Optimal spending of a wealth fund in the discrete time life cycle model

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Abstract

The paper analyses optimal spending of an endowment fund. We use the life cycle model for both expected utility and recursive utility in discrete time. First we find the optimal consumption and investment policies for both kinds of utility functions. This we apply to a sovereign wealth fund that invests broadly in the international financial markets. We demonstrate that the optimal spending rate, i.e., the consumption to wealth ratio, is significantly lower than the fund's expected real rate of return. Using the expected return as the spending rate, implies that the fund's value converges towards 0 with probability 1 and also in expectation, as time goes. For both kinds of long term convergence we find closed form threshold values. Spending below these values secures that the fund will last "forever". For reasonable values of the preference parameters, the optimal spending rate is demonstrated to satisfy these long term requirements.

KEYWORDS: Life cycle model, optimal spending rate, endowment funds, expected utility, recursive utility, risk aversion, EIS, consumption to wealth ratio, almost sure convergence, 1st mean convergence
JEL-Code: G10, G12, D9, D51, D53, D90, E21.

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1 Introduction

We consider optimal investment strategies and optimal spending from an endowment fund consistent with the life cycle model. We demonstrate that the optimal spending rate is, for reasonable values of the preference parameters, strictly smaller than the expected rate of return, and the difference is significant.

According to Eeckhoudt, Gollier and Schlesinger (2005) is preference for diversification intrinsically equivalent to risk aversion. Extracting the expected real rate of return on a fund is associated to risk neutrality.

We take the security market as given, assumed to be in equilibrium, and introduce a price taking agent in this market. In this setting we reconsider the problem of optimal consumption and portfolio selection to obtain closed form solutions. In the context of an endowment fund, the results from analyzing this more general problem can immediately be utilized in order to determine an optimal spending rate as the consumption to wealth ratio. We have considered both expected utility, in which case risk aversion plays a prominent role, and recursive utility where consumption substitution is separated from risk aversion, which is clarifying.

The microfoundation of adopting utility functions in contexts like this has been discussed in the literature, and seems well founded (see for example Blancard (1985)). In some interpretations this requires a constant population.

When the investment opportunity set is deterministic, there exist explicit and closed form solutions for optimal spending in the continuous-time model (Merton (1969-71)). This was utilized in Aase and Bjerksund (2021), who also extended the analysis to a stochastic investment opportunity set.

In the discrete-time model closed form solutions are hard to come across. The early papers by Mossin (1968), and Samuelson (1969) realized that dynamic programming can be used, but this did not lead to closed form solutions. We have used directional derivatives to first find the optimal consumption subject to a budget constraint. With a closed form solution to this problem, we next solved the optimal portfolio selection problem by maximizing the remaining utility at each time t.

When finding closed form solutions to the optimal consumption to wealth ratio, we relied on first having formulated an appropriate model of the financial market. This model is inspired from the no-arbitrage theory following the seminal paper by Black and Scholes (1973). This breakthrough was in a continuous-time setting, but soon afterwards several papers, and books, followed using a discrete time framework, see for example Cox and Rubinstein (1985) and Skiadas (2009).

If the extraction rate is the one of expected return, this normally means that the agent is *risk neutral* at the level of spending, and must then, to be consistent, be risk neutral at the level of optimal portfolio selection as well. But the consequence of such an investment strategy is rarely advocated by anyone responsible for an endowment fund, whatever its purpose.

We demonstrate that a *particular* spending policy, the expected real rate of the fund, is not consistent with a reasonable long term development of the fund, and will with probability one eventually deplete any fund that is managed by diversification. In addition, the expected value of the fund will converge to 0 as time t increases. We find closed form threshold values for both kinds of long term developments.

Most endowments have the perspective that they should last "forever". Consequently, there is a trade-off between current spending and future spending opportunities. Tobin (1974) develop sustainable spending rules in a deterministic world. It can be argued that it is sustainable to spend the real interest rate (or something slightly smaller) within this setting.

Uncertainty complicates this picture. It has been argued that it is sustainable for an endowment to spend the expected fund return, see e.g., Campbell (2012), who considered university endowments. Moreover, this idea motivates the current 3 % fiscal rule that applies to the 1 trillion USD Norwegian sovereign wealth fund.

Our article is concerned with optimal extraction of endowment funds in general, and has in particular been motivated by the Norwegian Government Pension Fund Global, earlier called the Norwegian Oil Fund or just the Norwegian sovereign wealth fund, which we consider as an example of the general theory. This is illustrated in the paper's last section.

1.1 Related literature

Dybvig and Qin (2019) consider a fund with normal iid log-returns. The authors find that for the fund to last "forever", spending must not exceed expected fund return subtracted by half the variance. The discrepancy between expected fund return and sustainable spending is far from negligible.

The two key decisions of an endowment fund that invests in the financial market is how much risk to take and how much to spend. From a theoretical point of view, the two decisions are closely related and must be determined jointly. To examine the questions one must address the issue of the objective function by which optimality is to be measured. Merton (1971) presents optimal portfolio and consumption rules for an investor who maximizes expected, additive and separable utility with constant relative risk aversion in a continuous-time world, where risky asset returns are iid. Recursive utility is a more generalized framework where the investor's risk aversion and consumption substitution are disentangled, see, e.g., Epstein and Zin (1991).

Campbell and Sigalov (2020) adopt the Merton model as well as Epstein-Zin preferences, and assume that there is a constraint on the spending rule. The authors examine two alternative constraints: (i) spending the expected return; and (ii) the maximum sustainable spending follows the assumption of Dybvig and Qin (2019). The authors find that the constraint induces increased risk taking (referred to as "reaching out for yield").

Campbell and Martin (2022) introduce a sustainability constraint that the representative agent may choose to impose on herself. The constraint imposes an upper bound on the consumption to wealth ratio, which is shown to lie between the riskless rate and the expected return on optimally invested wealth. The constraint requires that the time t-conditional expected utility should not be allowed to decline, in expectation, over time. Also we consider utility as a stochastic process, but we do not constrain it. Rather we study the long term developments of the fund value itself under various assumptions on spending.

In Merton (1990), ch 21, optimal investment strategies for university endowment funds are analyzed, where the objective is maximization of expected utility, related to several activities consistent with the purposes of the university. We limit the scope to how much to optimally spend in the numeraire unit of account, which is a purely financial question. How much to spend on each of several activities we consider as a political issue.

One purpose of this paper is to compare the optimal spending with the conventional wisdom of spending the expected return, or any other ad hoc rule, under various assumptions. For this reason, we adopt and develop the life cycle model, where we consider the recursive utility framework in addition to the standard expected utility, in the setting of discrete time. We find two threshold values that spending should not exceed, in order for the fund value to be maintained in the long run in a probabilistic sense. These values are independent of the agent's preferences. We then demonstrate that, when the agent is reasonably patient, the optimal consumption to wealth ratio passes the two long run tests. If the spending rate is set equal to the expected rate of return on the endowment fund, both tests fail. The implications of this is that the fund value W_t converges to zero with probability 1 as time t increases, and, moreover, that the expected value of the fund converges to zero as time t goes to infinity.

The paper is organized as follows: The basic discrete-time model is formulated in Section 2. In Section 3 we solve the optimal consumption and portfolio choice problem with expected utility, and in Section 4 we find the corresponding optimal consumption to wealth ratio. In Section 5 we consider the asymptotic behaviour of a sovereign wealth fund. Numerical illustrations for expected utility follow in Section 6. In Section 7 we solve the optimal consumption and portfolio choice problem for recursive utility, and in Section 7.2 we find the corresponding optimal consumption to wealth ratio. Numerical illustrations for recursive utility follow in Section 8. In Section 9 we consider the Norwegian SWF Government Fund Global, and Section 10 concludes. The paper contains x appendices where some of the the technical material and proofs can be found.

2 The basic financial model

In this paper we are concerned with the optimal spending from a sovereign wealth fund, which we interpret as finding the optimal consumption to wealth ratio. Towards this end, we first consider the optimal consumption and portfolio selection problem using the life cycle model.

We have an agent represented by the pair (U, e), where $U(c)$ is the agent's utility function over consumption processes c, and e is the agent's endowment process. The problem consists in maximizing utility subject to the agent's budget constraint

$$(2.1) \qquad \sup_{c,\varphi} U(c) \quad \text{subject to} \quad E\Big(\sum_{t=0}^{T} \pi_t c_t\Big) \leq E\Big(\sum_{t=0}^{T} \pi_t e_t\Big) := w,$$

where φ are the optimal fractions of wealth in the various risky investment possibilities facing the agent, and w is the current value of the agent's wealth. The quantity π_t is the state price at time t, i.e., the Arrow-Debreu state prices in units of probability. The horizon is $T \leq \infty$.

The consumer takes as given a dynamic financial market, consisting of N risky securities and one riskless asset, the latter with rate of return r_t, a stochastic process. The agent's actions do not affect market prices of the risky assets, nor the risk-free rate of return r_t.

2.1 Preliminaries

We first assume that U represented separable and additive expected utility.

We address the same basic problem as the continuous-time paper by Aase and Bjerksund (2021), but we deviate on several accounts. First, the agent's preferences are represented by expected additive and separable utility of the form

$$U(c) = E\Big(\sum_{t=0}^{T} u(c_t, t)\Big). \tag{2.2}$$

Here $u(c,t)$ is the agent's felicity index, which we assume to be of the CRRA-type, meaning that the real function $u(x,t) = \frac{1}{1-\gamma}x^{1-\gamma}\beta^t$, where γ is the agent's relative risk aversion and β is the agent's patience factor (the utility discount factor). The parameters γ and β are constants satisfying $0 < \gamma < \infty$ and $\beta \in (0,1]$. In continuous-time models $\beta = e^{-\delta}$, where $\delta = -\ln(\beta)$ is the impatience rate, $0 \leq \delta < \infty$. In our model we define the impatience rate more naturally from the relationship $\beta = \frac{1}{1+\delta}$. Recursive utility is treated in Section 7.

From the general theory in Appendix 1 we have that optimal consumption c_t^* and the optimal wealth at time W_t^* are connected as follows

$$W_t^* = \frac{1}{\pi_t} E_t\Big\{\sum_{s=t}^{T} \pi_s c_s^*\Big\}. \tag{2.3}$$

Here $E_t(X) := E(X|\mathcal{F}_t)$ is the conditional expectation of any random variable X given the information by time t, where $\mathcal{F}_t$, is the information filtration, $0 \leq t \leq T$, and π_t, the Arrow-Debreu state price in units of probability, will be characterized in Appendix 1.

The following model of the financial market is a discrete-time version of the theory that emerged after the no-arbitrage theory of contingent claims analysis had been established, motivated by the seminal paper of Black and Scholes (1973). Its aim is to characterize complete financial markets with no arbitrage possibilities. In such a market our agent operates as a price taker. There is by now an extensive literature on this topic, primarily in a continuous-time setting. The presentation in Appendix 1 is adapted from Skiadas (2009), see also Aase (2017). With the aid of this, we next focus on the predictions of the standard expected utility model.

3 Solution of the consumption and investment problem with expected utility

3.1 Optimal consumption choice

We want to solve problem (2.1), where $U(c)$ is given by (2.2). The Lagrangian for this problem is

$$\mathcal{L}(c;\lambda) = E\Big\{\sum_{t=0}^{T}\Big(u(c_t,t) - \lambda\pi_t(c_t - e_t)\Big)\Big\}, \tag{3.1}$$

where λ is the Lagrangian multiplier. We use Kuhn-Tucker with the Lagrange function given above, which reduces the problem to an unconstrained maximization problem. We find the first order condition using directional derivatives in function space (Gateaux derivatives), and finally we determine the Lagrange multiplier that yields equality in the budget constraint, which must hold since $u(x,t)$ is strictly increasing in x. The Saddle Point Theorem provides the final solution. Alternatively, we could have employed dynamic programming based on the wealth equation (11.6).

Denoting the directional derivative of $\mathcal{L}$ in the direction c by $\bigtriangledown\mathcal{L}(c^*,\lambda;c)$, the first order condition for this unconstrained problem is

$$\bigtriangledown\mathcal{L}(c^*,\lambda;c) = 0 \quad \text{for all} \quad c \in L, \tag{3.2}$$

where $\lambda > 0$ is the Lagrange multiplier for the wealth constraint. Here the optimal consumption path c^* is assumed to exist, and we can ignore the positivity constraint on c because of the behaviour of $u(x,t)$ when x approaches zero. We then obtain that

$$E\Big\{\sum_{t=0}^{T}\big(u'(c_t^*,t) - \lambda\pi_t\big)c_t\Big\} = 0 \quad \text{for all} \quad c \in L, \tag{3.3}$$

which implies that

$$u'(c_t^*,t) = \lambda\pi_t, \quad a.s., \quad t = 0,1,\cdots,T, \tag{3.4}$$

where prime means derivative with respect to the first variable. Using the functional form of u, we find

$$c_t^* = (\lambda \pi_t \beta^{-t})^{-\frac{1}{\gamma}}, \quad t = 0, 1, \cdots, T. \tag{3.5}$$

From (3.5) we see that the optimal consumption is exposed to market movements only: When the state price π_t is down, times are 'good' and consumption c_t^* is high, and vice versa when the state price π_t is up, consumption goes down. However, consumer spending tends to stay fairly stable, presumably because consumers use wealth to dampen the market variations. This is better explained by use of non-expected utility, which we return to below.

The property expressed in (3.5) is seen to hold for all consumers with CRRA utility, whatever the value of the γ-parameter so long as it is strictly positive. Equation (3.5) expresses a version of the *mutuality principle* (e.g., Borch (1960, 1962) and Wilson (1968)): When the market is down, it is down for *everyone* and everyone consumes less (but to a varying degree), and vice versa everyone consumes more when the market is up.

When there is no market uncertainty, i.e., when $\pi_t = \prod_{s=0}^{t}(1 + r_s)^{-1}$ for all $t \leq T$, the model is known as the Ramsey model, see Ramsey (1928), Koopmans (1960).

3.2 The Associated Optimal Portfolio Selection Problem

We now turn to the investment policy that goes along with the optimal consumption strategy of the the model presented in Appendix 1.

Mossin (1968) was one of the first to study the problem of optimal investments. He leaves out intermediate consumption, i.e., consumption only takes place in the final period, and considers two assets, one risky and one risk-free. With CRRA-utility, he demonstrated that the optimal fraction φ_t of wealth held in the risky asset is constant across time provided returns are iid. He uses dynamic programming, as does Samuelson (1969) for the same problem, except that the latter allows consumption in every period. Neither of these authors arrive at explicit formulas for general CRRA utility, but Samuelson is concerned with the special case $\gamma = 1$.

Let us return to the optimal consumption c_t^* solving problem (2.1) and characterized in Section 3.1. In the present model it is known that the the optimal consumption is proportional to wealth for expected utility, so that

$c_t^* = A_t W_t^*$ where W_t^* is optimal wealth. Provided we limit ourselves to a deterministic investment opportunity set $\mathcal{I}_t = (r_t, \eta_t, \nu_t)$, the factor A_t is deterministic, which we now assume.

Let $R_t = (R_t^{(1)}, R_t^{(2)}, \cdots, R_t^{(N)})'$ are the (simple) returns on the N risky assets, and r_t is the (simple) return on the risk-free asset. Since c_t^* solves the constrained optimization problem of Section 3.1, the optimal portfolio weights φ_{t+1} at time t for the next period solve the following problem $\sup_{\varphi_t} E_t(u(c_{t+1}^*))$, or

$$\sup_{\varphi} E_t\Big(\frac{1}{1-\gamma}\big(A_{t+1}W_{t+1}(\varphi_{t+1})\big)^{1-\gamma}\Big), \quad t = 0, 1, \ldots, T-1, \tag{3.6}$$

where

$$W_{t+1} = (W_t - c_t^*)(1 + r_{t+1} + \varphi_{t+1}(R_{t+1} - r_{t+1})).$$

From our assumptions, this problem reduces to solving the following

$$\sup_{\varphi} E_t\Big(\frac{1}{1-\gamma}\big(1 + r_{t+1} + \varphi_{t+1}(R_{t+1} - r_{t+1})\big)^{1-\gamma}\Big), \quad t = 0, 1, \ldots, T-1.$$

Implied by our notation is that φ_{t+1} is $\mathcal{F}_t$-measurable. The first order condition is

$$E_t\big\{\big(1+r_{t+1}+\varphi_{t+1}(R_{t+1}-r_{t+1})\big)^{-\gamma}\big(R_{t+1}-r_{t+1}\big)\big\} = 0, \quad t = 0, \ldots, T-1. \tag{3.7}$$

Provided the product $r_t E_t(R_{t+1} - r_t)$ is small enough, by a Taylor series approximation it follows that

$$\varphi_{t+1} \approx \frac{1}{\gamma}\big(E_t((R_{t+1} - r_{t+1})(R_{t+1} - r_{t+1})')\big)^{-1} E_t(R_{t+1} - r_{t+1}) = \frac{1}{\gamma}(M_t M_t')^{-1}\nu_t, \tag{3.8}$$

which, when return distributions are independent and stationary over time, implies that the ratios φ_{t+1} are constant over time. The inverse matrix $(M_t M_t')^{-1}$ in this expression is based on excess returns instead of the expected square deviations $(E_t(R_{t+1} - E_t(R_{t+1}))(R_{t+1} - E_t(R_{t+1}))')^{-1}$ and is therefore not identical to the covariance matrix $(\sigma_t \sigma_t')^{-1}$ found in the continuous time analysis. This means that φ_t will be lower for the discrete time model but the difference is small. However, as we see below, the corresponding adjustments in the matrix in (3.8) implies that the consumption to wealth ratio is approximately the same in both models, as the matter must be. Under the same type of assumptions, there should be no real economic difference between the predictions of these two approaches.

In the continuous-time model $\varphi_c = \frac{1}{\gamma}(\sigma\sigma')^{-1}\nu_t$, where $\eta'\eta = \nu'(\sigma\sigma')^{-1}\nu$ and the product can be shown to satisfy $\frac{1}{\gamma}\eta'\eta = \gamma\varphi_c'(\sigma\sigma')\varphi_c$. Similarly we can show that $\frac{1}{\gamma}\nu'(MM')^{-1}\nu = \gamma\varphi'(MM')\varphi$. We can also link the market-price-of-risk inner product $\eta'\eta$ to the optimal portfolio ratios φ_c for the continuous-time model and φ given in (3.8) via $\frac{1}{\gamma}\eta'\eta = \gamma\varphi_c'(MM')\varphi$. This is demonstrated as follows.

$$\frac{1}{\gamma}\eta'\eta = \frac{1}{\gamma}\nu'(\sigma\sigma')^{-1}(MM')(MM')^{-1}\nu = \gamma\nu'(\sigma\sigma')^{-1}\frac{1}{\gamma}(MM')\varphi = \gamma\varphi_c'(MM')\varphi.$$

In order to find M from σ we can use the concept of "bias" in statistical estimation theory, which gives the connection

$$MM' = \sigma\sigma' + (E(R_{t+1}) - r_{t+1})(E(R_{t+1}) - r_{t+1})'.$$

Having solved the optimal consumption and portfolio selection problem in the life cycle model for expected additive and separable utility, we can now use this to find the optimal spending rate as the consumption to wealth ratio for an endowment fund. This we do in the next section.

4 The optimal consumption to wealth ratio

We now address the optimal spending problem of a sovereign wealth fund. First we prove the following result:

Theorem 1. *The connection between the optimal wealth W_t^* and the optimal consumption c_t^* at any time t is given by the following relationship:*

$$W_t^* = c_t^* E_t\Big\{ \sum_{s=t}^{T} \beta^{(s-t)/\gamma} \prod_{v=t+1}^{s} \frac{(1-\eta_v'\triangle B_v)^{1-\frac{1}{\gamma}}}{(1+r_v)^{1-\frac{1}{\gamma}}} \Big\}. \tag{4.1}$$

The proof can be found in Appendix 2.

In order to progress further, we need some simplifying assumptions. From now on we adopt the assumption of Section 3.2 of a stationary and deterministic investment opportunity set $\mathcal{I}_t = (r, \eta, \nu)$ for all t. In order to compare our results to the associated continuous-time version, we also assume that

$\triangle B_t^i$ are independent for $i \neq j$, and independent and identically distributed across time t for each i.

Let $X(v) = (1 - \eta_v' \triangle B_v)$, for all $v \geq t$, $t = 1, 2, \ldots, T$. In order for the model to be complete, these variables are discretely distributed with state probabilities $p_k, k = 1 \ldots S$, summing to 1, and satisfying

$$(i) \sum_{k=1}^{S} \triangle B_v^i(k)\, p_k = 0, \quad (ii) \sum_{k=1}^{S} (\triangle B_v^i(k))^2\, p_k = 1,$$

$$\text{and} \quad \sum_{k=1}^{S} \triangle B_v^i(k) \triangle B_v^j(k) p_k = 0, \; i \neq j, \; \text{for all } v.$$

If we assume we can neglect moments of order three and higher, we can simplify using Taylor series approximations. This gives the following result:

Theorem 2. *(a) Consider the finite horizon case $T < \infty$. Under the above assumptions the optimal consumption to wealth ratio $c_t(T)$ can be written as follows:*

$$c_t(T) = \frac{c_t^*}{W_t^*} = \frac{1 - \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}} (1 + \frac{1-\gamma}{2\gamma^2} \eta' \eta)}{1 - \big(\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}} (1 + \frac{1-\gamma}{2\gamma^2} \eta' \eta)\big)^{T-t}}. \tag{4.2}$$

(b) Consider the infinite horizon case where $T = \infty$. The optimal consumption to wealth ratio is given by

$$c_t(\infty) = \frac{c_t^*}{W_t^*} = 1 - \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}} (1 + \frac{1-\gamma}{2\gamma^2} \eta' \eta), \tag{4.3}$$

provided $\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}} (1 + \frac{1-\gamma}{2\gamma^2} \eta' \eta) < 1$.

The proof can be found in Appendix 2.

Remark 1. By using Taylor approximations of the exponential function and the logarithmic function, we can write the above formula for $c_t(\infty)$ as follows

$$c_t(\infty) \approx \delta\, \big(\frac{1}{\gamma}\big) + \big(1 - \frac{1}{\gamma}\big)\big(r + \frac{1}{2\gamma} \nu' (\sigma\sigma')^{-1} \nu\big). \tag{4.4}$$

The right-hand side is the exact formula for the optimal spending found in equation (16) in Bjerksund and Aase (2021) for the continuous-time model, where the impatience rate $\delta = -\ln(\beta)$. This expression can be seen to be a convex combination of the impatience rate δ and the certainty equivalent rate of return on the fund. The derivation can be found in Appendix 1. □

Remark 2. For reasonable values of the preference parameters one can verify that the expected return on the fund is larger than the consumption to wealth ratio:

$$\mu_W := r + \varphi\nu > 1 - \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}\left(1 + \frac{1-\gamma}{2\gamma^2}\eta'\eta\right), \tag{4.5}$$

provided the agent is reasonably patient, i.e., β is large enough. In practice 'large enough' certainly holds provided $\beta \geq 0.96$, say. This is most conveniently demonstrated by use of the alternative formula (4.4) in Remark 1. The proof of this can be found in Appendix 2, and is similar to the corresponding demonstration in Aase and Bjerksund (2021) for the continuous-time model. The examples to follow below turn out to confirm this claim, where the inequality holds with good margin. □

5 The asymptotic behaviour of a sovereign wealth fund.

In this section we investigate what happens to the fund after a long time has elapsed from the present, under different spending scenarios. We take the model explained in Appendix 1 as given. The spending rate $c_0(\infty)$ is the consumption to wealth ratio. We take this rate and the expected return rate of the fund μ_W as exogenously given, and investigate the long term behaviour of the fund as a function of these two rates.

We consider two different types of convergence, $\mathcal{L}_1$-convergence, or convergence in 1st mean, and convergence almost surely (a.s.). It is well known that these two types of convergence do not imply each other. See, for example, Breiman (1968).

Our starting point is the dynamic equation for the fund given in equation (11.6) in Appendix 1. This equation can be written

$$W_{t+1} = (W_t - c_t)(1 + R^W_{t+1}), \quad t = 0, 1, \ldots$$

From our assumptions it follows that with an infinite horizon, the spending rate $c_0(\infty)$ is a constant. For simplicity of notation we call this rate c in this section. Accordingly, this relationship can be written

$$W^*_{t+1} = W^*_t(1-c)(1+R^W_{t+1}), \quad t = 0, 1, \ldots$$

and iterating, this becomes

$$W^*_{t+1} = W_0(1-c)^{t+1}\prod_{s=0}^{t}(1+R^W_{s+1}), \quad t = 0, 1, \ldots$$

5.1 Convergence in 1st mean; martingale theory.

Let us first look at convergence in first mean. Employing our assumption about iid returns and taking expectations, this gives

$$E(W^*_{t+1}) = W_0\big((1-c)(1+\mu_W)\big)^t, \quad t = 0, 1, \ldots \tag{5.1}$$

Let us tentatively see what happens if the extraction rate c is equal to the expected rate of return μ^W on the fund. This gives

$$E(W^*_{t+1}) = W_0(1-\mu_W^2)^t, \quad t = 0, 1, \ldots$$

Assuming $|\mu_W| < 1$, this means that $E(W^*_t) \to 0$ as $t \to \infty$.

On the other hand, if $c = 0$ in (5.1), then $E(W_{t+1}) = W_0(1+\mu_W)^t \to \infty$ as $t \to \infty$ (we drop the *-notation in what follows). Here the limit is not a random variable. In the first case $\{W_t\}_{t\geq 0}$ is a supermartingale, in the latter case $\{W_t\}_{t\geq 0}$ is a submartingale.

We are obviously most interested in the latter, namely situations where $\{W_t\}_{t\geq 0}$ is a submartingale, i.e., $E_t(W_{t+1}) \geq W_t$, $t = 0, 1, \ldots$

If $\{W_t\}_{t\geq 0}$ is a supermartingale, i.e., $E_t(W_{t+1}) \leq W_t$, $t = 0, 1, \ldots$ then the fund will deteriorate in expectation, and as we shall see below, also almost surely.

The wealth portfolio $\{W_t\}_{t\geq 0}$ will be a martingale, that is, $E_t(W_{t+1}) = W_t$, $t = 0, 1, \ldots$ when $(1-c)(1+\mu_W) = 1$ which happens when

$$c = \frac{\mu_W}{1+\mu_W} := m.$$

This means that when $c < \frac{\mu_W}{1+\mu_W}$, then $\{W_t\}_{t\geq 0}$ is a submartingale, and $E(W_t) \to +\infty$ as $t \to \infty$, and when $c > \frac{\mu_W}{1+\mu_W}$, then $\{W_t\}_{t\geq 0}$ is a supermartingale, and $E(W_t) \to 0$ as $t \to \infty$.

The standard submartingale convergence theorem does not apply here, since the conditions are not satisfied. If they were, there would exist a random variable X such that $E|W_t - X| \to 0$ as $t \to \infty$. One problem is that our wealth process $\{W_t\}_{t \geq 0}$ is not uniformly integrable.[1]

We can use the above results to get estimates for how long time it takes for the funds expected value to be equal to some fraction of the current value. Consider the following example.

Example 1. Suppose the expected return on the fund $\mu_W = 0.05$ and the spending rate is $c = 0.045$. Here $\frac{\mu_W}{1+\mu_W} = 0.0476$. Also suppose $W_0 = 10.000$ in some units. We consider $T = 100$.

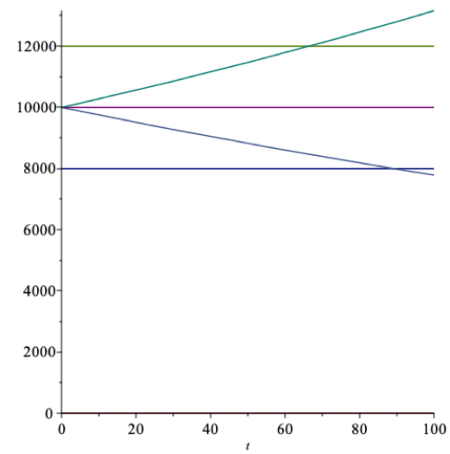

Fig. 1: Various expected developments of an endowment fund

In Figure 1 the line from W_0 represents the martingale $W_t^{(1)}$ where the spending rate is $c = \frac{\mu_W}{1+\mu_W}$. The increasing submartingale $W_t^{(2)}$ follows the spending rate assumed to be $c = 0.045$, while the decreasing curve is the supermartingale $W_t^{(3)}$ where the spending rate is $c = \mu_W = 0.05$.

Here we can answer various types of questions. To illustrate, suppose we ask the question: How long does it take before the expected value of the fund $E(W_t^{(2)})$ is up 20%? The answer is 66.39 years. Next, for the fund with $c = \mu_W$, it takes 89.15 years before the expected value $E(W_t^{(3)})$ is down 20%. □

We have proven the following result:

[1]The same situation occurs in continuous time with the geometric Brownian motion process.

Theorem 3. *Suppose c is the spending rate of an endowment fund W and μ_W is the expected real rate of return on W. Then we have the following three situations:*

1) If $c < m = \frac{\mu_W}{1+\mu_W}$, then $\{W_t\}_{t\geq 0}$ is a submartingale and $E(W_t) \to +\infty$ as $t \to \infty$.

2) If $c > m = \frac{\mu_W}{1+\mu_W}$, then $\{W_t\}_{t\geq 0}$ is a supermartingale and $E(W_t) \to 0$ as $t \to \infty$.

3) If $c = m = \frac{\mu_W}{1+\mu_W}$, then $\{W_t\}_{t\geq 0}$ is a martingale and $E_t(W_{t+1}) = W_t$ for $t = 1, 2, \dots$

In the continuous-time model with Brownian motion driven uncertainty, for the corresponding martingale result it was shown that the wealth eventually converges to zero with probability 1 (Aase and Bjerksund (2021), p11). Here we can appeal to a theorem of Jean Ville (1939) in the context of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{n=0}^{\infty}, P)$, suppose $E \in \mathcal{F}$. If a nonnegative martingale diverges to infinity when E happens, then $P(E) = 0$. This result does not need the iid assumption which, admittedly, is heroic. Together with this assumption, however, with the results of the next section it says roughly the same as the continuous-time result: The martingale will approach 0 with probability 1.

5.2 Almost sure convergence.

Let us move to convergence almost surely. Here we have the following result.

Theorem 4. *Let c be the spending rate and R^W the rate of return of the endowment fund (a random variable). Then we have the following:*

(a) If $\big(ln(1-c) + E(ln(1+R^W))\big) < 0$, then $W_t \to 0$ almost surely as $t \to \infty$.

(b) If $\big(ln(1-c) + E(ln(1+R^W))\big) > 0$, then W_t grows without limit almost surely, as t increases.

The proof, which can be found in Appendix 2, makes use of the strong law of large numbers (SLLN).

Suppose now that the spending rate $E(R^W) = \mu_W$ is being used, as advocated by some researchers and spokespeople.[2]

[2]This is the extraction rule for the Norwegian SWF Government Fund Global, determined by the Norwegian Parliament (Stortinget).

In this situation

$$\ln(1-c) + E(\ln(1+R^W)) < \ln(1-c) + \ln(1+E(R^W))$$

by Jensen's inequality since the logarithmic function is strictly concave. Furthermore

$$\ln(1-c) + \ln(1+E(R^W)) = \ln((1-c)(1+E(R^W))) = \ln(1-c^2) < 0$$

since $c \in (0,1)$. By Theorem 4, part (a), it follows that $W_t \to 0$ with probability 1 as $t \to \infty$. By the above theory it also follows that $E(W_t) \to 0$ as $t \to \infty$. From this we suggest that this policy is not a viable spending rule for an endowment fund.

Let us consider the case (b) of the theorem,

$$\ln(1-c) + E(\ln(1+R^W)) > 0.$$

This inequality holds if and only if the spending rate c satisfies

$$c < \hat{c} := 1 - \exp(-E\{\ln(1+R^W)\}).$$

Let us define

$$ce_1 := E(R^W) - \frac{1}{2}E\{(R^W)^2\}.$$

It turns out that $\hat{c}$ can be approximated by ce_1: By a Taylor series approximation of the logarithmic function we know that the standard approximation for financial return data for the term $E\{\ln(1+R^W)\}$ is $[E(R^W) - \frac{1}{2}\text{var}(R^W)]$. Also $\hat{c} \approx E\{\ln(1+R^W)\} - \frac{1}{2}(E\{\ln(1+R^W)\})^2$ by a Taylor series approximation of the exponential function. The latter can be written $E(R^W) - \frac{1}{2}\text{var}(R^W) - \frac{1}{2}(E(R^W) - \frac{1}{2}\text{var}(R^W))^2$, and this expression is seen to be well approximated by $E(R^W) - \frac{1}{2}E\{(R^W)^2\}) = ce_1$ to the fourth order.

This means that ce_1 is a threshold which an extraction rate c should not pass from below in order to have long term "viability" of the endowment fund.

How accurate is this approximation, and is it on the conservative side? To check this, we need the probability distribution of $(1+R^W)$. As a numerical illustration, suppose that $E(R^W) = \mu_W = 0.06$ and $\text{var}(R^W) = (0.14)^2$. This means that $ce_1 = 0.0484$.

To compute $E\{\ln(1+R^W)\}$ we the need the probability distribution of $(1+R^W)$. As an illustration, let $1+R^W = 1+\mu_W + \sigma_W \triangle B$ where $\triangle B$ is one-dimensional and discrete, taking two values (u, p) and $(d, (1-p))$ with $u = -0.79$, $d = 1.30$ and $p = 0.62$, so that $E\{\triangle B\} = 0$ and $E(\triangle B)^2 = 1$. (This distribution is used below in Section 6.1 as well.) Using this, we obtain $E\{\ln(1 + R^W)\} = 0.05016$. This means that $\hat{c} = 0.0489$, so the difference $(\hat{c} - ce_1) = 0.0005$ which is sufficiently small for most practical purposes and on the safe side. When the spending rate $c < ce_1$, this means that $c < \hat{c}$ as well and the spending rate c passes the long run test.[3]

By construction it is always true that $\mu_W > ce_1 = \mu_W - \frac{1}{2}E(R_W^2)$ and $\mu_W > \frac{\mu_W}{1+\mu_W}$. With this trivial, but important observation, we have the following reminder:

Corollary 1. *Let μ_W be the expected real rate of an endowment fund. Then we have the following:*

If the spending rate is set equal to the expected rate of return μ_W, then the fund value $W_t \to 0$ almost surely as $t \to \infty$, and the expected value $E(W_t) \to 0$ as $t \to \infty$.

Notice that this result is independent of any of the preference parameters. It depends on our statistical assumptions.

Next we illustrate this result by numerical examples. Our results can be compared to the corresponding results for the time-continuous model, see Aase and Bjerksund (2021); here we present extensions.

6 Numerical illustrations - Expected utility.

For reasonable market quantities, we compare the optimal spending rate for an endowment fund to the real expected rate of return from the fund. The optimal spending rate we interpret as the consumption to wealth ratio of the previous section. We also compare to the thresholds of the last section, and to a quantity related to the certainty equivalent rate of return of the fund.

One reason for such comparisons is the claim that it is both optimal and sustainable to spend the expected real return of a sovereign fund. For example, this is the rule, determined in parliament, for the Norwegian SWF Government Fund Global, one of the World's largest sovereign funds. Based on the last section, our claim is that this is not an optimal spending rate, it

[3]The standard approximation for $E\{\ln(1 + R^W)\}$ is $[E(R^W) - \frac{1}{2}\text{var}(R^W)]$, which is here 0.05020. Using the latter, we obtain $\hat{c} \approx 1 - \exp(-[E(R^W) - \frac{1}{2}\text{var}(R^W)]) = 0.0490$.

is too high, and will, if followed, deplete the fund at a final future time with probability 1. We consider this fund as an example in Section 9.[4]

Here we illustrate the above theory by use of real market data. We assume the agent takes the US-market as given, where the risky part of our fund is represented by the S&P-500 index. This corresponds to one of the best functioning securities markets in the World, and should be representative in construction of the underlying market quantities. The data are as follows.

In Table 1 we provide the key summary statistics of the data in Mehra and Prescott (1985) on the real annual return data related to the S&P-500, denoted by S, as well as for the annualized consumption data, denoted c, and the return on Government bills, denoted b [5].

	Expectat.	Standard dev.	Covariances
Return S&P-500	6.98%	16.54%	$\mathrm{cov}(S, b) = .001401$
Government bills	0.80%	5.67%	$\mathrm{cov}(c, b) = -.00016$
Equity premium	6.18%	16.67%	

Table 1: Key US-data for the time period 1889-1978. Discrete-time annual compounding.

Example 1.

Consider the above market data, and the following preference parameters: $\beta = 0.99$ and $\gamma = 2.5$.

For these parameters and with the market data of Table 1, the optimal portfolio fraction in the risky part of the market is $\varphi = 0.82$, the expected rate of return on the fund is $\mu_W = r + \varphi\nu = 0.06$ where $\nu = 0.062$ and $\eta = 0.37$ from the above table. Furthermore, $M = 0.1709$, where M is defined in equation (3.8), and the volatility of the return on the fund is $\sigma_W = \varphi\sigma = 0.14$, where $\sigma = 0.1654$ all follow from the above table.

For reasons to be clear below, we consider the following expression

$$ce_\gamma = \mu_W - \frac{1}{2}\gamma(\sigma_W'\sigma_W + \mu_W^2),$$

while the following quantity is the key for asymptotic comparisons, defined

[4] We assume the fund is fully funded, so that the external influx to the fund has come to an end. When this is not the case, see Section 9.1.

[5] There are of course newer data by now, but these retain the same basic features.

in Section 5,

$$ce_1 = \mu_W - \frac{1}{2}(\sigma_W'\sigma_W + \mu_W^2).$$

We see that when $\gamma > 1$, $ce_\gamma < ce_1$, and when $\gamma < 1$, $ce_\gamma > ce_1$.

When the extraction rate is below ce_1, the fund W_t grows in t with probability 1, while if it is above ce_1, the fund value converges to 0 with probability 1 as $t \to \infty$.

Normally we will have that $\gamma > 1$ in which case $ce_\gamma < ce_1$, so ce_γ may be a viable candidate for a spending rate. We will refer to this quantity as the certainty equivalent return in this paper (although this is standard terminology only when $E(R^W) = 0$).[6]

From the last section recall the interpretation of the threshold $m = \mu_W/(1 + \mu_W)$. For the above parameters the certainty equivalent rate of return $ce_\gamma = 0.032$, $ce_1 = 0.05$ and $m = 0.057$. The optimal spending rate is $c_0(\infty) = 0.026$. This value is seen to be consistent with long term sustainability of the fund. However, if the real rate of return ($= 0.06$) is used as the spending rate, this is not sustainable in the long run and the fund will converge to 0 almost surely as t goes to infinity. Moreover, $E(W_t)$ will converge to 0 at a geometric rate as $t \to \infty$.

With a finite horizon of $T = 500$ years, the extraction rate in equation (4.3) is time dependent, and will increase sharply as the horizon comes closer. In Figure 2 we present a graph of the optimal extraction rate, and in the same graph we also represent the real rate of return together with m, ce_1 and ce_γ.

[6]The Arrow-Pratt approximation to the certainty equivalent rate of return is given by $\mu_W - \frac{1}{2}\gamma(\sigma_W'\sigma_W)/(1 + \mu_W)$ when $\mu_W \neq 0$, a slightly larger quantity than ce_γ.

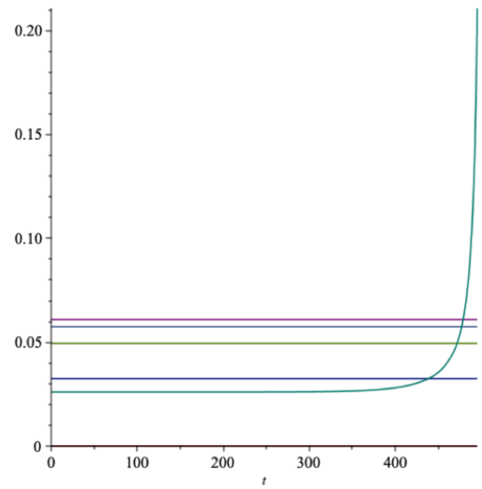

Fig. 2: The optimal extraction rate as a function of time (EU).

The hyperbolic type curve is the optimal extraction rate $c_t(500), 0 \leq t < 500$, the upper horizontal line is the expected rate of return on the fund, the next horizontal line is m, then follows ce_1, and the lowest horizontal line is the certainty equivalent rate of return ce_γ. The growth rate of the optimal consumption c_t^* is 0.04 with standard deviation 0.14. $\square$

Consider the following illustration: In Figure 2a we show five graphs as functions of γ for $\beta = 0.99$. The lowest one is the long term optimal spending rate $c_0(\infty)$ as a function of γ, the next lowest is the certainty equivalent ce_γ, then comes the threshold value $ce_1(\gamma)$, the next highest is the martingale threshold value $m(\gamma) = \mu_W/(1 + \mu_W)$ and the highest located curve is the expected rate of return on the fund $\mu_W(\gamma)$. These quantities are time independent, but will depend on γ via $\mu_W(\gamma)$, since they depend on μ_W.

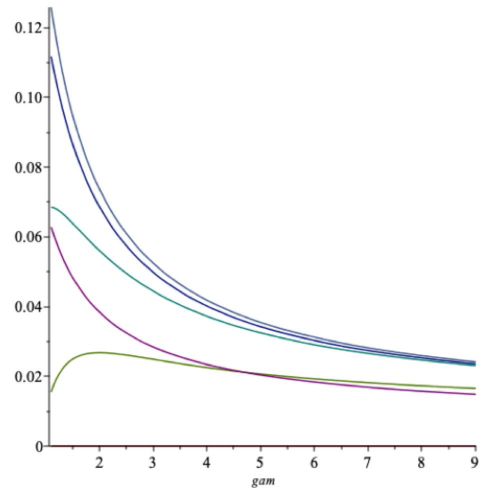

Fig. 2a: The functions $c_0(\infty)(\gamma)$, $\mu_W(\gamma)$, $m(\gamma)$, $ce_1(\gamma)$ and ce_γ as γ vary.

We see that the long term optimal spending rate is lower than the other quantities for any reasonable value of the relative risk aversion γ. This illustrates our above claim in this example, and moreover it indicates that the criterion of spending the real expected rate of return $\mu_W(\gamma)$ is not only larger than the optimal one, but also larger than the two threshold values $ce_1(\gamma)$ and $m(\gamma)$ for "any" values of γ and β. This means that the fund will 1) converge to 0 with probability 1 with this extraction policy regardless of the relative risk aversion γ, and 2) the expected value of the fund will converge to zero as $t \to \infty$, with μ_W as the spending rate.

6.1 Discrete state probabilities

As mentioned in Appendix 1, in order for a discrete time model to be complete, the set of states of the world must be finite. In the present situation we have one risky asset, the index, so let me suggest a simple model for the $\triangle B_t$ with two states of nature, "up" and "down" with probabilities p_u and p_d respectively. In this situation the formula for the spending rate is the following

$$c_0(\infty) = 1 - \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}\big((1-\eta u)^{(1-\frac{1}{\gamma})}p_u + (1-\eta d)^{(1-\frac{1}{\gamma})}p_d\big) \tag{6.1}$$

where u and d satisfy the two equations (i) $up_u + dp_d = 0$; and (ii) $u^2 p_u + d^2 p_d = 1$. We must estimate the two probabilities from return data in the stock market, and determine u and d from the two equations (i) and (ii).

Consider the following estimates: $p_u = 0.62$ and $p_d = 1 - p_u = 0.38$. This gives $u = -0.79$ and $d = 1.30$. This is calibrated to give the same value of the spending rate for $\beta = 0.99$ and $\gamma = 2$ as the above model based on truncation of Taylor series.

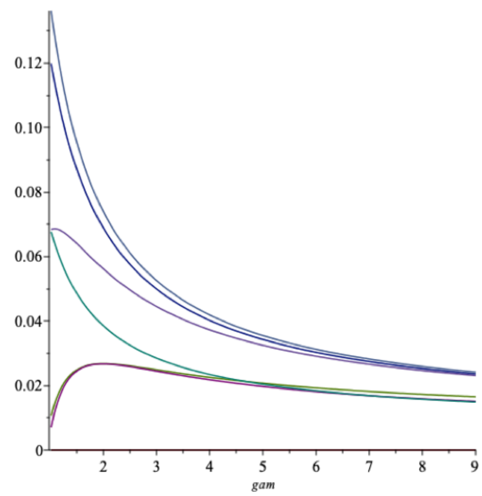

Fig. 2b: The functions $c_0(\infty)(\gamma)$, $\mu_W(\gamma)$, $m(\gamma)$, $ce_1(\gamma)$ and ce_γ as γ vary.

In Figure 2b we show the same graphs as in Figure 2a, with the addition that the spending rate in equation (6.1) is included together with the one in the previous figure. These two curves can be seen to be almost indistinguishable. It is noteworthy that the simple two-state Binomial model is this flexible.

6.2 Patience

What we mean by a reasonable patience factor β is illustrated next. In Figure 2c we show graphs of μ_W, $m(\beta)$, ce_1 and $c_0(\infty)$, as functions of β. Here $\gamma = 2.0$.

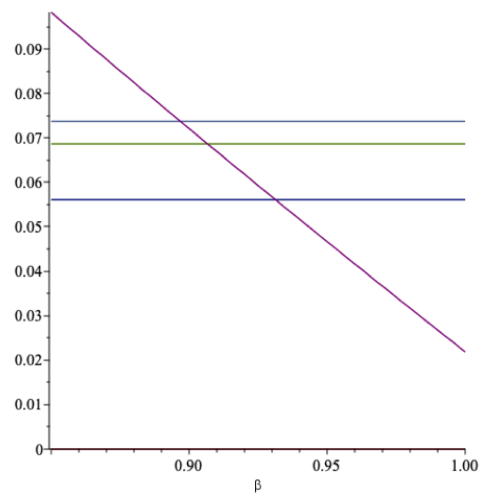

Fig. 2c: The functions $c_0(\infty)(\beta)$, $\mu_W(\beta)$, $m(\beta)$, $ce_1(\beta)$ as β vary.

The falling (convex) curve is the optimal spending rate $c_0(\infty)$ as a function of β. The next horizontal lines are independent of β, where the highest is the line μ_W showing the real rate of return on the fund, and the lowest one is the threshold ce_1 for a.s. convergence, while the one in the middle is the $m(\beta)$-threshold. The figure shows that when the agent is impatient enough, the optimal spending rate is larger than the expected rate of return on the fund, then comes a range where $c_0(\infty)(\beta)$ is lower than μ_W but larger than m, and when the agent is patient enough, the optimal spending rate is lower than both m and ce_1. For this example we see that this happens when $\beta > 0.935$. Normally one sets $\beta \geq 0.98$ in applied work. A nation as the agent must be considered patient.

In the expression for $c_t(T)$ the parameter β occurs together with the parameter γ as $\beta^{\frac{1}{\gamma}}$. When γ increases, so does $\beta^{\frac{1}{\gamma}}$. This has the effect that the agent appears as more "patient" with increasing γ as can be observed in Figure 2a, where the optimal spending rate decreases as γ increases. This decrease is primarily a consequence of increasing risk aversion, but impatience and risk aversion are not completely disentangled in this model.

With the tools of the asymptotics section we can, for example, calculate how many years it will take before $E(W_t)$ has deteriorated, or increased a certain fraction, depending on the spending policy (recall Figure 1). To illustrate, in this situation with an optimal spending rule $c_0(\infty) = 0.025$ and expected return on the fund $\mu_W = 0.06$, it takes about 12 years for $E(W_{12})$ to have increased 50% using the optimal spending rule, while it takes about

61 years for $E(W_{61})$ to have decreased 20% using μ_W as the spending rate.

7 Recursive utility

Recursive utility (RU) is considered to be a more realistic representation of preferences than expected, additive and separable utility (EU) that we have considered so far. In particular RU separates risk aversion from consumption substitution in temporal models, which is important, since since these two properties are rather different. Recursive preferences have an axiomatic underpinning in the basic work in the field by Kreps and Porteus (1978).

Again we want to solve the problem (2.1), where the utility function $U(c)$ is defined via the following "aggregator"

$$U_t = f(c_t, m_{t+1}) = v^{-1}((1-\beta)v(c_t) + \beta v(m_{t+1})), \ t < T, \ U_T = c_T, \tag{7.1}$$

where v is a felicity index with inverse function v^{-1}, m_{t+1} is a conditional certainty equivalent as of time t, and β is the patience factor defined as as before. In this case $U(c)$ in (2.1) is given by U_0.

So, where does such an aggregator come from? The standard separable and additive expected utility representation has an ordinally equivalent version which, when normalized, can be expressed in recursive form. For example, the representation

$$U_t = E_t\Big[\sum_{s=t}^{T-1} \beta^{s-t} v(c_s) + \frac{\beta^{T-t}}{1-\beta} v(c_T)\Big] \tag{7.2}$$

is ordinally equivalent to the recursive version in (7.1), provided the conditional certainty equivalent $m_{t+1} = v^{-1}(E_t(v(U_{t+1})))$ is the one of expected utility with felicity index v.

Thus, in order to deviate, in a non-trivial way, from the standard, additive representation of preferences, it is assumed that the conditional certainty equivalent can be represented as above, but with a *different* felicity index u: $m_{t+1} = u^{-1}(E_t(u(U_{t+1})))$, $u \neq v$. This turns out to be an important step, since consumption substitution in a deterministic world is something very different from risk aversion, where the latter only makes sense under uncertainty. This essential difference is taken into account by the recursive model.

On the one hand this approach stays close enough to the standard, additive representation of preferences to still benefit from many of its useful properties, insights and interpretations, on the other this step is significant enough to avoid some of its unrealistic and negative features. However, this generalization comes at a price of added complexity, as is naturally the case with most generalizations.

In this article we employ the two standard functions v and u, defined up to affine transformations as $v(w) = \frac{1}{1-\rho}(w^{1-\rho} - 1)$ and $u(w) = \frac{1}{1-\gamma}(w^{1-\gamma} - 1)$, with inverse functions $v^{-1}(y) = ((1-\rho)y+1)^{\frac{1}{\rho-1}}$ and $u^{-1}(y) = ((1-\gamma)y+1)^{\frac{1}{\gamma-1}}$ respectively. The following scale invariant aggregator results from (7.1)

$$U_t = f(c_t, m_{t+1}) = ((1-\beta)c_t^{1-\rho} + \beta m_{t+1}^{1-\rho})^{\frac{1}{1-\rho}}, \tag{7.3}$$

where the conditional certainty equivalent m is given by

$$m_{t+1} = (E_t[U_{t+1}^{1-\gamma}])^{\frac{1}{1-\gamma}}.$$

The parameter $\gamma \geq 0$ corresponds to the agent's relative risk aversion in the standard one-period model (the time-less model), and has the same interpretation here. Similarly, in a deterministic setting the parameter $\rho \geq 0$, where $\frac{1}{\rho}$ is the elasticity of intertemporal substitution (EIS) in consumption. These parameters correspond to different properties of the individual's preferences - and should be measured independently. In the standard, additive expected utility model, $\gamma = \rho$, which turns out to be rather restrictive.

When $\rho = 1$, the felicity index $v(x) = \ln(x)$, and $U_t = m_{t+1}^{\beta} c_t^{1-\beta}$, and when $\gamma = 1$, then we have $u(x) = \ln(x)$, and $m_{t+1} = \exp(E_t[\ln(U_{t+1})])$.

The parameter β is the 'patience' factor, where $0 \leq \beta \leq 1$ as for EU. The impatience rate $\delta = 1/\beta - 1$.

While preferences over deterministic consumption plans are solely determined by the function v, the limitation of the expected additive, discounted utility in the presence of uncertainty rests on the fact that the function determining risk aversion also governs the purely deterministic development.

RU overcomes this latter problem, and other problems, by simply separating v from u.

The version in (7.3) is known as the Epstein-Zin aggregator (see Epstein and Zin (1989-91), Chew and Epstein (1991)). For continuous-time see Duffie and Epstein (1992), and for risk premiums and the equilibrium interest rate see, for example, Aase (2016).

7.1 Optimal consumption and portfolio selection with recursive utility

In Appendix 3 we have relegated the analysis of the recursive model, where we derive a closed form expression for the optimal consumption in equation (13.11), and compare it to the corresponding expression for EU. Moreover, we find the optimal portfolio selection rule, and show that this is the same as for the EU-model, assuming a deterministic investment opportunity set.

These two results are the basics for our expression for the optimal spending rule for RU, which follows next.

7.2 Optimal consumption to wealth ratio (RU)

We now address the optimal spending problem of a sovereign wealth fund. Starting with the wealth equation (13.5) in Appendix 3, we proceed as before with the expression in (13.12) in Appendix 3 for the optimal consumption. We have the following result:

Theorem 5. *The connection between the optimal wealth W_t^* and the optimal consumption c_t^* at any time t is given by the following relationship:*

(7.4)

$$W_t^* = c_t^* E_t\Big\{ \sum_{s=t}^{T} \beta^{(s-t)/\rho} \prod_{v=t+1}^{s} \frac{(1-\eta_v' \triangle B_v)^{1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}} \, (1+\mu_{W,v}+\sigma_{W,v}\triangle B_v)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}}{(1+r_v)^{1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}} \Big\}.$$

The proof can be found in Appendix 4.

In order to progress further, we need som simplifying assumptions. From now on we adopt the assumption of Section 3.2 of a stationary and deterministic investment opportunity set $\mathcal{I}_t = (r, \eta, \nu)$ for all t. The same type of assumptions are made as in the case of the expected utility, from which we can characterize the conditional expectation on the right-hand side of equation (7.4).

Recall that $\mu_W = r + \varphi'\nu$ is the expected real rate of return on the wealth portfolio W, and $\sigma_W'\sigma_W = \varphi'\sigma\sigma'\varphi$ is the corresponding variance of the return rate of the fund.

We use Taylor series approximations and neglect moments of order three and higher. This leads to the following result:

Theorem 6. *(a) Consider the finite horizon case $T < \infty$. Define*

$$k(\beta,\rho,\gamma,\eta,\nu,r,\sigma,\varphi) = \beta^{\frac{1}{\rho}}(1+r)^{(\frac{1}{\rho}\frac{1-\rho}{1-\gamma}-1)}\Big\{1-(\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})\mu_W \tag{7.5}$$
$$+\frac{1}{2}(\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})(1+\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})(\mu_W^2+\sigma_W'\sigma_W)+(\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})\cdot$$
$$(1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma})\big[1-(1-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})\mu_W\big]\eta'\sigma_W-\frac{1}{2}(\frac{1}{\rho}\frac{1-\rho}{1-\gamma})(1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma})\cdot$$
$$\big[1-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}\mu_W-\frac{1}{2}(\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})(1+\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})\mu_W^2\big]\eta'\eta\Big\}.$$

Under the above assumptions the optimal consumption to wealth ratio $c_t(T)$ can be written as follows:

$$c_t(T) = \frac{c_t^*}{W_t^*} = \frac{k(\beta,\rho,\gamma,\eta,\nu,r,\sigma,\varphi)-1}{(k(\beta,\rho,\gamma,\eta,\nu,r,\sigma,\varphi))^{T-t}-1}. \tag{7.6}$$

(b) Consider the infinite horizon case where $T = \infty$. The optimal consumption to wealth ratio is given by

$$c_t(\infty) = \frac{c_t^*}{W_t^*} = 1 - k(\beta,\rho,\gamma,\eta,\nu,r,\sigma,\varphi), \tag{7.7}$$

provided $k(\beta,\rho,\gamma,\eta,\nu,r,\sigma,\varphi) < 1$.

The proof can be found in Appendix 4.
Notice that the optimal spending rate depends on the statistical dependence between the state price and the return on the wealth portfolio via the term $\text{cov}_t((1-\eta'\triangle B_v),(1+\mu_W+\sigma_W'\triangle B_v)) = -\eta'\sigma_W$ for $v > t$.

Remark 3. The optimal spending rate with recursive utility in continuous time with continuous price processes based on Brownian motion was presented in Aase and Bjerksund (2021). The exact expression for the spending rate is the following

$$\hat{c}_0(\infty) = \frac{\delta}{\rho} + (1-\frac{1}{\rho})(r+\frac{1}{2}\gamma\varphi_c(\sigma\sigma')\varphi_c), \tag{7.8}$$

again a convex combination of the impatience rate δ and the certainty equivalent rate of return from the fund. In the convex combination the parameter

ρ now plays the role that γ played in the corresponding formula for expected utility.

In Figure 3 we present a graph of the function in (7.8) together with the optimal extraction rate in equation (7.7) as functions of ρ when $\beta = 0.99$ and $\gamma = 2$.

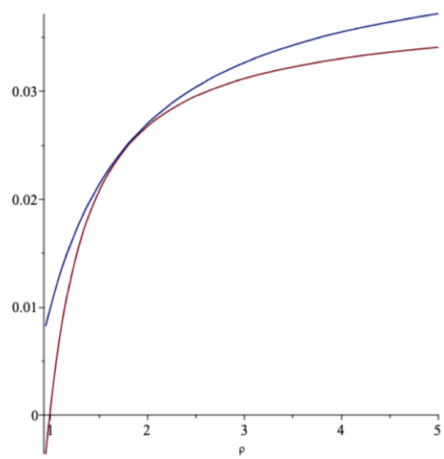

Fig. 3: The two optimal extraction rates as functions of ρ (RU).

The lowest graph is the spending rate in equation (7.8). The spending rate in equation (7.7) is seen to deviate for more extreme values of ρ, where the approximation may not be as good as for more central values of this parameter.

Next we compare these two spending rates as functions of γ when $\beta = 0.99$ and $\rho = 2.0$.

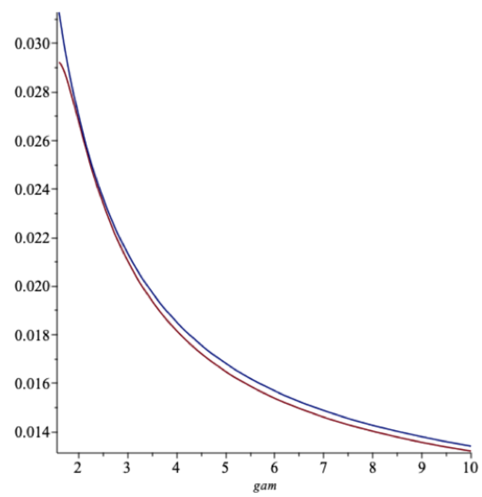

Fig. 4: The two optimal extraction rates as functions of γ (RU).

The lowest graph is the one in equation (7.7). The fit is seen to be reasonable, given γ is a bit larger than 1.

Below we also compare the continuous-time spending rate with the one based on the present discrete time model using the discrete state probabilities. The spending rate with RU and the Binomial version is the following:

(7.9)

$$c_0^B(\infty) = 1 - \beta^{\frac{1}{\rho}}(1+r)^{\frac{1}{\rho}\frac{1-\rho}{1-\gamma}-1}\Big\{(1-\eta u)^{1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}(1+\mu_W+\sigma_W u)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}p_u + (1-\eta d)^{1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}(1+\mu_W+\sigma_W d)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}p_d\Big\}.$$

For the recursive model we calibrate the discrete state version for $\rho = 2.5$, $\gamma = 2.0$ and $\beta = 0.99$ to $p_u = 0.51$, $p_d = 1 - p_u$, $u = -0.98$ and $d = 1.02$ using (7.7).

In Figure 5 we graph the spending rate in (7.8) and the Binomial one in (7.9) as functions of ρ when $\gamma = 2$ and $\beta = 0.99$:

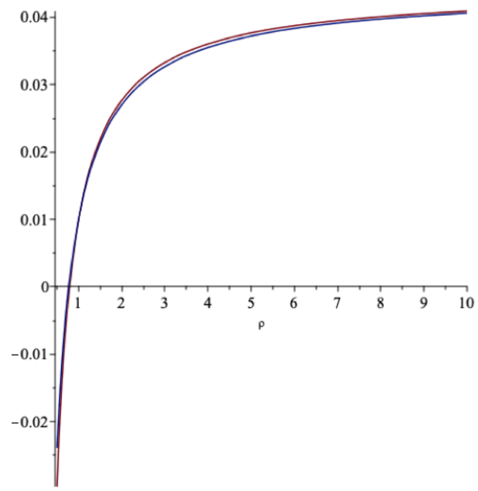

Fig. 5: The cont.time and the Binomial spending rates as ρ vary.

As we see, this fit is rather impressive. In Figure 6 we do the same comparison when γ vary for $\beta = 0.99$ and $\rho = 2.0$.

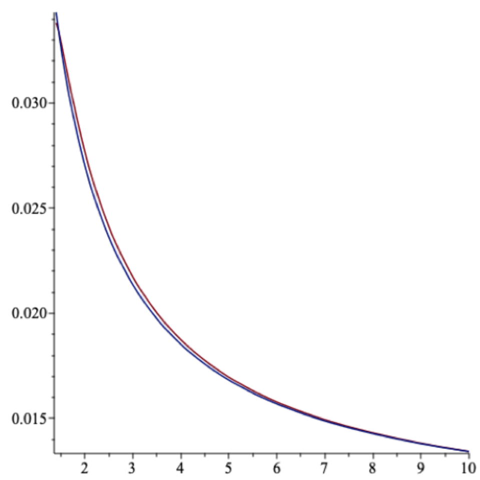

Fig. 6: The cont.time and the Binomial spending rates as γ vary.

Also here the fit is excellent. $\square$

7.3 The conditional expected consumption growth rate and the associated volatility

With the techniques of Theorem 6, we can readily find the expected growth rate of the optimal consumption and its standard deviation. Let

$$a := \frac{1}{\rho}\frac{1-\rho}{1-\gamma}, \quad b := \frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}.$$

The conditional expected consumption growth rate is

$$(7.10) \quad E_t\big(\frac{c^*_{t+1}}{c^*_t} - 1\big) = \beta^{\frac{1}{\rho}}(1+r_{t+1})^a \Big(1 - b\mu_W + \frac{1}{2}b(1+b)\big(\mu_W^2 + \sigma_W'\sigma_W\big) - ba\big(1-(1+b)\mu_W\big)\eta'\sigma_W + \frac{1}{2}a(1+a)\big(1 - b\mu_W - \frac{1}{2}b(1+b)\mu_W^2\big)\eta'\eta\Big) - 1.$$

Also

$$(7.11) \quad E_t\big((\frac{c^*_{t+1}}{c^*_t})^2\big) = \beta^{\frac{2}{\rho}}(1+r_{t+1})^{2a}\Big(1 - 2b\mu_W + b(1+2b)\big(\mu_W^2 + \sigma_W'\sigma_W\big) - 4ba\big(1-(1+2b)\mu_W\big)\eta'\sigma_W + a(1+2a)\big(1 - b\mu_W - b(1+2b)\mu_W^2\big)\eta'\eta\Big).$$

From these two quantities we can find the standard deviation of the conditional expected consumption growth rate as

$$\sigma_{c^*}(t) = \sqrt{E_t\big((\frac{c^*_{t+1}}{c^*_t})^2\big) - \big(E_t\big(\frac{c^*_{t+1}}{c^*_t}\big)\big)^2}.$$

The proof can be found in Appendix 4. By setting $a = \frac{1}{\gamma}$ and $b = 0$ we obtain the corresponding results for EU.

8 Numerical illustrations - Recursive utility.

First consider the same data as in Example 1, with preference parameters $\beta = 0.99$, $\gamma = 2.5$ and $\rho = 2.0$. This parameter constellation ($\gamma > \rho$) represents preference for early resolution of uncertainty. Only the the optimal extraction rate function changes, while the other data remain the same as in

Example 1. The long term optimal spending rate $c_0(\infty) = 0.023$, which is comparable to the optimal value 0.026 for EU in Example 1.

Next consider the following example.

Example 4. Let $\beta = 0.99$, $\gamma = 2.0$ and $\rho = 2.5$. The optimal portfolio weight $\varphi = 1.044$, representing a more risky portfolio than above, $\mu_W = 0.074$, $\sigma_W = 0.17$, $ce_\gamma = 0.0384$ and $ce_1 = 0.056$, and $m = 0.069$. The optimal extraction rate in the long run is $c_0(\infty) = 0.030$, while it is 0.027 for the expected utility model. The graphs corresponding to Figure 2 are here:

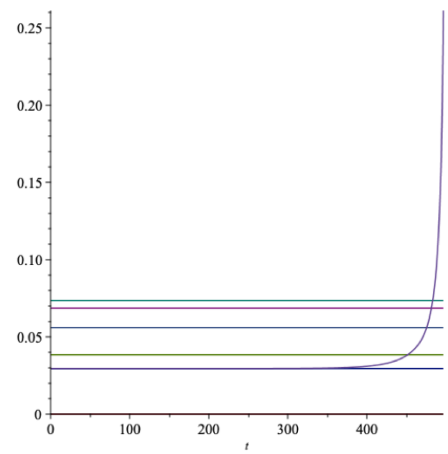

Fig. 7: The optimal extraction rate as a function of time (RU).

The upper line is the expected return on the fund, the next line corresponds to the threshold $m = \mu_W/(1 + \mu_W)$, then follows ce_1, and we notice that the optimal rate is below these two lines consistent with long term tests, while the certainty equivalent return rate is slightly above the optimal rate. The expected rate of return on the fund is seen to fail both the long term tests as a spending rate, in agreement with Corollary 1. The optimal spending rate $c_t(500), 0 \leq t < 500$, is the hyperbolic-type curve sharply increasing towards the horizon, which passes both long-run tests almost to the end of the horizon. The lowest line, tangent to this curve, is $c_0(\infty)$.

The growth rate of the optimal consumption c_t^* is 0.05 with standard deviation 0.18. Since $\gamma < \rho$, the agent has preference for late resolution of uncertainty.

When the parameter ρ decreases, the resistance to substitute consumption across time decreases and the optimal spending rate decreases. For example, when $\rho = 1.8$ and the other parameters are as above, then $c_0(\infty) = 0.025$. If the impatience increases, the optimal spending rate increases. For example, if $\beta = 0.90$, $\gamma = 2.0$ and $\rho = 1.8$, then $c_0(\infty) = 0.075$ for RU and 0.072 for EU. The growth rate of c_t^* is now -0.007 with standard deviation 0.17. Impatience does in general not help much on growth. $\square$

In Figure 8 we show a graph of the optimal extraction rate $c_0(\infty)(\rho)$ as a function of ρ for the values of $\beta = 0.99$ and $\gamma = 2.0$. Also included are the two thresholds m and ce_1, explained Section 6, as well as the expected rate of return on the fund μ_W and the certainty equivalent return rate ce_γ, all with the same numerical values as in Figure 7. These four quantities do not depend on ρ. This follows, since the portfolio fractions φ only depends on the relative risk aversion γ also for RU, and the expected rate of return on the fund $\mu_W = r + \varphi'\nu$ where ν is the vector of excess returns on the risky assets and r is the risk-free rate of return. Hence μ_W only depends on the preferences via the parameter γ also for recursive utility.

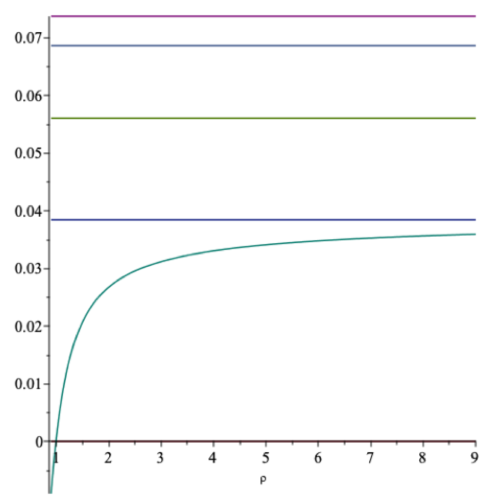

Fig. 8: The optimal long term spending rate as a function of ρ.

The optimal spending increases with ρ. By increasing the impatience, optimal spending increases for all values of ρ. Notice that nothing dramatic happens when ρ passes γ in value. We observe that the optimal spending rate passes both the long term tests with good margins.

For the truncated model the parameter β can not be too small, since then the curve $c_0(\infty)(\rho)$ starts out increasing, reaches a maximum and then continues with a decreasing convex shape. When the agent becomes impatient enough, say $\beta = 0.90$, this may seemingly happen. However, this feature is not real: When we use the Binomial model this pattern disappears (see for example Figure 5), and the curve is strictly increasing as in Figure 8.

In Figure 9 we show the graphs of optimal spending rate $c_0(\infty)(\gamma)$ as a function of γ, as well as the quantities $\mu_W(\gamma)$, $m(\gamma)$, $ce_1(\gamma)$ and ce_γ, when $\rho = 2$ and $\beta = 0.99$. The lowest graph represents the optimal long term spending rate. The highest falling curve is the expected rate of return on the fund, the next curve is the martingale threshold $m(\gamma)$, then the $ce_1(\gamma)$ graph and finally the curve representing the certainty equivalent ce_γ.

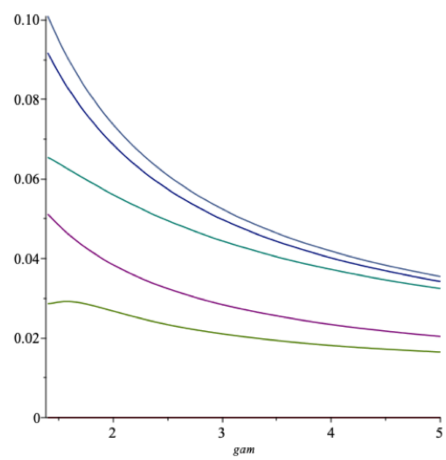

Fig. 9: $c_0(\infty)(\gamma)$, $\mu_W(\gamma)$, $m(\gamma)$, $ce_1(\gamma)$, ce_γ as functions of γ (RU).

The optimal spending rate $c_0(\infty)$ is seen to pass both the long run criteria for all vales of γ with good margin, while the expected rate of return on the fund fails both, in agreement with Corollary 1.

When the parameter γ is close enough to 1, the spending curve is large, which is due to the singularity in the coefficients a and b at $\gamma = 1$. In this case the truncations are not valid (but the discrete state probability version is not similarly affected).

For the data of this section and with an optimal spending rule $c_0(\infty) = 0.038$ with recursive utility and expected return on the fund $\mu_W = 0.074$, it takes about 12 years for $E(W_{12})$ to have increased 50% using the optimal spending rule, while it takes about 41 years for $E(W_{41})$ to have decreased 20% using μ_W as spending rule.

These examples and graphical illustrations tell us that the models, both of the *EU* and of the *RU* type, are fairly robust with respect to the size of the optimal spending rate. By changing the preference parameters within reasonable ranges, the optimal rate changes only moderately. This means that our results should have real world policy implications regarding optimal spending rates from endowment funds.

From Corollary 1 it follows that the spending rate can not be set equal to the expected real rate of the fund. This result is independent of all the preference parameters.

We round off with the case of the Norwegian SWT Government Fund Global.

9 The Norwegian SWF Government Fund Global

For this sovereign fund the Norwegian Ministry of Finance set down a commission in 2016 to consider the asset allocation problem. Table 2 below reflects the commission's market view on equity and risky bonds.[7]

	Expectation	Standard dev.	Covariance
Equity	4.83%	16.00%	0.00384
Bonds	0.68%	6.00%	
Equity premium	4.15%	14.67%	

Table 2: The commission's market view, Norwegian Ministry of Finance (2016).

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The commission recommends an equity share of $\varphi = 70\%$. Given a riskless rate of 0.68% and an equity premium with expectation 4.15% and standard deviation 14.67%, the excess return $\nu = 4.15\%$, $\sigma = 0.16$, the market-price-of-risk $\eta = \sigma^{-1}\nu = 0.2594$, and $M = 0.1627$.

This translates into an implicit relative risk aversion of $\gamma = 2.24$. This implies that the expected real rate of return $\mu_W = r + \varphi\nu = 3.6\%$ and the standard deviation $\sigma_W = \varphi\sigma = 11.20\%$.

[7]The report, which uses geometric returns, i.e., $E_t[\log\{X(t+1)/X(t)\}]$, is referred in our list of references. This is translated to simple returns in Table 2. The covariance reported in the table stems from the following calculation $0.16 \cdot 0.06 \cdot 0.4 = 0.00384$, where the intertemporal correlation coefficient is 0.4.

The certainty equivalent fund return is $ce_\gamma = 2.0\%$, $ce_1 = 2.9\%$ and $m = \mu_W/(1+\mu_W) = 3.5\%$. Observe that μ_W is larger than both ce_1 and m, thus not acceptable in the long run (with probability 1 and in 1th mean).

Let $\beta = 0.99$. Then the optimal long term spending rate with *expected utility* is $c_0(\infty) = 0.016$, which passes both the long run tests. This is 2.0% lower than the expected real return on the fund.

With *recursive utility,* assuming $\rho = 2.7$ and $\beta = 0.99$, where the other parameters are as above, then $c_0(\infty) = 0.017$ and we have preference for late resolution of uncertainty. When $\rho = 1.5$ we have $\gamma > \rho$, preference for early resolution of uncertainty, and the optimal long term spending is $c_0(\infty) = 0.014$, assuming the other parameters are as above.

For the data of this section and with an optimal spending rule $c_0(\infty) = 0.017$ with recursive utility and expected return on the fund $\mu_W = 0.036$, it takes about 12 years for $E(W_{12})$ to have increased 20% using the optimal spending rule, while it takes about 40 years for $E(W_{44})$ to have decreased 5% using μ_W as spending rule.

At the end of 2021 the market value of this fund was 1299 billions USD, and 5% decrease in 22 years amounts to 65 billions USD in expectation. If the optimal spending rule had been used, the fund would have been 11232 billions USD higher in expectation after 40 years. Should the young generations of Norwegians accept this?[8]

In Figure 10 we illustrate the optimal long term spending rate as a function of the parameter ρ. The parameter $\beta = 0.99$ and $\gamma = 2.24$. The upper line represents the expected rate of return on the fund $\mu_W = 0.036$. The next line is $m = 0.035$, then follows $ce_1 = 0.029$, and finally $ce_\gamma = 0.020$. The optimal spending rate $c_0(\infty)$ is the lowest increasing curve, and passes both the long run tests. In contrast, the expected rate of return on the fund, as a spending rate, does not pass either test (recall Corollary 1).

[8] In Norwegian kroner the amounts are, with the exchange rate of 10.0 NOK to the USD: Fund value at the end of 2021: 12299 billions NOK, the decline in expectation is 614 billions NOK in 40 years, and the difference between the two spending rules amounts to 112320 billions NOK in expectation.

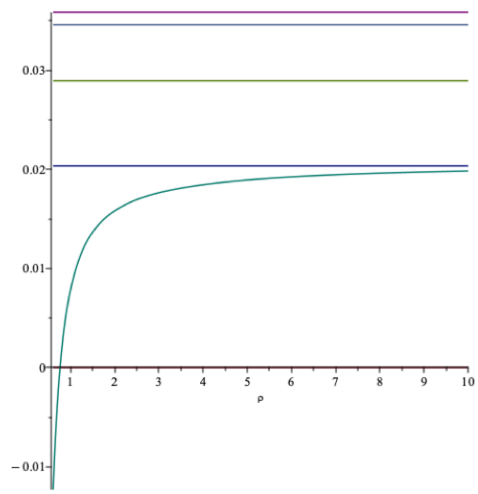

Fig. 10: The optimal spending rate $c_0(\infty)$ as a function of ρ.

Let us illustrate with some concrete values of ρ. In Figure 11 the parameter $\rho = 2.0$, $\beta = 0.98$ and $\gamma = 2.24$ as above. This implies preference for early resolution of uncertainty. Then we have the following picture in Figure 11: The horizontal lines are the same as in the previous figure (γ and φ are the same).

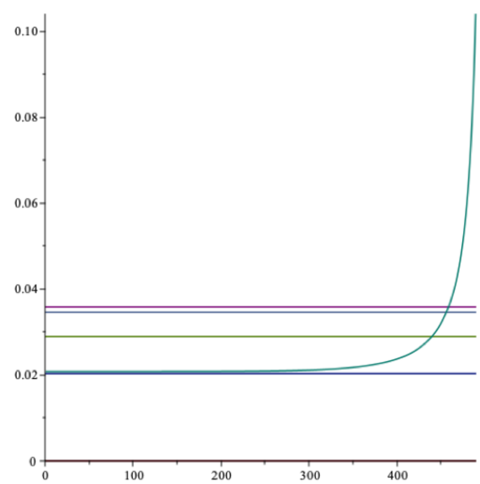

Fig. 11: The optimal extraction rate as a function of time (RU).

The optimal long run extraction rate is here $c_0(\infty) = 0.021$ for RU, larger than indicated in the previous figure, since the impatience rate has increased. The hyperbolic type curve is the optimal extraction rate $c_t(500), 0 \leq t < 500$.

This rate passes all the tests up up to around 400 years. In this example the growth rate of the optimal consumption c_t^* is 0.016 with a standard deviation of 0.11. The optimal long run spending rate with EU is here $c_0(\infty) = 0.022$.

Next we let γ vary: This corresponds to different values of the portfolio weight φ, which would fall from 0.98 to 0.31 when γ increase from 1.6 to 5, following a convex, hyperbolic curve. For $\rho = 2.0$ and $\beta = 0.99$, Figure 12 gives a graph of the optimal long term spending as a function of γ, the lowest curve in the figure.

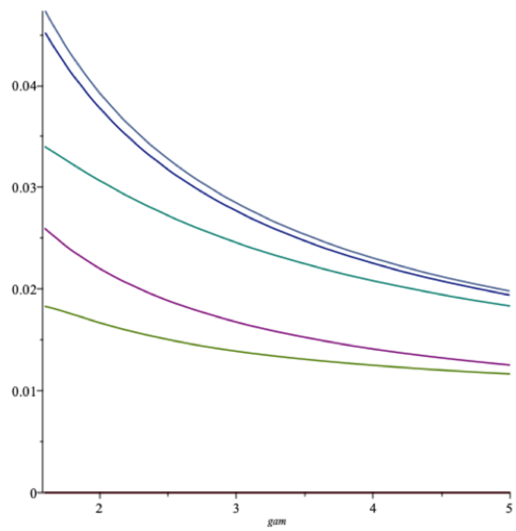

Fig. 12: The optimal spending rate $c_0(\infty)$ as a function of γ (RU).

In the Figure 12 is also included from the top, the expected rate of return, the martingale threshold m, the threshold ce_1 and the certainty equivalent ce_γ, all as functions of γ. Higher risk aversion leads to a lower optimal spending rate.

What if the optimal rate becomes very low or even negative? Suppose $EIS = 1.43$ corresponding to $\rho = 0.70$, $\gamma = 2.24$ and $\beta = .99$ so the agent is patient. Now the optimal long term spending rate is -0.0034 for recursive utility, i.e., negative. This must clearly be ruled out in the infinite horizon case, but still makes sense with a finite horizon. Say for instance that the fixed horizon is $T = 500$ years from now. Does this mean no spending at all?

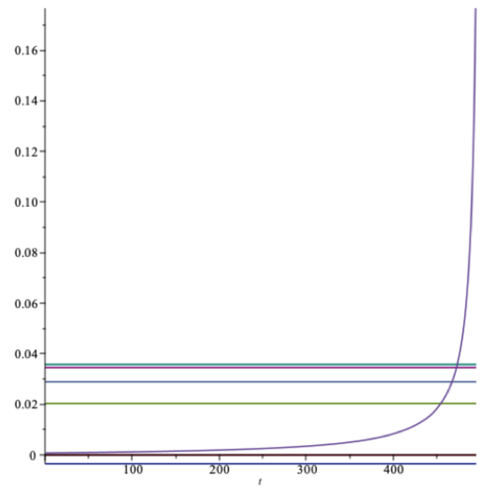

Fig. 13: The optimal spending rate as a function of time (RU).

Clearly not. See Figure 13, where the horizontal lines are the same as in Fig. 11, except the lowest line, which corresponds to $c_0(\infty) = -0.0034$. The optimal spending the first year is 0.00076 of the fund value, the optimal spending in year 30 is 0.00086, the optimal spending in year 70 is 0.0010, in year 100 it is 0.0012, in year 450 it is 0.0184 and in year 473 is is 0.0354 of the fund value, which is equal to the expected real rate of return on the fund in this situation, and so forth.

When the parameter ρ decreases further, the optimal spending rate $c_0(\infty)$ decreases, and still the optimal spending with a finite horizon is strictly positive, and increasing as the horizon comes closer. Provided $c_0(\infty) < ce_1$, where $ce_1 = 0.029$ here, the fund grows with probability one as t increases, but for $c_0(\infty) > \hat{c}$ the fund value converges to 0 almost surely.

The consumption growth rate in this example is 0.03 with a standard deviation of 0.15.

The test with almost sure convergence implies that a negative extraction rate $c_0(\infty)$ satisfies the long run tests, and the fund grows with time with probability one. This situation corresponds to an agent where the consumption substitution dominates.

9.1 Exogenous income streams

Suppose there is an exogenous income stream I_t added to the fund each year. The consequences of this will now be discussed. Let us assume that I_t is added to the fund in year t, where this stream of cash flows are assumed iid. We assume the amount I_t at time t is invested in the financial markets together with the rest of the fund W_t^*, where the new fund is denoted $W_t^I = W_t^* + I_t$ and the new spending we call c_t^I. Under this assumption the real expected return μ_W^I is equal to μ_W, since there is no reason that this addition to the fund will alter the expected return so long as the same optimal portfolio selection rule is used on the total. Recall that $\mu_W = r + \varphi\nu$ and $\sigma_W = \varphi'\sigma\sigma'\varphi$, that is, both these key parameters depend on market related quantities only. This means that the threshold values $ce_1^I = ce_1$ and $m_W^I = m_W$ are the same as before, and so is ce_γ.

What about the optimal spending rate $c_0(\infty)$? For expected utility we notice from the proof of Theorem 1 that the basic change happens in the budget constraint, where the Lagrange multiplier λ obtains a new value, but from the proof this is seen to have no consequence for the consumption to wealth ratio $c_0(\infty)$. For recursive utility we see from the proof of Theorem 5 that the optimal spending rate depends in addition to the budget constraint, also on the parameters μ_W and σ_W, which we have argued do not change by the added income stream. Accordingly, the spending rate $c_0(\infty)$ and the final horizon version $c_0(T)$ will both remain unchanged by I for both types of preferences.

However, optimal spending will naturally change, that is, increase, since we assume $I > 0$. This follows since the optimal spending with an exogenous income stream is still proportional to wealth:

$$c_t^I = c_0(\infty)W_t^I = c_0(\infty)(W_t^* + I_t) = c_0(\infty)W_t^* + c_0(\infty)I_t.$$

where $c_t^* = c_0(\infty)W_t^*$ is the optimal spending with no added income stream.

For the Norwegian SWF Government Fund Global this is of interest, since still an exogenous addition to the fund occurs each year. As explained, this will allow a larger spending, but the optimal "fiscal" rule, that is, the consumption to wealth *ratio*, remains unchanged by income I.

As an example, for the year 2021 the market value of this fund was 12.340 billion NOK, where the addition (I_{2021}) from the external oil-related activity was 2.942 billion NOK. This amount has been stable for several years, supporting our iid assumption. Supposing the optimal spending rule was 2% this year, this would amount to 188 billion NOK from the fund ex the direct oil supplement, and in addition 59 billion NOK from the latter.

10 Summary.

A central part of this paper has been to derive the optimal spending of an endowment fund, the consumption to wealth ratio. We show that this rate can not equal the expected real rate of the fund, since this would not be consistent with preference for diversification.

The rationale for this is that provided the fund is managed by diversification, this means that risk aversion, consumption substitution and impatience are essential in the optimal consumption and portfolio choice problem. To be consistent, the spending rate must also reflect this. As a consequence, the expected real rate of return is typically not an optimal spending rate, since this criterion is linked to risk neutrality.

We have developed two tests, with the property that if the optimal spending rate is below the corresponding threshold values, the fund will last "forever".

For this purpose, we adopted, and further developed the life cycle model to fit our purposes, where we consider both expected additive and separable utility as well as recursive utility in the setting of discrete time.

We demonstrate that when the agent is reasonably patient, the optimal consumption to wealth ratio passes the two long run tests, and is moreover strictly smaller than the expected rate of return on the fund. If the spending rate is set equal to the latter, both tests are demonstrated to fail. The implications of this is that the fund value W_t converges to zero with probability 1 as time t increases, and the expected value of the fund converges to zero as time t goes to infinity.

11 Appendix 1

11.1 The Financial Market

We consider a consumer who has access to a securities market, as well as a credit market. The security market consists of N risky and one risk-free security. The language and notation used here extends to continuous-time settings with a Brownian filtration. More details can be found in e.g., Skiadas (2009). The continuous-time analogue can be found in e.g., Duffie (2001).

The information filtration $\mathcal{F}_t$ is generated by a d-dimensional martingale

$B = (B^1, \cdots, B^d)'$, where prime means transpose, such that

$$E_{t-1}(\triangle B_t^i \triangle B_t^j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j, \end{cases} \tag{11.1}$$

for $t \in \{1, \cdots, T\}$, where $\triangle B_t^i = B_t^i - B_{t-1}^i$, and $B_0^i = 0$. Any martingale M can be uniquely expressed as $M_t = M_0 + \sum_{s=1}^t \theta_s' \triangle B_s$ for some predictable process θ. We shall assume that the number of risky assets $N = d$, where d is the spanning number of the filtration $\mathcal{F}_t, t = 0, 1, \ldots, T$, and T is the finite horizon of the economy. The vector stochastic process B is a dynamic orthonormal basis for the set of zero-mean martingales.

The Doob decomposition of any adapted stochastic process x_t is here a discrete-time, stochastic process that can be uniquely written as

$$x_t = x_0 + \sum_{s=1}^t \mu_s^x + \sum_{s=1}^t \sigma_s^x \triangle B_s,$$

or

$$\triangle x_t = \mu_t^x + \sigma_t^x \triangle B_t.$$

for μ^x and σ^x predictable processes, which can be expressed as

$$\mu_t^x = E_{t-1}(\triangle x_t) \quad \text{and} \quad \sigma_t^x = E_{t-1}(\triangle x_t \triangle B_t').$$

Price processes are denoted by S, and when adjusted for dividends they are called adjusted price process, or gains processes, denoted by X. The price process S of any risky asset is assumed nonzero at nonterminal nodes, and the return process R is defined by

$$R_t^n := \frac{\triangle X_t^n}{S_{t-1}^n} = \mu_t^{R,n} + \sigma_t^{R,n} \triangle B_t, \quad \text{for} \quad t = 1, \cdots, T,$$

where $\sigma_t^{R,n}$ is a $1 \times N$-vector at each time t related to each single asset n, $n = 1, 2, \cdots, N$. By summing this equation over t we obtain what is called the cumulative return process (R_0 is assumed to be arbitrary).

The securities market can now be described by the vector ν_t of expected returns of the N given risky securities in excess of the risk-less instantaneous return r_t, and σ_t^R is an $N \times N$ matrix associated to the risky asset prices, normalized by the asset process, so that $\sigma_t^R(\sigma_t^R)'$ is the covariance matrix for asset returns. Both μ_t^R and σ_t^R are assumed to be adaptive, measurable stochastic processes.

There is an underlying probability space $(\Omega, \mathcal{F}, P)$ and an increasing information filtration $\mathcal{F}_t$ generated by the d-dimensional orthonormal basis B. The parameter $\sigma^{(0)} = 0$, so that $r_t = \mu_0(t)$ is the risk-free interest rate (also a stochastic process). The state price $\pi(t)$ is connected to a density process ξ_t given by

$$\xi_t = \prod_{s=0}^{t} (1 - \eta_s' \triangle B_s). \tag{11.2}$$

The process η_t is called the market-price-of-risk process. The process ξ_t can be interpreted as a conditional density process $\xi_t = E_t(dQ/dP)$ of a probability measure Q equivalent to the given measure P. In our framework ξ_t is connected to the state price π_t as follows, $\pi_t = \pi_0 s_t^{-1} \xi_t$, where s_t is the price of the risk-less asset, with simple return $r_t = \triangle s_t / s_{t-1}$, so that $s_t = \prod_{s=0}^{t}(1 + r_s)$. Here r_t is the return on the risk-less asset in the time interval between $t-1$ and t, so it is $\mathcal{F}_{t-1}$-measurable. From this, using (11.2), we obtain the expression

$$\pi_t = \pi_0 \prod_{s=0}^{t} \frac{(1 - \eta_s' \triangle B_s)}{1 + r_s}. \tag{11.3}$$

We consider discounted price process $X_t s_t^{-1}$. The market-price-of-risk $\eta(t)$ satisfies, for each cumulative return process R^n, the equations

$$\mu_t^{R,n} - r_t = \sigma_t^{R,n} \eta_t, \quad n = 1, 2, \cdots, N,$$

for each t, where $\sigma_t^{R,n}$ is $1 \times N$. Alternatively, η_t satisfies the following system of equations

$$\sigma_t^R \eta_t = \nu_t, \quad t \in \{0, 1, \cdots, T\}, \tag{11.4}$$

where the nth component of ν_t equals $(\mu_t^{R,n} - r_t)$, the excess, conditional expected rate of return on security n at time t, $n = 1, 2, \cdots, N$. In (11.4) σ_t^R is a $N \times N$ matrix at each time t, assumed to be invertible ($d = N$). The matrix $\sigma_t' \sigma_t$ is the covariance matrix of the risky assets in units of prices at each time t, where $\sigma_t = \sigma_t^R$ for short, with a similar notational simplification for μ_t.

Equations (11.4) are the basic no-arbitrage restrictions for the financial market; when this system of equations hold, there exists a unique vector η_t for each t, modulo some technical conditions. We then think of the market as being in a 'dynamic equilibrium', where a price taker, our consumer, trades optimally resulting in an optimal consumption plan c and generating optimal wealth W.

Having determined the market-prices-of-risk η_t from (11.4), these in turn determine the state prices π_t in (11.3). The vector η_t gives a relationship between "risk" (conditional variances and covariances) and excess returns that must hold for there to be no arbitrage possibilities in the market, which is the basic message found in Black and Scholes' (1973) theory.

Let $(\theta^{(0)}, \theta)$ be a trading strategy, which finances the consumption plan c and generates the wealth W. Let

$$\varphi_t^{(j)} = \frac{\theta_t^{(j)} X_t^{(j)}}{W_t - c_t}, \quad j = 1, 2, \cdots, N$$

where $\varphi_t = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})'$ is the vector of portfolio ratios in the N risky assets. At the beginning of period t, the agent allocates the proportion $\frac{c_{t-1}}{W_{t-1}}$ of the wealth W_{t-1} to immediate consumption and invests the remaining amount $(W_{t-1} - c_{t-1})$ in the $1 + N$ assets, with proportion $\varphi_t^{(j)}$ going to asset $j \in \{1, 2, \cdots, N\}$ and the remaining proportion $(1 - \sum_{j=1}^N \varphi_t^{(j)})$ going to the risk-less asset.

The end-of-period wealth W_t is the result of this investment/consumption strategy and given by

$$W_t = (W_{t-1} - c_{t-1})\big((1 + r_t)(1 - \sum_{j=1}^{N} \varphi_t^{(j)}) + \varphi_t'(1 + R_t)\big),$$

where $R_t = (R_t^{(1)}, R_t^{(2)}, \cdots, R_t^{(N)})'$ are the (simple) returns on the N risky assets, and r_t is the (simple) return on the risk-free asset.

The consumer's problem is, for each initial wealth level w, to solve

$$\sup_{(c,\varphi)} U(c), \tag{11.5}$$

subject to an intertemporal budget constraint

$$W_t = \big(W_{t-1} - c_{t-1}\big)\big(1 + r_t + \varphi_t'(R_t - r_t)\big), \qquad W_0 = w. \tag{11.6}$$

The simple return R_t^W on the wealth portfolio is given by the relationship

$$1 + R_t^W = \frac{W_t}{W_{t-1} - c_{t-1}}, \tag{11.7}$$

since consumption c is contained in W, where c may be interpreted as "dividend".

The present problem is known as a *temporal* problem of choice.

12 Appendix 2.

Proof of Theorem 1.

From the relationship (2.3) we have

$$W_t = \frac{1}{\pi_t} E_t\Big(\sum_{s=t}^{T} \pi_s c_s^*\Big).$$

The optimal consumptions $c_s^* = (\lambda \pi_s \beta^{-s})^{-\frac{1}{\gamma}}$, $s = t, t+1, \ldots, T$ are found from (3.5). It follows from the expression (11.3) that

$$\frac{\pi_s}{\pi_t} = \prod_{v=t+1}^{s} \frac{1 - \eta_v' \triangle B_v}{1 + r_v}, \quad \text{where the product is 1 when } t = s,$$

which means that we can write

$$W_t = \frac{1}{\pi_t} E_t\Big(\sum_{s=t}^{T} \lambda^{-\frac{1}{\gamma}} \beta^{t/\gamma} \pi_t^{1-1/\gamma} \beta^{(s-t)/\gamma} \prod_{v=t+1}^{s} \frac{(1 - \eta_v' \triangle B_v)^{1-1/\gamma}}{(1 + r_v)^{1-1/\gamma}}\Big) =$$

$$c_t^* E_t\Big\{\sum_{s=t}^{T} \beta^{(s-t)/\gamma} \prod_{v=t+1}^{s} \frac{(1 - \eta_v' \triangle B_v)^{1-\frac{1}{\gamma}}}{(1 + r_v)^{1-\frac{1}{\gamma}}}\Big\}.$$

This shows the formula (4.1). □.

Proof of Theorem 2.

We use the result of Theorem 1, and first we compute the following expectation:

$$E_t\Big(\prod_{v=t+1}^{s} (1 - \eta_v' \triangle B_v)^{1-\frac{1}{\gamma}}\Big) = \prod_{v=t+1}^{s} E_t(1 - \eta_v' \triangle B_v)^{1-\frac{1}{\gamma}} \tag{12.1}$$

where we have used independence across time of the random variables $(1 - \eta_v' \triangle B_v)$, $v = 1, 2, \ldots$.

At this point we use a Taylor series approximations to the second order of the power function $(1 - \eta_v' \triangle B_v)^{1-\frac{1}{\gamma}}$, which gives for $v > t$

$$E_t\big\{(1-\eta_v' \triangle B_v)^{1-\frac{1}{\gamma}}\big\} = E_t\big\{1-(1-\frac{1}{\gamma})\eta' \triangle B_v + \frac{(1-\frac{1}{\gamma})(1-\frac{1}{\gamma}-1)}{1 \cdot 2}\eta' \triangle B_v \triangle B_v' \eta\big\} =$$

$$1 + \frac{1-\gamma}{2\gamma^2}\eta'\eta.$$

This means that equation (12.1) can be written

$$E_t\big(\prod_{v=t+1}^{s}(1 - \eta_v' \triangle B_v)^{1-\frac{1}{\gamma}}\big) = (1 + \frac{1-\gamma}{2\gamma^2}\eta'\eta)^{(s-t)}.$$

As a consequence, from Theorem 1 the optimal wealth to consumption ratio, $\frac{1}{c_t(T)}$, can be expressed as follows:

$$\frac{1}{c_t(T)} = \frac{W_t^*}{c_t^*} = \sum_{s=t}^{T}\big(\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}(1 + \frac{1-\gamma}{2\gamma^2}\eta'\eta)\big)^{(s-t)}.$$

The right hand side can be seen to be a geometric sum, which is given by the formula of the Theorem. Letting $T \to \infty$, this formula results in

$$\frac{1}{c_t(\infty)} = \frac{W_t^*}{c_t^*} = \frac{1}{1 - \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}\big(1 + \frac{1-\gamma}{2\gamma^2}\eta'\eta\big)},$$

provided the term $\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}(1+\frac{1-\gamma}{2\gamma^2}\eta'\eta\big) < 1$. For reasonable values of the parameters this inequality obviously holds true, which proves the theorem. $\square$.

Proof of the spending formula (4.4) in Remark 1.

We start with the optimal spending formula in Theorem 2, which can be approximated as

$$c_t(\infty) = 1 - \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}\big(1 + \eta'\eta\big)^{\frac{1-\gamma}{2\gamma^2}},$$

and rewritten as

$$1-\exp\Big(\frac{1}{\gamma}\big[\ln(\beta)+\ln(1+r)(1-\gamma)+\ln(1+\eta'\eta)\frac{1-\gamma}{2\gamma}\big]\Big).$$

Next we use a first order Taylor approximation of the two last logarithmic functions, which results in

$$1-\exp\Big(\frac{1}{\gamma}\big[\ln(\beta)+r(1-\gamma)+\eta'\eta\frac{1-\gamma}{2\gamma}\big]\Big).$$

Finally, we use a first order Taylor approximation of the exponential function, which reduces the formula to the following expression

$$\delta\,\big(\frac{1}{\gamma}\big)+\big(1-\frac{1}{\gamma}\big)\big(r+\frac{1}{2\gamma}\nu'(\sigma\sigma')^{-1}\nu\big),$$

where $\delta=-\ln(\beta)$. This is the formula in Remark 1. □

Demonstration of the formula (4.5) in Remark 2.

Here we make use of the formula (4.4) in Remark 1. We want to show that

$$r+\varphi'\nu>\delta\,\big(\frac{1}{\gamma}\big)+\big(1-\frac{1}{\gamma}\big)\big(r+\frac{1}{2\gamma}\nu'(\sigma\sigma')^{-1}\nu\big).$$

From our results in Section 3.2, this is the same as

$$r+\gamma\varphi'(MM')\varphi>\delta\,\big(\frac{1}{\gamma}\big)+\big(1-\frac{1}{\gamma}\big)\big(r+\frac{1}{2}\varphi_c'(MM')\varphi\big).$$

Since φ_c and φ are close, this inequality can be written

$$\frac{1}{2}\varphi'(MM')\varphi\geq\frac{\delta-r}{\gamma(1-\gamma)}.$$

Because $\delta\approx r$ and $\frac{1}{2}\varphi'(MM')\varphi>0$ with good margin, this inequality holds true, with good margin, unless the impatience rate δ is too large. □

Proof of Theorem 4.

Starting with the relationship

$$W_{t+1}^{*}=W_0(1-c)^{t+1}\prod_{s=0}^{t}(1+R_{s+1}^{W}),\quad t=0,1,\ldots$$

where c is a spending rate, we obtain the following

$$\frac{1}{t+1}\ln(W_{t+1}) = \frac{1}{t+1}\ln(W_0) + \ln(1-c) + \frac{1}{t+1}\sum_{s=0}^{t}\ln(1+R^W_{s+1})$$

With our iid assumption for $R_1, R_2, \ldots$ it follows by the SLLN that

$$\frac{1}{t+1}\sum_{s=0}^{t}\ln(1+R^W_{s+1}) \to E(\ln(1+R^W)) \text{ almost surely.}$$

This means that

$$\frac{1}{t+1}\ln(W_{t+1}) \to \ln(1-c) + E(\ln(1+R^W)) \text{ almost surely,}$$

or

$$\ln(W_{t+1}) - (t+1)(\ln(1-c) + E(\ln(1+R^W)) \to 0 \text{ almost surely as } t \to \infty.$$

This shows that when $(\ln(1-c) + E(\ln(1+R^W)) > 0$ then $\ln(W_{t+1}) \to +\infty$ as $t \to \infty$, which implies that W_t grows without limit, almost surely, as t increases. This proves (b) of the theorem.

When the term $(\ln(1-c) + E(\ln(1+R^W)) < 0$ on the other hand, then $\ln(W_{t+1}) \to -\infty$ as $t \to \infty$, which implies that $W_t \to 0$ almost surely, which proves (a). □

13 Appendix 3.

13.1 The first order conditions for optimal consumption: Recursive utility.

First we determine the optimal consumption. The agent is characterized by a utility function U and an endowment process $e \in L$. The agent's problem is

$$\sup_{c \in L_+} U(c) \text{ subject to } E\Big(\sum_{s=0}^{T}\pi_s c_s\Big) \le E\Big(\sum_{s=0}^{T}\pi_s e_s\Big),$$

where L is the space of adapted consumption processes, L_+ its positive cone, and π is the state price deflator.

The Lagrangian of the problem is

$$\mathcal{L}(c, \lambda) = U(c) - \lambda\, E\Big(\sum_{s=0}^{T} \pi_s (c_s - e_s) \Big),$$

where $\lambda > 0$ is the Lagrangian multiplier. Assuming U to be continuously differentiable, the gradient of U at c in the direction x is denoted by $\triangledown U(c; x)$. This directional derivative is a linear functional, and by the Riesz Representation Theorem and for example, dominated convergence, it is given by

$$\triangledown U(c; x) = E\Big(\sum_{s=0}^{T} y_s x_s \Big).$$

Here y is the Riesz representation of $\triangledown U(c; \cdot)$. The first-order condition is

$$\triangledown \mathcal{L}(c, \lambda; x) = 0 \text{ for all } x \in L.$$

This is equivalent to

$$E\Big\{ \sum_{s=0}^{t} (y_s - \lambda \pi_s) x_s \Big\} = 0 \text{ for all } x \in L.$$

This implies that $y_t = \lambda \pi_t$ for all $t \leq T$.

Our next task is to characterize the Riesz representation y of U. When this is done, by the above result we have the marginal rates of substitution in the economy equal to the price ratios, $y_{t+1}/y_t = \pi_{t+1}/\pi_t$.

13.2 The marginal rate of substitution

In order to find the optimal consumption we need to find the Riesz representation p associated with the utility function U as explained in the last section.

Using directional derivatives and backward induction, we can show that the utility gradient is given by the following expression

$$\triangledown\, U(c; x) = \triangledown U_0(c; x) = E\Big\{ \sum_{t=0}^{T} x_t\ f_c(c_t, m_{t+1}) \prod_{s=0}^{t-1} \frac{f_m(c_s, m_{s+1})}{u'(m_{s+1})} u'(U_{s+1}) \Big\}, \tag{13.1}$$

from which it follows that the Riesz representation is given as

$$y_t = f_c(c_t, m_{t+1}) \prod_{s=0}^{t-1} \frac{f_m(c_s, m_{s+1})}{u'(m_{s+1})} u'(U_{s+1}), \tag{13.2}$$

for $t = 0, 1, \cdots, T$, (see e.g., Aase (2021)).

The intertemporal marginal rate of substitution, or the stochastic discount factor, $\mathcal{M}_{t+1} := y_{t+1}/y_t$ is given by the formula

$$\mathcal{M}_{t+1} = \frac{f_c(c_{t+1}, m_{t+2})}{f_c(c_t, m_{t+1})} f_m(c_t, m_{t+1}) \frac{u'(U_{t+1})}{u'(m_{t+1})}. \tag{13.3}$$

Along the optimal consumption path $\mathcal{M}_{t+1} = \pi_{t+1}/\pi_t$, that is, $\mathcal{M}_{t+1}$ equals the price ratio.

In order to find a formula for the stochastic discount factor we must compute the quantities in (13.3), which are

$$\frac{\partial}{\partial c} f(c_t, m_{t+1}) = (1-\beta) U_t^{\rho} c_t^{-\rho}, \quad \frac{\partial}{\partial m} f(c_t, m_{t+1}) = \beta U_t^{\rho} m_{t+1}^{-\rho},$$

and

$$\frac{u'(U_{t+1})}{u'(m_{t+1})} = \frac{U_{t+1}^{-\gamma}}{m_{t+1}^{-\gamma}}.$$

This means that the stochastic discount factor takes the form

$$\mathcal{M}_{t+1} = \beta \Big(\frac{c_{t+1}}{c_t}\Big)^{-\rho} \Big(\frac{U_{t+1}}{m_{t+1}}\Big)^{\rho-\gamma}. \tag{13.4}$$

Let c signify optimal consumption, and W_t is the agent's wealth at time t, given by

$$W_t = \frac{1}{\pi_t} E_t\Big(\sum_{s=t}^{T} \pi_s c_s\Big). \tag{13.5}$$

Our definition of wealth W_t includes current consumption (dividend), so the simple real rate of return on the wealth portfolio over the period $(t, t+1)$ is R^W_{t+1} given by

$$1 + R^W_{t+1} := \frac{W_{t+1}}{W_t - c_t}, \tag{13.6}$$

as before. By the definition in (13.6), it now follows by a string of manipulations that

$$\mathcal{M}_{t+1} = \beta^{\frac{1-\gamma}{1-\rho}} \Big(\frac{c_{t+1}}{c_t}\Big)^{-\rho\frac{1-\gamma}{1-\rho}} \big(1+R^W_{t+1}\big)^{\frac{\rho-\gamma}{1-\rho}}. \tag{13.7}$$

This expression has been the starting point for much of the literature on RU in discrete time models; see for example, Mehra and Donaldson (2008) and Cochrane (2008). This is the stochastic discount factor, first derived by Epstein and Zin (1989-91) in their seminal papers based on dynamic programming techniques.

We finally find the optimal consumption c^* as follows: Below we assume $\rho > 0$ and $\gamma \neq 1$. Using that $y_t = \lambda \pi_t$, we obtain from the relationship in (13.7) that

$$\ln\Big(\frac{c^*_{t+1}}{c^*_t}\Big) = \frac{1}{\rho}\ln(\beta) - \frac{1}{\rho}\,\frac{1-\rho}{1-\gamma}\ln\Big(\frac{\pi_{t+1}}{\pi_t}\Big) - \frac{1}{\rho}\,\frac{\gamma-\rho}{1-\gamma}\ln(1+R^W_{t+1}). \quad \text{(RU)} \tag{13.8}$$

It is instructive to compare this relationship to the corresponding one for expected utility, which is

$$\ln\Big(\frac{c^*_{t+1}}{c^*_t}\Big) = \frac{1}{\gamma}\ln(\beta) - \frac{1}{\gamma}\,\ln\Big(\frac{\pi_{t+1}}{\pi_t}\Big). \qquad \text{(EU)} \tag{13.9}$$

Both these difference equations can be solved, using iteration and the properties of the logarithm. Starting with equation (13.8), it can be written

$$c^*_{t+1} = c^*_t \frac{\big(\beta(1+r_t)^{\frac{1-\rho}{1-\gamma}}\big)^{\frac{1}{\rho}}}{(1-\eta'_t\triangle B_t)^{\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}(1+\mu_{W,t}+\sigma'_{W,t}\triangle B_t)^{\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}}, \tag{13.10}$$

where $\mu_{W,s}$ is the expected simple return of the wealth portfolio at time s and $\sigma_{W,s}$ is the corresponding vector of volatilities. Using this recursive relation also for c^*_t and iterating, we obtain the following closed form solution for the optimal consumption

$$c^*_t = c_0\beta^{\frac{t}{\rho}} \prod_{s=0}^{t}(1+r_s)^{\frac{1}{\rho}\frac{1-\rho}{1-\gamma}} \prod_{s=0}^{t}(1-\eta'_s\triangle B_s)^{-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}} \cdot \prod_{s=0}^{t}(1+\mu_{W,s}+\sigma'_{W,s}\triangle B_s)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}, \tag{13.11}$$

while the corresponding solution for the conventional EU-model is

$$c_t^* = c_0\beta^{\frac{t}{\gamma}}\prod_{s=0}^{t}(1+r_s)^{\frac{1}{\gamma}}\prod_{s=0}^{t}(1-\eta_s'\triangle B_s)^{-\frac{1}{\gamma}}. \tag{13.12}$$

This last expressions is equivalent to (3.5) in Section 3.1, and has been used in the analysis of the EU model, see Appendix 1. Notice that $\mu_W = r + \varphi'\nu$ and $\sigma_W' = \varphi'\sigma$, where ν, r and σ are market related quantities and where φ are the optimal portfolio ratios, to be found in the next section for RU.

The formulas (13.8)-(13.12) first appeared in Aase (2017).

13.3 Optimal portfolio selection with RU

Let us consider the basic problem (2.1) at time t, where the optimal consumption has been inserted. Since the utility function is increasing, the constraint is binding and what remains is to solve the finite dimensional problem

$$\max_{\varphi} U_t = \max_{\varphi_{t+1}} f(c_t^*, m_{t+1}). \tag{13.13}$$

Since the portfolio weights for the next period only appear in the second term of the aggregator f, the first order condition is

$$\frac{\partial}{\partial\varphi} f(c_t^*, m_{t+1}) = f_m(c_t^*, m_{t+1})\frac{\partial}{\partial\varphi} m_{t+1} =$$

$$\beta\big(\frac{U_t}{m_{t+1}}\big)^{\rho}\frac{\partial}{\partial\varphi}\big(E_t(U_{t+1}^{1-\gamma})\big)^{\frac{1}{1-\gamma}} = \beta\big(\frac{U_t}{m_{t+1}}\big)^{\rho}\frac{1}{1-\gamma}\big(E_t(U_{t+1}^{1-\gamma}\big)^{\frac{\gamma}{1-\gamma}}\frac{\partial}{\partial\varphi}E_t(U_{t+1}^{1-\gamma}) = 0$$

where we have used one of the formulas appearing after equation (13.3). Now we employ the property of scale invariance, which here means that $U_t = f_c(c_t, m_{t+1})W_t$. With this, the above can be written

$$\beta\big(\frac{U_t}{m_{t+1}}\big)^{\rho}\frac{1}{1-\gamma}\big(E_t(U_{t+1}^{1-\gamma}\big)^{\frac{\gamma}{1-\gamma}}\frac{\partial}{\partial\varphi}E_t\big(f_c^{1-\gamma}W_{t+1}^{1-\gamma}\big) = 0.$$

Using the equation for the wealth dynamics in (11.6), the first order conditions for φ are

$$E_t\big(f_c(c,m)^{1-\gamma}(1 + r_t + \varphi_{t+1}(R_{t+1} - r_{t+1}))^{-\gamma}(R_{t+1} - r_{t+1})\big) = 0, \tag{13.14}$$

for $t = 0, 1, , \dots, T-1$.[9] In our case $f_c(c_t, m_{t+1}) = (1-\beta)U_t^{\rho} c_t^{-\rho}$. With a deterministic investment opportunity set it follows that $f_c(c_t, m_{t+1})$ is deterministic, since the consumption to wealth ratio is deterministic. This means that we obtain the same first order conditions as for expected utility given in equation (3.7), and under the same conditions φ_{t+1} is given by the formula (3.8) also for recursive utility. This agrees with Skiadas (2009). A continuous-time version can be found in Svensson (1989).

Having solved the optimal consumption and portfolio selection problem in the life cycle model for recursive utility, the solution can be used to find the optimal spending rate, which we do in the next section.

14 Appendix 4.

Proof of Theorem 5.

We again start from the relationship (2.3)

$$W_t = \frac{1}{\pi_t} E_t\Big(\sum_{s=t}^{T} \pi_s c_s^* \Big).$$

The optimal consumptions c_s^*, $s = t, t+1, \dots, T$ are found from (13.8). It again follows from the expression (11.3) that

$$\frac{\pi_s}{\pi_t} = \prod_{v=t+1}^{s} \frac{1 - \eta_v' \triangle B_v}{1 + r_v},$$

which means that we can write

$$W_t = \frac{1}{\pi_t} E_t\Big(\sum_{s=t}^{T} c_0 \beta^{t/\rho} \pi_t^{1 - \frac{1}{\rho}\frac{1-\rho}{1-\gamma}} \prod_{s=0}^{t} (1 + \mu_s^W + \sigma_s^W \triangle B_s)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}$$

$$\beta^{(s-t)/\rho} \prod_{v=t+1}^{s} \frac{(1 - \eta_v' \triangle B_v)^{1 - \frac{1}{\rho}\frac{1-\rho}{1-\gamma}}}{(1 + r_v)^{1 - \frac{1}{\rho}\frac{1-\rho}{1-\gamma}}} \prod_{v=t+1}^{s} (1 + \mu_v^W + \sigma_v^W \triangle B_v)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}} \Big) =$$

$$c_t^* E_t\Big\{ \sum_{s=t}^{T} \beta^{(s-t)/\rho} \prod_{v=t+1}^{s} \frac{(1 - \eta_v' \triangle B_v)^{1 - \frac{1}{\rho}\frac{1-\rho}{1-\gamma}} (1 + \mu_v^W + \sigma_v^W \triangle B_v)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}}{(1 + r_v)^{1 - \frac{1}{\rho}\frac{1-\rho}{1-\gamma}}} \Big\}.$$

[9] Existence and uniqueness of this problem has been proved by Skiadas (2008).

This shows the formula (7.4). $\square$.

Proof of Theorem 6.

We use the result of Theorem 5. We simplify the notation as follows in the first steps below:

$$a := \frac{1}{\rho}\frac{1-\rho}{1-\gamma}, \quad b := \frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}.$$

First we need to compute expectations of the following type:

$$E_t\Big((1-\eta'\triangle B_v)^{1-a}(1+\mu_W+\sigma_W'\triangle B_v)^{-b}\Big),$$

for $v \geq t+1$. By use of a Taylor series approximation to the second order, we obtain

$$E_t\Big(\big(1-(1-a)\eta'\triangle B_{v+1}+\frac{(1-a)(1-a-1)}{1\cdot 2}\eta'\triangle B_{v+1}\triangle B_v'\eta\big)\cdot$$
$$\big(1-b(\mu_W+\sigma_W'\triangle B_v)+\frac{(-b)(-b-1)}{1\cdot 2}(\mu_W+\sigma_W'\triangle B_v)^2\big)\Big) =$$
$$1-b\mu_W+\frac{1}{2}b(1+b)(\mu_W^2+\sigma_W'\sigma_W)+b(1-a)\eta'\sigma_W$$
$$-b(1-a)(1+b)\mu_W\eta'\sigma_W-\frac{1}{2}a(1-a)(1-b\mu_W)\eta'\eta-\frac{1}{4}a(1-a)b(1+b)\mu_W^2\eta'\eta.$$

Here we have carried out the multiplication, then ignored terms of order 3 and larger, and used the property of $\triangle B_v$ in (11.1).

We now apply the result of Theorem 5. By independence through time this gives that the reciprocal of the optimal spending rate, $c_t(T)^{-1} = \frac{W_t^*}{c_t^*}$, can be written

$$\frac{W_t^*}{c_t^*} = E_t\Big\{\sum_{s=t}^{T}\beta^{(s-t)/\rho}\prod_{v=t+1}^{s}\frac{(1-\eta_v'\triangle B_v)^{1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}(1+\mu_v^W+\sigma_v^W\triangle B_v)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}}{(1+r_v)^{1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}}\Big\} =$$

$$\sum_{s=t}^{T}\Big(\beta^{\frac{1}{\rho}}(1+r)^{(\frac{1}{\rho}\frac{1-\rho}{1-\gamma}-1)}\big\{1-b\mu_W+\frac{1}{2}b(1+b)(\mu_W^2+\sigma_W'\sigma_W)+b(1-a)\eta'\sigma_W$$
$$-b(1-a)(1+b)\mu_W\eta'\sigma_W-\frac{1}{2}a(1-a)(1-b\mu_W)\eta'\eta-\frac{1}{4}a(1-a)b(1+b)\mu_W^2\eta'\eta\big\}\Big)^{s-t} =$$

$$\frac{k(\beta,\rho,\gamma,\eta,\nu,r,\sigma,\varphi)^{T-t}-1}{k(\beta,\rho,\gamma,\eta,\nu,r,\sigma,\varphi)-1},$$

where

$$k(\beta,\rho,\gamma,\eta,\nu,r,\sigma,\varphi)=\beta^{\frac{1}{\rho}}(1+r)^{(\frac{1}{\rho}\frac{1-\rho}{1-\gamma}-1)}\big\{1-(\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})\mu_W + \frac{1}{2}(\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})(1+\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})(\mu_W^2+\sigma_W'\sigma_W)+(\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})\cdot (1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma})\big[1-(1-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})\mu_W\big]\eta'\sigma_W-\frac{1}{2}(\frac{1}{\rho}\frac{1-\rho}{1-\gamma})(1-\frac{1}{\rho}\frac{1-\rho}{1-\gamma})\cdot \big[1-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}\mu_W-\frac{1}{2}(\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})(1+\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma})\mu_W^2\big]\eta'\eta\big\}. \tag{14.1}$$

This proves Theorem 6. □

<u>Derivation of the expected consumption growth rate and its SD for RU.</u>

In this kind of analysis it can be of interest to find the expected consumption growth rate and the corresponding volatility. Here this can be done with the technique developed in the previous theorem. We start with the equation (13.10) from Section 13.2, which can be written:

$$\frac{c_{t+1}^*}{c_t^*}=\beta^{\frac{1}{\rho}}(1+r_t)^{\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}(1-\eta_t\triangle B_t)^{-\frac{1}{\rho}\frac{1-\rho}{1-\gamma}}(1+\mu_t^W+\sigma_t^W\triangle B_t)^{-\frac{1}{\rho}\frac{\gamma-\rho}{1-\gamma}}. \tag{14.2}$$

Following the steps in the proof of Theorem 6, the conditional expected consumption growth rate is

$$E_t\big(\frac{c_{t+1}^*}{c_t^*}-1\big)=\beta^{\frac{1}{\rho}}(1+r_t)^a\Big(1-b\mu_W+\frac{1}{2}b(1+b)\big(\mu_W^2+\sigma_W'\sigma_W\big) - ba\big(1-(1+b)\mu_W\big)\eta'\sigma_W+\frac{1}{2}a(1+a)\big(1-b\mu_W-\frac{1}{2}b(1+b)\mu_W^2\big)\eta'\eta\Big)-1. \tag{14.3}$$

Next we need the conditional expected value of the following ratio

$$(\frac{c_{t+1}^*}{c_t^*})^2=\beta^{\frac{2}{\rho}}(1+r_t)^{2a}\,(1-\eta_t\triangle B_t)^{-2a}(1+\mu_t^W+\sigma_t^W\triangle B_t^{-2b},$$

which is

$$E_t\big((\frac{c_{t+1}^*}{c_t^*})^2\big)=\beta^{\frac{2}{\rho}}(1+r_t)^{2a}\Big(1-2b\mu_W+b(1+2b)\big(\mu_W^2+\sigma_W'\sigma_W\big) - 4ba\big(1-(1+2b)\mu_W\big)\eta'\sigma_W+a(1+2a)\big(1-b\mu_W-b(1+2b)\mu_W^2\big)\eta'\eta\Big). \tag{14.4}$$

From these two quantities we can find the conditional standard deviation of the expected consumption growth rate as

$$\sigma_{c^*}(t) = \sqrt{E_t\big((\frac{c^*_{t+1}}{c^*_t})^2\big) - \big(E_t\big(\frac{c^*_{t+1}}{c^*_t}\big)\big)^2}.$$

□

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