, \tag{2.22}$$

$$\bar{K}(t) = \frac{\delta - r}{r}T\left(\frac{1}{2} + \frac{1}{e^{rT}-1} - \frac{1}{rT}\right) + \frac{e^{rT}K^0 - K^T}{e^{rT}-1} + \frac{K^T - K^0}{rT}. \tag{2.23}$$

For the assumptions made we get

$$\bar{C}(t) = \bar{C}([t_0, t_0+T]), \tag{2.24}$$

$$\bar{K}(t) = \bar{K}([t_0, t_0+T]). \tag{2.25}$$

However, equality relations (2.24) and (2.25) can fail to hold for more general models.

2.4 Calibration of the discount rate δ

For modeling it is necessary to choose the subjective rate of discount δ for the discount function (2.2). Here one can make use of the maximal loan value and the average quantities of the model such as the average loan value (2.15) and the loan payment (2.16). We suggest to specify a convex combination of these quantities as a multiplum of the annual income:

$$\alpha_1 \max_{t\in[t_0,t_0+T]} K(t,t_0) + \alpha_2\bar{K}([t_0,t_0+T]) + \alpha_3 P([t_0,t_0+T]) = \beta w,$$

$$\alpha_1, \alpha_2, \alpha_3 \geq 0, \qquad \alpha_1+\alpha_2+\alpha_3 = 1, \qquad \beta > 0.$$

Here all α_i and β are constants.

Several possibilities can be easily selected:

1. $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$: the maximal loan value

$$\max_{t\in[t_0,t_0+T]} K(t,t_0) = \beta w; \tag{2.26}$$

2. $\alpha_1 = \alpha_3 = 0.5$, $\alpha_2 = 0$: the sum of the maximal loan value and the loan payment

$$\max_{t\in[t_0,t_0+T]} K(t,t_0) + P([t_0,t_0+T]) = 2\beta w; \tag{2.27}$$

3. $\alpha_2 = \alpha_3 = 0.5$, $\alpha_2 = 0$: the sum of the average loan value and the loan payment

$$\bar{K}([t_0, t_0 + T]) + P([t_0, t_0 + T]) = 2\beta w. \tag{2.28}$$

The analytical expression for the maximal value of $K(t, t_0)$ for $t \in [t_0, t_0 + T]$ is provided in Appendix B.

In this section we provided the solution of the control problem and the associated average quantities for the utility function (2.3). One can employ the other utility functions (2.4) and (2.5) in a similar manner.

3 Interest rate changes

Now we incorporate interest rate changes into the basic model, presented in the previous section. The general description is supplemented by particular results for the exponential utility function (2.3).

First we consider the change of a pair (r, δ), which consists of the interest rate and the discount rate, from the original values (r_0, δ_0) to new values (r_1, δ_1). Determination of $\delta_1(r_1)$ is considered separately.

3.1 Particular age groups

We assume that the interest rate had a long time value r_0. Then at some time, which we denote by $t = 0$ for convenience, the interest rate gets changed from r_0 to a new value r_1, which will be held for a long time. There are three types of behavior for the consumption rate $C(t, t_0)$ and the loan $K(t, t_0)$:

1. only with the interest rate r_0 if $t_0 \leq -T$;
2. with both interest rates r_0 and r_1 if $-T < t_0 < 0$;
3. only with the interest rate r_1 if $t_0 \geq 0$.

The solutions of these three cases are the following:

1. Case $t_0 \leq -T$ (before the transition)

   The consumption rate $C_0(t, t_0)$ and the loan $K_0(t, t_0)$ are given by the basic model (2.12) and (2.13) with $r = r_0$ and $\delta = \delta_0$, namely

$$C_0(t, t_0) = C(t, t_0)|_{r=r_0, \delta=\delta_0}, \qquad K_0(t, t_0) = K(t, t_0)|_{r=r_0, \delta=\delta_0}. \tag{3.1}$$

2. The transition case $-T < t_0 < 0$

   The interval (t_0, t_0+T) includes the moment of the interest rate change $t = 0$. The interval gets split into two subintervals: $(t_0, 0)$ with the interest rate r_0 and $(0, t_0+T)$ with the interest rate r_1.

   (a) $t \in (t_0, 0)$

   For the interval $(t_0, 0)$ the results are the same as for the case $t_0 \le -T$:

   $$C_1(t, t_0) = C_0(t, t_0), \qquad K_1(t, t_0) = K_0(t, t_0), \qquad t_0 < t < 0,$$

   see (3.1) for $C_0(t, t_0)$ and $K_0(t, t_0)$.

   At the end of this interval, i.e. at time $t = 0$, the household has the loan

   $$\hat{K} = K_1(0, t_0) = K_0(0, t_0). \tag{3.2}$$

   (b) At the time $t = 0$ there is a change of the interest rate r and an induced change of the rate of discount δ. The new values are r_1 and δ_1. We assume that parameter δ_0 was fitted as described in point 2.4. Determination of the parameter δ_1 stands as a separate problem, which will be discussed in the next point.

   At $t = 0$ there is a switch from the control problem equations (2.10) with $r = r_0$, $\delta = \delta_0$ and boundary conditions (2.7) to the equations (2.10) with $r = r_1$, $\delta = \delta_1$ and the modified boundary conditions (3.3).

   (c) $t \in (0, t_0 + T)$

   For the rest, i.e. on the interval $(0, t_0 + T)$, the dynamics is given by equations (2.10) with r_1 and δ_1. The boundary conditions are

   $$K_1(0, t_0) = \hat{K}, \qquad K_1(t_0 + T, t_0) = K^T, \tag{3.3}$$

   where

   $$\hat{K} = \frac{\delta_0 - r_0}{r_0} T \left(-\frac{t_0}{T} - \frac{e^{-r_0 t_0} - 1}{e^{r_0 T} - 1} \right) + K^0 \frac{e^{r_0 T} - e^{-r_0 t_0}}{e^{r_0 T} - 1} + K^T \frac{e^{-r_0 t_0} - 1}{e^{r_0 T} - 1}.$$

   The solution is given by the consumption rate

   $$C_1(t, t_0) = w + (r_1 - \delta_1) \left(t + \frac{t_0 + T}{e^{r_1(t_0+T)} - 1} - \frac{1}{r_1} \right) + r_1 \frac{K^T - e^{r_1(t_0+T)} \hat{K}}{e^{r_1(t_0+T)} - 1}$$

and the loan value

$$K_1(t,t_0) = \frac{\delta_1 - r_1}{r_1}\left(t - (t_0+T)\frac{e^{r_1 t}-1}{e^{r_1(t_0+T)}-1}\right) + \hat{K}\frac{e^{r_1(t_0+T)} - e^{r_1 t}}{e^{r_1(t_0+T)}-1} + K^T\frac{e^{r_1 t}-1}{e^{r_1(t_0+T)}-1}.$$

Note that $\hat{K}$ is a function of several parameters and K^T is a constant.

3. Case $t_0 \geq 0$ (after the transition)

   Here $C_2(t,t_0)$ and $K_2(t,t_0)$ are given by the basic model (2.12) and (2.13) with $r = r_1$ and $\delta = \delta_1$, i.e. they are

$$C_2(t,t_0) = C(t,t_0)|_{r=r_1,\delta=\delta_1}, \qquad K_2(t,t_0) = K(t,t_0)|_{r=r_1,\delta=\delta_1}. \quad (3.4)$$

3.2 Determination of the new discount rate δ_1

When the interest rate gets changed from r_0 to r_1, it induces the change of the discount rate from δ_0 to δ_1. We suggest to find δ_1 from the following relation for quantities characterizing the old and new cases

$$\alpha_1 \max K_2 + \alpha_2 \bar{K}_2 + \alpha_3 P_2 = \alpha_1 \max K_0 + \alpha_2 \bar{K}_0 + \alpha_3 P_0,$$

$$\alpha_1, \alpha_2, \alpha_3 \geq 0, \qquad \alpha_1 + \alpha_2 + \alpha_3 = 1.$$

Here $\max K_i$ is the maximal loan value, $\bar{K}_i$ is the average loan value and P_i is the payment for the loan. Index $i = 2$ is used for the new values (upon completion of the transition)

$$\max K_2 = \max_{t\in[t_0,t_0+T]} K_2(t,t_0), \qquad \bar{K}_2 = \bar{K}([t_0,t_0+T]),$$
$$P_2 = P_2([t_0,t_0+T]), \qquad t_0 \geq 0.$$

Index $i = 0$ is used for the original values

$$\max K_0 = \max_{t\in[t_0,t_0+T]} K_0(t,t_0), \qquad \bar{K}_0 = \bar{K}([t_0,t_0+T]),$$
$$P_0 = P_0([t_0,t_0+T]) \qquad t_0 \leq -T.$$

The maximal values are discussed in Appendix B, the values $\bar{K}_i$ and P_i are given in (2.15) and (2.16). Note that a particular choice of t_0 for $t_0 \geq 0$ or $t_0 \leq -T$ does not matter for the considered model.

In the next section there will be tested the following methods to find $\delta_1(r_1)$:

1. $\alpha_1 = \alpha_3 = 0.5$, $\alpha_2 = 0$

   Here the new and original values (r, δ) are related by assumption that the sum of the maximal loan value and the loan payment is the same for new values r_1 and δ_1 as it was for the original values r_0 and δ_0, i.e.

$$\max K_2 + P_2 = \max K_0 + P_0; \tag{3.5}$$

2. $\alpha_2 = \alpha_3 = 0.5$, $\alpha_1 = 0$

   Similarly we can use the sum of the average loan value and the payment:

$$\bar{K}_2 + P_2 = \bar{K}_0 + P_0; \tag{3.6}$$

3. $\alpha_3 = 1$, $\alpha_1 = \alpha_2 = 0$

   For completeness we can also consider

$$P_2 = P_0. \tag{3.7}$$

   This approach does not suit for small values r_1 because

$$P_2 \approx \frac{T^3}{12}(\delta_1 - r_1)r_1 + K^0\left(1 + \frac{r_1 T}{2}\right) - K^T\left(1 - \frac{r_1 T}{2}\right)$$

   and

$$P_2 \to K^0 - K^T \qquad \text{as} \qquad r_1 \to 0.$$

Remark 3.1 *It might also be interesting to consider the unchanged discount rate:*

$$\delta_1 = \delta_0.$$

However, this makes sense only for very small changes of the interest rate $r_1 \approx r_0$.

3.3 Averaging for all age groups

The average consumption rate and the average loan value at some time t depend on contributions from households of the age groups with t_0 satisfying $t - T \leq t_0 \leq t$. We recall that there are no demographic changes. There are three cases for such averaging depending on time t.

1. Case $t \leq 0$ (before the interest rate change)

   Only households with $t_0 \leq -T$ contribute. In this case

   $$\bar{C}_0(t) = \bar{C}_0 = \frac{1}{T}\int_{t-T}^{t} C_0(t, t_0)dt_0, \qquad t_0 \leq -T \tag{3.8}$$

   and

   $$\bar{K}_0(t) = \bar{K}_0 = \frac{1}{T}\int_{t-T}^{t} K_0(t, t_0)dt_0, \qquad t_0 \leq -T \tag{3.9}$$

   are actually constant. These quantities are given by (2.22) and (2.23) with $r = r_0$ and $\delta = \delta_0$.

2. Case $0 < t < T$ (the transition of the consumption patterns)

   The households are divided into two groups: households with $t - T < t_0 < 0$, changing their behavior at $t = 0$, and households with $0 \leq t_0 < t$, joining labor force under the new conditions, i.e. with the interest rate r_1 and the discount rate δ_1. We get

   $$\bar{C}_1(t) = \frac{1}{T}\int_{t-T}^{0} C_1(t, t_0)dt_0 + \frac{1}{T}\int_{0}^{t} C_2(t, t_0)dt_0, \qquad 0 < t < T \tag{3.10}$$

   $$\bar{K}_1(t) = \frac{1}{T}\int_{t-T}^{0} K_1(t, t_0)dt_0 + \frac{1}{T}\int_{0}^{t} K_2(t, t_0)dt_0, \qquad 0 < t < T. \tag{3.11}$$

   The integrals with C_1 and K_1 cannot be found analytically and should be computed numerically. For the integrals with C_2 and K_2 both analytical and numerical approaches can be used.

3. Case $t \geq T$ (after the transition to the new conditions)

   There are only households with $t_0 \geq 0$ in labor force. The values

   $$\bar{C}_2(t) = \bar{C}_2 = \frac{1}{T}\int_{t-T}^{t} C_2(t, t_0)dt_0, \qquad t \geq T \tag{3.12}$$

and

$$\bar{K}_2(t) = \bar{K}_2 = \frac{1}{T}\int_{t-T}^{t} K_2(t, t_0)dt_0, \qquad t \geq T \qquad (3.13)$$

are constant. These values are provided by (2.22) and (2.23) with $r = r_1$ and $\delta = \delta_1$.

The results of this section can be easily adapted to models with demographic factors. It requires to modify only averaging (or aggregation) procedure. The results for the particular age groups remain the same.

4 Numerical simulation

Now we turn to numerical computations for the developed model.

4.1 Interest rate changes modeling

First, we discuss the effect of the interest rate change for particular age groups, described in point 3.1. Then, we consider the averaged results, described in point 3.3.

4.1.1 An interest rate decrease

The following parameters

$$T = 40, \qquad w = 1, \qquad K^0 = 1, \qquad K^T = -2,$$
$$r_0 = 0.04, \qquad \delta_0 = 0.055, \qquad r_1 = 0.02, \qquad \delta_1 = 0.044 \quad (4.1)$$

are taken for illustration. Nonzero values K^0 and K^T are chosen to consider the general case of the boundary conditions. The initial discount rate $\delta_0 = 0.055$ is chosen to fit the calibration condition $\max K = 3w$, i.e. according to relation (2.26) with $\beta = 3$. The new value of the discount rate $\delta_1 = 0.044$ is obtained as average of the three values given by the three particular methods (3.5), (3.6) and (3.7).

The results for particular age groups are given in Figures 4–7. Figure 4 compares the consumption pattern and the corresponding loan value dynamics before (for $t_0 = -T$, $r = r_0$, $\delta = \delta_0$) and after (for $t_0 = 0$, $r = r_1$, $\delta = \delta_1$) the transition caused by the interest rate change. The slope of the consumption rate gets steeper, indicating an increase of borrowing against

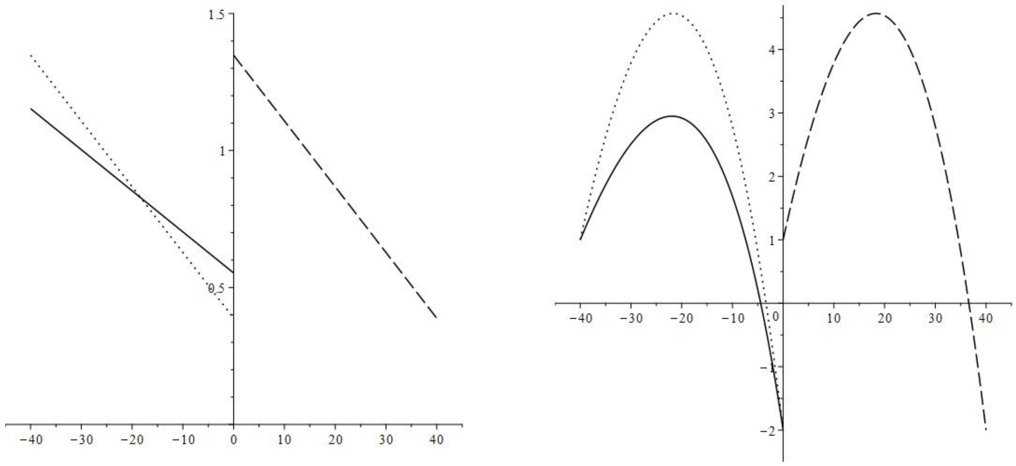


Figure 4: The interest rate decrease scenario. The consumption rate (left) and the loan value (right): the initial pattern for the interval $[-40, 0]$, i.e. for age group $t_0 = -40$, (solid line), the final pattern for the interval $[0, 40]$, i.e. for age group $t_0 = 0$, (dashed line) and the final pattern shifted to the initial interval (dotted line).

the future earnings for the lower interest rate r_1. The same is shown by the greater values of the loan.

Figures 5–7 show the evolution of the consumption rate and borrowing pattern for three particular age groups: $t_0 = -30$, $t_0 = -20$ and $t_0 = -10$. Comparison of the figures shows a very small change (almost no change of the borrowing) for $t_0 = -30$, a noticeable change for $t_0 = -20$ and a substantial change for $t_0 = -10$. It is reasonable to expect that the earlier during the working life the change of the interest rate takes place, the more substantial effect on the consumption and borrowing behavior it will have.

The averaged results are given in Figure 8. We observe that the consumption rate jumps by about 10% from $\bar{C}(0_-) = \bar{C}_0$ to $\bar{C}(0_+) = \bar{C}_1(0)$ at the moment of the interest rate change. Then, it decreases to the value $\bar{C}(T) = \bar{C}_2$, $\bar{C}_2 > \bar{C}_0$. During the transition the average loan value gets substantially increased from $\bar{K}(0) = \bar{K}_0$ to $\bar{K}(T) = \bar{K}_2$.

It is important to stress that the consumption rate gets its maximal value right after the interest rate change. It confirms that the decrease of the interest rates can be an efficient tool to support consumption in times of recession or economic crisis. We observe that the optimal consumption rates for the

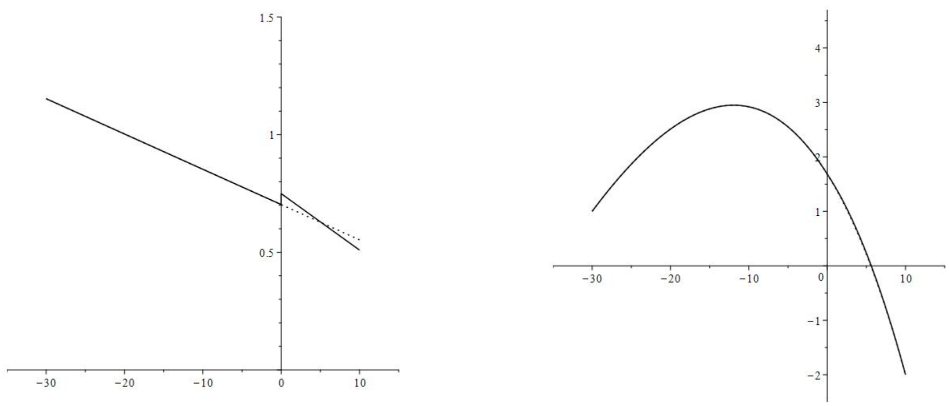


Figure 5: The interest rate decrease at $t = 0$. The consumption rate (left) and the loan value (right) for the interval $[-30, 10]$, i.e. for age group $t_0 = -30$. The dotted line shows how it would be without the change of the interest rate.

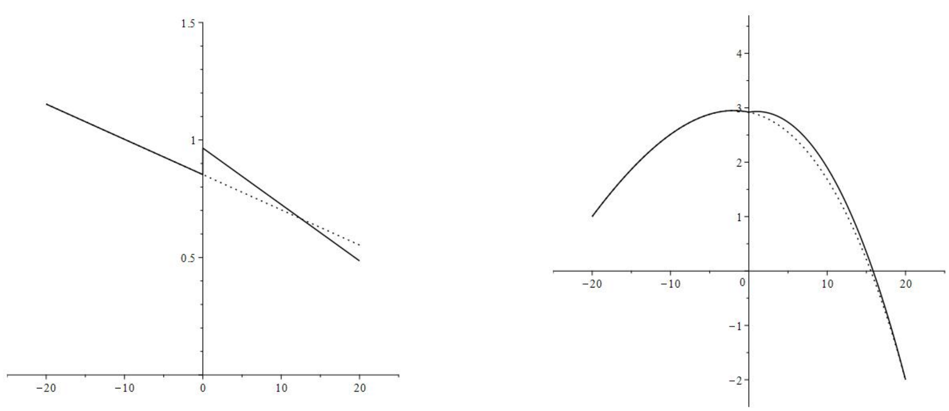


Figure 6: The interest rate decrease at $t = 0$. The consumption rate (left) and the loan value (right) for the interval $[-20, 20]$, i.e. for age group $t_0 = -20$. The dotted line shows how it would be without the change of the interest rate.

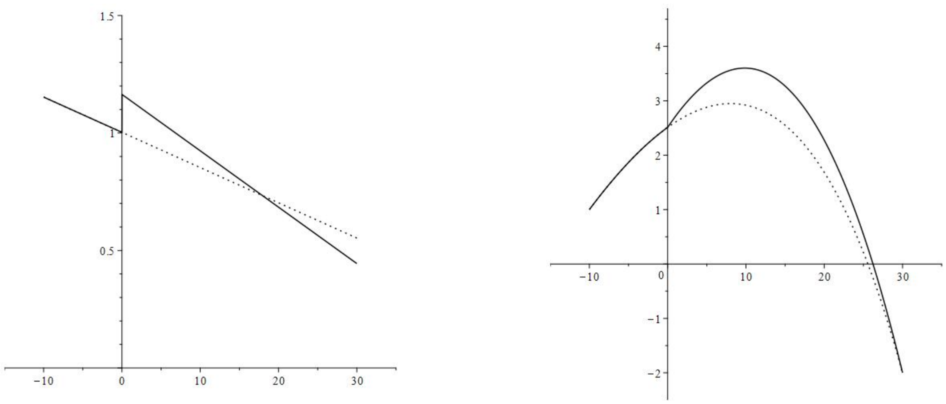


Figure 7: The interest rate decrease at $t = 0$. The consumption rate (left) and the loan value (right) for the interval $[-10, 30]$, i.e. for age group $t_0 = -10$. The dotted line shows how it would be without the change of the interest rate.

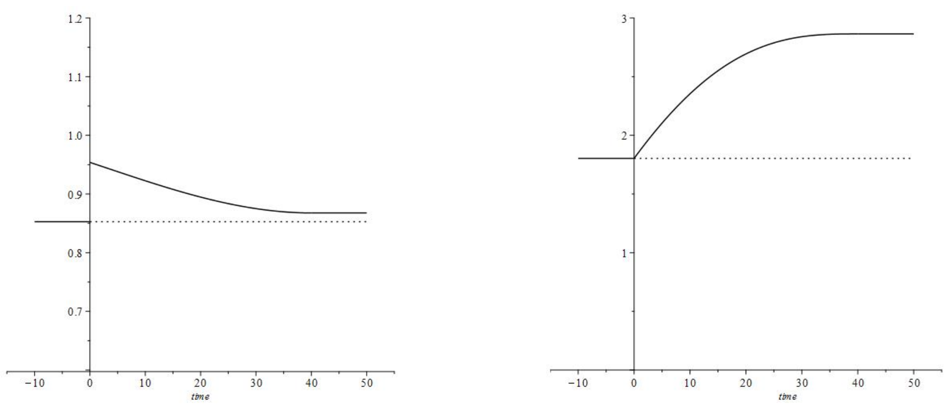


Figure 8: The transition for the interest rate decrease at $t = 0$. The average consumption rate (left) and the average loan value (right) for parameters (4.1). The transition happens on the interval $[0, 40]$. The dotted line shows how it would be without the change of the interest rate.

individual age groups and the average consumption rate are *discontinuous* at the moment of the interest rate change: they have jump increases. Of course, different age groups have different responses to the interest rate change.

4.1.2 An interest rate increase

Though the focus is the interest rate decrease, for completeness of the discussion we also provide a converse change of the interest rate. We use the same parameters as in the previous point and just interchange the values r and δ before and after the change, i.e. consider the parameters

$$\begin{aligned} &T = 40, \qquad w = 1, \qquad K^0 = 1, \qquad K^T = -2, \\ &\qquad\qquad r_0 = 0.02, \qquad \delta_0 = 0.044, \qquad r_1 = 0.04, \qquad \delta_1 = 0.055. \end{aligned} \tag{4.2}$$

The results are given in Figures 9–13. The changes are converse to those in the case of the interest rate decrease: An increase of the interest rate leads to an immediate jump decrease of the consumption rate is shown in Fig. 13. As time passes it increases to the value corresponding to the new interest rate. At the same time the loan value gets decreased to the value corresponding to the new interest rate. As it was observed in the previous point: the earlier in the life cycle the change happens, the more substantial impact it gives.

It should be noted that in reality it might be not so easy to reduce the debt load in the case of the interest rate increase as to increase borrowing in the case of the interest rate decrease. Therefore the simulation results of this point should be considered with some caution. Without loan decrease one can expect a more substantial decrease of the consumption rate.

4.2 Different values of the new interest rate

Now we consider how the changes of the consumption and loan depend on the new interest rate r_1. We keep parameters

$$T = 40, \qquad w = 1, \qquad r_0 = 0.04, \qquad \delta_0 = 0.055 \tag{4.3}$$

and remove the starting and final borrowing/saving

$$K^0 = K^T = 0 \tag{4.4}$$

to make the discussion more universal.

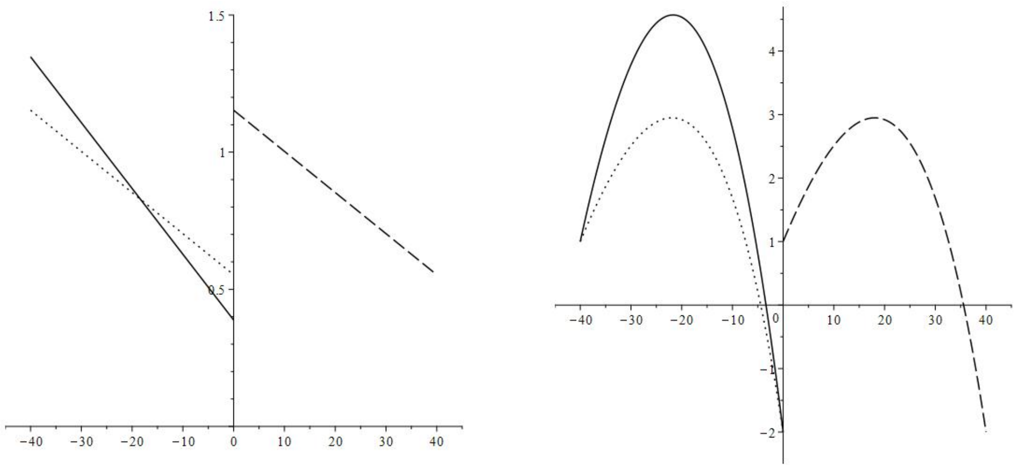


Figure 9: The interest rate increase scenario. The consumption rate (left) and the loan value (right): the initial pattern for the interval $[-40, 0]$, i.e. for age group $t_0 = -40$, (solid line), the final pattern for the interval $[0, 40]$, i.e. for age group $t_0 = 0$, (dashed line) and the final pattern shifted to the initial interval (dotted line).

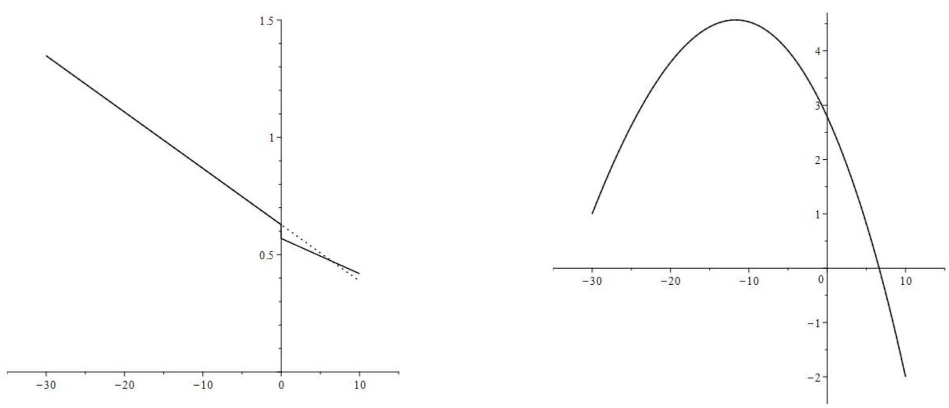


Figure 10: The interest rate increase at $t = 0$. The consumption rate (left) and the loan value (right) for the interval $[-30, 10]$, i.e. for age group $t_0 = -30$. The dotted line shows how it would be without the change of the interest rate.

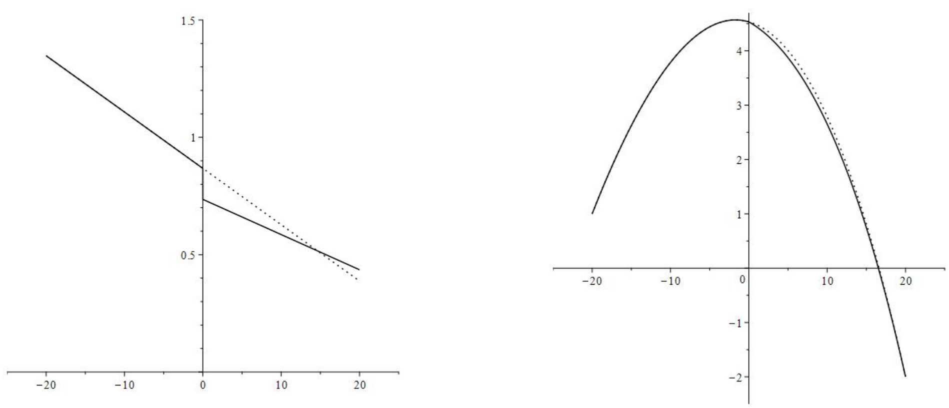


Figure 11: The interest rate increase at $t = 0$. The consumption rate (left) and the loan value (right) for the interval $[-20, 20]$, i.e. for age group $t_0 = -20$. The dotted line shows how it would be without the change of the interest rate.

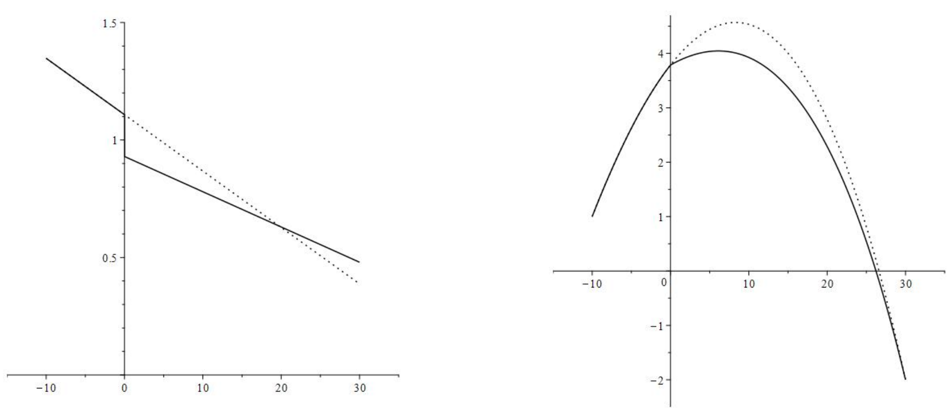
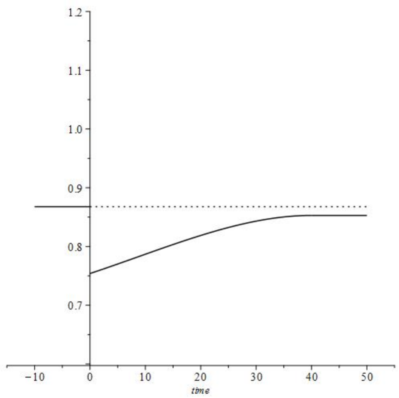


Figure 12: The interest rate increase at $t = 0$. The consumption rate (left) and the loan value (right) for the interval $[-10, 30]$, i.e. for age group $t_0 = -10$. The dotted line shows how it would be without the change of the interest rate.



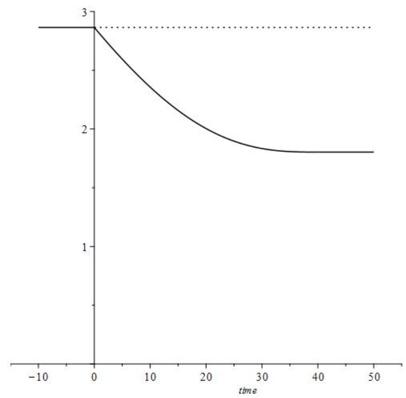


Figure 13: The transition for the interest rate increase at $t = 0$. The average consumption rate (left) and the average loan value (right) for parameters 4.2). The transition happens on the interval $[0, 40]$. The dotted line shows how it would be without the change of the interest rate.

First we compute the values $\delta_1(r_1)$ provided by the methods (3.5), (3.6) and (3.7). Figure 14 presents the simulation results. It shows that for sufficiently large values of r_1 (roughly for $r_1 \geq 0.03$) all three method provide close values. However for small values r_1 the methods differ. The approach based on the loan payment (3.7) does not work here. At the same time the other two methods remain similar. The method based on the maximal loan value and the loan payment (3.5) seems the most reasonable.

According to [4] the discount rate can be described as a constant plus the interest rate. For $r_1 \geq 0.03$ all three method seem to correspond to this statement. The phenomena of low interest rates is recent and it is much less understood what happens to the discount rate in this case.

Figures 15 and 16 show the immediate change of the average consumption rate

$$\frac{\bar{C}(0_+) - \bar{C}(0_-)}{\bar{C}(0_-)} = \frac{\bar{C}_1(0) - \bar{C}_0}{\bar{C}_0} \tag{4.5}$$

(at the moment of the interest rate change) and the final change of the average consumption rate

$$\frac{\bar{C}(T) - \bar{C}(0_-)}{\bar{C}(0_-)} = \frac{\bar{C}_2 - \bar{C}_0}{\bar{C}_0} \tag{4.6}$$

for different values r_1. Figures also provide the final change of the average

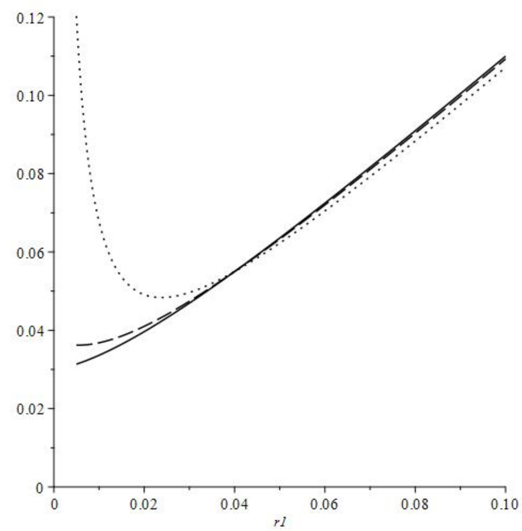
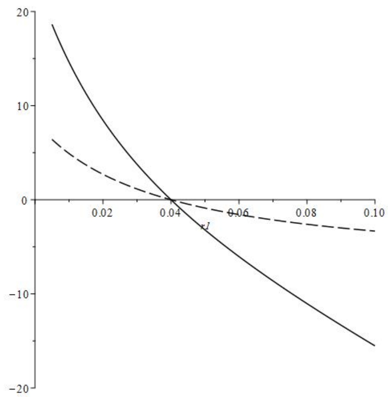


Figure 14: Determination of $\delta_1(r_1)$ for parameters (4.3), (4.4). Three approaches are given: the method based on the maximal loan value and the loan payment (solid line), the method based on the average loan value and the loan payment (dashed line) and the method based on the loan payment (dotted line).



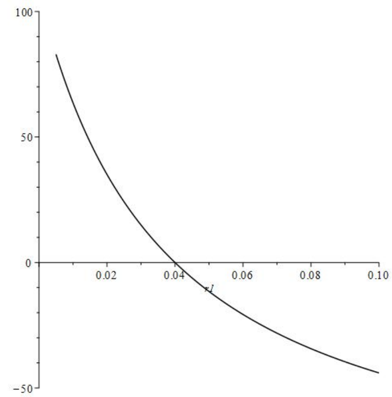


Figure 15: Left plots: The immediate change of the average consumption rate $(\bar{C}(0_+) - \bar{C}_0)/\bar{C}_0$ in percent (solid line) and the final change of the average consumption rate $(\bar{C}_2 - \bar{C}_0)/\bar{C}_0$ in percent (dashed line). Right plot: The final change of the average loan value $(\bar{K}_2 - \bar{K}_0)/\bar{K}_0$ in percent. The plots use determination of δ_1 based on the maximal loan value and the loan payment.

loan

$$\frac{\bar{K}(T) - \bar{K}(0)}{\bar{K}(0)} = \frac{\bar{K}_2 - \bar{K}_0}{\bar{K}_0} \tag{4.7}$$

for different values r_1. We recall that there is no immediate change to the loan value: it is continuous. As it can be expected, decrease (increase) of the interest rate leads to greater (smaller) average consumption rate in the short run and final average loan values. The final average consumption rate is also greater (smaller). The figures correspond to two methods of $\delta_1(r_1)$ determination: (3.5) and (3.6). The second method gives greater changes for both the consumption rates and the loan values.

5 Discussion and concluding remarks

In this paper we considered the effect of interest rate changes on the household consumption and borrowing behavior. The main motivation is to understand the stimulating effect of the interest rate decrease. The analysis is based on a simple model for particular age groups, which was called the basic model. The results obtained for different age groups shape aggregate consumption. The debt load is considered as a part of the model.

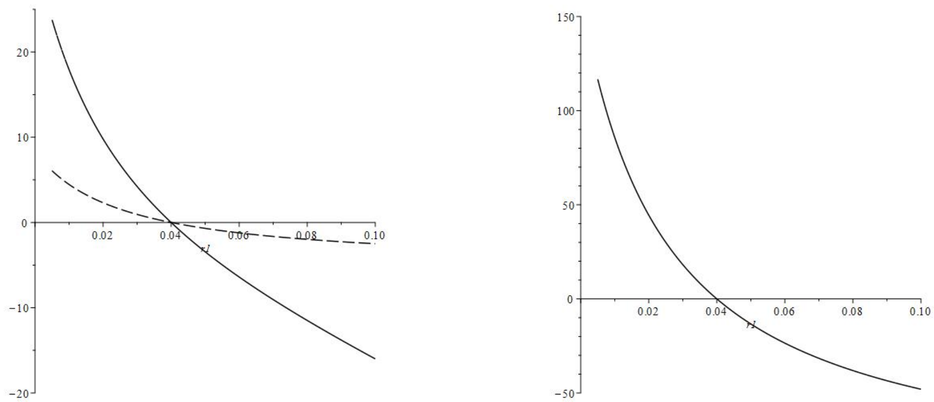


Figure 16: Left plots: The immediate change of the average consumption rate $(\bar{C}(0_+) - \bar{C}_0)/\bar{C}_0$ in percent (solid line) and the final change of the average consumption rate $(\bar{C}_2 - \bar{C}_0)/\bar{C}_0$ in percent (dashed line). Right plot: The final change of the average loan value $(\bar{K}_2 - \bar{K}_0)/\bar{K}_0$ in percent. The plots use determination of δ_1 based on the average loan value and the loan payment.

It is assumed that the other factors get adjusted to the changes of the consumption rate (it can be expected that in reality the other factors would diminish the effect of the interest rate changes on consumption).

The suggested approach is based on the following building elements:

1. A simple basic model, which describes the consumption pattern for a household representing a certain age group. At this stage the interest rate is constant.

2. Incorporation of an interest rate change into the basics model as a parameter change. This change of the parameters effects the control problem system of equations (2.10). The two cases of this system (before and after the interest rate change) are connected by continuity of the loan value and by a relation determining the new value of the discount rate. The consumption rate is discontinuous at the moment of the interest rate change.

3. The results concerning the consumption rates and the loan values for different age groups derive the aggregate values.

Such age-structured models seem appropriate for household consumption and borrowing decisions: households plan over a long (but finite) horizon. A number of simplifying assumptions allows partially analytical treatment of the model.

Numerical simulation was employed to understand transition coursed by an interest rate change. Since no real data was used the obtained results are qualitative. Let us review the main observations. To be specific we consider a decrease of the interest rate.

1. The increase of consumption rate is most substantial right after the interest rage decrease. Both the consumption rates for individual age groups and the aggregate consumption rate show jump increases.

2. After the immediate jump increase the average consumption rate decreases to the value corresponding to the new interest rate and discount factor (parameters r_1 and δ_1). During the transition the consumption rate is greater than upon the completion of the transition

$$\bar{C}_1(t) > \bar{C}_1(T) = \bar{C}_2, \qquad 0 < t < T$$

   and $\bar{C}_1(t)$ is decreasing. In the numerical example the new consumption rate (upon completion of the transition) is greater than the original value

$$\bar{C}_2 > \bar{C}_0.$$

3. High values of the consumption rate during the transition are achieved by an increase of the debt load:

$$\bar{K}_0 = \bar{K}_1(0) < \bar{K}_1(t) < \bar{K}_1(T) = \bar{K}_2, \qquad 0 < t < T \tag{5.1}$$

   and $\bar{K}_1(t)$ is increasing.

There is the inverse relation between change of the interest rate and the consumption rate. In case of the interest rate increase, the average consumption rate has a jump decrease at the moment of the interest rate change and then shows a smooth increase to the final value. At the same time the average loan has gradual decrease to the final value.

Particular remarks should be made about practice of interest rate reductions and lows interest rates as well as potential increases of low interest rates. It should be noted that the interest rate decreases are used for economic stimulation in times of recession or economic crisis. However, if used excessively, they can undermine sustainable economic and technological development.

Several concerns as well as negative effects related to low values of the interest rates were recently reported in the literature. The are mainly related to efficiency of macroeconomic policy. Low interest rates can lead to record high pubic and private debt that presents a frigile balance with high costs of debt [13]. It gives rise to the question: What is a save level of public debt when interest rates are low [14]? It is natural to expect consumption decrease in case of the interest rate increase [8].

Low level interest rates make monetary policy less effective in stimulating bank lending growth [15]. In a particular empirical study for Japan [16] it was found that the effectiveness of monetary policy is lower when interest rates are close to zero. A rise of the interest rate can have an indirect effect on the deterioration of the property market [17], including the growth of housing wealth and the growth rate of housing prices.

Finally, it is worth mentioning that several simplifying assumptions, which were made to achieve analytical tractability, can be reconsidered to improve modeling.

Probably the most important correction can come from a better presentation of the income profile and the corresponding profile of the consumption pattern during the life cycle. Income $w(t, t_0)$ has inverted U-shape (also called hump shape) and declines at retirement [18, 19]. It affects the consumption profile because consumption follows the changes of income.

Modeling of intertemporal preferences can benefit from use of a nonconstant discount factor δ. The constant discount factor is standard in the economic literature. The nonconstant factor can take into account the evolution of individual needs. Many other factors such as inflation, households investments, changes to higher economic and technological development paths, etc. can also be included. However, it would require application of more comprehensive models [20, 21].

It is possible to extend the approach presented in this paper to model gradual changes of the interest rates. Such gradual changes can be approximated as sequences of jump changes from one value to another. Each particular change can be treated as the paper suggests.

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Appendix A. Comparison of utility functions

There are several possibilities to chose a utility function. Three popular options are the exponential

$$U^a(C) = 1 - e^{-C},$$

logarithmic

$$U^b(C) = \ln C$$

and power

$$U^c(C) = \frac{1}{1-\gamma} C^{1-\gamma}, \quad 0 < \gamma < 1$$

functions. In the paper the exponential utility function is used. It is interesting to compare it with the other possible functions for the basic model, which is described point 2.1 and solved in point 2.2.

The control problem leads to the system of equations (2.10), namely

$$K'(t) = rK(t) + C(t) - w,$$

$$\lambda'(t) = -r\lambda(t),$$

$$e^{-\delta t} U'(C(t)) + \lambda(t) = 0$$

with the boundary conditions (2.7), i.e.

$$K(t_0) = K^0, \qquad K(t_0 + T) = K^T,$$

where K^0 and K^T are chosen constant.

Solutions of the control problem have the forms given by the following functions (function $\lambda(t)$ is omitted).

- Exponential utility function $U^a(C) = 1 - e^C$:

$$C^a(t, t_0) = A + (r - \delta)t,$$

$$K^a(t, t_0) = \frac{w - A}{r} + (\delta - r)\left(\frac{t}{r} + \frac{1}{r^2}\right) + Be^{rt};$$

- Logarithmic utility function $U^b(C) = \ln C$:

$$C^b(t, t_0) = Ae^{(r-\delta)t},$$

$$K^b(t, t_0) = \frac{w}{r} - \frac{A}{\delta} e^{(r-\delta)t} + Be^{rt};$$

- Power utility function $U^c(C) = \frac{1}{1-\gamma}C^{1-\gamma}$, $0 < \gamma < 1$:

$$C^c(t, t_0) = Ae^{\frac{r-\delta}{\gamma}t},$$

$$K^c(t, t_0) = \frac{w}{r} - \frac{A}{\delta + \frac{(r-\delta)(\gamma-1)}{\gamma}} e^{\frac{r-\delta}{\gamma}t} + Be^{rt}.$$

The integration constants A and B are to be determined from the boundary conditions. Already at this stage it is easy to see that the exponential utility U^a provides a simpler solution for the consumption rate: $C^a(t, t_0)$ is linear in time t.

Application of the boundary conditions provides the final results.

- Exponential utility function $U^a(C) = 1 - e^C$:

$$C^a(t, t_0) = w + (r - \delta)T \left(\frac{t - t_0}{T} + \frac{1}{e^{rT} - 1} - \frac{1}{rT} \right) + r\frac{K^T - e^{rT}K^0}{e^{rT} - 1},$$

$$K^a(t, t_0) = \frac{\delta - r}{r} T \left(\frac{t - t_0}{T} - \frac{e^{r(t-t_0)} - 1}{e^{rT} - 1} \right) + K^0 \frac{e^{rT} - e^{r(t-t_0)}}{e^{rT} - 1} + K^T \frac{e^{r(t-t_0)} - 1}{e^{rT} - 1};$$

- Logarithmic utility function $U^b(C) = \ln C$:

$$C^b(t, t_0) = \delta \left(\frac{w}{r} \frac{e^{rT} - 1}{e^{rT} - e^{(r-\delta)T}} + \frac{K^T - K^0 e^{rT}}{e^{rT} - e^{(r-\delta)T}} \right) e^{(r-\delta)(t-t_0)},$$

$$\begin{aligned} K^b(t, t_0) = {} & \frac{w}{r} \left(1 - \frac{(1 - e^{(r-\delta)T})e^{r(t-t_0)} + (e^{rT} - 1)e^{(r-\delta)(t-t_0)}}{e^{rT} - e^{(r-\delta)T}} \right) \\ & + K^0 \frac{e^{rT}e^{(r-\delta)(t-t_0)} - e^{(r-\delta)T}e^{r(t-t_0)}}{e^{rT} - e^{(r-\delta)T}} + K^T \frac{e^{r(t-t_0)} - e^{(r-\delta)(t-t_0)}}{e^{rT} - e^{(r-\delta)T}}; \end{aligned}$$

- Power utility function $U^c(C) = \frac{1}{1-\gamma}C^{1-\gamma}$, $0 < \gamma < 1$:

$$C^c(t, t_0) = \left(\delta + \frac{(r-\delta)(\gamma-1)}{\gamma} \right) \left(\frac{w}{r} \frac{e^{rT} - 1}{e^{rT} - e^{\frac{r-\delta}{\gamma}T}} + \frac{K^T - K^0 e^{rT}}{e^{rT} - e^{\frac{r-\delta}{\gamma}T}} \right) e^{\frac{r-\delta}{\gamma}(t-t_0)},$$

$$\begin{aligned} K^c(t, t_0) = {} & \frac{w}{r} \left(1 - \frac{(1 - e^{\frac{r-\delta}{\gamma}T})e^{r(t-t_0)} + (e^{rT} - 1)e^{\frac{r-\delta}{\gamma}(t-t_0)}}{e^{rT} - e^{\frac{r-\delta}{\gamma}T}} \right) \\ & + K^0 \frac{e^{rT}e^{\frac{r-\delta}{\gamma}(t-t_0)} - e^{\frac{r-\delta}{\gamma}T}e^{r(t-t_0)}}{e^{rT} - e^{\frac{r-\delta}{\gamma}T}} + K^T \frac{e^{r(t-t_0)} - e^{\frac{r-\delta}{\gamma}(t-t_0)}}{e^{rT} - e^{\frac{r-\delta}{\gamma}T}}. \end{aligned}$$

It is easy to see that the general solution for the utility function $U^b(C)$ is included into the general solution for the utility function $U^c(C)$ as a particular case corresponding to $\gamma = 1$. This can also be concluded from

$$(U^b(C))' = \frac{1}{C}, \qquad (U^c(C))' = \frac{1}{C^\gamma}.$$

The careful comparison of the solutions for utility functions U^b and U^c shows that these solutions are the same provided that

$$r - \delta^b = \frac{r - \delta^c}{\gamma},$$

where δ^b and δ^c are the discount factors for U^b and U^c, respectively. All other parameters t_0, T, r, w, K_0 and K^T are assumed to be the same. Therefore, if the discount rate is obtained by calibration, for example, using the condition (2.26) which is used for computations in the paper, the utility functions U^b and U^c provide the same result. Taking this into account, we discard U^c from further consideration.

Finally, we present a numerical comparison of $C(t)$ and $K(t)$ for the exponential utility U^a and the logarithmic utility U^b. We take $t_0 = 0$ and the same parameters as used point 4.1.1:

$$T = 40, \qquad w = 1, \qquad K^0 = 1, \qquad K^T = -2, \qquad r = 0.04.$$

The discount rate δ is calibrated according to the condition (2.26) for $\max K = 3w$. We obtain

$$\delta^a = 0.055, \qquad \delta^b = 0.0571.$$

The consumption rate and the loan value are given in Figure 17. The comparison shows that the obtained values $C(t, t_0)$ and $K(t, t_0)$ for the utility functions U^a and U^b are close. At the same time U^a gives the linear consumption rate that might be preferable for visualization.

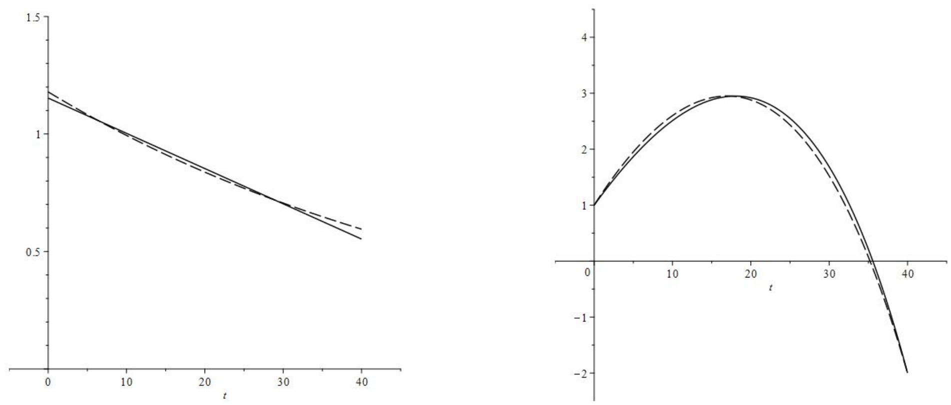

Figure 17: The basic model. The consumption rate (left) and the loan value (right). Solutions for utilities U^a and U^b are given by solid and dashed lines, respectively.

Appendix B. The maximal loan value for the basic model

We consider the solution of the control problem obtained in point 2.2, namely the loan value

$$K(t,t_0) = \frac{\delta - r}{r} T \left(\frac{t-t_0}{T} - \frac{e^{r(t-t_0)}-1}{e^{rT}-1} \right) + K^0 \frac{e^{rT} - e^{r(t-t_0)}}{e^{rT}-1} + K^T \frac{e^{r(t-t_0)}-1}{e^{rT}-1}, \qquad t_0 \le t \le t_0 + T.$$

The maximal value of $K(t,t_0)$ is reached at the time $\tilde{t}$ given by the equation

$$K_t(\tilde{t}, t_0) = 0$$

provided that it satisfies $t_0 < \tilde{t} < t_0 + T$. We obtain

$$\tilde{t} = t_0 + \frac{1}{r} \ln \left(\frac{e^{rT} - 1}{rT + \frac{r^2}{\delta - r}(K^0 - K^T)} \right),$$

which provides

$$\begin{aligned}
&\max_{t\in[t_0,t_0+T]} K(t,t_0) = K(\tilde{t},t_0) \\
&= \frac{\delta-r}{r^2}\left(\ln\left(\frac{e^{rT}-1}{rT+\frac{r^2}{\delta-r}(K^0-K^T)}\right)+\frac{rT}{e^{rT}-1}-\frac{rT}{rT+\frac{r^2}{\delta-r}(K^0-K^T)}\right) \\
&\qquad\qquad\qquad\qquad + \frac{K^0e^{rT}-K^T}{e^{rT}-1}+\frac{K^T-K^0}{rT+\frac{r^2}{\delta-r}(K^0-K^T)}.
\end{aligned}$$

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