Existence and Uniqueness of Equilibrium in a Reinsurance Syndicate

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Abstract

In this paper we consider a reinsurance syndicate, assuming that Pareto optimal allocations exist. Under a continuity assumption on preferences, we show that a competitive equilibrium exists and is unique. Our conditions allow for risks that are not bounded, and we show that the most standard models satisfy our set of sufficient conditions, which are thus not too restrictive. Our approach is to transform the analysis from an infinite dimensional to a finite dimensional setting.

KEYWORDS: existence of equilibrium, uniqueness of equilibrium, Pareto optimality, reinsurance model, syndicate theory, risk tolerance, exchange economy, probability distributions, Walras' law

I Introduction

We consider the reinsurance syndicate introduced by Borch (1960-62), a model closely related to the exchange economy studied by Arrow (1954). Bühlmann (1984) shows that, provided that there are Pareto optimal risk

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exchanges, an equilibrium exists for bounded risks. While this result may be of interest for practical purposes (since the accumulated wealth in the World is obviously bounded), in modeling contexts this precludes many probability distributions that are of interest, but which may just happen to have unbounded supports.

Bühlmann 's arguments are based on affine contracts, but we shall extend to arbitrary contracts in this paper. We basically swap his assumption of bounded risks and a Lipschitz condition with a continuity requirement on preferences. The latter we demonstrate is satisfied for the most common exchange economies studied within the "finance contexts". Under this condition we demonstrate both existence and uniqueness of equilibrium.

When Parto optimal risk exchanges exist, there will be competitive equilibria after a redistribution of the initial endowments $X_i, i \in \mathcal{I} := \{1, 2, \cdots, I\}$, here a set of random variables referred to as the initial portfolio allocation of the I members of the reinsurance syndicate (by the Second Welfare Theorem). We provide a set of sufficient conditions for the existence of an equilibrium for a *given* set of initial portfolios $X = (X_1, X_2, \cdots, X_I)$. Since the set of sufficient conditions for a Pareto optimal exchange to exist are very weak indeed for the model that we consider (see e.g., DuMouchel (1968)), our approach is not restrictive for this reason. In fact, if there are no Pareto optimal contracts, there can not be a competitive equilibrium either, by the First Welfare Theorem.

The existence of equilibrium in infinite dimensional models is, of course, extensively studied in the mathematical economics literature. Bewley (1972) is an early reference of existence in infinite-dimensional spaces, and later this topic has been extensively studied by many authors, including Mas-Colell (1986), Mas-Colell and Zame (1991), Araujo and Monteiro (1989), and Dana (1993) among others. Uniqueness of equilibrium is a lesser explored subject in infinite dimensional settings.

Our approach will be based to a large extent on "risk theory", which requires us to first define what is meant by a reinsurance syndicate. This essentially enables us to transform problems from the infinite dimensional space of L^2, to finite dimensional Euclidian space.

In Section 2 we present some of the basic properties of such a market. In Section 3 we discuss existence of equilibrium in a reinsurance syndicate, and give the basic existence theorem of the paper. Our exposition rely mainly on the results of Section 2, and a fixed point theorem. Here one can also find several examples, and we prove uniqueness of equilibrium. Section 4 compares our result to a corresponding theorem emerging from a more general theory of an exchange economy, and Section 5 concludes.

II The reinsurance Syndicate

Consider a one-period model of a syndicated market with two time points, zero and one. The initial portfolio allocation of the members is denoted by $X = (X_1, X_2, \cdots, X_I)$, i.e., the one which realizations would result at time one if no reinsurance exchanges took place. At time zero X is a random vector defined on a probability space $(\Omega, \mathcal{F}, P)$ with a probability distribution function $F(x) = P[X_1 \leq x_1, \cdots, X_I \leq x_I]$. After reinsurance at time zero the random vector $Y = (Y_1, Y_2, \cdots, Y_I)$ results, the final portfolio allocation, satisfying $\sum_{i \in \mathcal{I}} Y_i = \sum_{i \in \mathcal{I}} X_i$, since nothing "disappears" or is added in a pure exchange of risks.

One difference between a syndicate and and the general exchange economy of Arrow (1954) is that the variables X_i signify economic gains or losses measured in some unit of account, not consumption, which implies that negative values are allowed. When this happens to a member, this person may be interpreted to be bankrupt.

Consider the problem of each member i of the syndicate

$$\sup_{Z_i \in L^2} Eu_i(Z_i) \qquad \text{subject to} \qquad \pi(Z_i) \leq \pi(X_i), \tag{1}$$

for $i \in \mathcal{I}$; the members maximize expected utility subject to their budget constraints.

Let us call a treaty Y *feasible* if it satisfies $\sum_{i=1}^{I} Y_i \leq \sum_{i=1}^{I} X_i := X_M$, where by X_M we mean the "market portfolio", which is just the aggregate of the initial portfolios of the members. Our definition of equilibrium is:

Definition 1 *A competitive equilibrium is a collection* $(\pi; Y_1, Y_2, \ldots, Y_I)$ *consisting of a price functional* π *and a feasible allocation* $Y = (Y_1, Y_2, \ldots, Y_I)$ *such that for each i,* Y_i *solves the problem (1).*

An important feature of this syndicate is that there are no restrictions on contract formation. As a consequence it can be shown that *the pricing functional* π *must be linear and strictly positive if and only if there does not exist any arbitrage* (e.g., Aase (2002)).

We shall restrict attention to initial portfolios X_i and sharing rules Y_i, all in $L^2 := L^2(\Omega, \mathcal{F}, P)$, that involve no arbitrage.

Since any (strictly) positive, linear functional on L^2 is also continuous, by the Riesz Representation Theorem there exists a unique random variable $\xi \in L^2_{++}$, the interior of the positive cone of L^2, such that

$$\pi(Z) = E(Z\xi) \qquad \text{for all} \qquad Z \in L^2.$$

Notice that the system is closed by assuming *rational expectations*. This means that the market clearing price π implied by the members behavior is assumed to be the same as the price functional π on which the members decisions are based.

Formally our definition of (strong) Pareto optimality is the following

Definition 2 *A feasible allocation $Y = (Y_1, Y_2, \ldots, Y_I)$ is called Pareto optimal if there is no feasible allocation $Z = (Z_1, Z_2, \ldots, Z_I)$ with $Eu_i(Z_i) \geq Eu_i(Y_i)$ for all i and with $Eu_j(Z_j) > Eu_j(Y_j)$ for some j.*

The following characterization of Pareto optimal allocations is well known:

Proposition 1 *Suppose u_i are concave and increasing for all i. Then Y is a Pareto optimal allocation if and only if there exists a nonzero vector of member weights $\lambda \in R_+^I$ such that $Y = (Y_1, Y_2, \ldots, Y_I)$ solves the problem*

$$\sup_{(Z_1,\ldots,Z_I)} \sum_{i=1}^{I} \lambda_i E u_i(Z_i) \qquad \textit{subject to} \qquad \sum_{i=1}^{I} Z_i \leq X_M. \tag{2}$$

If the allocation Y is Pareto optimal, then the problem (2) defines a utility function $u_\lambda(\cdot) : R \to R$ for this λ, such that

$$E u_\lambda(X_M) = \sum_{i=1}^{I} \lambda_i E u_i(Y_i). \tag{3}$$

Notice that the existence of the member weights λ is a consequence of the Separating Hyperplane Theorem applied to Euclidian R^I. As it turns out, these member weights determine state prices via the marginal utility $u'_\lambda(X_M)$ of the representative member computed at the aggregate portfolio X_M. Thus, despite of the unfortunate fact that the interior of L_+^2 is empty, there is still hope to get supportability of preferred sets via the construction in Proposition 1.

Pareto optimal allocations can be further characterized under the above conditions, the following is known as Borch's Theorem (see e.g., Borch (1960-62)):

Proposition 2 *A Pareto optimum Y is characterized by the existence of non-negative member weights $\lambda_1, \lambda_2, \ldots, \lambda_I$ and a real function $u'_\lambda(\cdot) : R \to R$, such that*

$$\lambda_1 u'_1(Y_1) = \lambda_2 u'_2(Y_2) = \cdots = \lambda_I u'_I(Y_I) := u'_\lambda(X_M) \qquad a.s. \tag{4}$$

Proposition (2) can be proven from Proposition (1) by the Kuhn-Tucker theorem and a variational argument (see e.g., Aase (2002)).

Karl Borch's characterization of a Pareto optimum $Y = (Y_1, Y_2, \cdots, Y_I)$ simply says that there exist positive "member" weithts λ_i such that the marginal utilities at Y of all the members are equal modulo these constants.

Because of the smoothness assumptions of Proposition 1 which we maintain in this paper, both sides of the equations (4) are real, differentiable functions (the right-hand side because of the implicit function theorem), i.e., $Y_i(\cdot) : B \to R$ and $u'_\lambda(\cdot) : B \to R$ for some subset $B \subseteq R$ of the reals, so taking derivatives of both sides gives

$$u''_i(Y_i(x))Y'_i(x) = \lambda_i^{-1} u''_\lambda(x), \qquad x \in B \subseteq R.$$

Dividing the second equation by the first, we obtain the following non-linear differential equation for the Pareto optimal allocation function $Y_i(x)$:

$$\frac{dY_i(x)}{dx} = \frac{R_\lambda(x)}{R_i(Y_i(x))}, \qquad Y_i(x_0) = b_i, \qquad x, x_0 \in B, \tag{5}$$

where $R_\lambda(x) = -\frac{u''_\lambda(x)}{u'_\lambda(x)}$ is the absolute risk aversion function of "the representative member", and $R_i(Y_i(x)) = -\frac{u''_i(Y_i(x))}{u'_i(Y_i(x))}$ is the absolute risk aversion of member i at the Pareto optimal allocation function $Y_i(x)$, $i \in \mathcal{I}$.

Since $\sum_{i \in \mathcal{I}} Y'_i(x) = 1$, we now get by summation in (5) that

$$\rho_\lambda(x) = \sum_{i \in \mathcal{I}} \rho_i(Y_i(x)), \; x \in B,$$

or

$$\rho_\lambda(X_M) = \sum_{i \in \mathcal{I}} \rho_i(Y_i(X_M)) \qquad a.s. \tag{6}$$

as an equality between random variables. This allows us to rewrite the differential equations (5) as follows

$$\frac{dY_i(x)}{dx} = \frac{\rho_i(Y_i(x))}{\rho_\lambda(x)}, \qquad Y_i(x_0) = b_i, \qquad x, x_0 \in B. \tag{7}$$

In other words, provided Pareto optimal sharing rules exist, we have the following results, which we shall utilize later:

Proposition 3 *(a) The risk tolerance of the syndicate $\rho_\lambda(X_M)$ equals the sum of the risk tolerances of the individual members in a Pareto optimum.*

(b) The real, Pareto optimal allocation functions $Y_i(x) : R \to R$, $i \in \mathcal{I}$ satisfy the first order, ordinary nonlinear differential equations (7).

(c) The following relationships hold

$$\frac{\partial}{\partial \lambda_i} u'_\lambda(x) = \frac{1}{\lambda_i} \frac{dY_i(x)}{dx} u'_\lambda(x), \qquad x \in B, \qquad i \in \mathcal{I}. \tag{8}$$

The result in (a) was found by Borch (1985); see also Bühlmann (1980) for the special case of exponential utility functions, and also Gerber (1978), among others. The result in (c) is contained in Theorem 10 p. 130 in Wilson (1968).

It is well-known that if an equilibrium exists, then the first order necessary and sufficient conditions are given by the equations (4). If this is the case, then the Riesz representation ξ, also called the state price deflator, is given by $\xi = u'_\lambda(X_M)$ a.s. This is our next result:

Assume that $\pi(X_i) > 0$ for each i. It seems reasonable that each member of the syndicate is required to bring to the market an initial portfolio of positive value. In this case we have the following (a proof can be found in Aase (2002)):

Theorem 1 *Suppose that $u'_i > 0$ and $u''_i \leq 0$ for all $i \in \mathcal{I}$, and assume that a competitive equilibrium exists, where $\pi(X_i) > 0$ for each i. The equilibrium is then characterized by the existence of positive constants α_i, $i \in \mathcal{I}$, such that for the equilibrium allocation $Y = (Y_1, Y_2, \dots, Y_I)$*

$$u'_i(Y_i) = \alpha_i u'_\lambda(X_M), \qquad a.s. \qquad \text{for all} \qquad i \in \mathcal{I}, \tag{9}$$

Here α_i are the Lagrange multipliers associated with the problem (1), and the relation between these and the member weights λ_i is seen to be $\alpha_i = \lambda_i^{-1}$ for all $i \in \mathcal{I}$.

III Existence and Uniqueness of Equilibrium

Will there always exist prices such that the budget constraint all hold with equality? We will now analyze this question for the reinsurance syndicate just descrecibed.

The problem of existence of equilibrium in an infinite dimensional setting has been extensively discussed in the literature. Several difficulties are identified, among them that the interior of the orthant L^2_+ is empty, so calculus becomes rather difficult. Normally the Separating Hyperplane Theorem guarantees that it will be possible to separate a convex set C from a point $x \notin C$, provided that the interior of C is not empty. Hence, if consumption sets have non-empty interior, then the continuity and convexity of preferences will guarantee that preferred sets can be price supported.

As commented after Proposition 1, despite of this difficulty we obtain the member weights by a separation argument, which provides us with state prices via the representative member's marginal utility at X_M. It should thus be possible to use this construction to show existence of equilibrium. As it turns out, all we have to do is to make an extra smoothness assumption on preferences. In this section we make this precise by utilizing the results of the previous section to essentially transform the problem from an infinite dimensional to a finite dimensional setting.

To this end we start with the initial portfolios X_i, which are supposed to satisfy $X_i \in L^2$, $i \in \mathcal{I}$. The final portfolios Y_i and the state price deflator ξ are supposed to be in L^2 and L^2_{++} respectively, according to this theory, the latter because L^2 is its own dual space, where the two plusses stems from the absence of arbitrage. However, both the probability distribution of X and the utility functions are exogenously given, and it is not clear at the outset that any choice of these, satisfying $X_i \in L^2$, will have these properties. From the results of the previous section, it follows that $|Y_i| \leq |X_M|$ for all i, so if $X_M \in L^2$, then $Y_i \in L^2$ for all $i \in \mathcal{I}$. However it is far from clear that $\xi = u'_\lambda(X_M) \in L^2$, which this theory requires to be internally consistent. That is, will there exist state prices $\xi = u'_\lambda(X_M)$ having finite variances such that the budget constraints are all satisfied? These are the problems we now address.

First we notice a few facts about about the existence problem. The state prices $u'_\lambda(X_M)$ are determined by the member weights λ, and the budget sets remain unchanged if we multiply all these weights by any positive constant, so each member's demand function $Y_i(X_M) := Y_i^{(\lambda)}$ is accordingly homogeneous of degree zero in λ. Hence we can restrict attention to member weights belonging to the $(I-1)$ dimensional unit simplex

$$S^{I-1} = \{\lambda \in R^I_+ : \sum_{i=1}^{I} \lambda_i = 1\}.$$

Since we consider a pure exchange economy with strictly increasing utility functions, an equilibrium will exist if there exists some $\lambda \in S^{I-1}$ such that

$$E(u'_\lambda(X_M)(Y_i^{(\lambda)} - X_i)) = 0, \qquad \text{for } i = 1, 2, \cdots, I, \tag{10}$$

where we have chosen to parameterize the optimal allocations $Y_i(X_M)$ by the member weights λ. The existence problem may be resolved if one can identify these budget constraints with a continuous function $f : S^{I-1} \rightarrow S^{I-1}$ and then employ Brower's fixed-point theorem.

The idea is perhaps best illustrated by a few examples: In the first one the utility functions are negative exponentials.

Example 1: Suppose $u_i'(x) = e^{-\frac{x}{a_i}}, i \in \mathcal{I}$. It is a consequence of Proposition 2 that the Pareto optimal allocations are affine in the aggregate wealth X_M, i.e.,

$$Y_i^\lambda := Y_i(X_M) = \frac{a_i}{A} X_M + b_i,$$

where the constants a_i are the risk tolerances of the members, $A = \sum_{i\in\mathcal{I}} a_i$ by the result (6), so that A the risk tolerance of the representative member or the syndicate, and b_i are zero-sum side-payments, corresponding to $Y_i(x_0) = b_i$ for $x_0 = 0$.

By imposing the normalization $E(u_\lambda'(X_M)) = 1$ (corresponding to a zero risk-free interest rate), the budget constraints of the members correspond to the equations

$$\lambda_i = \frac{e^{\frac{b_i}{a_i}}}{E\{e^{-\frac{X_M}{A}}\}}, \qquad i \in \mathcal{I}, \tag{11}$$

where the zero-sum side-payments b_i are given by

$$b_i = \frac{E\{X_i e^{-X_M/A} - \frac{a_i}{A} X_M e^{-X_M/A}\}}{E\{e^{-X_M/A}\}}, \qquad i \in \mathcal{I}. \tag{12}$$

Since there is a one to one connection between the member weights λ_i and the side-payments b_i, the latter could alternatively be used in the fixed-point argument. $\square$

The second example is that of constant relative risk aversion:

Example 2: Preferences represented by power utility means that $u_i(x) = (x^{1-a_i} - 1)/(1 - a_i)$ for $x > 0$, $a_i \neq 1$ and $u_i(x) = \ln(c_i x + d_i)$ for $x > 0$ and $a_i = 1$, for positive constants c_i and d_i, where the natural logarithm results as a limit when $a_i \to 1$. This example only makes sense in the no-bankruptcy case where $X_i > 0$ a.s. for all i.

Let us assume that the supports of the initial portfolios are $(0, \infty)$, and $Y_i(x_0) = b_i$ for some $x_0 > 0$. The parameters $a_i > 0$ are the *relative risk aversions* of the members, here given by positive constants, and we consider the HARA-case where $a_1 = a_2 = \ldots = a_I = a$.

The marginal utilities of the members are given by $u_i'(x) = x^{-a}$, and the Pareto optimal allocations Y_i^λ are found from Proposition 2 to be

$$Y_i(X_M) = \frac{\lambda_i^{1/a}}{\sum_{j\in\mathcal{I}} \lambda_j^{1/a}} X_M, \qquad i \in \mathcal{I}. \tag{13}$$

The differential equations (5) for these allocations are

$$\frac{dY_i(x)}{Y_i(x)} = \frac{dx}{x}, \qquad Y_i(x_0) = b_i \qquad i \in \mathcal{I}, \tag{14}$$

showing that $Y_i(X_M) = \frac{b_i}{x_0} X_M$, where b_i is member i's share of the market portfolio when the latter takes on the value x_0, where $\sum_{j\in\mathcal{I}} b_i = x_0$.

Comparing the two versions of the Pareto optimal allocations, we notice that $\frac{b_i}{x_0} = \frac{\lambda_i^{1/a}}{\sum_{j\in\mathcal{I}} \lambda_j^{1/a}}$, again giving a one to one correspondence between the constants b_i of the differential equations (5) and the member weights λ_i. The member weights λ_i are determined by the budget constraints, implying that

$$\lambda_i = k \left(\frac{E(X_i X_M^{-a})}{E(X_M^{1-a})} \right)^a, \qquad i \in \mathcal{I}, \tag{15}$$

or, λ_i is determined modulo the proportionality constant $k = (\sum_{j\in\mathcal{I}} \lambda_j^{1/a})^a$ for each i. $\square$

For both these examples we have computed the respective equilibria, where it is understood that the expectations appearing in the expressions for the member weights exist. This must accordingly follow from any set of sufficient conditions for existence of equilibrium.

The reason that the existence of the λ_i, or, equivalently the b_i, is not automatic, is that both the probability distribution of X and the utility functions are given exogenously, as explained in the introduction. Although it is clear that if $X_M \in L^2$, then also $Y_i \in L^2$, it is still not obvious that $\xi = u'_\lambda(X_M)$ is in L^2. This has to be checked separately.

While the first order conditions for an optimal exchange of risks do not depend on the probability distribution of the vector X of the initial endowments, clearly the equilibrium allocation $Y^{(\lambda)}$ does depend on this distribution through the budget constraints, and only if this probability distribution allows for the computation of the moments appearing in the expressions for the member weights λ_i, as e.g., in (11) and (15), the relevant equilibrium will stand a chance to exist.

These examples indicate that instead of focusing attention on the member weights λ_i, we might as well consider the constants b_i of the differential equations (5), and try to associate with the budget constraints a fixed-point for these. This observation turns out to be quite general, and is the line of attack we choose to follow.

A natural condition to impose for the constants b_i to exist, might be that all the risks are bounded. Often this is too strong. For example if X is multinormally distributed, and thus possesses unbounded supports, certainly the moments in (12) can still be computed, and are well defined. This is also the case for many other distributions with unbounded supports.

However, even in the case with bounded supports it is not clear that the pricing functional π is continuous. To see this, consider Example 2 with $B = (0, 1]$. Here the state prices represented by the function $u'_\lambda(X_M) = cX_M^{-a}$ for some constant c depending on the member weights λ and a. Suppose that X_M is uniformly distributed on $(0, 1)$. Then all the initial portfolios have bounded supports, but it is seen that $u'_\lambda(X_M)$ is not a member of L^2 if $a > 1/2$, e.g., in the log utility case there would be no equilibrium. Empirical research indicate that the parameter a is in the range between 1 and 20, so for this particular example there is no equilibrium in the interesting parameter range.[1]

III-A A basic fixed point argument

As observed in the previous section, instead of focusing attention on the member weights λ_i (because these determine prices via $u'_\lambda(X_M)$), we restrict attention to the constants b_i of the differential equations (7). The optimal allocations, now parameterized by b instead of λ, are functions of the aggregate risk X_M, i.e., $Y_i^{(b)} := Y_i(X_M)$, where $Y_i(\cdot) : B \to R$. Likewise the state price deflator ξ also depends on b through Proposition (2), allowing us write $\xi = u_b(X_M)$ to emphasize this.

Returning to the first order, non-linear differential equations (7) for the optimal allocations $Y_i^{(b)}$, in order to use the standard theory of differential equations of this type, Bühlmann (1984) used the following assumption:[2]

(A1) The risk tolerance functions $\rho_i(y)$ satisfy the Lipschitz condition $|\rho_i(y) - \rho_i(y')| \leq M|y - y'|$ for all i.

Let us check some of the most used examples, and see if this requirement seems plausible: For negative exponential utility, the marginal utility is given by $u'_i(x) = \frac{1}{a_i}e^{-x/a_i}$ and the risk tolerance $\rho_i(y) = a_i$, so $|\rho_i(y) - \rho_i(y')| = 0$, and the condition is trivially satisfied.

For power utility $u_i(x) = \frac{1}{(1-a_i)}x^{(1-a_i)}$ with constant relative risk aversion $a_i \neq 1$, the risk tolerance $\rho_i(y) = (1/a_i)y$ and $|\rho_i(y) - \rho_i(y')| = (1/a_i)|y - y'|$,

[1] Bühlmann's (1984) overlooked this possibility, and confined his analysis to situations of the type described by Example 1.

[2] In Bühlmann (1984), the assumption (A1) was made for the absolute risk aversions $R_i(y)$ instead of the risk tolerances $\rho_i(y)$. In this case we do not obtain that e.g., power, or logarithmic utility functions satisfy Bühlmann's assumption H. It is not clear that the differential equation (7) has a solution under H. But (A1) is what we think he meant.

so here the condition is satisfied using $M = \max_i\{\frac{1}{a_i}\}$.

When the relative risk aversion equals one, the logarithmic utility function is appropriate, i.e., $u_i(x) = \ln(c_i x + d_i)$ for positive constants c_i and d_i. In this case the risk tolerance $\rho_i(y) = y + \frac{d_i}{c_i}$ in which case (A2) holds with $M = 1$.

Our basic assumption is that $X_i \in L^2$ for all $i \in \mathcal{I}$. By Minkowski's inequality also $X_M \in L^2$, but what about the optimal portfolios Y_i? Recall from (6) that $\rho(x) = \sum_{i=1}^{I} \rho_i(Y_i(x))$, implying that

$$|Y_i(X_M) - Y_i(0)| \leq |X_M|, \tag{16}$$

which means that and $Y_i \in L^2$ for all $i \in \mathcal{I}$ as well.

Bühlmann's assumptions of finite supports of the X_i together with assumption (A1) allowed him to use standard, global results of ordinary, non-linear differential equations to guarantee that the optimal allocations are continuous in the constants b_i. In order to relax this condition, observe that the differential equations given by (7) are indeed very "nice", since the non-linear functions

$$F_i(y_i, x) := \frac{\rho_i(y_i)}{\rho(x)}$$

satisfiy $|F_i(y_i, x)| \leq 1$ for all i due to (6). Thus Witner's condition of global existence is satisfied for the differential equations (7). In this case we do indeed have global existence and uniqueness of solutions for these equations, over the entire region $(x, y_i) \in R^2$. In order for the solutions $Y_i(x)$ to be *continuous* functions of the constants b_i, the following is sufficient:

(A2) The functions $F_i(y_i, x)$ and $\frac{d}{dy_i} F(y_i, x)$ are continuous for all (x, y_i).

This assumption also replaces (A1). Let us check (A2) for the standard cases. For the negative exponential utility function we can use the domain B of the X_i to be all of $R = (-\infty, \infty)$, and $F_i(y_i, x) = \frac{a_i}{A}$ so the condition is trivially satisfied.

For the power utility function the quantity $a_i > 0$ now means the relative risk aversion of member i, and the function $F(y_i, x)$ is given by

$$F(y_i, x) = \frac{\frac{1}{a_i} y_i}{\rho(x)},$$

where $\rho(x)$ is a smooth function of x, so again (A3) is satisfied and the domain B of the X_i can be taken to be $B = R_{++} = (0, \infty)$.

For the logarithmic utility function we obtain that

$$F(y_i, x) = \frac{y_i + \frac{d_i}{c_i}}{x + \sum_j \frac{d_j}{c_j}},$$

so $\frac{d}{dy_i}F(y_i, x) = (x + \sum_j \frac{d_j}{c_j})^{-1}$ which is continuous for $x > -\sum_j \frac{d_j}{c_j}$ Here $B = (b, \infty)$ where $b = \max_i\{-d_i/c_i\}$.

We conclude that the assumption (A2) is not really restrictive, since it does not rule out any of the most common examples.

A closer examination of Assumption (A2) reveals that the only additional requirement it imposes on the preferences of the members is that the third derivative of the utility functions must exist and be continuous.[3]

Let us now assume that the moments implied by the budget conditions given in (10) exist. Sufficient for this to be the case is that $E\{(u_b'(X_M))^2\} < \infty$. From Pareto optimality it follows that $\lambda_i u_i'(Y_i) = u_b'(X_M)$, implying that it is also sufficient that $E\{(u_i'(Y_i))^2\} < \infty$ for all i.

Finally notice that the state price deflator $u_b'(X_M)$ is also a continuous function of b for the same reason, since $u_i'(\cdot)$ is a continuous function for each i, and Y_i^b is continuous in b for all i.

We are then in position to prove the following:

Theorem 2 *Suppose $u_i' > 0$, $u_i'' \leq 0$, u''' are continuous for all i, and $E\{(u_b'(X_M))^2\} < \infty$. Then an equilibrium exists.*

Proof: Consider the mapping $f : R^I \rightarrow R^I$ which sends $b = (b_1, b_2, \cdots, b_I)$ into $c = (c_1, c_2, \cdots, c_I)$ by the rule

$$E(u_b'(X_M)\big(X_i - (Y_i^{(b)} - b_i)\big) = c_i, \qquad \text{for } i = 1, 2, \cdots, I. \tag{17}$$

By (16) it follows that $|Y_i^{(b)} - b_i| \leq |X_M|$, so $E(Y_i - b_i)^2 \leq EX_M^2 = M < \infty$, and $EY_i^2 < M_i < \infty$ implies that $b_i \in G$ for some compact rectangle G in R^I. Also

$$|c_i| \leq E(u_b'(X_M)|X_i - (Y_i^{(b)} - b_i)|) \leq$$
$$\Big\{E(u_b'(X_M))^2\Big\}^{\frac{1}{2}}\Big\{E(X_i^2) + E(Y_i - b_i)^2\Big\}^{\frac{1}{2}} < K_i < \infty$$

for any $b \in G$ by first applying the Schwarz inequality and then Minkowski's inequality. This establishes $c \in H$ where H is is a rectangle like G. Let J be the rectangle in R^I containing both G and H. Denote the hyperplane $\sum_{i=1}^I b_i = x_0$ by F. Note that the intersection $F \cap J$ is non-empty, compact

[3]This allows us to check whether the members are prudent or not.

and convex. The mapping $b \to c$ defined by f in (17) maps $F \cap J$ into $F \cap J$ since by Walras' law

$$\sum_{i=1}^{I} c_i = E\Big(u_b'(X_M)\big(\sum_{i=1}^{I} X_i - (\sum_{i=1}^{I} Y_i^{(b)} - \sum_{i=1}^{I} b_i)\big)\Big) = \sum_{i=1}^{I} b_i.$$

By our above observation that the optimal allocations $Y_i^{(b)}$ and the state price deflator $u_b'(X_M)$ are all continuous functions of b, and since the linear functional $\pi(Z) = E(u_b'(X_M)Z)$ is continuous in L^2 from our assumption that $\xi = u_b'(X_M) \in L^2$, the mapping f is continuous and hence has a fixed-point by Brower's theorem. Therefore there exist b_i^* such that

$$E(u_{b^*}'(X_M)(X_i - (Y_i^{(b^*)} - b_i^*)) = b_i^*, \qquad \text{for } i = 1, 2, \cdots, I$$

and consequently

$$E(u_{b^*}'(X_M)(Y_i^{(b^*)} - X_i)) = 0, \qquad \text{for } i = 1, 2, \cdots, I.$$

This completes the proof. $\square$

Let us consider some illustrations where Theorem 2 is conclusive, but where the assumption of bounded risks is not satisfied.

Example 3. Returning to the situation in Example 1 where the utility functions are negative exponential, consider the case where there exists a feasible allocation Z, in which the components Z_i are i.i.d. exponentially distributed with parameter θ. Let $X = DZ$ where D is an $I \times I$-matrix with elements $d_{i,j}$ satisfying $\sum_i d_{i,j} = 1$ for all j, so that $X_M = \sum_{i=1}^{I} Z_i := Z_M$.

This gives an initial allocation X of dependent portfolios, which seems natural in a realistic model of a reinsurance market. Here it means that the X_i portfolios are mixtures of exponential distributions with a fairly arbitrary dependence structure.

In this case X_M has a Gamma distribution with parameters I and θ. According to Theorem 2 all we have to check for an equilibrium to exist is that $E\{(u_i'(Y_i))^2\} < \infty$ for all i, or equivalently that $E\{(u_\lambda'(X_M))^2\} < \infty$. Since $u_\lambda'(X_M) = Ke^{-X_M/A}$ for some constant K, we have to verify that the following integral is finite:

$$E\big(e^{-\frac{2X_M}{A}}\big) = \int_0^\infty e^{-2x/A}\theta e^{-\theta x}\frac{(\theta x)^{(I-1)}}{(I-1)!}dx.$$

This is indeed the case, since by the moment generating function of the Gamma distribution it follows that

$$E\big(e^{-\frac{2X_M}{A}}\big) = \Big(\frac{\theta}{\theta + \frac{2}{A}}\Big)^I < 1$$

because both the parameter θ and the risk tolerance A of the syndicate are strictly positive.

Instead of the assumption of the exponential distributions, suppose that the Z_i are independent, each with a Pareto distribution, i.e., with probability density function

$$f_{Z_i}(x) = \frac{\alpha_i c_i^{\alpha_i}}{z^{1+\alpha_i}}, \qquad c_i \leq z < \infty, \qquad \alpha_i, c_i \in (0, \infty).$$

This is known as the Pareto distribution of the first kind [4]. In this case $\mathrm{E}Z_i$ exists only if $\alpha_i > 1$, and $\mathrm{var}Z_i$ exists only if $\alpha_i > 2$, etc. The moment generating functions $\varphi_i(\beta) = Ee^{\beta Z_i}$ of these distributions exist for $\beta \leq 0$, since the random variables $e^{\beta Z_i}$ are then bounded. Carrying out the same construction as above, we notice that

$$E\big(e^{-\frac{2}{A}X_M}\big) = \prod_{i=1}^{I} E\big(e^{-\frac{2}{A}Z_i}\big) < \infty$$

since each of the factors has finite expectation. Accordingly, for these distributions a competitive equilibrium exists by Theorem 2.

Here the X_i are mixtures of Pareto distributions, but we should exert some caution, since our theory is developed for risks belonging to L^2. We are outside this domain regarding the Z_i if $\alpha_i < 2$ for some i, in which case $X_j \notin L^2$ for any j. However, as long as the initial risks are in L^2, an equilibrium exists.

Finally consider the normal distribution in this example, and assume that each X_i is $\mathcal{N}(\mu_i, \sigma_i)$-distributed and that X is jointly normal, where $\mathrm{cov}(X_i, X_j) = \rho_{ij}\sigma_i\sigma_j$ for $i, j = 1, 2, \cdots, I$. By the moment generating function of the normal distribution we have that

$$E\left(u'_\lambda(X_M)\right)^2 = E\left(e^{-\frac{2}{A}X_M}\right) = \exp\left(2\big(\frac{\sigma}{A}\big)^2 - 2\frac{\mu}{A}\right) < \infty \qquad \forall\, i,$$

where $\mu = \sum_{i=1}^{I} \mu_i$ and $\sigma^2 = \sum_{i=1}^{I} \sigma_i^2 + 2\sum_{i>j} \sigma_i\sigma_j\rho_{ij}$. Thus an equilibrium exists.

Even if the positivity requirements are not met, still all the computations of the equilibrium are well defined, the state price deflator $\xi(X_M)$ is an element of L^2_{++}, prices can readily be computed, and an equilibrium exists.

It may admittedly be unclear what negative wealth should mean in a one period model, but aside from this there are no formal difficulties with this

[4]This distribution borrows its name from the Italian-born Swiss professor of economics, Vilfredo Pareto (1848-1923).

case as long as utility is well defined for all possible values of wealth. In the reinsurance syndicate we usually interpret $X_i = w_i - V_i$ where w_i are initial reserves and V_i are claims against the ith reinsurer, or member. In this case negative values of X_i have meaning, in that when this occurs, reinsurer i is simply bankrupt, or in financial distress. $\Box$

In the above example with the Pareto distributions, if the parameters α_i satisfy $1 < \alpha_i < 2$ for all i, expectations exist, but not variances. Still $u'_\lambda(X_M) = e^{-\frac{2}{A}X_M} \in L^2_{++}$, however L^2 is not the relevant dual for L^1, which is L^∞. We notice that $u'_\lambda(X_M) \in L^\infty_{++}$ as well, which means that this case is now well defined. This is so because our development in Theorem 2 is easily seen to be valid for L^1 replacing L^2, in fact any L^p-space will do, for $1 \leq p < \infty$, with dual space L^q, where $\frac{1}{p} + \frac{1}{q} = 1$.

The space L^∞ is well behaved from the point of view of supporting preferred sets since the positive cone has a non-empty interior, but neither does L^1 furnish all the continuous linear functionals on L^∞, nor do we know that the strictly positive functionals on L^∞ are continuous.

We now turn to the case where the relative risk aversions of all the syndicate members are constants, as in Example 2:

Example 4. Consider the model of Example 2, where $u_i(x) = (x^{1-a_i} - 1)/(1-a_i)$ for $x > 0$, $a_i \neq 1$. We again restrict attention to the case where $a_1 = a_2 = \ldots = a_I = a$.

Recall that the weights λ_i are determined by the budget constraints, implying that

$$\lambda_i = k\left(\frac{E(X_i X_M^{-a})}{E(X_M^{1-a})}\right)^a, \qquad i \in \mathcal{I},$$

or, λ_i is determined modulo the proportionality constant $k = (\sum_{j\in\mathcal{I}} \lambda_j^{1/a})^a$ for each i.

Let us again consider a situation where there exists a feasible allocation Z, where the Z_i components are i.i.d. exponentially distributed with parameter θ. Let $X = DZ$ where D is an $I \times I$-matrix with elements $d_{i,j}$ satisfying $\sum_i d_{i,j} = 1$ for all j, so that $X_M = \sum_{i=1}^I Z_i := Z_M$.

Regarding existence of equilibrium, according to Theorem 2 it is sufficient to check that $u'_\lambda(X_M) \in L^2$. In this case X_M has a Gamma distribution with parameters I and θ, and all we have to check is if the expectation

$$E(X_M^{-2a}) = \int_0^\infty x^{-2a}\theta e^{-\theta x}\frac{(\theta x)^{I-1}}{(I-1)!}dx$$

is finite. The possible convergence problem is seen to occur around zero, and the standard test tells us that when $(-2a + I - 1) > -1$, or when $I > 2a$, this integral is finite. Thus, for example if $a = 10$, then equilibrium exists in this syndicate if the number of members exceeds 20.

One may wonder if the member weights λ_i can be computed when $I > 2a$. To check this consider the two expectations $E(X_M^{1-a})$ and $E(Z_i X_M^{-a})$. In order to verify that these expectations exist, we have to find the joint distribution of Z_i and X_M. It is given by the probability density

$$f(z_i, x) = \theta^2 e^{-\theta x} \frac{(\theta(x - z_i))^{I-2}}{(I-2)!}, \qquad z_i \leq x < \infty,\ 0 \leq z_i < \infty.$$

So we have to check if the integral

$$E(Z_i X_M^{-a}) = \int_0^\infty \int_{z_i}^\infty z_i x^{-a} \theta^2 e^{-\theta x} \frac{(\theta(x - z_i))^{I-2}}{(I-2)!} dz_i dx$$

is finite. The possible convergence problem is again seen to occur around zero, and the standard test requires that $(1 - a + I - 2) > -1$, i.e., when $I > a$ this integral is finite. From this it is obvious that the expectations $E(X_i X_M^{-a})$ also converge in the same region, by linearity of expectation, since $X_i = \sum_j d_{i,j} Z_j$.

Similarly we have to check the following expectation:

$$E(X_M^{1-a}) = \int_0^\infty x^{1-a} \theta e^{-\theta x} \frac{(\theta x)^{I-1}}{(I-1)!} dx.$$

Near zero the possible problem again occurs, and the standard comparison test gives convergence when $(1 - a + I - 1) > -1$, or when $I > a - 1$. To conclude, when $I > \max\{a, a-1\} = a$, both expectations exist, showing that the member weights exist in the parameter range $(I > 2a)$ where state prices are known to exist.

Notice that an equilibrium will exist with a fairly low number of participants in the interesting region for the parameter a. Consider e.g., the value $a = 1$ corresponding to a logarithmic utility function, then an equilibrium exists with only two members in the syndicate. When the relative risk aversion is two, only four members are required, and so on.

Finally consider the case of Pareto distributions for the initial portfolios X_i directly, assuming $\alpha_i > 2$ for all i. The integrals

$$E(X_i^{-2a}) = \left(c_i^{2a} (1 + \frac{2a}{\alpha_i}) \right)^{-1} < \infty.$$

Since $\min_{i \in \mathcal{I}} \alpha_i > 0$ there are no problems with convergence, and an equilibrium exists in this case regardless of the values of the relative risk aversion parameter $a, (a > 0)$ or its relationship to I, since $E(X_M^{-2a}) \leq \sum_i E(X_i^{-2a})$. In this latter case all the portfolios are bounded away from zero, which helps with the existence problem for power utility, while the exponential distribution has more probability mass near zero, potentially causing problems with existence in certain parameter ranges, as we have seen above. □

The result of this example is in line with the spirit of a competitive equilibrium, which generally implies that the theory may work better the *more* individuals that participate. Recall that classical economics sought to explain the way markets coordinate the activities of *many* distinct individuals each acting in their own self-interest.

III-B Uniqueness of Equilibrium

The question of uniqueness of equilibrium is largely unexplored in the infinite dimensional setting. However, given our smoothness assumptions one would expect equilibrium to be unique, provided one exists. In this section we show that this conjecture holds.

Approaches that take preferences and endowments as primitives seem to encounter many difficulties, in addition to the usual difficulty of doing calculus in infinite dimensional spaces. As mentioned before the natural domain of prices is a subset of the dual space of L^2, the positive orthant L^2_+, but this set has empty interior, which is very inconvenient for doing calculus. In general are excess demand functions typically not defined, and are not smooth even when they are defined. Araujo (1987) argues that excess demand functions can be smooth only if the "commodity" space is a Hilbert space, which is noticed to be the case in our model.

Inspired by our approach in Theorem 2, where we basically transformed the infinite dimensional problem into a finite dimensional one represented by the member weights λ, or equivalently, the constants b, we attempt the same line of reasoning regarding the uniqueness question.

Going back to the first order, non-linear differential equations in (7), to each point $(x_0, b_1, b_2, \cdots, b_I)$ there is only one solution $Y = (Y_1, Y_2, \cdots, Y_I)$ to these equations under the assumption (A2). However, there could be several fixed-points and thus one possible equilibrium associated with each of them.

Arguing in terms of the member weights λ instead of the b's, let us define the individual demands of the I members by $Z_i^{(\lambda)} = (Y_i^{(\lambda)} - X_i)$ and the excess demand $Z^{(\lambda)} = \sum_{i \in \mathcal{I}} Z_i^{(\lambda)}$. Below we show that these are well defined and smooth functions of the member weights λ_i, $i \in \mathcal{I}$.

One reason we consider the member weights here instead of the constants b, is due to Proposition 3 (c), equation (8), where is was shown that the state price $\xi(\lambda)$ is an increasing function of the weights λ_i. As a consequence, by increasing λ_i, member i's demand for reinsurance will decrease, since, loosely speaking, this can be associated with a strengthening of member i's initial "reserve" X_i, while all the other members' demands will decrease. This will be formalized below.

The excess demand is zero at the possible equilibrium points λ^*, corresponding to the points b^* of Theorem 2. If the excess demand curve as a function of each member weight λ_i is downward sloping for all i at all equilibria where Theorem 2 holds, there can only be one equilibrium. It is enough that $Z^{(\lambda)}$ is downward sloping in $(I-1)$ of the λ's because of the normalization of the weights. Because of the smoothness of the excess demand function in λ, this will be a sufficient condition for uniqueness.

By investigating the marginal effect on the excess demand Z^{λ^*} from a marginal increase in λ_i^*, making sure that the resulting λ is still on the simplex S^{I-1}, we may use this procedure to check for uniqueness. As real functions the demands $Z_i^{\lambda}: R \rightarrow R$ can be expressed as $Z_i^{\lambda} = Y_i^{\lambda}(x) - x_i$ where $\sum_i x_i = x$, and thus, in the language of calculus, we must therefore consider the quantities

$$Z^{\lambda^*} - \alpha(\sum_{i \in \mathcal{I}} \lambda_i - 1),$$

where α is the Lagrange multiplyer associated with the constraint of remaining on the simplex. Since any marginal change in one of the member weights will necessarily bring the resulting vector of weights outside the simplex unless the other weights are correspondingly lowered, $\alpha > 0$. Thus we compute the following

$$\frac{\partial Z^{\lambda^*}}{\partial \lambda_i} - \alpha \qquad \text{for} \qquad i = 1, 2, \cdots, (I-1)$$

at any equilibrium point λ^*, and check wether all these have the same sign for all $x \in B$.

In order to compute the quantities $\frac{\partial Z^{\lambda^*}}{\partial \lambda_i}$, we must find $\frac{dY_j^{\lambda^*}(x)}{d\lambda_i}$ for all $i, j \in \mathcal{I}$. It follows by differentiation of the first order conditions

$$\lambda_i u_i'(Y_i^{\lambda}(x)) = u_{\lambda}'(x) \qquad \text{for any } i$$

that

$$\frac{dY_i^{\lambda}(x)}{d\lambda_i} = \frac{1}{\lambda_i u_i''(Y_i^{\lambda}(x))} \big(\frac{\partial}{\partial \lambda_i} u_i' \lambda(x) - u_i'(Y_i^{\lambda}(x)) \big) \qquad \text{for } i = j,$$

for all $x \in B$, and using equation (8), and the first order conditions, we obtain

$$\frac{dY_i^{\lambda}(x)}{d\lambda_i} = \frac{1}{\lambda_i} \rho_i(Y_i^{\lambda}(x)) \Big(1 - \frac{dY_i^{\lambda}(x)}{dx} \Big) \qquad \text{for } i = j, \tag{18}$$

for all $x \in B$. Similarly we get

$$\frac{dY_j^\lambda(x)}{d\lambda_i} = -\frac{1}{\lambda_i}\rho_j(Y_j^\lambda(x))\frac{dY_i^\lambda(x)}{dx} \qquad \text{for } j \neq i, \tag{19}$$

for all $x \in B$. Notice that $\frac{dY_i^\lambda(x)}{dx} \in (0,1)$ by equation (7), in other words, an increase in the market portfolio leads to an increase in all the members portfolios Y_i, and no member assumes the entire increase because they are all risk averse. It follows that $\frac{dZ_i^\lambda(x)}{d\lambda_i} > 0$ for all i and $\frac{dZ_j^\lambda(x)}{d\lambda_i} < 0$ for all $j \neq i$, demonstrating what was explained above for the individual demands.

We are now in position to compute the required marginal changes in excess demand within the simplex. It is

$$\frac{\partial Z^{\lambda^*}}{\partial \lambda_i} - \alpha = \sum_{j \in \mathcal{I}} \frac{\partial Z_j^{\lambda^*}}{\partial \lambda_i} - \alpha = \frac{\partial Y_i^{\lambda^*}}{\partial \lambda_i} + \sum_{j \neq i} \frac{\partial Y_j^{\lambda^*}}{\partial \lambda_i} - \alpha =$$

$$\frac{1}{\lambda_i}\rho_i(Y_i^{\lambda^*}(x))\Big(1 - \frac{dY_i^{\lambda^*}(x)}{dx}\Big) - \sum_{j \neq i} \frac{1}{\lambda_i}\rho_j(Y_j^{\lambda^*}(x))\frac{dY_i^{\lambda^*}(x)}{dx} - \alpha,$$

for all $x \in B$, where we have used (18) and (19). Continuing, we get

$$\frac{\partial Z^{\lambda^*}}{\partial \lambda_i} - \alpha = \frac{1}{\lambda_i}\Big(\rho_i(Y_i^{\lambda^*}(x)) - \frac{dY_i^{\lambda^*}(x)}{dx}\rho_{\lambda^*}(x)\Big) - \alpha$$

for all $x \in B$, where we have used that

$$\rho_{\lambda^*}(x) = \sum_{i \in \mathcal{I}} \rho_i(Y_i(x)),\ x \in B,$$

according to Proposition 3(a). Finally using (7) we observe that

$$\frac{\partial Z^{\lambda^*}}{\partial \lambda_i} - \alpha = -\alpha < 0 \text{ for all } x \in B \text{ and } i \in \mathcal{I}.$$

The conclusion is formulated in the following theorem:

Theorem 3 *Under the assumptions of Theorem 2, the existing equilibrium in the reinsurance syndicate is unique.*

Thus our conjecture is confirmed. Notice that in the examples we have presented we were able to find the equilibrium by direct calculation, and the weights λ_i were uniquely determined (modulo multiplication by a positive constant) from the budget constraints. Thus these equilibria are all unique.

IV Comparison with a more general theory

Drawing on the results of a more general theory of an exchange economy, as in e.g., in Mas-Colell and Zame (1991) and Araujo and Monteiro (1989), based on proper preference relations (Mas-Colell (1986)), Aase (1993) formulated the following existence theorem for equilbibium in an exchange economy in L^2_+:

Theorem 4 *Assume $u_i(\cdot)$ continuously differentiable for all i. Suppose that $X_M \in L^2_{++}$ and there is any allocation $V \geq 0$ a.s. with $\sum_{i=1}^{I} V_i = X_M$ a.s., and such that $E\{(u_i'(V_i))^2\} < \infty$ for all i, then there exists a quasi-equilibrium.*

If every member i brings something of value to the market, in that $E(\xi \cdot X_i) > 0$ for all i, which seems like a reasonable assumption in most cases of interest, and is in fact one of our assumptions in Theorem 1, we have that a quasi-equilibrium is also an equilibrium, which then exists under the above stipulated conditions.

We notice that these requirements put joint restrictions on both preferences and probability distributions that are rather similar to the ones of Theorem 2. Although we have stronger requirements on the utility functions u_i, our requirement on X_M is weaker. In addition we also have demonstrated uniqueness of equilibrium. An example may illustrate the differences between the two theories:

Example 5. Consider the case of power utility of Example 4, where $u_i(x) = (x^{1-a_i} - 1)/(1 - a_i)$ for $x > 0$, $a_i \neq 1$. In this example the exponentially distributed Z_i's satisfy the assumptions of the allocation V in Theorem 4, and $X_M \in L^2_{++}$ since X_M has a Gamma distribution. Provided $E(\xi \cdot X_i) > 0$ for all i, an equilibrium will exist if

$$E(Z_i^{-2a_i}) = \int_0^\infty x^{-2a_i} \theta_i e^{-\theta_i x} dx < \infty,$$

which holds true when $a_i < 1/2$. As we demonstrated in Example 4, in the case where where $a_1 = a_2 = \ldots = a_I := a$, an equilibrium exists for $I > 2a$. Thus our previous result is stronger, or perhaps more relevant, since empirical studies suggest that the interesting values of a_i may be in the range between one and 20, say.

Here it is simple to verify existence also when the parameters a_i are unequal, and provided $E(\xi \cdot X_i) > 0$ for all i, an equilibrium will exist in the region $a_i < 1/2$ for all i according to the above theorem.[5]

[5]The explicit computation of the state price deflator ξ is not straightforward when the parameters are no longer equal equal across the agents. In this case sharing rules are

In the case where all the $d_{j,i}$ are equal (to $\frac{1}{I}$), the initial portfolios all have the same Gamma $(\theta I, I)$-distribution, in which case the allocation X satisfy the requirements of the allocation V of Theorem 4. In this case we get existence in the region $I > 2 \max_i\{a_i\}$, which is quite similar to the result of Example 4, □

We see that the two theories give comparable results, albeit they guarantee existence in slightly different regions depending upon circumstances.

V Summary

Classical economics sought to explain the way markets coordinate the activities of many distinct individuals each acting in their own self-interest. An elegant synthesis of two hundred years of classical thought was achieved by the general equilibrium theory. The essential message of this theory is that when there are markets and associated prices for all goods and services in the economy, no externalities or public goods and no informational asymmetries or market power, then competitive markets allocate resources efficiently.

In this paper the idea of general equilibrium has been applied to a reinsurance syndicate, where many of the idealized conditions of the general theory may actually hold. The most critical assumption seems to be that of no informational asymmetries. Reinsurers like to stress that their transactions are carried out under conditions of "utmost good faith" - *uberrima fides.* This means that the reinsurers usually accept, without question, the direct insurer's estimate of the risk and settlement of claims. The mere existence of rating agencies in this industry is an indication that there may be both adverse selection, and also elements of moral hazard in these markets. Nevertheless, the above theory may still give a good picture of what goes on in syndicated markets.

In models of such markets properties of competitive equilibria have only academic interest so long as it is not clear under what conditions they exist. Uniqueness is clearly also an issue of great importance.

The advantage with the existence and uniqueness theorems of this paper is that they rest largely on results in risk theory, or the theory of syndicates, which implies that we may essentially restrict attention to the member weights in Euclidian I-dimensional space, thus reducing the dimensionality of the problems. In contrast Theorem 4 requires a rather demanding, infinite dimensional equilibrium theory.

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certainly not linear.

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