NHH

INSTITUTT FOR FORETAKSØKONOMI

DEPARTMENT OF BUSINESS AND MANAGEMENT SCIENCE

FOR 16 2015

ISSN: 1500-4066

April 2015

Discussion paper

Shapley-Based Stackelberg Leadership Formation in Networks

BY

Ivan Belik AND **Kurt Jörnsten**

Norges Handelshøyskole

NORWEGIAN SCHOOL OF ECONOMICS

SHAPLEY-BASED STACKELBERG LEADERSHIP FORMATION IN NETWORKS

Ivan Belik, Kurt Jörnsten

Norwegian School of Economics, Helleveien 30, 5045 Bergen, Norway
Email address: Ivan.Belik@nhh.no, Kurt.Jornsten@nhh.no

Abstract
In the given research we study a leadership formation of the most influential nodes in networks. Specifically, we analyze the competition between a leader and a follower based on the Stackelberg leadership model. Applying the concept of Shapley value to measure node's importance, we represent the mechanism of Shapley-based Stackelberg leadership formation in networks. The approach is tested and represented in tabular and graphical formats.

Keywords: Stackelberg competition, Shapley value, leadership, networks analysis

1. INTRODUCTION[i]

The investigation of competition between network's leaders is at the core of the leadership formation analysis in social networks. The problem is closely correlated with the centrality measurement that is based on the different evaluation methods. Degree (Freeman, 1979), betweenness (Anthonisse, 1971; Freeman, 1977), and closeness (Beauchamp, 1965; Sabidussi, 1966) are the most widely known metrics that assess the structural centralities of nodes. The algorithmic measures of node's authority are well represented in Kleinberg (1999) and Page, Brin, Motwani, & Winograd (1999), where the notion of authority is given based on the analysis of link structures. An interesting approach to characterize the role of nodes within the networks is given by Scripps & Esfahanian (2007), where the community-based metric in the symbiosis with the degree-based measure is introduced in the context of the classification of nodes' roles. Another methodology for analyzing node's leadership and importance in networks is based on a game theoretic approach. Specifically, we employ the Shapley value concept developed by Aadithya, Ravindran, Michalak, & Jennings (2010) in order to analyze how the nodes' leadership positions in networks can be strengthened by establishing new links.

In the given research, we analyze the leadership formation in terms of the competition between the most influential leader and its follower in a network. We interpret the interaction between the leader and the follower in social networks based on the Stackelberg competition principle (Von Stackelberg, 2010) from the game theoretic domain. More details about the Stackelberg model can be found in Simaan, & Cruz (1973), Basar, Olsder, Clsder, Basar, Baser, & Olsder (1995), and He, Prasad, Sethi, & Gutierrez (2007).

Applying Shapley value to measure an agent's (i.e., nodes) leadership position in a network, we represent the mechanism of the Shapley-based Stackelberg leadership formation. It reflects the competition between the most influential agent (i.e., leader) and the second most influence agent (i.e., follower) in the network.

---

[i] The section uses partly or exclusively text and data from Belik, I., & Jornsten, K. (2015)

2. SHAPLEY VALUE AS AN AGENT'S IMPORTANCE MEASURE[ii]

The Shapley value (Shapley, 1952) is a game theoretic approach that provides the solution for computing of players' gains in cooperative games where the players' contributions are non-equal. The formal definition of the Shapley value (SV) is well-described in Littlechild & Owen (1973) and Gul (1989). Specifically, the SV equation for the player *i* in the coalition game with *n* players is the following:

$$SV_i(v) = \sum_{S \subseteq N\setminus\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right), \quad (1)$$

where:
N is a set of *n* players;
S is a coalition of players;
v is the characteristic function: $2^N \rightarrow \mathbb{R}$; $v(\emptyset) = 0$.

The Shapley value is characterized by different properties such as efficiency, symmetry, linearity etc. (Hart, 1989).
The special interest for employing the SV approach in the area of socio-economic network analysis is based on its use to measure the importance of nodes. In other words, SV is interpreted as the level of the nodes' importance within a network (Suri & Narahari, 2008; Gomez, González-Arangüena, Manuel, Owen, del Pozo, & Tejada, 2003).
We employ the computational approach of the Shapley value as a centrality measure in networks that was developed by Aadithya et al. (2010). They introduced the idea of Shapley value "in the domain of networks, where it is used to measure the importance of individual nodes, which is known as game theoretic network centrality" (Aadithya et al., 2010).
Consider graph *G(V,E)* and $v_i \in V$. All nodes (i.e., neighbors), which are reachable from v_i at most one hop within *G(V,E)* are denoted by $N_G(v_i)$. The degree of node v_i is defined by $deg_G(v_i)$. The SV interpretation for node v_i in *G(V,E)*, according to Aadithya et al. (2010), is the following:

$$SV(v_i) = \sum_{v_j \in \{v_i\} \cup N_G(v_i)} \frac{1}{1+deg_G(v_j)}, \quad (2)$$

Based on equation (2) Aadithya et al. (2010) introduced an algorithm to calculate SVs for all nodes in the network:

SV-COMPUTING:
Input: Unweighted graph G(V,E)
Output: SVs of all nodes in V(G)
for each v ∈ V(G) **do**
    ShapleyValue [v]= $\frac{1}{1+deg_G(v)}$;
    For each u ∈ N_G(v) **do**
        ShapleyValue [v] += $\frac{1}{1+deg_G(u)}$;
    end
end
return ShapleyValue;

[ii] The section uses partly or exclusively text and data from Belik, I., & Jornsten, K. (2015)

The advantage of the given algorithm is the polynomial running time *O(V+E)* (Cormen, Leiserson, Rivest, & Stein, 2003) to compute SVs for all nodes in *G(V,E)* based on equation (2).
To illustrate how the algorithm works we consider the trivial example (see Figure 1).

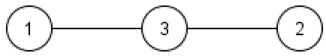

Figure 1. Example network for SVs computation

We calculate SVs for three nodes following the algorithm:

I. *For node 1:*

1) $\mathrm{SV}'(v_1) = \frac{1}{1+deg_G(v_1)} = \frac{1}{1+1} = \frac{1}{2}$

2) $u \in N_G(v_1)\colon u \in \{\text{"}node\ 3\text{"}\}$:

$\mathrm{SV}(v_1) = \mathrm{SV}'(v_1) + \frac{1}{1+deg_G(\text{"}node\ 3\text{"})} = \frac{1}{2} + \frac{1}{1+2} = \frac{5}{6}$

Thus, $\mathrm{SV}(v_1) = \frac{5}{6}$

II. *For node 2:*

1) $\mathrm{SV}'(v_2) = \frac{1}{1+deg_G(v_2)} = \frac{1}{1+1} = \frac{1}{2}$

2) $u \in N_G(v_2)\colon u \in \{\text{"}node\ 3\text{"}\}$:

$\mathrm{SV}(v_2) = \mathrm{SV}'(v_2) + \frac{1}{1+deg_G(\text{"}node\ 3\text{"})} = \frac{1}{2} + \frac{1}{1+2} = \frac{5}{6}$

Thus, $\mathrm{SV}(v_2) = \frac{5}{6}$

III. *For node 3:*

1) $\mathrm{SV}'(v_3) = \frac{1}{1+deg_G(v_3)} = \frac{1}{1+2} = \frac{1}{3}$

2) $u \in N_G(v_3)\colon u \in \{\text{node } 1, \text{"}node\ 2\text{"}\}$:

$\mathrm{SV}(v_3) = \mathrm{SV}'^{(v_3)} + \frac{1}{1+deg_G(\text{node } 1)} + \frac{1}{1+deg_G(\text{node } 2)} = \frac{1}{2} + \frac{1}{1+1} + \frac{1}{1+1} = \frac{4}{3}$

Thus, $\mathrm{SV}(v_3) = \frac{4}{3}$

Obviously, node 3 has the highest SV, and nodes 1 and 2 have equal SVs. The given results satisfy the efficiency requirement (Hart, 1989).

3. STACKELBERG LEADERSHIP FORMATION BASED ON SHAPLEY VALUE

In terms of networks, we consider Shapley-based Stackelberg competition as the strategic competition between two most powerful agents (i.e., nodes). Accordingly, leader is a node with the first largest Shapley value, and follower is a node with the second largest Shapley value. First step, we detect the leader and the follower in the network G, following the initialization procedure:

```
INITIALIZATION (G):
    SV-COMPUTING (G);
    NODES-SORTING (G);
    Leader = first largest in SL;
    Follower = second largest in SL;
```

Notation:

SV-COMPUTING (G): calculating SVs for all nodes in network G based on the algorithm represented in Section 2.

NODES-SORTING (G):

For graph G(V,E) we have SVs for all n=|G.V| nodes based on the SV-COMPUTING (G) results. Applying one of the sorting algorithms, such as Quick Sort, Heap Sort or Merge Sort (Cormen et al., 2003), we sort all nodes in descending order based on the corresponding SVs. NODES-SORTING(G) returns the sorted list of nodes (SL), where SL[1] is the node with the max SV-value, and SL[n] is the node with the min SV-value.

Based on the INITIALIZATION-procedure we have a list of nodes sorted by SV values, detected leader and follower, and their initial SVs, i.e., $SV_{initial}$(Leader) and $SV_{initial}$(Follower). Shapley-based Stackelberg competition starts with the leader's first move based on the LEADERSHIP (x) – procedure represented below. When the LEADERSHIP (x) – procedure is completed for the leader, then the follower acts following the LEADERSHIP (x) – procedure. Therefore, leader and follower act sequentially, following the LEADERSHIP (x) – procedure on each step, where x is a currently acting agent (i.e., the leader or the follower).

```
LEADERSHIP (x):
    k = 1;
    FOR i = k to n in SL:
        IF [edge (x, SL[i]) does not exist in G] AND [x ≠ SL[i]]:
        THEN: previous_SV(i) = SV(i);
              previous_SV(x) = SV(x);
              Establish test-edge (x, SL[i]);
              SV-COMPUTING (G);
              IF [previous_SV(i) < SV(i) AND previous_SV(x) < SV(x)]:
              THEN: approve edge (x, SL[i]);
                    Return: stop & save SV(x);
              ELSE: Erase edge (x, SL[i]);
                    Roll back to SV- and SL-results that exclude edge (x, SL[i]);
                    k = k + 1;
        ELSE: k = k + 1;
    return: save SV(x)
```

Shapley-based Stackelberg competition stops in one of two cases:

1) One of the agents (i.e., leader or follower) cannot improve its SV value. It means that on one of the iterations of the Shapley-based Stackelberg competition, the agent checked all possible links, but none of them was approved.
2) Node's degree (i.e., agent's number of links) achieves the value of (n-1).

When the Shapley-based Stackelberg competition is finished, we get the final SVs for the leader and for the follower, i.e. SV_{final}(Leader) and SV_{final}(Follower).

Shapley-based Stackelberg leadership equilibrium in networks:
The competing agents (i.e., the leader and the follower) are in the state of Shapley-based Stackelberg leadership equilibrium if the following conditions are satisfied:

1) $\mathrm{SV}_{\mathrm{final}}(\mathrm{Leader}) - \mathrm{SV}_{\mathrm{initial}}(\mathrm{Leader})$ is positive.
2) $\mathrm{SV}_{\mathrm{final}}(\mathrm{Follower}) - \mathrm{SV}_{\mathrm{initial}}(\mathrm{Follower})$ is positive.

4. TESTING ON NETWORKS

We show Shapley-based Stackelberg leadership formation based on two networks. First, we employ the network with a symmetric mixed topology. It combines different types of trivial topologies, such as "point-to-point", "star" and "ring". Second, we show our approach based on the real-life network that is represented by the connected network's component of the Department of Economics at NHH (Belik & Jornsten, 2014).

4.1 Symmetric network with mixed topology

The initial structure of the network G with symmetric mixed topology is represented in Figure 2. Based on the $\mathrm{INITIALIZATION}\ (G)$ – procedure we get two potential leaders (specifically, nodes 3 and 5) and four potential followers (specifically, nodes 1, 2, 6, and 7). SV results for the initial structure are represented in Table 1.

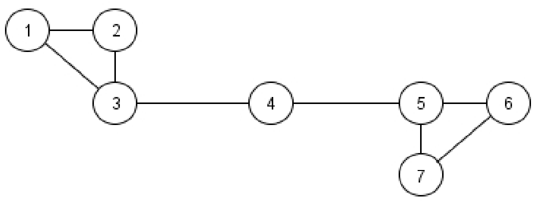

Figure 2. Symmetric mixed topology in the initial state

Table 1. Initial SVs

node	Shapley value
1	0.92
2	0.92
3	1.25
4	0.83
5	1.25
6	0.92
7	0.92

Since nodes SV("node 3") = SV("node 5"), we can assign any of them to be a leader. Similarly, we can assign any of the nodes 1, 2, 6 or 7 to be a follower.
We choose node 3 to be a leader and node 5 to be a follower.
According to the Shapley-based Stackelberg competition, the leader moves first, following the $\mathrm{LEADERSHIP}\ (x)$ – procedure and, accordingly, connecting to node 5. The results are represented in Figure 3 and Table 2.

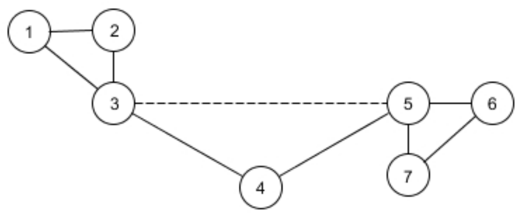

Figure 3. First modification of the symmetric mixed topology

Table 2. SVs after the first modification

node	Shapley value
1	0.87
2	0.87
3	1.40
4	0.73
5	1.40
6	0.87
7	0.87

Link “node 3 – node 5” improves the leadership positions for both nodes: ΔSV(“node 3”) = +0.15 and ΔSV(“node 5”) = +0.48. Therefore, link “node 3 – node 5” is approved based on the LEADERSHIP (x) – procedure.

Next, the follower (i.e., node 6) makes its move. Following the LEADERSHIP (x) – procedure, it links to node 3. As the result, we get a graph represented in Figure 4 (see Table 3).

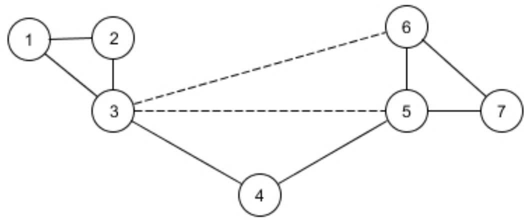

Figure 4. Second modification of the symmetric mixed topology

Table 3. SVs after the second modification

node	Shapley value
1	0.83
2	0.83
3	1.62
4	0.70
5	1.28
6	0.95
7	0.78

We approve the link “node 6 – node 3”, because the SVs for both nodes have been improved: ΔSV(“node 6”) = +0.08 and ΔSV(“node 3”) = +0.22. Since the increments are positive, we approve the given link.

Following the Shapley-based Stackelberg competition, the leader connects to node 7. The resulting network is represented in Figure 5 and the recalculated SVs are given in Table 4.

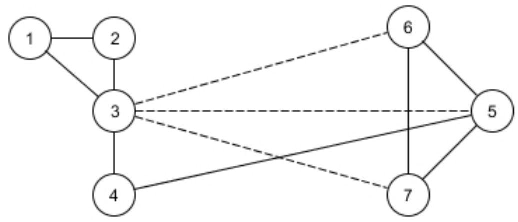

Figure 5. Third modification of the symmetric mixed topology

Table 4. SVs after the third modification

node	Shapley value
1	0.81
2	0.81
3	1.84
4	0.68
5	1.18
6	0.84
7	0.84

Link “node 3 – node 7” is approved, because the SVs for both nodes have been improved.

Sequentially, node 6 makes its move connecting to node 1. Based on the LEADERSHIP (x) – procedure we get the updated structure (see Figure 6) and the resulting SVs that are represented in Table 5.

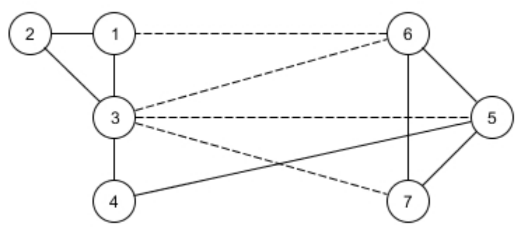

Figure 6. Fourth modification of the symmetric mixed topology

Table 5. SVs after the fourth modification

node	Shapley value
1	0.93
2	0.73
3	1.71
4	0.68
5	1.13
6	1.04
7	0.79

Link "node 6 – node 1" improves the leadership positions for both nodes: ΔSV("node 6") = +0.20 and ΔSV("node 1") = +0.12. Therefore, we approve link "node 6 – node 1".
Next, the leader has to make its move, but its degree (i.e., number of links) is equal to (n-1) in the given network. It corresponds to the second case of the Shapley-based Stackelberg competition that specifies the stopping procedure. Specifically, no more links can be established for the current node. The stop+conditions are described in Section 3. Therefore, the Shapley-based Stackelberg leadership formation for the current network competition is finished on this step
Finally, we check if the leader and the follower achieved the Shapley-based Stackelberg leadership equilibrium:

1) $\mathrm{SV}_{\mathrm{final}}(\mathrm{Leader}) - \mathrm{SV}_{\mathrm{initial}}(\mathrm{Leader})$ = 1.71 – 1.25 = +0.46
2) $\mathrm{SV}_{\mathrm{final}}(\mathrm{Follower}) - \mathrm{SV}_{\mathrm{initial}}(\mathrm{Follower})$ = 1.04 – 0.92 = +0.12

Since both increments are positive, we conclude that the leader and the follower are in the Shapley-based Stackelberg leadership equilibrium for the given network.

4.2 Real-life network

We illustrate our approach based on the co-authorship network of the Department of Economics at the Norwegian School of Economics (NHH) (Belik, & Jornsten, 2014). Specifically, we analyze the connected component of the departmental graph (see Figure 7). The given network is based on the scientific collaboration between the faculty members and their academic publication records according to the ISI Web of Science for the period 1950 – Spring, 2014.
Network's nodes correspond to the departmental faculty members, and the edges correspond to the existing joint publications. The initial SVs calculated for the network are represented in Table 6.

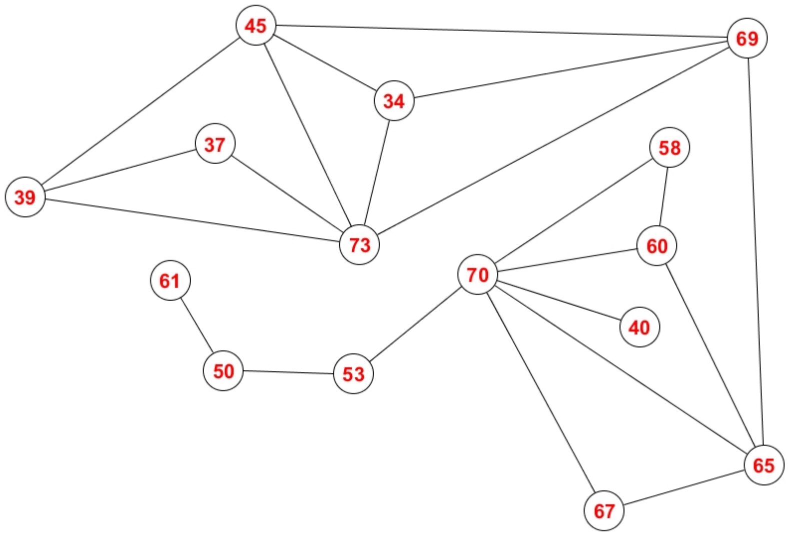

Figure 7. Departmental connected component

Table 6. The initial SVs for the departmental connected component

NODE	SV	NODE	SV
node 34	0.82	node 60	0.93
node 37	0.75	node 61	0.83
node 39	0.95	node 65	1.13
node 40	0.64	node 67	0.68
node 45	1.07	node 69	1.02
node 50	1.17	node 70	2.09
node 53	0.81	node 73	1.40
node 58	0.73		

Based on the INITIALIZATION (G) – procedure we detect the leader and the follower that are node 70 and node 73, respectively. Iteratively running Shapley-based Stackelberg competition based on the LEADERSHIP (x) – procedure we get the results represented in Table 7.

Table 7. SVs based on the established links in the Shapley-based Stackelberg competition

LINK			Shapley Value			
Iteration	Start node	End node	Start node		End node	
			before	after	before	after
1	node 70	node 73	2.09	2.22	1.40	1.50
2	node 73	node 50	1.50	1.73	1.17	1.21
3	node 70	node 50	2.39	2.20	1.21	1.27
4	node 73	node 65	1.67	1.82	1.09	1.17
5	node 70	node 45	2.34	2.49	1.01	1.08
6	node 73	node 60	1.78	1.97	0.85	0.90
7	node 70	node 69	2.43	2.59	0.88	0.94
8	node 73	node 53	1.92	2.17	0.62	0.63
9	node 70	node 39	2.50	2.69	0.84	0.87

It is important to notice that during the given Shapley-based Stackelberg competition the link "node 15 – node 10" was not approved on iteration 8, because the increment of SV ("node 10") was negative. When link "node 70 – node 39" was established on iteration 9, then node 73 (i.e., the follower) did four trials to establish links on iteration 10, but none one of the links were approved due to the reasons represented in Table 8:

Table 8. Trials to establish links for the follower on iteration 10

TRIAL	REASON
Link «node 15 – node 10»	ΔSV("node 10") is negative
Link «node 15 – node 8»	ΔSV("node 8") is equal to zero
Link «node 15 – node 4»	ΔSV("node 4") is negative
Link «node 15 – node 12»	ΔSV("node 12") is equal to zero

Since the follower could not improve its SV value on iteration 10, and, sequentially, no link was established, then the Shapley-based Stackelberg competition was finished based on the first case of the stop conditions described in Section 3.
Therefore, on the final ninth iteration of the Shapley-based Stackelberg leadership formation we get $\mathrm{SV_{final}}(\mathrm{Leader}) = 2.69$ and $\mathrm{SV_{final}}(\mathrm{Follower}) = 2.11$.
The updated network's structure is represented in Figure 8.

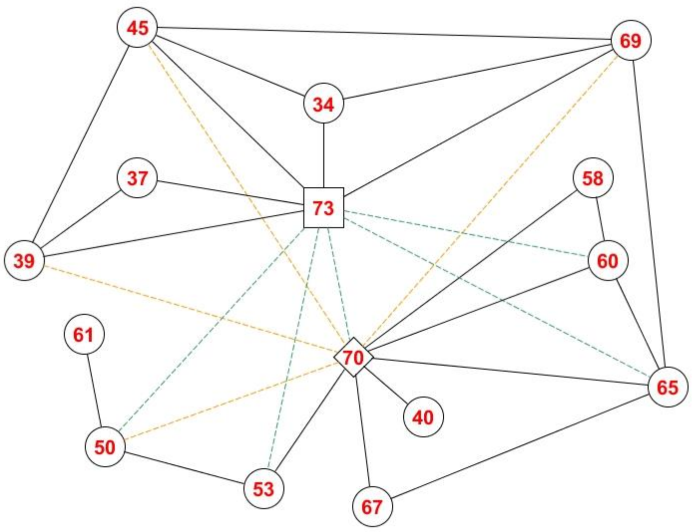

Figure 8. The updated structure of the departmental connected component

Finally, we check if the leader and the follower achieved the Shapley-based Stackelberg leadership equilibrium:

3) $\mathrm{SV_{final}}(\mathrm{Leader}) - \mathrm{SV_{initial}}(\mathrm{Leader}) = 2.69 - 2.09 = +0.60$
4) $\mathrm{SV_{final}}(\mathrm{Follower}) - \mathrm{SV_{initial}}(\mathrm{Follower}) = 2.11 - 1.40 = +0.71$

Since both increments are positive, we conclude that the leader and the follower are in the Shapley-based Stackelberg leadership equilibrium for the given network.

5. CONCLUSION

Generally, the analysis of leadership formation is an important problem in terms of the strategic network formation. It becomes even more important if the most influential agents (i.e., the leader and the follower) are competing for the influence within a network. Since they are initially the most powerful in a network, the analysis of their actions is critically important in terms of understanding the prospective network modifications.

In the given research, we represented the integrative approach that reflects the mechanism of the leadership formation in networks. Specifically, we employ the Shapley value concept to measure the influential power of nodes (i.e., agents) and the concept of the Stackelberg competition to formalize a leader-follower behavior in networks. The resulting formalization of the Shapley-based Stackelberg competition reflects the leadership formation in networks in terms of the leader-follower sequential acting. In addition, we provided the interpretation of the Shapley-based Stackelberg leadership equilibrium in networks.

The represented mechanism of the Shapley-based Stackelberg leadership formation is tested based on the symmetric network with mixed topology and the real-life network retrieved from the NHH co-authorship network. The results are tested and represented in tabular and graphical formats.

REFERENCES

Aadithya, K. V., Ravindran, B., Michalak, T. P., & Jennings, N. R. (2010). Efficient computation of the Shapley value for centrality in networks. In *Internet and Network Economics* (pp. 1-13). Springer Berlin Heidelberg.

Anthonisse, J. M. (1971). *The Rush in a Directed Graph.* Tech. Rep. BN 9/71, Stichting Mathematisch Centrum, 2e Boerhaavestraat 49 Amsterdam.

Basar, T., Olsder, G. J., Clsder, G. J., Basar, T., Baser, T., & Olsder, G. J. (1995). *Dynamic noncooperative game theory* (Vol. 200). London: Academic press.

Beauchamp, M. A. (1965). An Improved Index of Centrality. *Behavioral Science, 10*, 161–163.

Belik, I., & Jornsten, K. (2014). The Comparative Analysis of the NHH and BI Networks. *NHH Dept. of Business and Management Science Discussion Paper No. 2014/34*. Available at SSRN:http://ssrn.com/abstract=2510292 or http://dx.doi.org/10.2139/ssrn.2510292

Belik, I., & Jornsten, K. (2015). The Analysis of Leadership Formation in Networks Based on Shapley Value. *NHH Dept. of Business and Management Science Discussion Paper No. 2015/2.* Available at SSRN: http://ssrn.com/abstract=2549618 or http://dx.doi.org/10.2139/ssrn.2549618

Cormen, T. H., Leiserson, C. E., Rivest, R. L., & Stein, C. (2003). *Introduction to Algorithms*. MIT Press. Cambridge, MA.

Freeman, L. C. (1977). A Set Of Measures of Centrality Based upon Betweenness. *Sociometr*y, *40*, 35–41.

Freeman, L. C. (1979). Centrality in Social Networks: Conceptual Clarification. *Social Networks, 1*, 241–256.

Gomez, D., González-Arangüena, E., Manuel, C., Owen, G., del Pozo, M., & Tejada, J. (2003). Centrality and power in social networks: a game theoretic approach. *Mathematical Social Sciences*, *46*(1), 27-54.

Gul, F. (1989). Bargaining foundations of Shapley value. *Econometrica: Journal of the Econometric Society*, 81-95.

Hart, S. (1989). Shapley Value. In *The New Palgrave: Game Theory*, J. Eatwell, M. Milgate and P. Newman (Editors), Norton, pp. 210–216.

He, X., Prasad, A., Sethi, S. P., & Gutierrez, G. J. (2007). A survey of Stackelberg differential game models in supply and marketing channels. *Journal of Systems Science and Systems Engineering*, *16*(4), 385-413.

Kleinberg, J. M. (1999). Authoritative sources in a hyperlinked environment. *Journal of the ACM (JACM)*, *46*(5), 604-632.

Littlechild, S. C., & Owen, G. (1973). A simple expression for the Shapley value in a special case. *Management Science*, *20*(3), 370-372.

Page, L., Brin, S., Motwani, R., & Winograd, T. (1999). The PageRank citation ranking: Bringing order to the web. Stanford Digital Libraries SIDL-WP-1999-0120.

Sabidussi, G. (1966). The Centrality Index of a Graph. *Psychometrika, 31*, 581–603.

Scripps, J., Tan, P. N., & Esfahanian, A. H. (2007, August). Node roles and community structure in networks. In *Proceedings of the 9th WebKDD and 1st SNA-KDD 2007 workshop on Web mining and social network analysis* (pp. 26-35). ACM.

Shapley, L. S. (1952). *A value for n-person games* (No. RAND-P-295). RAND CORP SANTA MONICA CA.

Simaan, M., & Cruz Jr, J. B. (1973). On the Stackelberg strategy in nonzero-sum games. *Journal of Optimization Theory and Applications*, *11*(5), 533-555.

Suri, N. R., & Narahari, Y. (2008, May). Determining the top-k nodes in social networks using the Shapley value. In *Proceedings of the 7th international joint conference on Autonomous agents and multiagent systems-Volume 3* (pp. 1509-1512). International Foundation for Autonomous Agents and Multiagent Systems.

Von Stackelberg, H. (2010). *Market structure and equilibrium*. Springer Science & Business Media.