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The Value of Petroleum Exploration under Uncertainty

A Real Options Approach

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Abstract

We develop a framework for valuation and optimal decision making in oil exploration projects with uncertain surroundings. In particular, we construct a real options valuation framework that incorporates the stochastic process of the oil price, on-going exploration costs subjected to an uncertain time to completion of the exploration, and the total amount of oil in the field. First, we outline a model including an abandonment option. If abandoned, we let the owner sell the project to another player in the market. Then, we extend the model by including an option to delay the final investment cost when the exploration is completed. Furthermore, we find the optimal threshold levels for the oil price at which the project should be invested in or abandoned. Despite the elements of flexibility involved, and the uncertainty in oil prices and time to completion, we are able to obtain simple closed form solutions. We find that by allowing the owner to sell the project at abandonment, the total project value and the abandonment threshold level increase. However, the effect is diminishing with the oil price. The option to delay the final investment increases the project value but lowers the abandonment threshold level. Finally, we perform sensitivity analysis by changing important input parameters, such as the expected time to completion of the exploration stage and the total amount of oil in the field. The focus of the thesis is on the petroleum industry, and the Norwegian continental shelf in particular. However, the framework and the possible closed form solutions are applicable to a wide range of investment cases, such as R&D projects and projects involving other natural resources.

Keywords: investment analysis, real options, oilfield development

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1. Introduction

The valuation and assessment of a petroleum exploration project is a complex and challenging operation, due to simultaneous uncertainty in several input factors. Uncertainty is here defined as deviations from the expected outcome. Fluctuations in the oil price play a significant role in deciding the value of a project. Furthermore, the uncertain time to completion of the exploration stage will also lead to uncertainty in the on-going exploration costs of a project. As the cost level of petroleum exploration and investment has grown rapidly over the last decade, optimal decision making concerning new petroleum investment projects is crucial. In this thesis we present a real options framework for evaluating the exploration stage of a petroleum investment project with uncertain surroundings. Based on the cost of exploration, the final investment cost and the oil price, among other factors, we estimate the value of the project in the exploration stage. Thus, unlike traditional NPV models, we consider the value of managerial flexibilities concerning abandonment of the project, as well as timing the final investment optimally. Accordingly, we find optimal oil price threshold levels for investment and abandonment. First, we will introduce the Norwegian continental shelf, the area of focus in our thesis, by briefly reviewing its history and discussing the situation on the shelf today.

1.1 Exploring the Norwegian Continental Shelf

There has been a huge development on the Norwegian continental shelf since the first major oil discovery by Philips Petroleum back in 1968. The oil and gas sector have become Norway´s most important element in the economy, constituting approximately 20% and 50% of the GDP and export in 2013 (Norwegian Ministry of Petroleum and Energy a, 2014), respectively. An increasing demand for oil together with a surge in oil prices from year 2000 have resulted in full employment, a nation in wealth and a $870,000bn government pension fund (Norges Bank Investment Management, 2014).

Today, more than 50 petroleum-related companies operate on the Norwegian continental shelf, from big and international players to smaller Scandinavian-wide companies. Norway is the largest oil producer and exporter in Western Europe and the world's 3rd largest producer of gas (U.S. Energy Inormation Administration, 2014). The activity on the Norwegian shelf is widespread over 2.039.951 square kilometers from the core-areas in the southern region of the

North Sea, which contains the first major oil field, Ekofisk, through the Norwegian Sea and to the Barents Sea some miles outside the northern coastline of Norway.

Figure 1 - The Norwegian continental shelf (Norwegian Ministry of Petroleum and Energy a, 2014)

The expansion into new areas of exploration illustrate the constant race of finding new profitable oil fields on the Norwegian continental shelf. This has been the situation for the last 40 years, as illustrated in Figure 2. This figure shows the number of exploration wells drilled on the Norwegian shelf since the start in 1966, and separates between wildcat- and appraisal

wells. Wildcat wells are drilled in search for a new reservoir, while an appraisal well is drilled to determine the size of a reservoir that is already discovered. The development in the number of exploration wells drilled has been steady, with some peaks along the way. The increase in the last decade is mainly due to changes in the oil price. Thus we see that the oil price plays an important role in deciding the activity level of exploration projects on the Norwegian shelf, and it is also a crucial factor in the process of exploration which we will turn to in the next sub-section.
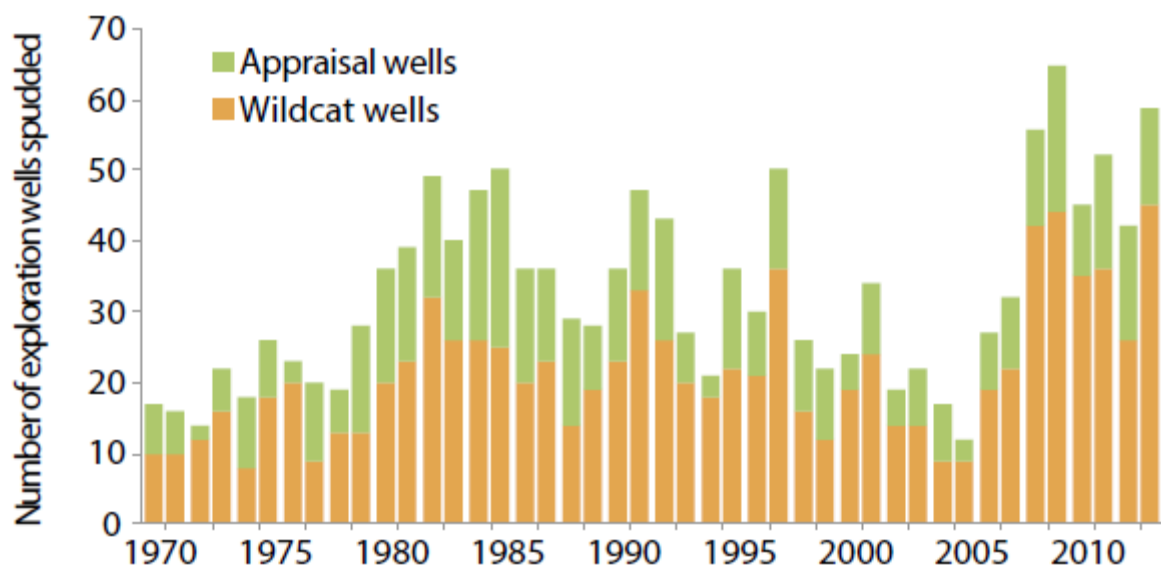
Figure 2 - Spudded exploration wells on the Norwegian continental shelf 1970-2013 (Norwegian Ministry of Petroleum and Energy a, 2014)

1.2 The Process of Exploration

The process of exploration initially starts when the licenses for exploring in a new area are issued. These are issued by the Norwegian Ministry of Petroleum and Energy (NMPE) in cooperation with the Norwegian petroleum directorate (NPD) and other governmental instances (Norwegian Ministry of Petroleum and Energy b, 2007). The licenses are awarded to the companies based on geological understanding, technical expertise and financial strength. Normally, several companies share an exploration license, with one of the companies being the responsible operator. The companies owning the license then cooperate in the exploration stage. The first steps are basic groundwork in the form of acquiring and interpreting seismic data of the area. These are 3D maps of the ground below the seabed, which will allow the geologists insight to whether, and possibly where, it is likely to find oil in the area of the exploration license. When the owners of the license agree that the possibility of

discovering oil in the area is satisfactory, they drill a wildcat well where they think the oil reservoir is located. Given the data from the seismic surveys, the owner of the project are quite certain that there should be oil in the field when they decide to drill a wildcat well. At this stage, substantial expenditures are made and the process is therefore referred to as an exploration project. The owners of the exploration license are likely to drill several exploration wells before they have enough information about the potential oilfield. If the exploration drilling is successful and there are proofs of a substantial amount of oil in the field, then the next step is to drill appraisal wells to determine the size of the field more precisely. However, it is common that some exploration wells are dry, even if there is oil or gas located somewhere in the field. In such cases, the wildcat well may have missed the potential reservoir by only a few meters. There may also be discoveries of oil, but where the reservoir is too small or in too complex geographical surroundings, to be profitable enough for the oil to be extracted. Consequently, there is a need for continuous valuation of the exploration project to see whether it is profitable to continue the exploration process or not. This source of flexibility should not be neglected when evaluating the project.

Should there be a discovery that is large- and well-suited enough to make extraction profitable, thus covering the final investment costs of infrastructure, equipment and other costs from extraction, then the owner of the project must decide when it is optimal to make the final investment. At this point, the project is regarded as an investment project. Historically, the profitability on the Norwegian shelf has been at a high level which has led to immediate development. However, changes in the cost level and oil price may create a value from timing the final investment decision. If the decision to make the final investment to develop the oilfield is made, the owner of the exploration license must submit a Plan for Development and Operations (PDO) to the NMPE (Norwegian Petroleum Directorate a, 2010). This plan contains data about the field and a detailed description of how the oilfield is thought to be developed, with several analysis and NPV-calculations of profitability. If the PDO is approved, then the owners of the license are free to develop the field within a certain period of time.

Should, however, the owner of the exploration license not find it optimal to make the final investment and extract the oil in the field, there is also an opportunity to sell the license, or at least parts of it to another company. The project may be of more value to another company if there is a possibility to tie the project into other projects and installations in the same area which in turn will reduce the exploration-, investment- and production costs. An example of a

company selling an undeveloped field, is the sale of Noreco's share in the oil discovery Flyndre in 2011 (Offshore.no, 2011). Flyndre is a small field that was discovered as early as in 1974, but was calculated to not being large enough for further development and production. However, new technology and nearby infrastructure has changed the situation, and Maersk Oil UK decided to buy Noreco's share in 2011. Evidently, Noreco's share of the field was not valueless after all. In 2014, Maersk Oil UK received a permission from the NMPE to go through with the process of developing the field, and production is expected to start in 2016 (Norwegian Ministry of Petroleum and Energy c, 2014). This case illustrates that there is an active market for trading proved and probable reserves, or in other words, oil reserves that are discovered but not yet developed. There are several examples of similar transactions, too. From a real options point of view, the option to sell an undeveloped reserve and the opportunity to delay the final investment, add value to the oil exploration projects.

1.3 The Situation Today

Recently, exploration- and investment projects on the Norwegian shelf have been subject to increased demands regarding profitability. First of all, the increasing cost level in general plays a crucial role when evaluating new projects. The high profitability in the sector over
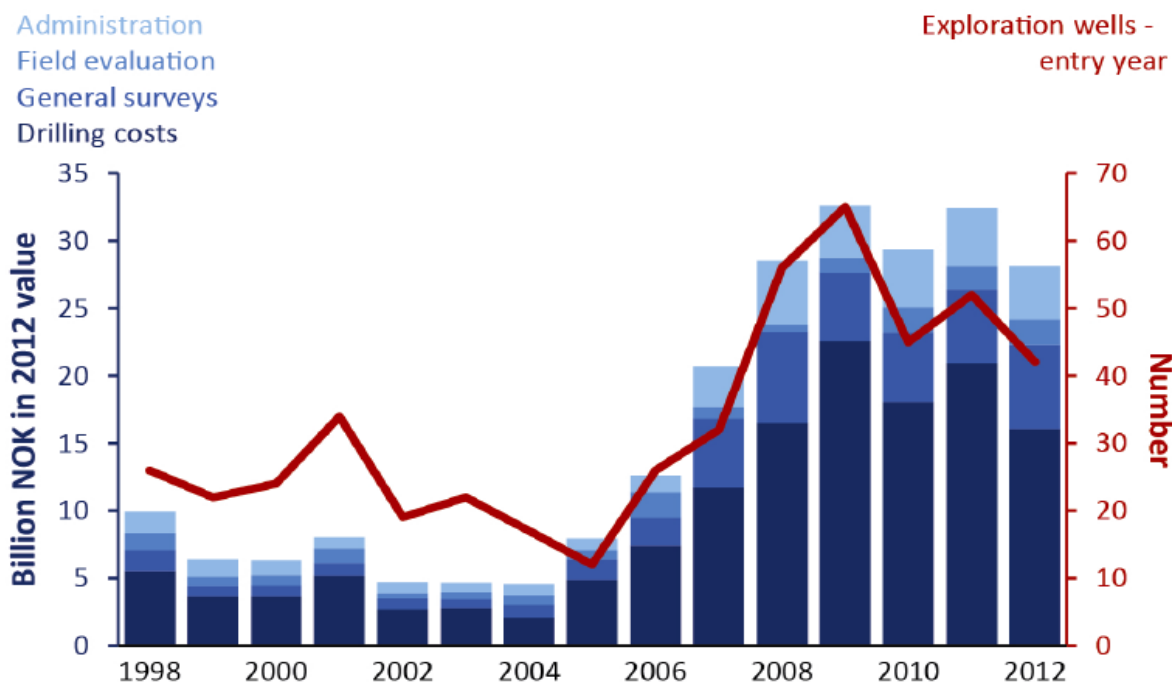
Figure 3 – Developments in exploration costs and number of wells spudded, 1998-2012 (Norwegian Petroleum Directorate b, 2013)

the last 40 years has allowed costs for staff, equipment and services, to grow rapidly. Figure 3 illustrates the development of total exploration costs spent on the Norwegian shelf over the last decade. It is easy to observe an increase in both cost- and exploration level from 2006, which is mainly due to changes in the oil price in this period. Although the total expenditures on the Norwegian shelf are growing accordingly to the increase in number of exploration wells drilled, we also see a rapid growth in the cost per well. This is shown in Figure 4, which illustrates how the exploration cost per well has doubled in the period from 1998 to 2012.
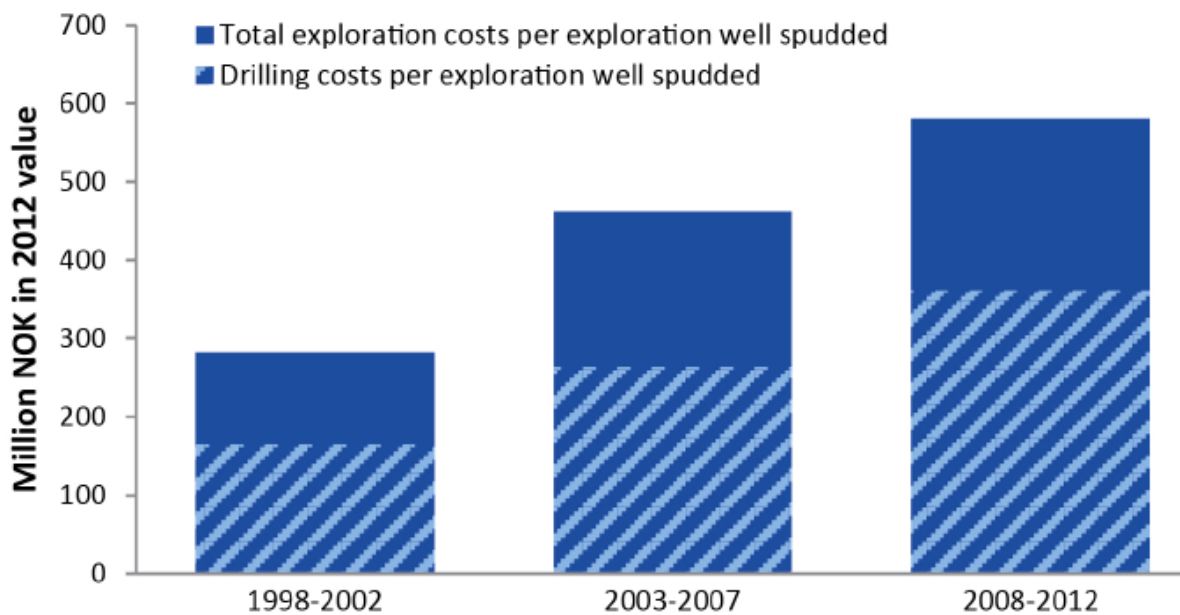
Figure 4 - Developments in exploration costs per well spudded (Norwegian Petroleum Directorate b, 2013)

A second factor that affects the value of both new and existing projects is the oil price. Larger downward fluctuations may be enough to turn a profitable project into a loss (Hegnar.no, 2014). It is, in fact, claimed that the oil price is the most important source of uncertainty for investment projects in the petroleum sector (Limperopoulos, 1995). The Brent crude oil price has increased steadily over the last 40 years, especially in the latter part of the 2000s as mentioned in sub-section 1.1. However, it is subject to larger fluctuations, as of the end of 2014, which is illustrated in Figure 5. The recent fluctuations are likely to have a crucial impact on the value of several development projects on the Norwegian shelf. This challenge is emphasized by the leading player on the Norwegian shelf, Statoil, which has claimed that several projects in the company's portfolio will not be accepted for investment should the oil price stay low for a longer period of time (Sysla.no, 2014).

A third factor that affects the profitability in oil exploration- and investment projects is the diminishing number of profitable fields and the consequences from this. Intuitively, it is easy to see how a non-renewable energy source, such as oil, has become more difficult and complex to discover and extract as the total level of oilfield reserves on the Norwegian shelf
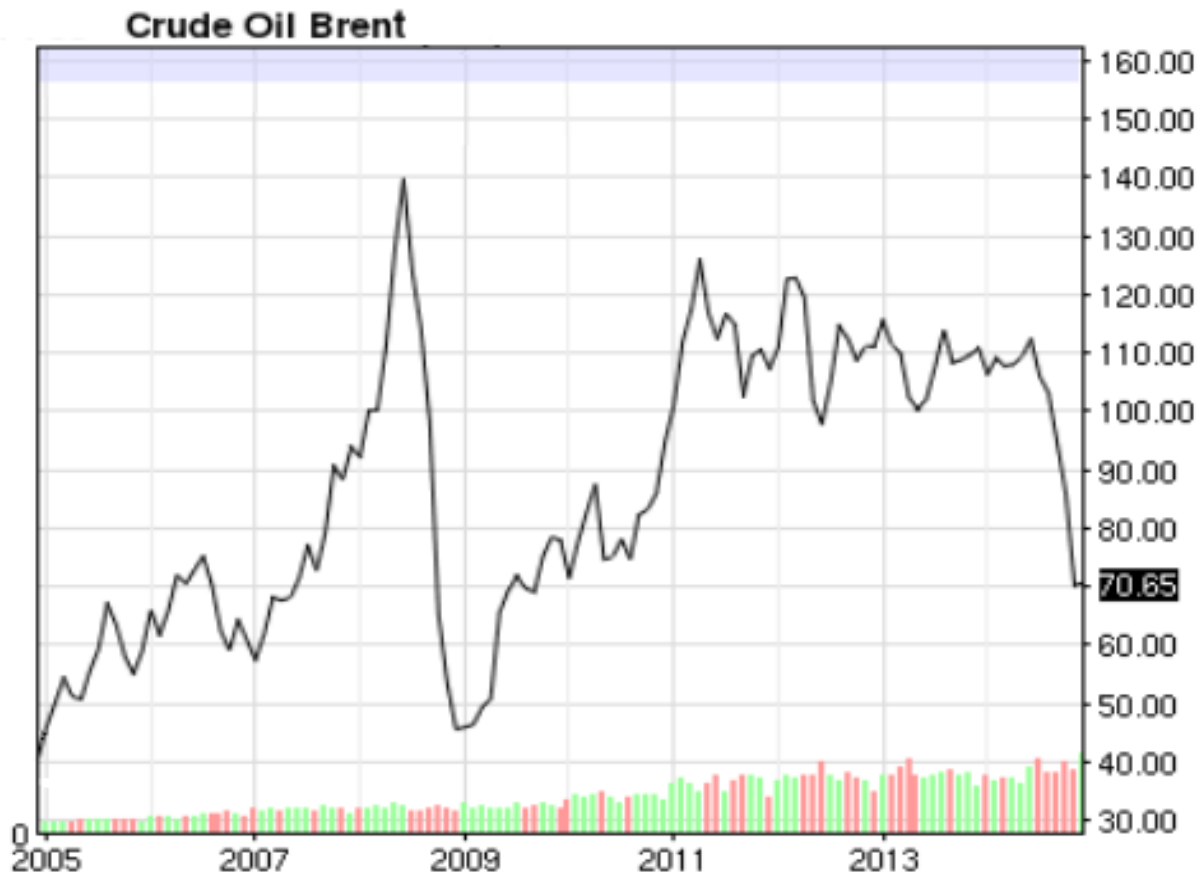
Figure 5 - Development in oil price from 2005 to 2014 (Nasdaq.com a, 2014)

is reduced. Smaller fields provide less cost-savings due to economies of scale, and will come at a higher cost for the operator. In addition, operators are forced to explore in more complex areas as the largest and most apparent fields on the shelf are already discovered (Lund, 1999).

All factors mentioned above illustrate the importance of advanced development strategies and sophisticated valuation tools for new oil investment projects on the Norwegian continental shelf. An obstacle to precise valuation is the high degree of uncertainty, in all factors mentioned in this sub-section. Therefore, the value of flexibility in such projects should not be neglected, as in traditional NPV-models (Limperopoulos, 1995). As the cost level and the degree of uncertainty is increasing, it becomes vital to provide valuations of new projects that are as close to their expected actual value as possible. The main target will always be to invest in the projects that are profitable, and to avoid those that bear losses.

In this thesis, we develop a real options framework that incorporates the value of flexibility in the evaluation of an offshore petroleum project. We consider a situation where the owner of the project is exploring for oil, and is in need of a precise valuation to make the optimal decisions. More precisely, he must decide whether to continue the exploration or not. Furthermore he must decide if, and possibly *when*, to make the final investment after the exploration is completed. In the next two sections we present an overview of the concept of options and real options, and relevant literature. Section four presents the assumptions and notations behind our framework, while we in section five present the analytical formulations. In section six we apply the framework in a case study. Finally, we conclude and offer directions for future research.

2. Overview of Options and Real Options

In this part we give a brief introduction to options and real options, and in particular how real option analysis differ from the traditional NPV-approach to capital budgeting and valuation of projects. Furthermore, we discuss the concepts of dynamic programming and contingent claims analysis.

2.1 Financial Options

An option is a derivative, i.e. a financial instrument where the value is derived from the value of an underlying investment. Most frequently, the underlying investment on which an option is based is the equity shares in a publicly listed company (Nasdaq.com b, 2014). Options are widely used in financial markets as an instrument for risk management or speculation.

There are two types of options – call options and put options. A call option gives the holder the right, but not the obligation, to buy a specified quantity of an underlying asset at a predetermined price, called the strike price or exercise price, within a set time period. A put option is similar, but now the holder has the right, but not the obligation, to sell a specified quantity of an underlying asset at a predetermined exercise price, before or at a predefined maturity date. In any case, the holder pays a price for this right called the premium, and the option is "in the money" when the spot price is above (below) the strike price for a call option (put option), meaning that the holder can exercise the option and get a positive profit in return.

The most common option styles are American and European options. The difference between them lies in how they (potentially) are exercised. American options can be exercised at any time from purchase until the expiration date, while European-style options can only be exercised at expiration date. The possibility of early exercise makes American options more valuable, and generally harder to analyze, than corresponding European options. However, in most cases, there is a time premium associated with the remaining life of an option that makes early exercise sub-optimal (Damodaran, 2014).

Taking a long position in an option means buying a call option or a put option. Opposite, a short position in an option means selling either one of them. The figures on the next page describe the payoffs from different European options.
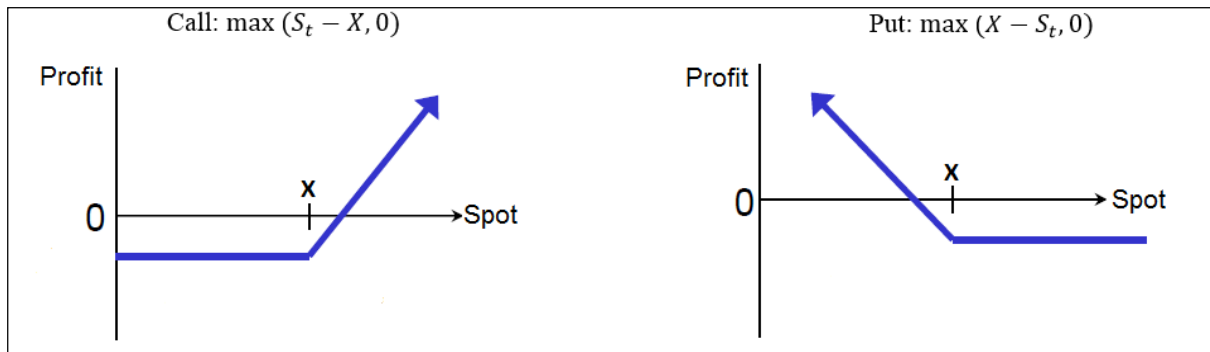
Figure 6 - Long position in a call or put option

The blue lines in Figure 6 indicates a long position. A long position in a call option returns a positive payoff when the spot price is higher than the exercise price at maturity. Otherwise, the payoff is zero and the owner has a loss equal to the premium paid for the option. It is the opposite situation for a long position in a put option, where the payoff is positive when the option is exercised at a time when the spot price is below the exercise price, X. Otherwise, the value is zero and the owner has a loss equal to the premium paid for the option.
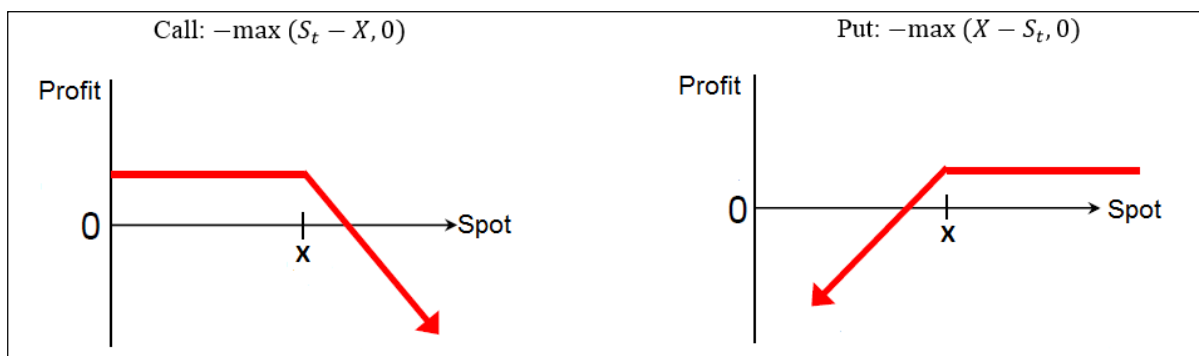
Figure 7 - Short position in a call or put option

The red lines in Figure 7 indicates a short position, where the owner has an obligation to either sell or buy the underlying asset to the exercise price at maturity. A short position in a call option or a put option is attractive as long as the spot price is below or above the exercise price, X, respectively. Then, the payoff is equal to the premium received for the option. On the other hand, a short position in a call option, or a put option, is unprofitable when the owner of the option exercise it at a time when the spot price is above, or below, the exercise price, X, respectively.

2.2 Real Options

In definition, a real option is the application of derivatives theory to the operation and valuation of real investment projects and assets (McDonald, 2006). In comparison to a financial option, any real investment can be seen as a call option with strike price equal to the investment cost and the present value of future cash flows equal to the price of the underlying asset. In this context, the present value of future cash flows is then compared to the investment cost in order to make a decision to invest in the project or not, i.e. to exercise the option.

In traditional approaches to capital budgeting, e.g. the NPV-method, the decision to invest or not depends solely on the sum of discounted future (expected) cash flows compared to the investment cost. In other words, if we assume an irreversible investment, it is a go or no go decision considered today based on the level of future discounted cash flows. Thus, when an irreversible investment is made, the firm somewhat "kills" the option to invest. Consequently, it gives up the possibility of waiting for new information to arrive. In other words, there is an opportunity cost of making the investment that is not included in the NPV-analysis. Another drawback of the NPV-method, is that a wrongfully set discount rate may put a good project to an end or cause the firm to undertake bad projects (Dixit & Pindyck, 1994, p. 6).

Real-option valuation, however, captures value obtained by managerial flexibility to invest at the right time. Put another way, uncertainty and the firm`s ability to respond to it (flexibility) are the source of value of an option. It gives the management the opportunity to make a decision based on how events unfold in time (Schwartz, 2012).

For many cases, e.g. an investment project, the main types of real options are abandonment, timing, expansion, and temporarily suspension. The option to abandon is valuable if the firm is losing money, thus the managers are not obligated to continue with the business plan if it becomes unprofitable. It can be analyzed as having the investment project together with a long put option; if the value drops under a certain threshold level, the best decision is to shut down or abandon the investment project. Moreover, having the option to restart a temporarily suspended project can be viewed as having a long position in a call option.

A positive net present value does not necessarily imply an instant investment. In many cases it is optimal to "wait and see", obtain better information or hope for improved market conditions. This is the so-called timing option. Also, the option to expand a successful project or invest in sequential stages, can be valuable depending of the market scenario. In the case of

petroleum production, we can argue that more wells should be added to the production system if the price of oil is high. In this case, the optimal strategy to the firm is to exercise their option to expand the production if there is available capacity.

2.2.1 Dynamic Programming vs. Contingent Claims Analysis

Sequential investment under uncertainty is generally dealt with using one of two approaches; dynamic programming or contingent claims analysis. Dynamic programming is based on the mathematical theory of sequential decisions, while contingent claims analysis is inspired by option valuation methods from financial markets. These two methods lead to identical results in many cases, but make different assumptions about financial markets and the investor's discount rates.

Contingent claims analysis value the investment project by finding an asset or a combination of assets in financial markets with similar, future returns. By doing this, the value of the investment project today will be similar to the value of the financial asset today, assuming perfect markets and a risk free discount rate. Dynamic programming, on the other hand, relax these assumptions and simply states that a whole sequence of decisions may be broken down into the immediate decision, and a value function of all future decisions. Thereafter, if the project has a finite end, the final decision is found using optimization methods. From this point, it is possible to work backwards to the initial decision to find the value of the project today. Thus, dynamic programming states that the project value may be found through the sum of the instantaneous profits from the project and the expected value of the project from all future periods. This concept is commonly known as the Bellman equation. (Dixit & Pindyck, 1994, pp. 93-100).

3. Literature Review

Dixit & Pindyck (1994) develop an analytical framework to general problems of investment under uncertainty. They draw the analogy with the theory of options in financial markets to valuation of investments in real assets that involve irreversible expenditures and uncertain future payoffs depending on one or more stochastic underlying variables. The authors challenge the standard neoclassical investment models such as the NPV-rule, as irreversibility and the possibility of delay are very important characteristics of most investments in reality. They outline and explain two approaches to dynamic optimization under uncertainty, dynamic programming and contingent claims analysis, and how they relate and differ.

McDonald & Siegel (1986) examine real options related to an investment project where the project value and the final investment cost both are dependent on stochastic variables. The variables follow a stochastic geometric Brownian motion (GBM) process, and the authors derive an optimal investment rule and a formula for valuation of the project. Their main findings are that the option to wait increases project value, and that it is optimal to postpone the final investment until the project value is twice the size of the investment cost.

Bjerksund & Ekern (1990) allow the output price to follow a continuous stochastic process, and thereby let the value of the investment project to be a function of the output price. They examine a productive investment opportunity that is irrevocable once undertaken. The fundamental source of uncertainty is the output price, which in turn is governed by a GBM process. The paper outlines a number of models with increasing degree of flexibility - both traditional accept/reject-models but also models incorporating the possibility of delaying the investment decision. The latter allows for optimal "wait and see" strategies, which in turn show significant additional value to the investment opportunity. Key takeaways are that break-even prices and values from the traditional NPV-analysis have to be reconsidered, as price uncertainty and decision flexibility from real-world settings are not accounted for in such analysis. Smith & McCardle (1999), however, examine option-pricing methods in relation to the traditional NPV methods, and their findings reveal an interesting conclusion. The two approaches are both equally capable of modelling flexibility. Although this flexibility is usually overlooked when setting up a decision problem using NPV methods, the authors argue that the two approaches should be viewed and used as complementary modelling approaches, and may even be integrated.

Cortazar & Schwartz (1997) present a model to value an undeveloped oil field, using contingent claim analysis. Unlike Bjerksund & Ekern (1990), Cortazar & Schwartz assume an output price that follows a mixed GBM and mean-reversion (MR) process, where a MR process assumes that the price reverts towards a historical mean. The solution identifies a critical spot price that triggers development of the field. They find that the option to delay investment accounts for a significant portion of the oil field`s total value. Brennan & Schwartz (1985), on the other hand, evaluate the production phase of an investment project illustrated by a copper-mine case. The model shows how assets with uncertain cash flows may be valued and how optimal decisions accounts for abandonment, temporary shutdowns and delayed investments. Moreover, it considers variation in risk and discount rate, and the benefits of storing the real asset, also known as net convenience yield.

Lund (1997) develops a framework for evaluation of offshore oilfield projects by modelling the output price with both GBM- and MR processes. The main idea is to value the flexibility in the projects, as the projects faces uncertainty in oil prices, costs and the properties of the oil field. Lund defines four project phases; exploration, conceptual study, engineering- and construction, and production. He also emphasizes the fact that the project owner is entitled to several sources of flexibility in all phases, such as timing the investment and switching the intensity levels in production. Stochastic dynamic programming resolves the optimization problem where the reservoir volume, the production capacity and the oil price are stochastic variables modelled by both GMB- and MR processes. Thus, Lund includes several sources of uncertainty and focuses on the interrelations between these sources. By doing this, Lund illustrates the lack of detail in many commonly used valuation models for oil development projects, and he concludes by explaining how flexibility is an important element of the value of an oil development project.

Miltersen & Schwartz (2007) develop an advanced valuation framework for investment projects using real options theory and the dynamic-programming approach. They consider a situation where the owner of the projects pays a certain amount of on-going exploration costs per unit of time until completion of the project. The time to completion is governed by an independent exponential distributed random variable, as there is uncertainty to *when* the exploration oilfield or the pharmacy product is found or completed. Thus, the value of the project will be the solution to an ordinary differential equation, which makes it possible to obtain closed form solutions to the problem. There is also uncertainty about the value of the outcome, which follows a geometric Brownian motion process. Miltersen & Schwartz add real

options to the analysis, such as abandonment at and before the exploration is completed, and the option to delay the final development cost. These features, among others, make the model highly relevant for oilfield-development projects.

Inspired by Miltersen & Schwartz (2007), we develop a framework for evaluating an oil exploration project. However, we include a compensation from selling the project at abandonment. Furthermore, to account for oil price volatility and its strong effect on the profitability of oil development projects, we let the project value be a function of the oil price. Unlike Brennan & Schwartz (1985), among others, but similar to Miltersen & Schwartz (2007), we evaluate the project in the exploration stage and not in the production stage. As Miltersen & Schwartz, we also extend the framework by including the option to delay the final development cost.

4. Assumptions and Notations

We assume that the owner has a monopoly right to the exploration project, and holds only one exploration project at a time. The expected time to completion, i.e. the time period of exploration until the last exploration well is drilled, follows a Poisson process. If developed, the project is assumed to have an expected finite lifetime similar to real oilfield projects. In the continuation of the thesis, we define a developed project as an active project. Furthermore, we assume that the expected lifetime of the active project also follows a Poisson process.

A Poisson process is a process subject to "events", for which the arrival time of such events follows a Poisson distribution (Dixit & Pindyck, 1994, p. 85). A Poisson distribution expresses the probability of a certain number of events occurring in a given period of time. In our framework, at any time T, there is therefore a probability $\lambda_n dT$ that the exploration is completed, or that the active project will end, is during the next short interval of time dT. We let the Poisson death parameters, λ_1 and λ_2, define the mean arrival intensity rate for the exploration to be completed and the active project to end, respectively. We assume that $\lambda_n > 0$, where $n = 1,2$, due to the discussion regarding seismic data in the introduction and a finite lifetime of the active project. Formally, we regard the exploration and the active project as infinite-lived, so dT becomes dt. The reason for this is that future profits from exploration and the active project are discounted by a rate that includes the Poisson death parameter, λ_n (Dixit & Pindyck, 1994, p. 200). We let q denote the Poisson process, and have that

$$dq = \begin{cases} 1 & \text{with probability } \lambda_n dt, \\ 0 & \text{with probability } 1 - \lambda_n dt. \end{cases}$$

The notations and interpretations of the Poisson death parameters are summarized in Table 1.

State	Exploration	Active project
Notation	λ_1	λ_2
Interpretation	Completion	Ending

Table 1 – Overview of notations for Poisson death parameters

Furthermore, we have that $\lambda_n = \frac{1}{T_n}$. Thus, the parameters $T_n = \frac{1}{\lambda_n}$, where $n = 1,2$, which give the expected years to completion of the exploration stage, and the end of the active project, respectively.

The Poisson process is assumed to be independent of the value process for the active project. This feature states that the Poisson death parameters should not be affected by changes in the value of the active project, thus the probability of the active project ending is not larger when the project value is high, vice versa.

The output, a barrel of extracted oil, is assumed sold at a price, *P,* where *P* is observable at any time *t* by the owner of the project. The dynamics of the output price, and thus the value of the active project, is assumed to follow a geometric Brownian motion as defined in equation (1). This is a continuous time stochastic process with three important properties; (i) only current value is useful for forecasting the future path (Markov property), (ii) independent increments, and (iii) changes over any finite time interval are normally distributed. The term *geometric* Brownian motion means that the drift and variance coefficients are functions of the current state and time (Dixit & Pindyck, 1994, p. 71). The dynamics of *P* are given by

$$(1) \quad dP_t = P_t \mu dt + P_t \sigma dW_t,$$

where σ is the instantaneous volatility of the price process, μ is its instantaneous drift parameter and W represent the increment of the standard Brownian motion (Wiener process).

We assume that it is impossible to create a portfolio that perfectly replicates the cash flow of a single oil investment project. Hence, we apply the dynamic-programming approach in order to evaluate the exploration project, according to the discussion in sub-section 2.2.1. If oil futures were used to replicate the value of the project, it would only allow for price-risk hedging, thus excluding project specific risks such as production rates and capacity. The strategy of investing in publicity traded companies, on the other hand, would also be insufficient as these companies hold a portfolio of numerous projects varying in style and dimensions. Thus, it is unlikely that the securities of such companies are highly correlated with only a single (real) investment project. Hence, the properties of dynamic programming seem more realistic to our case. Consequently, we will use a subjective discount rate and not a risk-free rate in our framework. The subjective discount rate, ρ, is assumed to be constant and strictly larger than the instantaneous drift, μ, to avoid infinite values of the project.

Otherwise, waiting longer would always be a better policy, and thus the optimum would not exist.

One important feature of our framework is that the expected time to completion and the expected end of the active project does not depend on calendar time. Thus, at any given time, *t,* the value of the project in the exploration stage depends only on the value of the active project, and not on the calendar date *t* itself. This will greatly simplify the analysis and allow for closed form solutions to many important and interesting cases.

The framework is developed in two steps, which we denote Model 1 and Model 2. In the first model, the owner of the exploration project must, at completion date, consider the net present value of the active project, denoted $V(P)$, and make an instant choice to invest or not. In Model 2 we extend the first model by letting the owner, at completion date, obtain a perpetual American call option, denoted $F(P)$, on the value of the active project. This option allows the owner to delay the final investment. In both models, the owner has the option to abandon the project at any time until and at completion of the exploration stage. The value of the project in the exploration stage is denoted $\Phi^{(m)}(P)$. To distinguish, we use superscript $m = 1$ for Model 1, and $m = 2$ for Model 2. The dynamics of the models are summarized in the flow chart below.
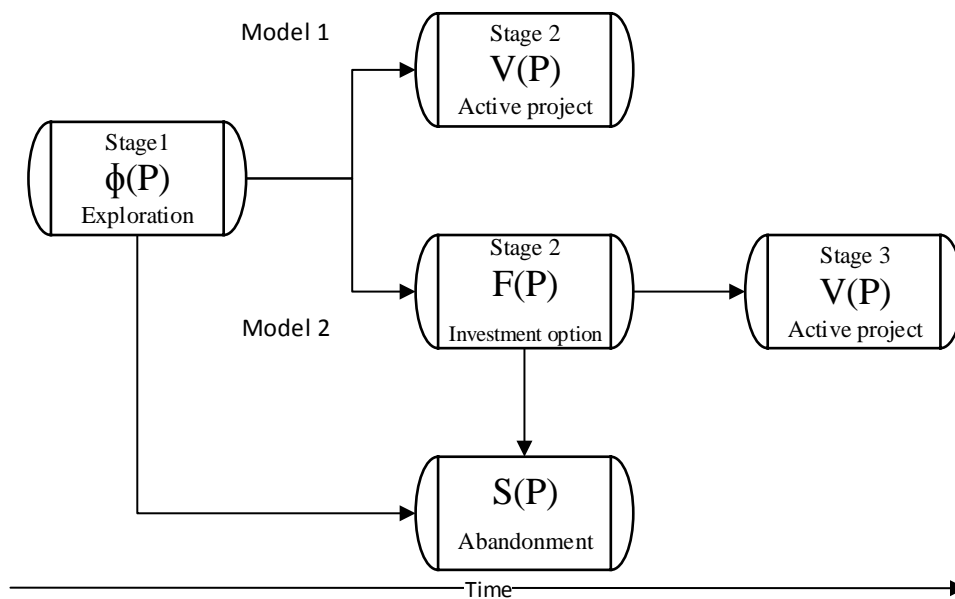
Figure 8 - Flow chart overviewing the dynamics of Model 1 and Model 2

$S(P)$ denotes the value of selling the exploration project at abandonment, as opposed to making the final investment and developing the oil field. We assume that the option to sell the projects always exist, and that the project is sold as a whole. Furthermore, we assume that the value from selling the project is received *instantly* at abandonment, as a one-time cash payment dependent on the oil price at the moment of abandonment. Thus it is not uncertain and not discounted, as opposed to the value of the active project, $V(P)$. If abandoned, the project is sold as an oil reserve that is not yet developed. Such a reserve is defined as a proved and probable reserve. The exogenous parameter, s, describes a constant value ratio of a proved and probable oil barrel to a barrel of extracted oil sold in the market.

The value of the active project, $V(P)$, fluctuates with the dynamics of the oil price, P, but it also depends on the expected average annual production of oil barrels, Q_1, which is correlated to the total expected amount of oil barrels in the field, Q_2. We assume that Q_2 and Q_1 are constants, and that they are observable at all times. Hence, the value of the active project is observable at any time to the owner. To simplify, we assume that the oilfield contains only oil, and not other petroleum resources. The final fixed investment cost is denoted K, and includes the discounted production costs as well. On-going exploration costs are denoted, k, and, hence, there will be abandonment threshold levels. These threshold levels are denoted $\underline{p}_m$. In Model 1, the owner of the exploration project invests when $NPV \geq 0$. This happens at an output price, $\overline{p}_1$. In Model 2, however, there will be an optimal investment threshold, $\overline{p}_2$, where the timing feature of the option to delay is accounted for.

All notations mentioned in this section are summarized in Table 2.

Overview of Notations	
Value functions	
The value of the exploration project, $m = 1,2$	$\Phi^{(m)}(P)$
The value of the investment option	$F(P)$
The value of selling the project	$S(P)$
The value of the active project	$V(P)$
Threshold levels	
Abandonment thresholds, $m = 1,2$	$\underline{p}_m$
NPV threshold, Model 1	$\overline{p}_1$
Investment threshold, Model 2	$\overline{p}_2$
Parameters for the dynamics of the price process	
Instantaneous drift	μ
Instantaneous volatility	σ

Parameters for the project	
Poisson death parameter, exploration stage	λ_1
Expected time to completion	T_1
Poisson death parameter, active project	λ_2
Expected lifetime of the active project	T_2
Expected total amount of oil	Q_2
Expected annual production of oil	Q_1
On-going exploration costs	k
Final fixed investment cost	K
Other parameters	
Subjective discount rate	ρ
Value ratio of proved and probable oil reserves	s

Table 2 – Overview of Notations

5. Analytical Formulations

In this section we present the analytical formulations of our framework. First we introduce Model 1 where the owner has the option to abandon and sell the project, but where the option to delay is not included. In Model 2, we extend Model 1 by including the option to delay the final investment when the exploration is completed.

5.1 Model 1: Investment Without the Option to Delay

We look at a situation where the owner incurs on-going exploration costs, k, while exploring for oil in an undeveloped oilfield. When the exploration is completed, there is an opportunity to pay the final investment cost, K, to obtain the value of the active project. If investing is not profitable at this point due to a negative NPV, the owner has the option to abandon the project. The project may also be abandoned at any time until the exploration is completed, as shown in Figure 6 below.
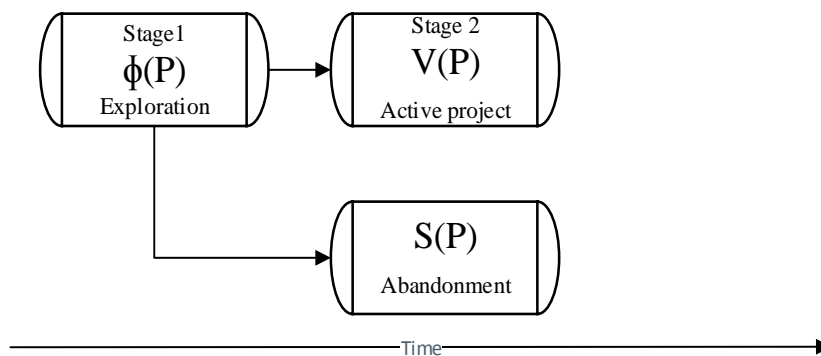
Figure 9 - Flow chart of Model 1

Thus, there is a trade-off in the exploration stage between the expected value of the active project, and the savings of future on-going exploration costs, k. In other words, the yearly spending of exploration costs, k, may be avoided if the expected benefits from proceeding are too small. Should this be the case, it will be optimal to abandon the exploration project. This happens when the oil price drops below the abandonment threshold level, $\underline{p}_1$. Should the project be abandoned, the owner obtains a compensation given by $S(P)$.

5.1.1 The Value of the Active Project

To find the value of the exploration project (stage 1), which incorporates the value of exploration with the potential value of the active project, we work backwards and start by finding the expected value of the active project (stage 2). Using dynamic programming methodology, the value of the active project is given by the Bellman equation

(2) $V(P) = PQ_1 dt - \rho K dt + (1 - \lambda_2 dt)(1 - \rho dt)E_P[V(P + dP)]$

The first two term on the right hand side of the equation are the instantaneous profits within the time interval, dt. The profits are simply given by the oil price, P, multiplied by the amount of oil barrels extracted per unit of time, Q_1. As this profit is earned instantaneously, it is not affected by the Poisson death parameter, λ_2, which gives us the possibility of the active project to end *after* the short time interval, dt. This gives us the intuition for the next term, which describes the expected profit after dt. As the profits from future time intervals, $E_P[V(P + dP)]$, are only earned if the active project is still alive, they are multiplied by the probability for the project to stay alive, given by $(1 - \lambda_2 dt)$. This probability is the inverse of the probability for the project to die in the next time interval. The term $(1 - \rho dt)$ denotes the continuous discounting of the future profits from the active project. Rearranging the equation and expanding using Ito's Lemma we get the ordinary differential equation (ODE) in (3). More detailed derivations are shown in Appendix A.

(3) $\frac{1}{2}\sigma^2 P^2 V''(P) + \mu P V'(P) - (\rho + \lambda_2)V(P) + PQ_1 - K = 0$

This ODE determines the value of the active project. The two first terms considers the uncertain development in the oil price, P. The third term illustrates the drop in future expected benefits from ending the project, while the last two terms are the net benefits from the project. The general solution to the ODE, with its transformations and eliminations, allows us to manipulate the equation with the result shown in equation (4), when we solve for $V(P)$. Further details are shown in Appendix A.

(4) $V(P) = \frac{PQ_1}{(\rho - \mu + \lambda_2)} - K$

The first term on the right hand side of the equation illustrates the present value of the earnings from the project. These are discounted by the subjective discount rate less the oil price drift parameter, ρ-μ, and the probability, λ_2, of the project ending in the next short time interval. As a result, the value of the active project will increase if the probability of it ending in the

next short time period is reduced. The project value will also increase with a lower discount rate and a larger positive drift in the oil price. These properties seem intuitive.

5.1.2 The Value of Selling the Project

If the owner of the project finds abandonment to be the optimal action, he or she will be compensated with the value of the undeveloped oil field, given by

(5) $S(P) = sPQ_2$.

This value is a function of the value ratio of a proved and probable oil barrel to a barrel of extracted oil sold in the market, s, the oil price, P, at the moment of abandonment and the total amount of oil barrels in the field, Q_2. The intuition behind this expression is that the value of a proved and probable oil field will depend on its size, the oil price, and a factor to adjust the value due to the fact that the field is not yet developed. As the owner of the project receives this value when he or she finds it optimal to abandon the project, it may be interpreted as a compensation from abandonment. Consequently, changes in the compensation from selling the project should affect the threshold level at which the project is abandoned.

5.1.3 The Value of the Exploration Project

We continue going backwards to find the value of the exploration project. The value of the project in this stage, and at completion, is governed by the abandonment threshold level, $\underline{p_1}$. Should the oil price drop below this level during the exploration or at completion, then the project will be abandoned and the owner will receive the compensation of $S(P)$ by selling the project. If, however, the oil price stays above this level during the exploration, the project is kept alive and the project value will include the potential value of the active project and the value of continued exploration. Thus the value of the project in the exploration stage, $\Phi^{(1)}(P)$, is given by the following equation.

(6) $\Phi^{(1)}(P) =$
$$\begin{cases} (1-\rho dt)\lambda_1 dt \mathrm{E}_P[V(P+dP)] + (1-\rho dt)(1-\lambda_1 dt)\mathrm{E}_P[\Phi^{(1)}(P+dP)] - kdt & \text{when } P \geq \underline{p_1} \\ S(P) & \text{when } P < \underline{p_1} \end{cases}$$

The top line in equation (6) is the Bellman equation and describes the value of the project when it is not abandoned. As the owner does not have any instantaneous profits in the exploration stage, this Bellman equation will only consist of expected future profits. In this case, the owner gets the value of the active project, by the probability of completing the

exploration, given by λ_1. Note again that we separate between the probability of completing the exploration in the next time interval, λ_1, and the probability of ending the active project in the next time interval, λ_2. Thus the first term in the top line of equation (6) describes the expected value from completing the exploration and obtaining the value of the active project. Similarly, the second term gives the value of the exploration project when it is not yet completed and the owner continues with the exploration. Hence, the value of exploration is multiplied by the probability of the project not being completed, $(1 - \lambda_1 dt)$. The last term, kdt, describes the on-going exploration cost per short unit of time, dt. We expand the right hand side of the top line in equation (6) using Ito's Lemma, and obtain the ODE

(7) $\frac{1}{2}\sigma^2 P^2 \Phi^{(1)''}(P) + \mu P \Phi^{(1)'}(P) - (\rho + \lambda_1)\Phi^{(1)}(P) + \lambda_1 V(P) - k = 0.$

More detailed derivations are found in Appendix A. Again the two first terms considers the uncertain development in the oil price, P, while the third term illustrates the drop in future expected benefits from completing the exploration stage. The value of the fourth term depends on the level of the oil price. The oil price that gives a NPV-value of the active project of zero, is $\overline{p}_1 = \frac{K(\rho - \mu + \lambda_2)}{Q_1}$. Should the oil price be above this level at completion, then the active project is in the money and thus accepted for investment. In other words, the term $\lambda_1 V(P)$ in equation (7) is only relevant if the oil price is above the NPV threshold level, $\overline{p}_1$. If the oil price is in the interval between the abandonment threshold level, $\underline{p}_1$ and the NPV threshold level, $\overline{p}_1$, during the exploration stage, the owner of the project will continue exploring for oil. Then the term, $\lambda_1 V(P)$, is not relevant and consequently eliminated from the equation[1]. Thus we have two ODEs to determine the value of the exploration project when it is not abandoned. Together with the value function from abandonment, we get the following set of equation to determine the value of the exploration project

(8) $sPQ_2 - \Phi^{(1)}(P) = 0$ \hfill when $P < \underline{p}_1$

(9) $\frac{1}{2}\sigma^2 P^2 \Phi^{(1)''}(P) + \mu P \Phi^{(1)'}(P) - (\rho + \lambda_1)\Phi^{(1)}(P) - k = 0$ \hfill when $\underline{p}_1 \leq P < \overline{p}_1$,

(10) $\frac{1}{2}\sigma^2 P^2 \Phi^{(1)''}(P) + \mu P \Phi^{(1)'}(P) - (\rho + \lambda_1)\Phi^{(1)}(P)$
$$+ \lambda_1 \left(\frac{PQ_1}{(\rho - \mu + \lambda_2)} - K \right) - k = 0 \qquad \text{when } \overline{p}_1 \leq P.$$

---

[1] Should, however, the oil price be in this interval at completion, the project is abandoned and the project value drops to the certain cash payment, $S(P)$, that is obtained from selling the project.

Note that the model is applicable both during the exploration and at the completion date. Equation (8) gives the value of the project when $P < \underline{p}_1$ and the project is not worth the cost of keeping it alive. Equation (9) gives the value of the project when it is out of the money, but since $\underline{p}_1 \leq P$ it is not abandoned before completion. However, should this situation occur when the project is completed, the owner will choose to abandon the project. When $\overline{p}_1 \leq P$, the project is in the money at completion, and it is optimal to make the investment and pay the final investment cost K to receive the future benefits of the investment. In equation (10) this is illustrated by the term $\lambda_1 \left(\frac{PQ_1}{(\rho - \mu + \lambda_2)} - K \right)$, which tells us that the value of the exploration project increases by the expected net present value from making the investment. In other words, this term describes the expected value of the *active* project.

To obtain the value functions of the project, we utilize the standard solutions to the homogenous parts of the ODEs above and solve for $\Phi^{(1)}(P)$. Since we have that $\lim_{P \to \infty} \frac{\Phi_2^{(1)}(P)}{V(P)}$ must be less than one and that $y_1 > 1$ we are able to eliminate the term including the second constant, $B_1^{(1)}$ from equation (13). Using this, together with $\lambda_1 = \frac{1}{T_1}$, and $\lambda_2 = \frac{1}{T_2}$ we get the value of the exploration project as

$$(11) \quad \Phi_1^{(1)}(P) = sPQ_2 \qquad\qquad \text{when } P < \underline{p}_1$$

$$(12) \quad \Phi_2^{(1)}(P) = A_1^{(1)}P^{y_1} + A_2^{(1)}P^{y_2} - \frac{kT_1}{1 + \rho T_1} \qquad\qquad \text{when } \underline{p}_1 \leq P < \overline{p}_1$$

$$(13) \quad \Phi_3^{(1)}(P) = B_2^{(1)}P^{y_2} + \frac{PQ_1}{(1 + (\rho - \mu)T_1)(\rho - \mu + \frac{1}{T_2})} - \frac{kT_1 + K}{1 + \rho T_1} \qquad\qquad \text{when } \overline{p}_1 \leq P.$$

Where the powers are the roots of $\frac{1}{2}\sigma^2 y(y-1) + \mu y - (\rho + \lambda_1)$ and are given by

$$(14) \quad y_1 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2(\rho + \lambda_1)\sigma^2}}{\sigma^2} > 1$$

and

$$(15) \quad y_2 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2(\rho + \lambda_1)\sigma^2}}{\sigma^2} < 0.$$

Equation (11) simply states that the value of the project, when the oil price is lower than the abandonment threshold, $\underline{p}_1$, is given by sPQ_2 as the project is abandoned and sold. Equation (12) gives the value of the exploration project when it is kept alive and the active project is

out of the money. The term $A_1^{(1)} P^{y_1}$ in equation (12) includes the effect from the possibility of the oil price to increase and thus the project value, $\Phi_2^{(1)}(P)$, to change to $\Phi_3^{(1)}(P)$. This effect is also found in an opposite manner in the term $A_2^{(1)} P^{y_2}$ which gives the effect from the possibility for the value of the project to change to $\Phi_1^{(1)}(P)$. The last term in equation (12) is the discounted on-going exploration costs. Equation (13) gives the value of the exploration project when the active project is in the money. The term $B_2^{(1)} P^{y_2}$ may be interpreted as the effect from the possibility for the price to drop below $\overline{p}_1$ and for the value of the exploration project to change to $\Phi_2^{(1)}(P)$. This also provides some intuition to why the constant $B_1^{(1)}$ is assumed to be zero, as there is no higher price interval with a corresponding value function in this set of equations. The second term in equation (13) illustrates the present value from the oil produced from the active field, while the last term is the discounted costs of exploration and final development of the active project. The price intervals with corresponding value functions and optimal actions are summarized in Table 3.

Oil Price interval	$P < \underline{p}_1$	$\underline{p}_1 \leq P < \overline{p}_1$	$\overline{p}_1 \leq P$
Project value	$\Phi_1^{(1)}(P)$	$\Phi_2^{(1)}(P)$	$\Phi_3^{(1)}(P)$
Optimal action	Abandon	Keep exploring	Invest

Table 3 – Overview of threshold levels and optimal actions in Model 1.

We solve the equations (11), (12) and (13) for given boundary conditions, which tie the general solutions into one differentiable and continuous value function for the exploration project $\Phi^{(1)}(P)$, contingent on the value of P. The four boundary conditions are

$$(16) \quad \Phi_1^{(1)}\left(\underline{p}_1\right) = \Phi_2^{(1)}\left(\underline{p}_1\right),$$

$$(17) \quad \Phi_1^{(1)'}\left(\underline{p}_1\right) = \Phi_2^{(1)'}\left(\underline{p}_1\right),$$

$$(18) \quad \Phi_2^{(1)}\left(\overline{p}_1\right) = \Phi_3^{(1)}\left(\overline{p}_1\right),$$

$$(19) \quad \Phi_2^{(1)'}\left(\overline{p}_1\right) = \Phi_3^{(1)'}\left(\overline{p}_1\right)$$

Condition (16) and (18) are the value matching conditions. Condition (16) reflect that the value of selling the oilfield and the value of keeping it alive but not investing yet, should be equal when the oil price is at the abandonment threshold level, $\underline{p}_{(1)}$. Similarly condition (18) ensures

that the value of the project when it is in the money, $\Phi_3^{(1)}$, is equal to the value of the project when it is kept alive but not invested in, at the break-even point where $P = \overline{p}_{(1)}$. The smooth pasting conditions (17) and (19) ensure that the value functions have identical slopes when they intersect. Thus we get a continuous and differentiable value function for the exploration project. We solve this set of equations for the four unknowns $A_1^{(1)}, A_2^{(1)}, B_2^{(1)}$ and $\underline{p}_1$. The derivations of these constant are found in Appendix A, with the results given by

$$(20) \quad A_1^{(1)} = \frac{(1-y_1)Q_1\overline{p}_1^{1-y_1}}{(y_1-y_2)(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{y_1\overline{p}_1^{-y_1}K}{(y_1-y_2)(1+\rho T_1)} + \frac{Q_1\overline{p}_1^{1-y_1}}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{\overline{p}_1^{-y_1}K}{1+\rho T_1}$$

$$(21) \quad A_2^{(1)} = \frac{sQ_2}{y_2\underline{p}_1^{y_2-1}} - \frac{y_1A_1^{(1)}p_{(1)}^{y_1-y_2}}{y_2}$$

$$(22) \quad B_2^{(1)} = A_2^{(1)} + \frac{(1-y_1)Q_1\overline{p}_1^{1-y_2}}{(y_1-y_2)(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{y_1\overline{p}_1^{-y_2}K}{(y_1-y_2)(1+\rho T_1)}$$

The equation for the abandonment threshold level, $\underline{p}_1$, is solved numerically using the closed form solutions of the set of equations above. The remaining boundary condition used to find $\underline{p}_1$ is,

$$(23) \quad \Phi_1^{(1)}\left(\underline{p}_1\right) = \Phi_2^{(1)}\left(\underline{p}_1\right).$$

The value of the project can be summed up as

$$(24) \quad \Phi^{(1)} = \begin{cases} sPQ_2 & \text{when } P < \underline{p}_1 \\ A_1^{(1)}P^{y_1} + A_2^{(1)}P^{y_2} - \frac{kT_1}{1+\rho T_1} & \text{when } \underline{p}_1 \leq P < \overline{p}_1 \\ B_2^{(1)}P^{y_2} + \frac{PQ_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{kT_1+K}{1+\rho T_1} & \text{when } \overline{p}_1 \leq P. \end{cases}$$

5.2 Model 2: Investment With the Option to Delay

In this section we extend Model 1 by letting the owner delay the final investment decision when the exploration is completed. This implies a timing feature, which in most cases is present in the real world. In Model 1, we assumed that the owner must abandon the project at completion if it is not in the money, or keep on exploring if the exploration is not yet completed. Now, he gets a perpetual American call option on the value of the active project at completion, and thus the option to delay paying the final investment cost. This is illustrated in stage 2 in Figure 10. Considering the fluctuations in oil prices, as well as the cost level, such an option could potentially be of significant value to the investor.
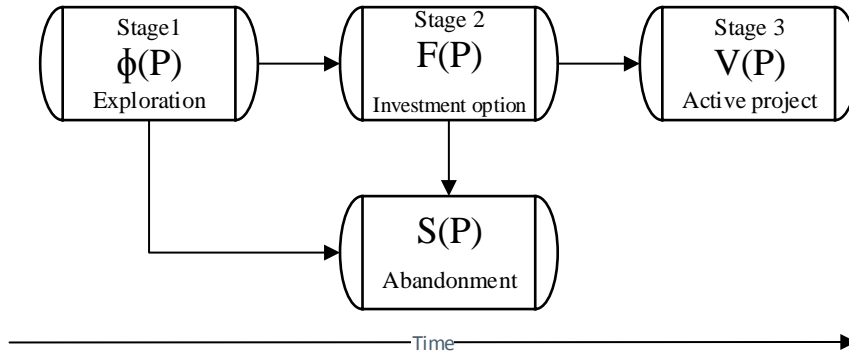
Figure 10 - Flow chart of Model 2

The value of the perpetual American call option is denoted $F(P)$. To separate this model from the previous one, we denote the value of the exploration project in Model 2, as $\Phi^{(2)}$. There will still be an abandonment threshold level, as the owner still has to pay on-going exploration costs. The threshold level is denoted $\underline{p_2}$, and the owner abandons the project for oil prices below this level. Comparing it to the abandonment threshold level in the first model, we can state that $\underline{p_2}$, will always be lower than $\underline{p_1}$ due to the fact that $F(P)$ is always greater than (or equal to) the value of the active project. Furthermore, we have an optimal investment threshold level, $\overline{p}_2$. This threshold level gives the value of the oil price when it is optimal to exercise the investment option and invest in the active project.

To find the value of the exploration project, $\Phi^{(2)}$, we again work backwards from the value of the active project, $V(P)$, now in stage 3. The value of $V(P)$ is, as before, given by equation (25). Further details and derivations are found in the sub-section 5.1.1 and in Appendix A.

(25) $\quad V(P) = \frac{PQ_1}{(\rho - \mu + \lambda_2)} - K$

The value of abandoning and selling the project is also unchanged and given by equation (5) in sub-section 5.1.2.

5.2.1 The Value of the Option to Delay

From the value of the active project, we continue backwards to the new feature of the model, the Amercian perpetual call option. The value of this option is given by

(26) $\quad F(P) = \begin{cases} C_1 P^{x_1} & \text{when } P < \overline{p}_2, \\ \dfrac{PQ_1}{(\rho-\mu+\lambda_2)} - K & \text{when } \overline{p}_2 \leq P. \end{cases}$

For oil prices equal to the optimal investment threshold level, $\overline{p}_2$, the owner of the project is indifferent between holding the value of the investment option, $C_1 P^{x_1}$, and the value of the active project. Thus he will invest in the active project when the oil price reaches this level. Consequently the value of the investment option, $F(P)$, is equal to the value of the active project for oil prices higher than $\overline{p}_2$. By using the boundary conditions for value matching and smooth pasting at the exercise threshold level, we obtain the optimal investment threshold level, $\overline{p}_2$ and the constant, C_1 as

(27) $\quad \overline{p}_2 = \dfrac{x_1}{(x_1-1)} \dfrac{K(\rho-\mu+\lambda_2)}{Q_1}$

and

(28) $\quad C_1 = \dfrac{\overline{p}_2^{1-x_1} Q}{x_1(\rho-\mu+\lambda_2)}.$

The power, x_1, is given by

(29) $\quad x_1 = \dfrac{\left(\frac{1}{2}\sigma^2-\mu\right)+\sqrt{(\mu-\frac{1}{2}\sigma^2)^2+2\rho\sigma^2}}{\sigma^2} > 1.$

The derivations of these constant are found in Appendix B and C.

5.2.2 The Value of the Exploration Project

By continuing going backwards, we now find the value of the exploration project, $\Phi^{(2)}(P)$, from the value of the call option, the value of the active project, and the value of abandoning the project. Hence, the Bellman equation becomes

(30) $\quad \Phi^{(2)}(P) =$
$\begin{cases} (1-\rho dt)\lambda_1 dt E_P[F(P+dP)] + (1-\rho dt)(1-\lambda_1 dt)E_P[\Phi^{(2)}(P+dP)] - kdt & \text{when } P \geq \underline{p}_2 \\ S(P) & \text{when } P < \underline{p}_2 \end{cases}$

The first term in the top line expression describes the value of the project when it is not abandoned as the oil price stays above the abandonment threshold level $\underline{p}_2$. Intuitively, the value is given by the probability of completion in the next time interval, λ_1, multiplied by the value from completing the project which is the expected present value of the American call option $F(P)$. Thus the next term is the probability of not completing the project in the next

time interval, $(1 - \lambda_1 dt)$, multiplied by the value of continuing with the exploration. As before, we see that the owner of the project obtains the value of $S(P)$ at abandonment. We expand the second term in the top line of equation (30) using Ito's Lemma, and obtain the ODE in (31). More detailed derivations are found in Appendix B.

$$(31) \quad \frac{1}{2}\sigma^2 P^2 \Phi^{(2)''}(P) + \mu P \Phi^{(2)'}(P) - (\rho + \lambda_1)\Phi^{(2)}(P) + \lambda_1 F(P) - k = 0.$$

All derivations are found in Appendix B. As in Model 1, the third term in this equation indicates the drop in the project value from completing the exploration stage. However, from the fourth term, we now see that the owner obtains the value of the American call option $F(P)$, with the probability λ_1. Thus, if we have that the oil price is below the investment threshold level at completion, $P < \overline{p}_2$, he or she will now delay the final investment as opposed to abandoning the project. By combining the ODE equation above with the possible values of $F(P)$ given by equation (26), we now have three equations to determine the value of the exploration project

$$(32) \quad sPQ_2 - \Phi^{(2)}(P) = 0 \qquad\qquad\qquad\qquad \text{when } P < \underline{p}_2$$

$$(33) \quad \frac{1}{2}\sigma^2 P^2 \Phi^{(2)''}(P) + \mu P \Phi^{(2)'}(P) - (\rho + \lambda_1)\Phi^{(2)}(P)$$
$$+ \lambda_1 C_1 P^{x_1} - k = 0 \qquad \text{when } \underline{p}_2 \leq P < \overline{p}_2$$

$$(34) \quad \frac{1}{2}\sigma^2 P^2 \Phi^{(2)''}(P) + \mu P \Phi^{(2)'}(P) - (\rho + \lambda_1)\Phi^{(2)}(P)$$
$$+ \lambda_1 \left(\frac{PQ_1}{(\rho - \mu + \lambda_2)} - K \right) - k = 0 \qquad \text{when } \overline{p}_2 \leq P.$$

Equation (32) gives the value when $P < \underline{p}_2$, i.e. the exploration project is not worth keeping alive. Equation (33) describes the situation when the oil price is below the optimal exercise threshold level, $\overline{p}_2$, but above the abandonment threshold level, $\underline{p}_2$. Because of the timing feature, the owner will now obtain an American perpetual call option on the value of the active project, given by the term $\lambda_1 C_1 P^{x_1}$, and can delay to pay the final investment cost in hope of e.g. improved market conditions. Again, the parameter λ_1 denotes the probability for completing the project and obtaining the value of the investment option. In equation (34), the price is above the optimal threshold level, $\overline{p}_2$, resulting in an immediate execution of the option to invest when the exploration is completed. In other words, the value of the active project is obtained with probability λ_1. The owner of the project is entitled to sell the project at any time and at any values of the oil price, but this is only *optimal* when the oil price drops below the abandonment threshold level.

Using $\lambda_1 = \frac{1}{T_1}$, and $\lambda_2 = \frac{1}{T_2}$, and the standard solutions to the ODEs above, we have three equations to determine the value of the project when solving for $\Phi^{(2)}(P)$. Note that the constant $B_1^{(2)}$ is eliminated for the same reasoning as in Model 1.

(35) $\quad \Phi_1^{(2)}(P) = sPQ_2$ $\qquad\qquad$ when $P < \underline{p}_2$

(36) $\quad \Phi_2^{(2)}(P) = A_1^{(2)}P^{y_1} + A_2^{(2)}P^{y_2} - \frac{kT_1}{1+\rho T_1} + \frac{C_1 P^{x_1}}{1+\left(r-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}$ $\qquad$ when $\underline{p}_2 \leq P < \overline{p}_2$

(37) $\quad \Phi_3^{(2)}(P) = B_2^{(2)}P^{y_2} + \frac{PQ_1}{(1+(r-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{kT_1+K}{1+\rho T_1}$ $\qquad$ when $\overline{p}_2 \leq P$.

Again, more detailed derivations are found in Appendix B. The interpretation of the terms in these equations are similar to the one presented in the Model 1 in sub-section 5.1.3. The new element is the last term in $\Phi_2^{(2)}(P)$, in equation (36), which denotes the value of the call option on the active project. This increases the value of the project in the intermediate price interval, and consequently in the two other price intervals through the value matching and smooth pasting conditions. Thus, it indicates a lower abandonment threshold level, $\underline{p}_2$, and a higher investment threshold level, $\overline{p}_2$, compared to $\underline{p}_1$ and the NPV threshold, $\overline{p}_1$, respectively. The project values and corresponding optimal choices are summarized in Table 4.

Oil Price interval	$P < \underline{p}_2$	$\underline{p}_2 \leq P < \overline{p}_2$	$\overline{p}_2 \leq P$
Project value	$\Phi_1^{(2)}(P)$	$\Phi_2^{(2)}(P)$	$\Phi_3^{(2)}(P)$
Optimal action	Abandon	Keep exploring/wait	Invest

Table 4 – Overview of threshold levels and optimal actions in Model 2.

Note that in the intermediate interval, the optimal action depends on whether the project is completed or not. If it is not completed, the owner of the project will decide to keep exploring. On the other hand, if the project is completed in this price interval, it is optimal to delay the final investment as the oil price is higher than the abandonment threshold, but lower than the investment threshold level.

The boundary conditions for the usual value matching and smooth pasting at the abandonment threshold level, $\underline{p}_2$, and continuity and differentiability at the optimal exercise threshold level for the American perpetual call option, $\overline{p}_2$, are set up as

$$(38) \quad \Phi_1^{(2)}\left(\underline{p}_2\right) = \Phi_2^{(2)}\left(\underline{p}_2\right),$$

$$(39) \quad \Phi_1^{(2)\prime}\left(\underline{p}_2\right) = \Phi_2^{(2)\prime}(\underline{p}_2),$$

$$(40) \quad \Phi_2^{(2)}(\overline{p}_2) = \Phi_3^{(2)}(\overline{p}_2),$$

$$(41) \quad \Phi_2^{(2)\prime}(\overline{p}_2) = \Phi_3^{(2)\prime}(\overline{p}_2),$$

allowing us to solve the coefficients $A_1^{(2)}, A_2^{(2)}$ and $B_2^{(2)}$ analytically. $\underline{p}_2$ is found numerically using the closed form solutions from the system of boundary conditions above and particularly the boundary condition in equation (38). The coefficients $A_1^{(2)}$, $A_2^{(2)}$ and $B_2^{(2)}$ are derived in Appendix B, and are given by

$$(42) \quad A_1^{(2)} = \frac{\frac{-(x_1-y_2)C_1}{\left(1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1\right)\overline{p}_2^{1-x_1}}+\frac{y_2K}{(1+rT_1)\overline{p}_2}+\frac{Q_1(1-y_2)}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})}}{(y_1-y_2)\overline{p}_2^{y_1-1}}$$

$$(43) \quad A_2^{(2)} = \frac{sQ_2-\left(y_1A_1^{(2)}\underline{p}_{(2)}^{y_1-1}+\frac{x_1C_1\underline{p}_{(2)}^{x_1-1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}\right)}{y_2\underline{p}_{(2)}^{y_2-1}}$$

and

$$(44) \quad B_2^{(2)} = A_2^{(2)} + \left(A_1^{(2)}\overline{p}_2^{y_1} + \frac{K}{1+\rho T_1} - \frac{\overline{p}_2Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{C_1\overline{p}_2^{x_1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}\right)\frac{1}{\overline{p}_2^{y_2}}$$

The value of the project in the exploration stage can be summarized as

$$(45) \quad \Phi^{(2)}(P) = \begin{cases} sPQ_2 & \text{when } P < \underline{p}_2, \\ A_1^{(2)}P^{y_1} + A_2^{(2)}P^{y_2} - \frac{kT_1}{1+rT_1} + \frac{C_1P^{x_1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1} & \text{when } \underline{p}_2 \le P < \overline{p}_2 \\ B_2^{(2)}P^{y_2} + \frac{PQ_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{kT_1+K}{1+\rho T_1} & \text{when } \overline{p}_2 \le P. \end{cases}$$

6. Case Study

In this part we construct a case to illustrate the framework numerically. The case is based on the undeveloped Knarr oilfield located in the North Sea, which has an expected production start in 2015. Most exogenous input values are obtained from Knarr`s Plan for Development and Operation (PDO). We will analyze the Knarr project using our framework and link the analysis to the NPV-calculations from the project's official PDO. We will also study how changes in important input parameters, such as the expected amount of oil discovered and the expected time to completion, affects the project value and its threshold levels. The PDO estimated a NPV for the project of NOK8.1 Billion (2010-kroners) with a corresponding break-even oil price of $47/bbl (Norwegian Ministry of Petroleum and Energy d, 2011).

6.1 Inputs

Expected time to completion of the exploration stage, T_1:
We regard the year of the first wildcat (exploration) well, which in the Knarr project was 2008, to be the start of the exploration project. To simplify, we set the expected completion date equal to the year when the last well was drilled, which was in December, 2011. Thus the exploration was conducted in the years 2008-2011. Therefore, we set the expected time to completion of the exploration project to $T_1 = 4$.

Expected lifetime of the active project, T_2:
The expected lifetime of the active project is set to six years based on the PDO of the project. The interpretation of this input parameter is the number of years in which the field produces oil.

Final fixed investment cost, K:
To make the costs in our framework comparable to the real expected NPV, we need to discount the costs for operations, production and disposal back to completion date, and add them to the final fixed investment cost. We use the same discount rate as in the original PDO which result in $K = \$2,364m$.

On-going investment costs, k:
In order to calculate k, we have estimated a total cost of exploration based on the number of exploration wells drilled and corresponding cost of exploration in the relevant time period.

Statistics from the NPD show that the average exploration cost per exploration well in the years 2008 to 2011 was about of $100m (Norwegian Petroleum Directorate b, 2013). As four exploration wells were drilled over four years during the exploration process of Knarr, we set the yearly on-going exploration cost to $100m.

The value ratio of proved and probable oil reserves, s:

To find s, we divide the price of a proved and probable oil barrel by the price of a barrel of extracted oil sold in the market. Using data from 2007, we estimate a value ratio of $s = 0,0625$ (Financial Times, 2008).

All input parameters are summarized in Table 5:

Inputs Base Case		
Parameters for the dynamics of the price process, P		
Instantaneous drift	μ	0,5% per year
Instantaneous volatility	σ	40% per year
Parameters for the exploration project		
Expected time to completion	T_1	4 years
Expected lifetime of the active project	T_2	6 years
Expected total amount of oil	Q_2	50.32m barrels
Expected annual production of oil	Q_1	8.39m barrels
On-going exploration costs	k	$ 100m per year
Final fixed investment cost	K	$2,364m
Other parameters		
Subjective discount rate	ρ	7% per year
Value ratio of proved and probable oil reserves	s	0.0625 per barrel

Table 5 – Input parameters used in the Base Case

6.2 Model 1: Investment Without the Option to Delay

First, we analyze the case using Model 1 where the project owner, at completion date, faces a decision to invest immediately or abandon the project based on the expected NPV of the active project. As outlined earlier, the owner can at any time abandon the project during the exploration stage in light of the abandonment threshold level.

Figure 11 illustrates this situation. The figure is composed by the oil price, P, on the x-axis with corresponding project values on the y-axis. The value of the active project is illustrated by the $V(P)$-line while the value of the exploration project is given by the $\Phi^{(1)}(P)$-line. Note
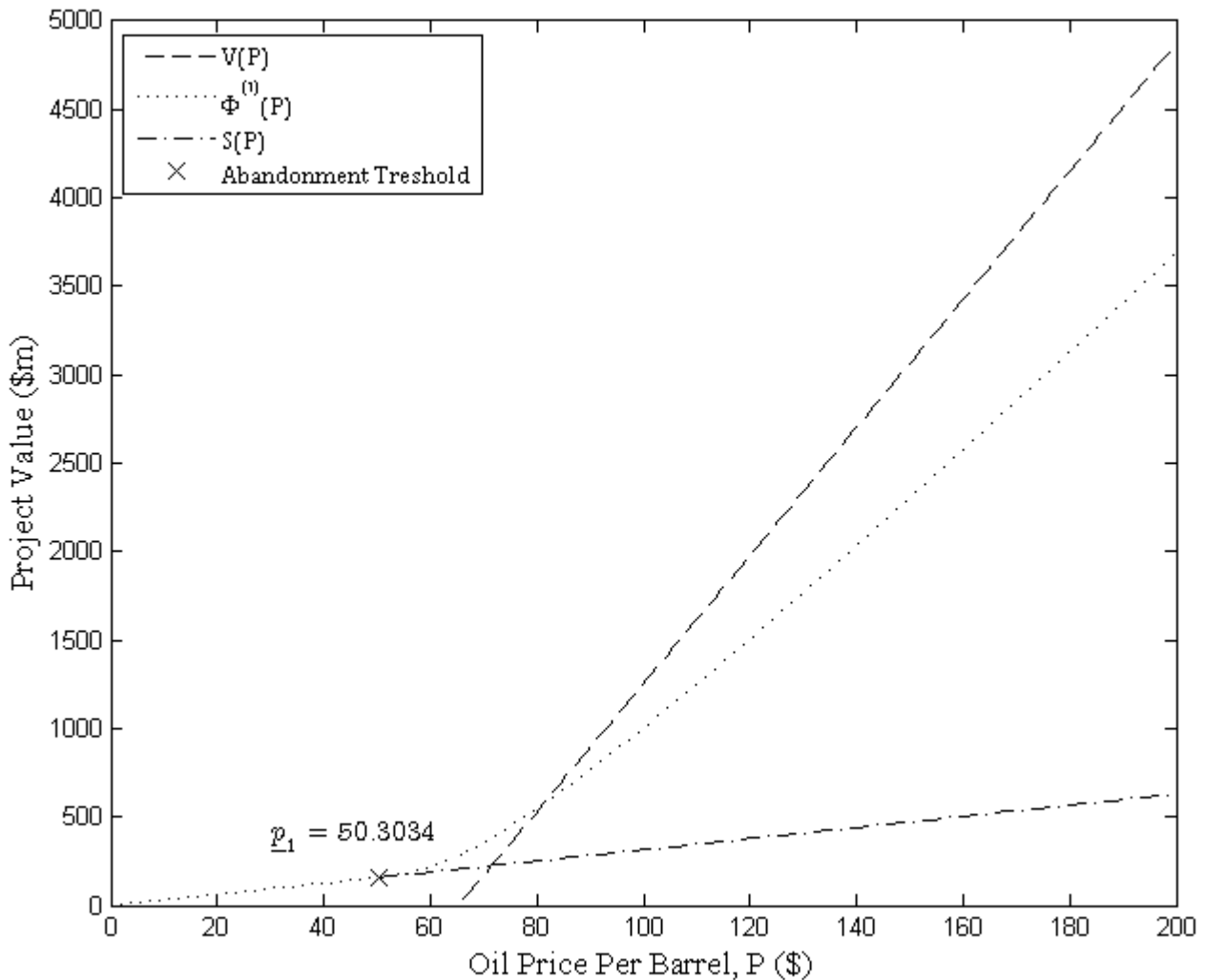
Figure 11 - Model 1 applied on the base case. The figure shows the value of the exploration project and the active project as a function of the oil price. It also shows that the exploration project is abandoned for oil prices below $50.30.

that the latter is positive for all oil prices above zero. The possibility of selling the project is technically available at all oil-price levels, which is illustrated by the wide range of the $S(P)$-line. However, this action is only optimal when the price drops below the threshold level. The break-even price for the active project, for which the final investment is made, is $65 per barrel of oil. This is illustrated by the $V(P)$-line intersecting with the x-axis at this point. Thus, the active project will return a positive NPV for any oil price above this level at completion date.

The cross in the figure indicates an optimal abandonment level at $\underline{p}_1 = 50.33$. Hence, if the oil price drops below this threshold level during the time of exploration, then the project should be abandoned immediately. This is illustrated by the $\Phi^{(1)}(P)$-line and the $S(P)$-line converging at this point, hence a convergence of the project value in the exploration stage and the value of selling the project. The reasoning behind the abandonment is that the oil price,

and thus the expected value of the active project, is too low compared to the future on-going exploration costs. The value of the project at abandonment is $\Phi^{(1)}(\underline{p_1}) = 0.0625 * \$50.33 * 50.32\text{m} = \$158.3\text{m}$.

6.3 Model 2: Investment With the Option to Delay

Now, we proceed to analyze the case using Model 2, where the owner of the exploration project, at completion date, obtains an American perpetual call option on the value of the active project. This option allows the owner to delay the final investment to develop the field.

In Figure 12, the value of the investment option is illustrated with the solid line, $F(P)$. Thus the difference between this line and the NPV line, $V(P)$, is the value of waiting. The value of
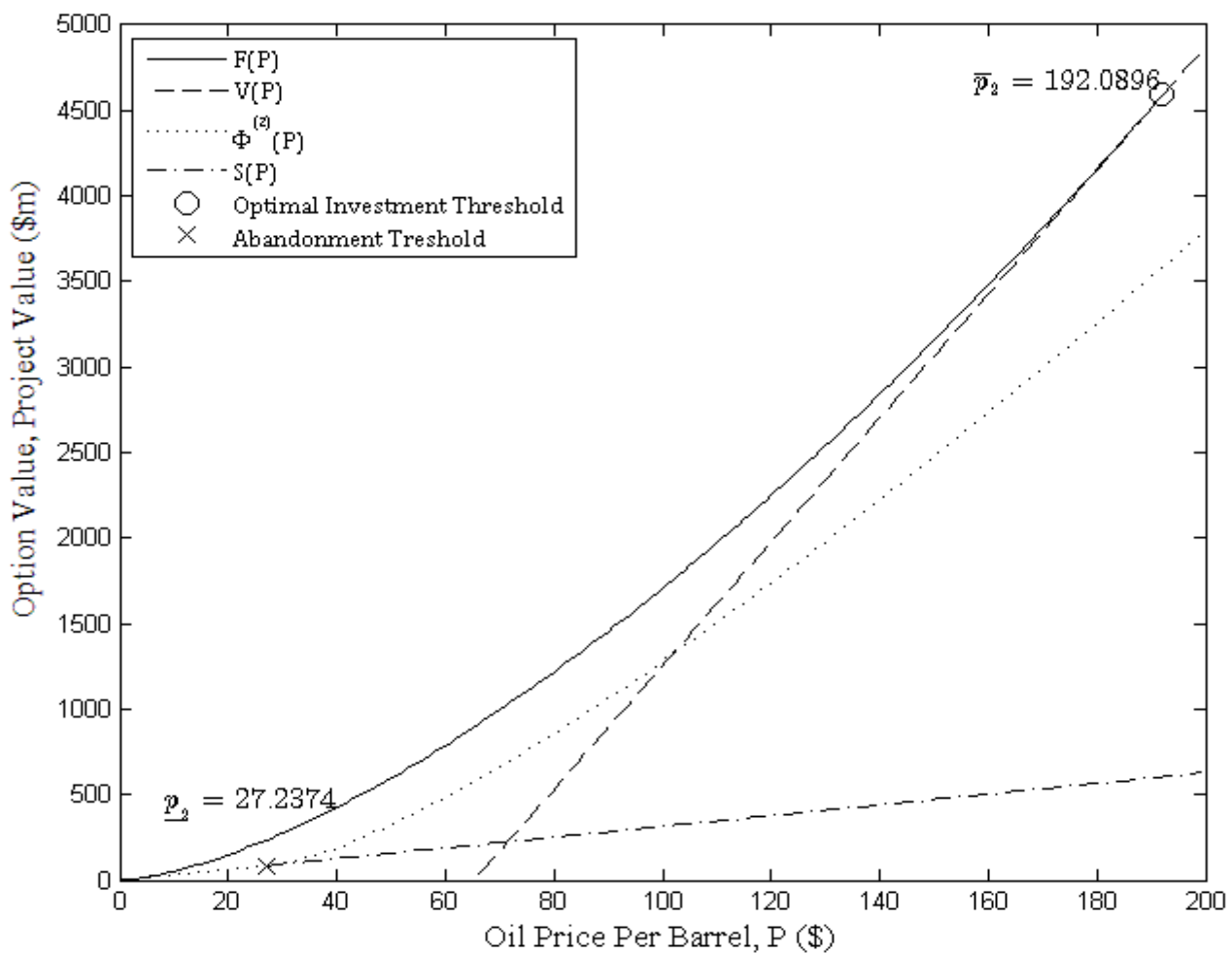

Figure 12 - Model 2 applied on the base case. The figure shows the value of the investment option, the exploration project and the active project as a function of the oil price. It also shows that the exploration project is abandoned for oil prices below $27.24 and that it is optimal to invest in the active project when the oil price reaches $192.09.

the exploration project, the active project and the value of selling the project are shown as before. Inclusion of the option to delay leads to severe impacts on the optimal time to invest and the abandonment threshold level. As it was optimal to invest at a break-even price of \$65 in Model 1, the owner now optimally invest when $\overline{p}_2 = 192.1$, indicated with a circle in the figure. This point is found where the value of the investment option converges with the value of the active project, as explained in sub-section 5.2.1. In reality, however, oil companies would invest for oil prices much lower than $\overline{p}_2$, as projects on average have a significant, positive NPV for oil prices lower than the investment threshold level. Still, this is technically the optimal price for developing the project. The abandonment threshold level is now found at $\underline{p}^{(2)} = 27.2373$ with a corresponding project value of $\Phi^{(2)}(\underline{p}_2) = 0.0625 * \$27.24 * 50.32m = \$85.67m$

Figure 13 (shown on the next page) includes both models, and summarizes the case-study so far. The figure illustrates how the value of the exploration project increases when the option to delay the final investment is included. This is shown by the $\Phi^{(2)}(P)$-line which is located above the $\Phi^{(1)}(P)$-line for all oil price higher than the abandonment threshold level in Model 2. The intuition behind this difference is that the owner of the exploration project with the option to delay enjoys the advantages of greater flexibility, which is reflected in the project value. Another effect from including the option to the delay, and thus, the increased project value, is that the abandonment threshold level decreases. Hence, the exploration project is kept alive at lower oil prices.
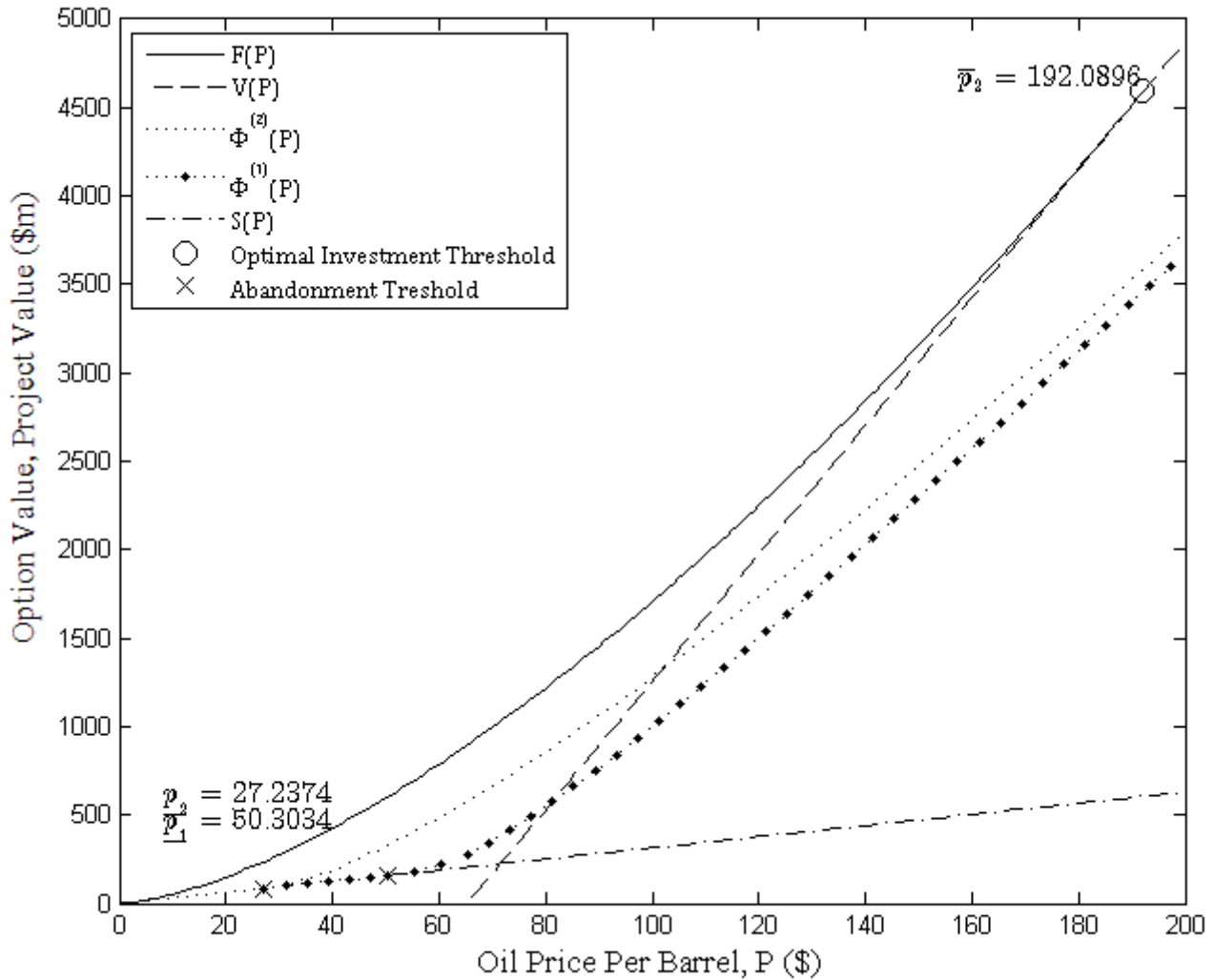
Figure 13 - Overview of Model 1 and Model 2 applied on the base case. The figure shows the investment option, the value of the exploration project in the two models, and the active project as a function of the oil price. Thus, the figure illustrates the increase in project value in Model 2, from including the option to delay the final investment. Furthermore, it illustrates how this increase is diminishing with the oil price and how it effects the abandonment threshold level.

6.4 Sensitivity Analysis

Here we look at the effect on project value and threshold levels from changes in some critical input parameters. In the real world, it is an important exercise for companies to perform sensitivity analysis in order to obtain greater insight and management control due to uncertainties in the business-environment.

6.4.1 Compensation

In Figure 14 we illustrate to what extent the value from selling the exploration project, $S(P)$,
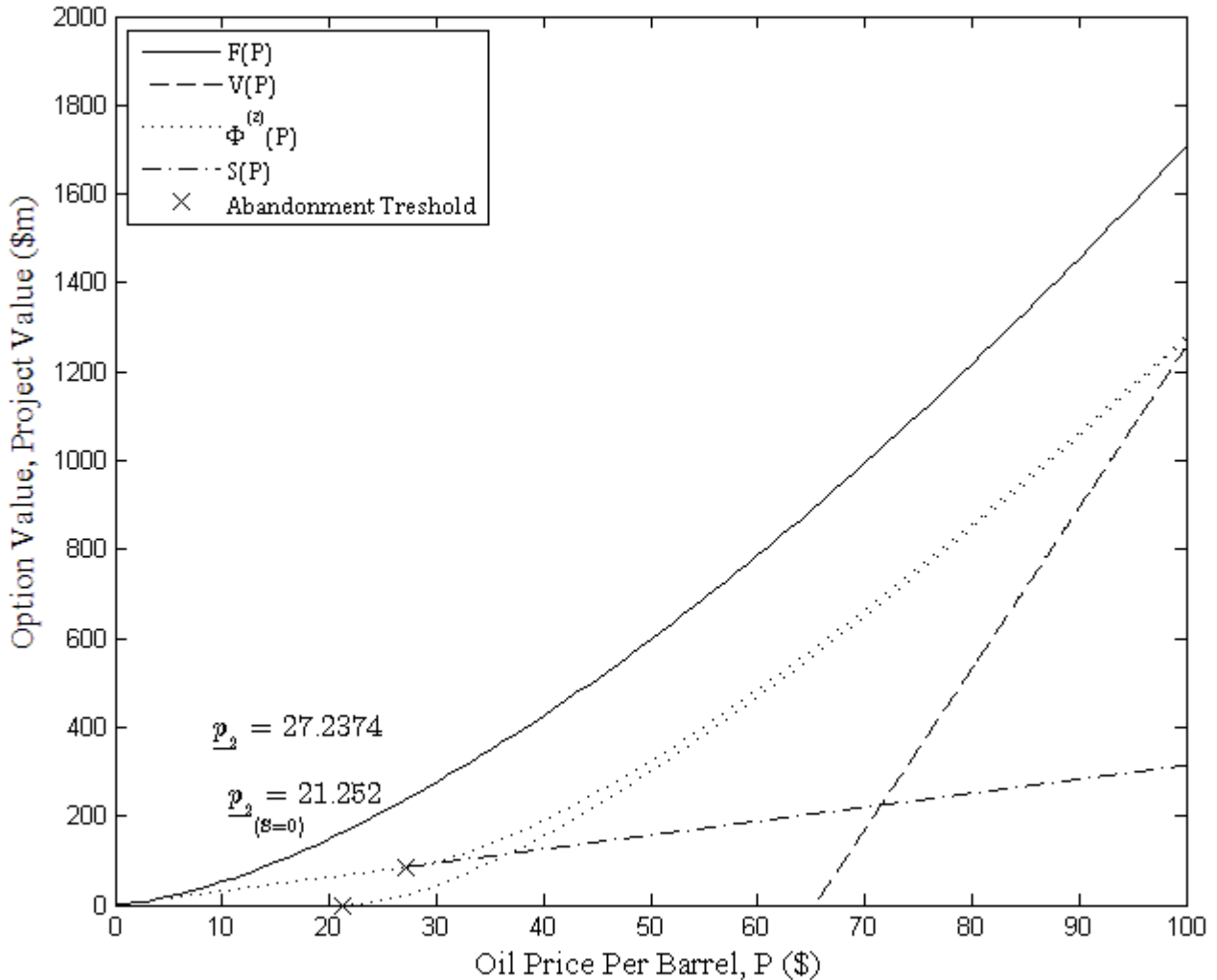
Figure 14 - Model 2 applied on the case without the possibility of selling the project. The figure shows the value of the investment option, the exploration projects with and without the value of S(P) and the active project as a function of the oil price. It also shows how the reduction in the project value, and how the abandonment threshold level decreases from $27.24 to $21.25.

affects the project value and the abandonment threshold level, by reducing s, and consequently $S(P)$, to zero. No compensation from selling the project is shown by the lower value function of $\Phi^{(2)}(P)$, with an abandonment threshold level labeled $(s = 0)$. When the compensation from abandonment *is* included, the value of the exploration project is given by the upper value function of $\Phi^{(2)}(P)$. Note, again, that the value functions of the exploration project, $\Phi^{(2)}(P)$, converge with the value from selling the project, $S(P)$ for oil prices below the abandonment threshold levels.

We observe that the abandonment threshold level decreases from $\underline{p}_2 = 27.2374$ to $\underline{p}_2 = 21.252$ when the value from selling the exploration project is removed. The reason for this drop is that the option of abandoning the project now is less lucrative. Thus, the option of abandoning the project yield less value and become the optimal choice when the oil price is at a lower level than before. In other words, the value from selling the project increases the demands of profitability put on the project, and may be crucial when a decision of continuing exploration or not, at times with low oil prices, is made. However, the effect on $\Phi^{(2)}(P)$ from having $s > 0$, is diminishing as the oil price increases. This is seen by the convergence of the two project-value functions, $\Phi^{(2)}(P)$, for higher oil price levels.

6.4.2 Total Amount of Oil

As we have incorporated total amount of oil, Q_2, in the framework, and consequently the annual average production, Q_1, changes to this parameter will certainly affect the results obtained. An increase in Q_2 should increase the value of both the exploration project and the active project, while a decrease in Q_2 should have similar, negative effects. In conjunction, it will affect the abandonment threshold level, the investment threshold level and the value of selling the project.

To illustrate the effects, we increase the total amount of oil in the field to $Q_2 = 75m$, and compare it to the base case (denoted BC). As illustrated in Figure 15, the levels for abandonment and optimal investment threshold are now $\underline{p}_2 = 18.2$ and $\overline{p}_2 = 128.33$, respectively. In other words, an increase in total output results in a decrease in both the abandonment threshold level and the exercise threshold level. The theoretic explanation of this result is due to the time-value of money, so that a larger annual production (and sale) of oil can be economically viable for a lower output price. The same reasoning can explain the decrease in the abandonment threshold level. As the value of the active project increase, the exploration project is worth holding on to for lower output prices.
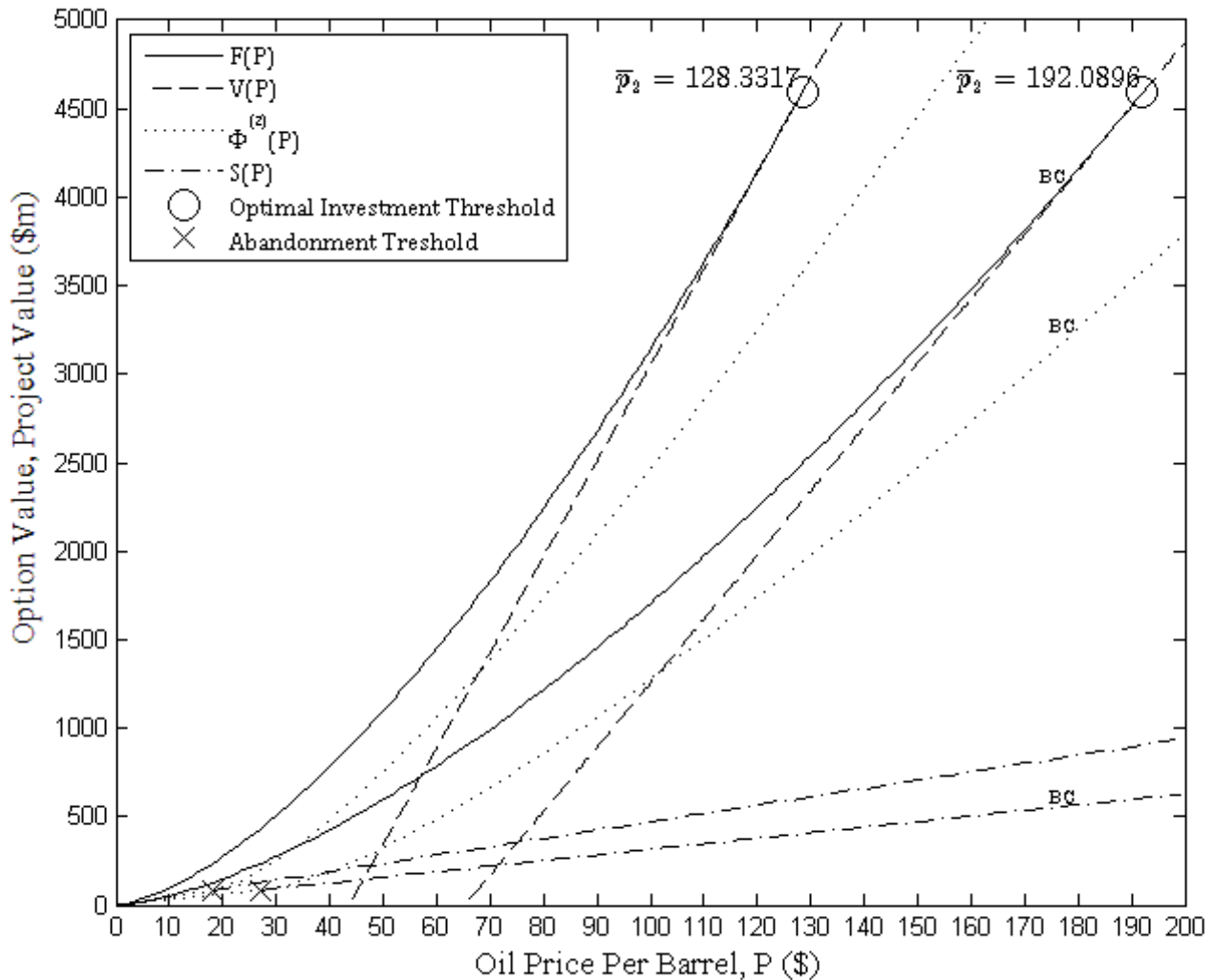
Figure 15 - Model 2 applied on the case with an increase in total amount of oil in the field. The figure shows the value of the investment option, the exploration project and the active project as a function of the oil price. It also shows how the abandonment threshold level decreases from $27.24 to $18.20, and the investment threshold level from $192.09 to $128.33, as a result of increasing the total amount of oil in the field.

6.4.3 Expected Time to Completion, Model 2

We first study the effect of changes to the expected time to completion of the exploration project, T_1, using Model 2. In Figure 16, the project value in the exploration stage is described for different scenarios of time to completion of the exploration project compared to the base case, where $T_1 = 4$. As illustrated, a decrease in the expected time to completion increases the value of the project for all oil prices. Consequently, the abandonment threshold level is reduced to $\underline{p}_2 = 21.2$ as there at fewer annual on-going exploration costs involved. This also

illustrates the importance of abandoning a project at the right time, as well as the importance of an efficient exploration process.

Figure 16 - Model 2 applied on the case with changes in the expected time to completion of the exploration stage. The figure shows the value of the investment option, the exploration project and the active project as a function of the oil price. It also shows how the project value increases, and how the abandonment threshold level decreases, from $35.70 to $21.20, when the expected time to completion is reduced from 8 to 2 years, respectively.

An increase in the expected time to completion yields the opposite effect. Now, the value of the project decreases for all oil prices and the abandonment threshold level increases to $\underline{p}_2 = 35.7$. This is due to the additional annual on-going exploration costs. In particular, the additional on-going exploration costs require a higher profitability of the active project for the exploration project to be kept alive, and thus it is abandoned at a higher oil price than before. Note that the optimal investment threshold, $\overline{p}_2$, is unaffected by changes in the expected time to completion.

6.4.4 Expected Time to Completion, Model 1

It should be stressed, however, that the relationship between the abandonment threshold level and the expected time to completion is not always as illustrated in Model 2. Depending on the size of the on-going exploration costs, k, and the time to completion, T_1, as well as the properties of the value functions, it is possible to obtain a different relationship between the time to completion and the abandonment threshold level. There is a trade-off between the value-decreasing effect from the on-going exploration costs, and the positive value of delaying the final investment cost, K. When we change the expected time to completion in Model 1, we find that there is no constant relationship between the time to completion and the project value, as illustrated in Figure 17.
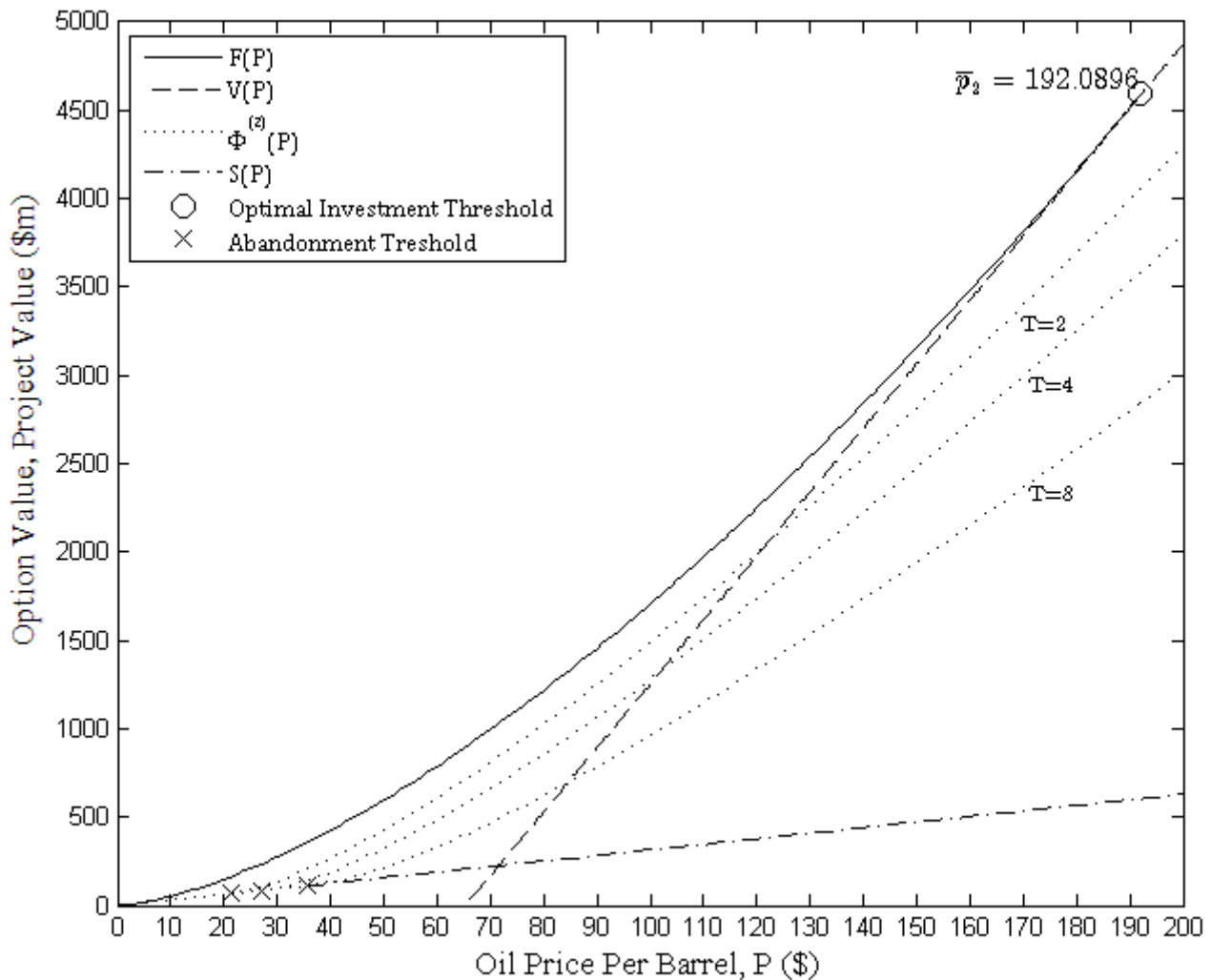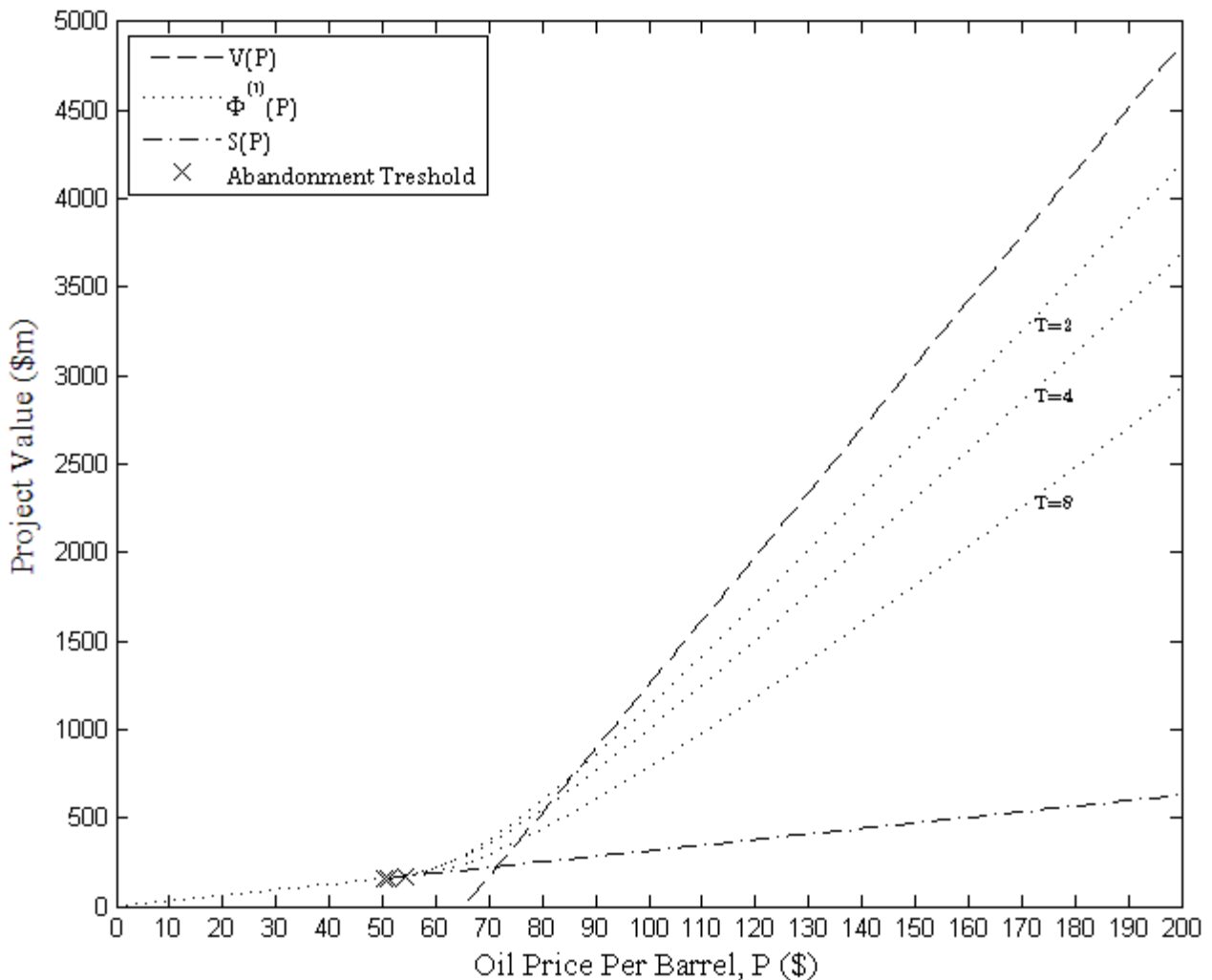

Figure 17 - Model 1 applied on the case with changes in the expected time to completion of the exploration stage. The figure shows the value of the exploration project and the active project as a function of the oil price. It also shows the non-monotonic relationship between the project value and changes in the time to completion for oil prices close to the abandonment threshold levels. Decreasing the time to completion yields an abandonment threshold level of $51.22 while increasing it yields $54.32.

We find that the relationship between the project values of different times to completion is interchangeable at oil prices close to the abandonment threshold level. A decrease in the expected time to completion, to $T_1 = 2$, increases the abandonment threshold level to $\underline{p}_1 = 51.22$, from the base case level of $\underline{p}_1 = 50.3$. This is the opposite of the effect found in Model 2. However, we obtain the same result as in Model 2 for higher oil-price levels. It is also worth noticing that the relative change in the threshold levels are smaller for Model 1 than for Model 2.

6.5 Our Framework vs. the Traditional NPV Approach

The difference between the PDO`s estimate of the break-even price for Knarr ($47), and our estimate ($65), is due to differences in calculating the net present value of the active project, $V(P)$. Among other things, we discount the project continuously and assume that the oil price is governed by a GBM process. Furthermore, we let λ_2 define the expected lifetime of the active project, whereas the PDO set a finite time.

However, there are some interesting aspects to emphasize. First, our framework illustrates to a great extent the optimal exercise and abandonment threshold levels. In theory, if the owner only uses the traditional NPV-analysis, he will not be able to see the exact oil price at which they should abandon (and sell) the exploration project. Thus, it is reasonable to believe that this project is abandoned at a sub-optimal level should the oil price drop. The same point apply for price levels where they optimally should have "timed" their investment, and waited for e.g. improved market conditions. The intuition behind is that oil sold for a considerably lower price today is less valuable than oil sold for an optimal higher price later.

In the real world, however, this is necessarily not the case as companies owning an oilfield project can put the investment decision on hold, for a period given by the terms of the license, if the current output price is lower than the break-even price. Hence, "real options" are present to companies at all times, but they are rarely emphasized and evaluated.

A main advantage of our framework is that it incorporates the value of several real options in a single value function, and provides the user with guidelines for optimal decision-making throughout the exploration stage and until development of the project. Thus, we argue that the value of the project should be assessed continuously, and that the project values and the optimal decisions should be derived from a framework that incorporates the value of

managerial flexibility. This can reduce the probability of undertaking unprofitable projects, and keeping those that are of value, which is the prime target for all investment managers in profit-seeking companies.

7. Conclusion

In this thesis we construct a framework for evaluation of complex oil exploration projects. We account for uncertainty in important input variables such as the oil price, exploration costs and the time to completion of the exploration. By applying real options theory, we manage to incorporate the value of managerial flexibility from the option to abandon the project, and the option to delay the final investment when the exploration is completed. Despite the complexity these features impose on the framework, we are able to obtain closed form solutions. Thus optimal threshold levels for abandonment and investment are easily derived. Investment approaches and analytical formulations that consider these properties simultaneously are limited.

The case study illustrates that our framework is consistent with theory. We find that the abandonment threshold level decreases when the option to delay is introduced, as the project becomes more valuable. Moreover, including the possibility of selling the project at abandonment, increases the abandonment threshold level. We also perform sensitivity analysis on important input parameters such as the total amount of oil barrels in the field, and the expected time-span of the exploration stage (which in turn affect the sum of exploration costs). These analysis show that the project value increases and the abandonment- and investment threshold levels decreases with the total amount of oil barrels in the field, as expected. Furthermore, the project value decreases when the exploration is more time-consuming and thus more expensive.

There are several areas of applications of our framework. It is particularly applicable to investment cases that are characterized by an exploration- or research stage with on-going costs before a final investment is made, and where future profits are closely related to fluctuations in the output price. Examples are pharmaceutical R&D investments and natural resource investments, such as wind- and mine exploration projects.

An interesting addition to our framework would clearly be to develop the evaluation of the active project. To limit the scope of this thesis, we let the value of the active project be given by a NPV calculation. In the real world, however, there are several flexibilities and possible options to be valued in the active stage of an oil project. Among these are possible tie-ins from other oil fields, switching options for the production intensity of the project, and abandonment

options when further production is no longer optimal. Another interesting extension would be to develop the value of selling the project to a more advanced level. Finally, it would be interesting to allow for a different price process, e.g. mean reversion.

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Appendix

A – Model 1

We solve the project using dynamic programming, where the value of the active project with a Poisson death parameter, λ_2, is given by

(1) $V(P) = PQ_1 dt - \rho K dt + (1 - \lambda_2 dt)(1 - \rho dt)\mathrm{E}_P[V(P + dP)]$

We remove the brackets and draw together similar terms

(2) $V(P) = PQ_1 dt - \rho K dt + V(P) - (\rho + \lambda_2)dt V(P) + \mathrm{E}_P[dV(P)]$

Expanding the term $\mathrm{E}_P[dV(P)]$ using Itos Lemma and simplifying yields

(3) $\frac{1}{2}\sigma^2 P^2 V''(P) + \mu P V'(P) - (\rho + \lambda_2)V(P) + PQ_1 - K = 0$

The general solution to the differentiated terms on the left hand side, with its transformations and eliminations, allows us to manipulate the equation with the following result when we solve for $V(P)$

(4) $V(P) = \frac{PQ_1}{(\rho - \mu + \lambda_2)} - K$

This expression gives the net present value of the active project.

Furthermore, we have that $S(P) = sPQ_2$. To obtain the value of the exploration stage, we need to take the possibility of abandonment into account, and thus include $S(P)$ in the value function in addition to the value of the project when it is not abandoned. Thus we have that the value of the project in the exploration stage is given by

(5) $\Phi^{(1)}(P) =$
$\begin{cases} (1 - \rho dt)\lambda_1 dt\mathrm{E}_P[V(P + dP)] + (1 - \rho dt)(1 - \lambda_1 dt)\mathrm{E}_P\big[\Phi^{(1)}(P + dP)\big] - k dt & \text{when } P \geq \underline{p_1} \\ S(P) & \text{when } P < \underline{p_1} \end{cases}$

We expand the right hand side of the equation using Ito's Lemma and obtain the following

(6) $\frac{1}{2}\sigma^2 P^2 \Phi^{(2)''}(P) + \mu P \Phi^{(2)'}(P) - (\rho + \lambda_1)\Phi^{(2)}(P) + \lambda_1 V(P) - k = 0$

However, the active project is only chosen for investment if $V(P) \geq 0$. This NPV threshold value is found at the price given by $\overline{p}_1 = \frac{K(\rho - \mu + \lambda_2)}{Q_1}$

Thus we have two ODEs to determine the value of the project when it is not abandoned. Together with the value function from abandonment, we get the following set of equation to determine the value of the project in the exploration stage

(7) $sPQ_2 - \Phi_1^{(1)}(P) = 0$ 	 when $P < \underline{p}_1$

(8) $\frac{1}{2}\sigma^2 P^2 \Phi^{(1)''}(P) + \mu P \Phi^{(1)'}(P) - (\rho + \lambda_1)\Phi^{(1)}(P) - k = 0$ 	 when $\underline{p}_1 \leq P < \overline{p}_1$

(9) $\frac{1}{2}\sigma^2 P^2 \Phi^{(1)''}(P) + \mu P \Phi^{(1)'}(P) - (\rho + \lambda_1)\Phi^{(1)}(P) + \lambda_1\left(\frac{PQ_1}{(\rho-\mu+\lambda_2)} - K\right) - k = 0$ when $\overline{p}_1 \leq P$

We now utilize the standard solution to the homogenous part of the ODEs above, and solve for $\Phi^{(1)}(P)$. Consequently, when including the alternative to abandon and sell the project, we now have three equations to determine the value of the project

(10) $\Phi_1^{(1)}(P) = sPQ_2$ 	 when $P < \underline{p}_1$

(11) $\Phi_2^{(1)}(P) = A_1^{(1)} P^{y_1} + A_2^{(1)} P^{y_2} - \frac{k}{\rho+\lambda_1}$ 	 when $\underline{p}_1 \leq P < \overline{p}_1$

(12) $\Phi_3^{(1)}(P) = B_1^{(1)} P^{y_1} + B_2^{(1)} P^{y_2} + \frac{\lambda_1 PQ_1}{(\rho-\mu+\lambda_1)(\rho-\mu+\lambda_2)} - \frac{k+\lambda_1 K}{\rho+\lambda_1}$ 	 when $\overline{p}_1 \leq P$.

Since we have that $\lim_{P\to\infty}\frac{\Phi_2^{(1)}(P)}{V(P)}$ must be less than one and that $y_1 > 1$ we are able to eliminate the term including the second constant, $B_1^{(1)}$ from $\Phi_2^{(1)}(P)$. Using this, along with $\lambda_1 = \frac{1}{T_1}$, and $\lambda_2 = \frac{1}{T_2}$ we have the three equations for the project value in the exploration stage as

(13) $\Phi_1^{(1)}(P) = sPQ_2$ 	 when $P < \underline{p}_1$

(14) $\Phi_2^{(1)}(P) = A_1^{(1)} P^{y_1} + A_2^{(1)} P^{y_2} - \frac{kT_1}{1+\rho T_1}$ 	 when $\underline{p}_1 \leq P < \overline{p}_1$

(15) $\Phi_3^{(1)}(P) = B_2^{(1)} P^{y_2} + \frac{PQ_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{kT_1+K}{1+\rho T_1}$ 	 when $\overline{p}_1 \leq P$.

Where the powers are the roots of

(16) $\frac{1}{2}\sigma^2 y(y-1) + \mu y - (\rho + \lambda_1) \Rightarrow \frac{1}{2}\sigma^2 y^2 - y\left(\mu - \frac{1}{2}\sigma^2\right) - (\rho + \lambda_1)$

and are given by

(17) $y_1 = \dfrac{\left(\frac{1}{2}\sigma^2-\mu\right)+\sqrt{(\mu-\frac{1}{2}\sigma^2)^2+2(\rho+\lambda_1)\sigma^2}}{\sigma^2} > 1$

and

(18) $\quad y_2 = \dfrac{\left(\frac{1}{2}\sigma^2 - \mu\right) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2(\rho + \lambda_1)\sigma^2}}{\sigma^2} < 0.$

With boundary conditions

$$(19) \quad \Phi_1^{(1)}\left(\underline{p}_1\right) = \Phi_2^{(1)}\left(\underline{p}_1\right),$$

$$(20) \quad \Phi_1^{(1)'}\left(\underline{p}_1\right) = \Phi_2^{(1)'}\left(\underline{p}_1\right),$$

$$(21) \quad \Phi_2^{(1)}(\overline{p}_1) = \Phi_3^{(1)}(\overline{p}_1),$$

$$(22) \quad \Phi_2^{(1)'}(\overline{p}_1) = \Phi_3^{(1)'}(\overline{p}_1)$$

We derive the equations for $A_1^{(1)}, A_2^{(1)}, B_2^{(1)}$ and $\underline{p}_1$ using the set of equations above and the boundary conditions.

Deriving the equation for $B_2^{(1)}$

We set $\Phi_2^{(1)}(\overline{p}_1) = \Phi_3^{(1)}(\overline{p}_1)$ and solve for $A_1^{(1)}$

$$A_1^{(1)}\overline{p}_1^{y_1} + A_2^{(1)}\overline{p}_1^{y_2} - \frac{kT_1}{1+\rho T_1} = B_2^{(1)}\overline{p}_1^{y_2} + \frac{\overline{p}_1 Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{kT_1 + K}{1+\rho T_1}$$

$$A_1^{(1)} = -A_2^{(1)}\overline{p}_1^{y_2 - y_1} + B_2^{(1)}\overline{p}_1^{y_2 - y_1} + \frac{\overline{p}_1^{1-y_1} Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{\overline{p}_1^{-y_1} K}{1+\rho T_1}$$

We set the 1st derivatives equal to each other, $\Phi_2^{(1)'}(\overline{p}_1) = \Phi_3^{(1)'}(\overline{p}_1)$ and insert $A_1^{(1)}$

$$y_1\overline{p}_1^{y_1-1}\left(-A_2^{(1)}\overline{p}_1^{y_2-y_1} + B_2^{(1)}\overline{p}_1^{y_2-y_1} + \frac{\overline{p}_1^{1-y_1}Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{\overline{p}_1^{-y_1}K}{1+\rho T_1}\right) + y_2 A_2^{(1)}\overline{p}_1^{y_2-1} =$$

$$y_2 B_2^{(1)}\overline{p}_1^{y_2-1} + \frac{Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})}$$

We resolve the bracket and collect and draw similar terms

$$(y_1 - y_2)B_2^{(1)}\overline{p}_1^{y_2-1} = (y_1 - y_2)A_2^{(1)}\overline{p}_1^{y_2-1} + \frac{(1-y_1)Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{y_1\overline{p}_1^{-1}K}{1+\rho T_1}$$

We divide by $(y_1 - y_2)\overline{p}_1^{y_2-1}$ to get $B_2^{(1)}$ alone

$$(23) \quad B_2^{(1)} = A_2^{(1)} + \frac{(1-y_1)Q_1\overline{p}_1^{1-y_2}}{(y_1-y_2)(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{y_1\overline{p}_1^{-y_2}K}{(y_1-y_2)(1+\rho T_1)}$$

Deriving the equation for $A_1^{(1)}$

We insert $B_2^{(1)}$ in the above term for $A_1^{(1)}$

$$A_1^{(1)} = -A_2^{(1)}\overline{p}_1^{y_2-y_1} + \overline{p}_1^{y_2-y_1}\left(A_2^{(1)} + \frac{(1-y_1)Q_1\overline{p}_1^{1-y_2}}{(y_1-y_2)(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{y_1\overline{p}_1^{-y_2}K}{(y_1-y_2)(1+\rho T_1)}\right) +$$

$$\frac{\overline{p}_1^{1-y_1}Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{\overline{p}_1^{-y_1}K}{1+\rho T_1}$$

We rearrange and collect terms and get the solution

$$(24) \quad A_1^{(1)} = \frac{(1-y_1)Q_1\overline{p}_1^{1-y_1}}{(y_1-y_2)(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{y_1\overline{p}_1^{-y_1}K}{(y_1-y_2)(1+\rho T_1)} + \frac{Q_1\overline{p}_1^{1-y_1}}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{\overline{p}_1^{-y_1}K}{1+\rho T_1}$$

Deriving the equation for $A_2^{(1)}$

We set the 1$^\text{st}$ derivative equal to each other $\Phi_1^{(1)\prime}\left(\underline{p}_1\right) = \Phi_2^{(1)\prime}\left(\underline{p}_1\right)$ and solve for $A_2^{(1)}$

$$sQ_2 = y_1A_1^{(1)}\underline{p}_1^{y_1-1} + y_2A_2^{(1)}\underline{p}_1^{y_2-1}$$

$$sQ_2 - y_1A_1^{(1)}\underline{p}_1^{y_1-1} = y_2A_2^{(1)}\underline{p}_1^{y_2-1}$$

We divide by $y_2\underline{p}_1^{y_2-1}$ to get to the solution

$$(25) \quad A_2^{(1)} = \frac{sQ_2}{y_2\underline{p}_1^{y_2-1}} - \frac{y_1A_1^{(1)}\underline{p}_1^{y_1-y_2}}{y_2}$$

The abandonment threshold level, $\underline{p}_1$, is found numerically using the boundary condition and equation given below

$$\Phi_1^{(1)}\left(\underline{p}_1\right) = \Phi_2^{(1)}\left(\underline{p}_1\right),$$

$$s\underline{p}_1Q_2 = A_1^{(1)}\underline{p}_1^{y_1} + A_2^{(1)}\underline{p}_1^{y_2} - \frac{kT_1}{1+\rho T_1}$$

B – Model 2

The value of the exploration project in stage 3 is derived in Appenix A

(26) $V(P) = \frac{PQ_1}{(\rho - \mu + \lambda_2)} - K$

The value of the exploration project in stage 2 is given by

(27) $F(P) = \begin{cases} (1 - \rho dt)\lambda_1 dt E_P[F(P + dP)] & \text{when } P < \overline{p}_2, \\ V(P) & \text{when } \overline{p}_2 \leq P. \end{cases}$

If the oil price is below the investment threshold level, $\overline{p}_2$, the optimal action for him is to wait for a higher oil price before making the final investment. The interpretation of the top line in the equation above is, thus, the value of delaying the final investment after the exploration stage is completed. Consequently, is the probability of completion, λ_1, and the future expected values of the option to delay included. The bottom line in the equation is the value of the option to delay when the oil price exceeds the investment threshold. Should this happen, the owner will pay the final investment cost to develop the field and obtain the value of the active project. Expanding the first equation on the right hand side using Ito's Lemma yields the differential equation

(28) $\frac{1}{2}\sigma^2 P^2 F''(P) + \mu P F'(P) - \rho F(P) = 0$

For $P < \overline{p}_2$, the general solution to this differential equation is

(29) $F(P) = C_1 P^{x_1} + C_2 P^{x_2}$

Where the power, x_1, is given by

(30) $x_1 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\rho\sigma^2}}{\sigma^2} > 1.$

The second term on the right hand side can be eliminated since as P approaches infinity, the value of the project decreases towards zero. Consequently, we have that $C_2 P^{x_2}$ grows to infinity along with P, since $x_2 < 0$. Thus $C_2 = 0$ and

(31) $F(P) = \begin{cases} C_1 P^{x_1} & \text{when } P < \overline{p}_2, \\ V(P) & \text{when } \overline{p}_2 \leq P. \end{cases}$

In this value function, C_1 and $\overline{p}_2$ are determined by the value-matching and smooth-pasting conditions between the two branches of $F(P)$, and are given by

$$C_1\overline{p}_2^{x_1} = \frac{\overline{p}_2 Q_1}{(\rho-\mu+\lambda_2)} - K, \ x_1 C_1 \overline{p}_2^{x_1-1} = \frac{Q_1}{(\rho-\mu+\lambda_2)}, \Rightarrow \overline{p}_2 = \frac{x_1}{(x_1-1)}\frac{K(\rho-\mu+\lambda_2)}{Q_1}, C_1 = \frac{\overline{p}_2^{1-x_1}Q}{x_1(\rho-\mu+\lambda_2)}$$

Derivation of these constant are given in Appendix C.

In stage 1, we have the value of the exploration process before the project is completed. There is also an option to abandon and sell the project in this state. The value of selling the project at the abandonment threshold level is, as in Model 1, given by

(32) $S(P) = sPQ_2$

Thus, the value of the project in stage 1 is given by

(33) $\Phi^{(2)}(P) =$

$$\begin{cases} (1-\rho dt)\lambda_1 dt E_P[F(P+dP)] + (1-\rho dt)(1-\lambda_1 dt)E_P[\Phi^{(2)}(P+dP)] - kdt & \text{when } P \geq \underline{p}_2 \\ S(P) & \text{when } P < \underline{p}_2 \end{cases}$$

Taking the first expression on the right hand side in the value function above and expanding using Ito's Lemma, we obtain

(34) $\Phi^{(2)}(P) = (1-(\rho+\lambda_1)dt)\left[\frac{1}{2}\sigma^2 P^2 \Phi^{(2)''}(P)dt + \mu P \Phi^{(2)'}(P)dt + \Phi^{(2)}(P)\right] +$

$(1-\rho dt)\lambda_1 dt \left[\frac{1}{2}\sigma^2 P^2 F''(P)dt + \mu P F'(P)dt + F(P)\right] - kdt$

After simplifying this equation, we obtain the ODE for $\Phi^{(2)}(P)$

(35) $\frac{1}{2}\sigma^2 P^2 \Phi^{(2)''}(P) + \mu P \Phi^{(2)'}(P) - (\rho+\lambda_1)\Phi^{(2)}(P) + \lambda_1 F(P) - k = 0$

However, the solution depends on the value of $F(P)$ as shown in equation (31)

(36)

$$\begin{cases} \frac{1}{2}\sigma^2 P^2 \Phi^{(2)''}(P) + \mu P \Phi^{(2)'}(P) - (\rho+\lambda_1)\Phi^{(2)}(P) + \lambda_1 C_1 P^{x_1} - k = 0 & \text{when } P < \overline{p}_2 \\ \frac{1}{2}\sigma^2 P^2 \Phi^{(2)''}(P) + \mu P \Phi^{(2)'}(P) - (\rho+\lambda_1)\Phi^{(2)}(P) + \lambda_1\left(\frac{PQ_1}{(\rho-\mu+\lambda_2)} - K\right) - k = 0 \\ \qquad\qquad\qquad\qquad \text{when } \overline{p}_2 \leq P. \end{cases}$$

We now utilize the standard solution to the homogenous part of the ODEs above, and solve for $\Phi^{(2)}(P)$. Consequently, when including the alternative to sell the project, we have three equations to determine the value of the project in stage 1, given by $\Phi^{(2)}(P)$:

(37) $\quad \Phi_1^{(2)}(P) = sPQ_2$ $\hspace{5cm}$ when $P < \underline{p}_2$

(38) $\quad \Phi_2^{(2)}(P) = A_1^{(2)}P^{y_1} + A_2^{(2)}P^{y_2} - \dfrac{k}{\rho+\lambda_1} + \dfrac{\lambda_1 C_1 P^{x_1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)}$ $\quad$ when $\underline{p}_2 \leq P < \overline{p}_2$

(39) $\quad \Phi_3^{(2)}(P) = B_1^{(2)}P^{y1} + B_2^{(2)}P^{y2} + \dfrac{\lambda_1 PQ_1}{(\rho-\mu+\lambda_1)(\rho-\mu+\lambda_2)} - \dfrac{k+\lambda_1 K}{\rho+\lambda_1}$ $\hspace{2cm}$ when $\overline{p}_2 \leq P$

Again, since we have that $\lim_{P\to\infty} \dfrac{\Phi_2^{(2)}(P)}{V(P)}$ must be less than one and that $y_1 > 1$ we are able to eliminate the term including the second constant, $B_1^{(2)}$ from $\Phi_2^{(2)}(P)$. Using this, along with $\lambda_1 = \dfrac{1}{T_1}$, and $\lambda_2 = \dfrac{1}{T_2}$ we have the three equations for the project value in the exploration stage as

(40) $\quad \Phi_1^{(2)}(P) = sPQ_2$ $\hspace{5cm}$ when $P < \underline{p}_2$

(41) $\quad \Phi_2^{(2)}(P) = A_1^{(2)}P^{y_1} + A_2^{(2)}P^{y_2} - \dfrac{kT_1}{1+rT_1} + \dfrac{C_1 P^{x_1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}$ $\quad$ when $\underline{p}_2 \leq P < \overline{p}_2$

(42) $\quad \Phi_3^{(2)}(P) = B_2^{(2)}P^{y_2} + \dfrac{PQ_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \dfrac{kT_1+K}{1+\rho T_1}$ $\hspace{2cm}$ when $\overline{p}_2 \leq P$.

Where the powers are given by

(43) $\quad y_1 = \dfrac{\left(\frac{1}{2}\sigma^2-\mu\right)+\sqrt{(\mu-\frac{1}{2}\sigma^2)^2+2(r+\lambda)\sigma^2}}{\sigma^2} > 1$

and

(44) $\quad y_2 = \dfrac{\left(\frac{1}{2}\sigma^2-\mu\right)-\sqrt{(\mu-\frac{1}{2}\sigma^2)^2+2(r+\lambda)\sigma^2}}{\sigma^2} < 0.$

And the boundary conditions are given by

(45) $\quad \Phi_1^{(2)}\left(\underline{p}_2\right) = \Phi_2^{(2)}\left(\underline{p}_2\right),$

(46) $\quad \Phi_1^{(2)\prime}\left(\underline{p}_2\right) = \Phi_2^{(2)\prime}\left(\underline{p}_2\right),$

(47) $\quad \Phi_2^{(2)}\left(\overline{p}_2\right) = \Phi_3^{(2)}\left(\overline{p}_2\right),$

(48) $\quad \Phi_2^{(2)\prime}\left(\overline{p}_2\right) = \Phi_3^{(2)\prime}\left(\overline{p}_2\right)$

We derive the equations for $A_1^{(2)}$, $A_2^{(2)}$, $B_2^{(2)}$ and $\underline{p}_2$ using the set of equations above and the boundary conditions.

Deriving the equation for $B_2^{(2)}$

We set $\Phi_2^{(2)}(\overline{p}_2) = \Phi_3^{(2)}(\overline{p}_2)$,

$$A_1^{(2)}\overline{p}_2^{y_1} + A_2^{(2)}\overline{p}_2^{y_2} - \frac{kT_1}{1+\rho T_1} + \frac{C_1\overline{p}_2^{x_1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1} = B_2^{(2)}\overline{p}_2^{y_2}$$

$$+ \frac{\overline{p}_2 Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} - \frac{kT_1+K}{1+\rho T_1}$$

Dividing by $\overline{p}_2^{y_2}$ and collecting terms gives $B_2^{(2)}$ as

(49) $\quad B_2^{(2)} = A_2^{(2)} + \left(A_1^{(2)}\overline{p}_2^{y_1} + \frac{K}{1+\rho T_1} - \frac{\overline{p}_2 Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{C_1\overline{p}_2^{x_1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}\right)\frac{1}{\overline{p}_2^{y_2}}$

Deriving the equation for $A_1^{(2)}$

We set $\Phi_2^{(2)\prime}(\overline{p}_2) = \Phi_3^{(2)\prime}(\overline{p}_2)$,

$$y_1 A_1^{(2)}\overline{p}_2^{y_1-1} + y_2 A_2^{(2)}\overline{p}_2^{y_2-1} + \frac{x_1 C_1 \overline{p}_2^{x_1-1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1} = y_2 B_2^{(2)}\overline{p}_2^{y_2-1} + \frac{Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})}$$

We insert for $B_2^{(2)}$

$$y_1 A_1^{(2)}\overline{p}_2^{y_1-1} + y_2 A_2^{(2)}\overline{p}_2^{y_2-1} + \frac{x_1 C_1 \overline{p}_2^{x_1-1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1} = y_2 \overline{p}_2^{y_2-1}\left(A_2^{(2)} + \left(A_1^{(2)}\overline{p}_2^{y_1} + \right.\right.$$

$$\left.\left. \frac{K}{1+\rho T_1} - \frac{\overline{p}_2 Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{C_1\overline{p}_2^{x_1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}\right)\frac{1}{\overline{p}_2^{y_2}}\right) + \frac{Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})}$$

We remove the brackets

$$y_1 A_1^{(2)} \overline{p}_2^{y_1-1} + y_2 A_2^{(2)} \overline{p}_2^{y_2-1} + \frac{x_1 C_1 \overline{p}_2^{x_1-1}}{1+\left(\rho - x_1\mu - \frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1} = y_2 A_2^{(2)} \overline{p}_2^{y_2-1} + y_2 A_1^{(2)} \overline{p}_2^{y_1-1} +$$

$$\frac{y_2 K \overline{p}_2^{-1}}{1+\rho T_1} - \frac{y_2 Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{y_2 C_1 \overline{p}_2^{x_1-1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1} + \frac{Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})}$$

We collect similar terms

$$y_1 A_1^{(2)} \overline{p}_2^{y_1-1} - y_2 A_1^{(2)} \overline{p}_2^{y_1-1} += -\frac{x_1 C_1 \overline{p}_2^{x_1-1}}{1+\left(\rho - x_1\mu - \frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1} + \frac{y_2 C_1 \overline{p}_2^{x_1-1}}{1+\left(\rho - x_1\mu - \frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1} +$$

$$\frac{y_2 K \overline{p}_2^{-1}}{1+\rho T_1} - \frac{y_2 Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})} + \frac{Q_1}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})}$$

We draw together similar terms and solve for $A_1^{(2)}$

$$(50) \quad A_1^{(2)} = \frac{\frac{-(x_1-y_2)C_1}{\left(1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1\right)\overline{p}_2^{1-x_1}} + \frac{y_2 K}{(1+\rho T_1)\overline{p}_2} + \frac{Q_1(1-y_2)}{(1+(\rho-\mu)T_1)(\rho-\mu+\frac{1}{T_2})}}{(y_1-y_2)\overline{p}_2^{y_1-1}}$$

Deriving the equation for $A_2^{(2)}$

We set $\Phi_1^{(2)'}\left(\underline{p}_2\right) = \Phi_2^{(2)'}\left(\underline{p}_2\right)$,

$$sQ_2 = y_1 A_1^{(2)} \underline{p}_2^{y_1-1} + y_2 A_2^{(2)} p_2^{y_2-1} + \frac{x_1 C_1 \underline{p}_2^{x_1-1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}$$

We solve for $A_2^{(2)}$

$$(51) \quad A_2^{(2)} = \frac{sQ_2 - \left(y_1 A_1^{(2)} \underline{p}_2^{y_1-1} + \frac{x_1 C_1 \underline{p}_2^{x_1-1}}{1+\left(\rho-x_1\mu-\frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}\right)}{y_2 \underline{p}_2^{y_2-1}}$$

The value of the abandonment threshold level, A_O, is found numerically using the boundary condition and equations given below

$$\Phi_1^{(2)'}\left(\underline{p}_2\right) = \Phi_2^{(2)'}\left(\underline{p}_2\right),$$

$$sp_2 Q_2 = A_1^{(2)} p_2^{y_1} + A_2^{(2)} p_2^{y_2} - \frac{kT_1}{1+\rho T_1} + \frac{C_1 p_2^{x_1}}{1+\left(\rho - x_1\mu - \frac{1}{2}x_1(x_1-1)\sigma^2\right)T_1}$$

C – Derivation of the option to delay

From Appendix B, we have that the possible values of $F(P)$ are

(52) $F_1(P) = C_1 P^{x_1}$ when $P < \bar{p}_2$

(53) $F_2(P) = \frac{PQ_1}{(\rho-\mu+\lambda_2)} - K$ when $\bar{p}_2 \leq P$.

$F_1(P)$ is the value of the option to delay the final investment in the relevant interval of oil prices below the investment threshold level, $\bar{p}_2$. $F_2(P)$ gives the value of the option when the oil price is equal to, or above the investment threshold level. In other words; when the owner exercises the option and obtains the value of the active project. Thus is the value of the investment option equal to the value of the active project when $\bar{p}_2 \leq P$. The corresponding boundary conditions are

(54) $F_1(\bar{p}_2) = F_2(\bar{p}_2)$

(55) $F_1'(\bar{p}_2) = F_2'(\bar{p}_2)$

The first boundary condition is the value matching condition, and states that the value of the call option should be equal to the value of the active project at the intersection where the oil price is at the exercise threshold level, $\bar{p}_2$. The second boundary condition is the smooth pasting condition, and simply states that the intersection between the two value functions should be smooth.

Deriving the equation for C_1

We set $F_1'(\bar{p}_2) = F_2'(\bar{p}_2)$

$$x_1 C_1 \bar{p}_2^{x_1-1} = \frac{Q_1}{(\rho-\mu+\lambda_2)}$$

We solve for C_1

(56) $C_1 = \frac{\bar{p}_2^{1-x_1} Q_1}{(\rho-\mu+\lambda_2)x_1}$

Deriving the equation for $\overline{p}_2$

We set $F_1\left(\overline{p}_2\right) = F_2\left(\overline{p}_2\right)$

$$C_1 \overline{p}_2^{x_1} = \frac{\overline{p}_2 Q_1}{(\rho - \mu + \lambda_2)} - K$$

We insert for C_1

$$\frac{\overline{p}_2 Q_1}{(\rho - \mu + \lambda_2)x_1} = \frac{\overline{p}_2 Q_1}{(\rho - \mu + \lambda_2)} - K$$

We multiply by $-x_1$

$$-\frac{\overline{p}_2 Q_1}{(\rho - \mu + \lambda_2)} = x_1 K - \frac{x_1 \overline{p}_2 Q_1}{(\rho - \mu + \lambda_2)}$$

We collect and draw together similar terms, and solve for $\overline{p}_2$

(57) $\quad \overline{p}_2 = \frac{x_1}{(x_1 - 1)} \frac{K(\rho - \mu + \lambda_2)}{Q_1}$