Intuitive probability of non-intuitive events

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Intuitive probability of non-intuitive events.

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Abstract

Quantitative probability in the subjective theory is assumed to be finitely additive and defined on all the subsets of an underlying state space. Functions from this space into an Euclidian n-space create a new probability space for each such function. We point out that the associated probability measures, induced by the subjective probability, on these new spaces can not be finitely additive and defined on all the subsets of Euclidian n-space, for $n \geq 3$. This is a consequence of the Banach-Tarski paradox. In the paper we show that subjective probability theory, including Savage's theory of choice, can be reformulated to take this, and similar objections into account. We suggest such a reformulation which, among other things, amounts to adding an axiom to Savage's seven postulates, and then use a version of Carathéodory's extension theorem.

KEYWORDS: The Banach-Tarski paradox, the axiom of choice, Savage's theory of choice, monotone continuity, countable additivity, Carathéodory's extension theorem, syndicates, contingent claims.

1 Introduction.

The classical representation of expected utility is due to John von Neumann and Oskar Morgenstern (von Neumann and Morgenstern (1944)). Several

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years earlier, Frank P. Ramsey outlined a theory of subjective probability and expected utility (Ramsey (1931)). Subjective probability was developed independently by de Finetti (1937), and is also treated by Borel (1924), Koopman (1940a,b), Venn (1886), and others. Émile Borel (1924) has an interesting discussion of a theory of probability put forward by John Maynard Keynes (Keynes (1921)). Much of this received little notice until the appearance of Savage's classic on the foundations of statistics (Savage (1954)). Building on Ramsey as well as von Neumann and Morgenstern for expected utility and de Finetti for subjective probability, Savage presented the first complete axiomatization of subjective expected utility.

Bruno de Finetti insisted that probability should be finitely additive and defined on all subsets of a state space. Koopman and de Finetti derived a probability measure from a qualitative probability under the assumption that, for any integer n, there are n mutually exclusive, equally probable events. Savage showed that this strong assumption is unnecessary, in that if a qualitative probability is just fine and tight, then there is one and only one probability measure compatible with it. Also Savage restricted attention to finite additivity on the set of all subsets in the state space. This is in contrast to the famous fundaments of probability theory set forth by A. N. Kolmogorov (Kolmogorov (1933)), where a probability measure is defined on the measurable events of a σ-algebra, and is countably additive.

In this paper we point out that if the range space in the theory of Savage (1954) is Euclidian n-space, with $n \geq 3$, one must follow Kolmogorov's choice in order to avoid serious paradoxes, like the Banach-Tarski paradox. To accomplish this, we suggest that the theory is reformulated to take this objection into account. We demonstrate an extension which involves an assumption of monotone continuity of the subjective probability, which we add as a postulate to Savage's 7 other postulates.

With this in place we first use the result that the quantitative probability measure P is countably additive as a result of this postulate, in which case we can the Hahn-Kolmogorov theorem, a version of the Carath*é*odory's extension theorem, which is a rather deep result in measure theory. This gives us a unique countable additive probability measure on the sigma-algebra generated by the algebra in Savage's model, which agrees with the subjective probability on the sigma-algebra. This structure can be carried over to the range spaces, in which case the paradoxes mentioned are avoided. As a consequence the basic other structures of Savage's approach remain. The importance of this can hardly be exaggerated, considering the high standing of this theory, in particular in economics of uncertainty, decision theory and related fields.

In the last section we present two examples that connect the subjective probability theory to the theory of syndicates as well as contingent claims evaluation. We indicate that probability, like prices, should be internalized, and how this interpretation could be applied more widely.

2 Subjective expected utility.

In this section we give a brief sketch of the subjective expected utility theory of Leonard Savage. In addition to Savage (1954), we also follow Fishburn (1970, 82 and 86). The basic primitives are a set Z of consequences, a set S of states of the world, and a preference relation $\succ$, a weak order, on the set H of all functions $f, g, \dots$ from S into Z. The elements of $H := Z^S$ are denoted by acts. If the agent selects f and state $s \in S$ occurs, then she will experience "consequence" $f(s) \in Z$. The agent is facing uncertainty about what state will occur, which is quantified by a finitely additive probability measure P on the set $\mathcal{A}^0$, the algebra of all subsets of S. An element $A \in \mathcal{A}^0$ is called an event, and $P(A)$ is a quantitative measure of the agent's degree of belief that event A occurs. That is, $P(A)$ is the individual's personal, quantitative probability of the event A that agrees with $\succ$ (see below).

Savage sets up seven postulates for $\succ$ on H. These contain a typical ordering axiom, several independence axioms, a tightness axiom saying that the state space is fine and tight, and a dominance axiom. From these it follows that there exists a unique, finitely additive probability measure P defined on all the subsets $\mathcal{A}^0$ of S and a bounded, unique up to a positive affine transformation, real-valued function $u : Z \to \mathbb{R}$ such that

$$f \succ g \Leftrightarrow \int_S u(f(s))dP(s) > \int_S u(g(s))dP(s). \tag{1}$$

For any $f \in H$, $P \circ f^{-1}(\cdot)$ is a probability measure on an algebra $\mathcal{B}_0$ of subsets of Z defined by

$$P_f(B) := P \circ f^{-1}(B) = P\{s \in S : f(s) \in B\}, \quad \text{for any } B \in \mathcal{B}_0.$$

We return to a specification of the set $\mathcal{B}_0$ and its extension below, which depends on the structure of Z. We call the probability space $(Z, \mathcal{B}_0, P_f)$ the *range* space of f induced by P.

With this connection in mind, the numerical representation in (1) can be reformulated as

$$f \succ g \Leftrightarrow \int_Z u(z)dP_f(z) > \int_Z u(z)dP_g(z), \tag{2}$$

where the integrals are in the sense of Lebesgue-Stieltjes. In the classical interpretation we would thus associate the acts with stochastic variables.

Savage's axioms also imply that the events in $\mathcal{A}^0$ are continuously divisible in the sense that, for any $A \in \mathcal{A}^0$ and any $\alpha \in [0, 1]$ there exists a $B \subseteq A$ such that $P(B) = \alpha P(A)$. Despite the fact that this forces S to be uncountably infinite, Z can have as few as two elements.

This representation clearly draws from de Finetti and von Neumann and Morgenstern. Consider the binary relation $\succ$ defined on $\mathcal{A}^0$ instead, with $A \succ B$ interpreted as "A is more probable than B". The construction is such that $A \succ B$ if and only if $f \succ g$ whenever c and d are consequences in Z such that c is preferred to d, $f(s) = c$ for all $s \in A$, $f(s) = d$ for all $s \in A^c$, $g(s) = c$ for all $s \in B$, and $g(s) = d$ for all $s \in B^c$. That is, $A \succ B$ if the individual would rather take her chances on A than B to obtain the desired consequence.

It is then proven that the axioms imply that there is a unique, atomless, finitely additive probability measure P on $\mathcal{A}^0$ for which

$$A \succ B \Leftrightarrow P(A) > P(B), \quad \text{for all } A, B \subseteq S. \tag{3}$$

Whenever this is the case, we say that P on $\mathcal{A}_0$ *agrees with* the comparative, or qualitative probability relation $\succ$.

The axioms are found reasonable by many persons provided that the state that obtains does not depend on the act that is actually implemented.

The theory of Savage has, as mentioned, a high standing in decision theory, in particular in economics where uncertainty plays a role. In his book on the theory of choice, David Kreps (1988) calls this theory "the crowning achievement of single-person decision theory".

We next turn to our main objection against the two choices in subjective probability theory; finite additivity on the set of *all subsets* of a range space.

3 Intuitive probability of non-intuitive events: The Banach-Tarski Paradox.

Consider Euclidian spaces. In the present theory this would mean that Z is Euclidian. If one seeks a measure which is a generalization of distance on $[0,1]$ in $\mathbb{R}$, one should restrict attention to a σ-algebra of subsets, for example the Borel-sets $\mathcal{B}$ or the Lebesgue measurable sets $\mathcal{L}$. In tis case it is impossible to extend the Lebesgue measure to all subsets $2^{(0,1]}$ of $[0,1]$. This refers to a countably additive extension, of course. If one is content with finite additivity, there is an extension to $2^{(0,1]}$. The same is true for the unit square in in $\mathbb{R}^2$ (see for example Billingsley (1995)). However, by the Banach-Tarski paradox (S. Banach and A. Tarski (1924)), for the unit cube in $\mathbb{R}^3$ it is not even possible with finite additivity if it is required that the measure is defined on all subsets of the unit cube in $\mathbb{R}^3$. Non-measurable sets in the domain of a probability measure may thus lead to strange situations, and hard paradoxes. Euclidian n-spaces, where $n \geq 3$ are important and common in probability theory, be it subjective, objective or otherwise.

For example, suppose we have a finitely additive probability measure P on Euclidian 3-space based on volume. Let V be the volume of a ball with radius $r = 1/2$. Then it is possible, by the Banach-Tarsky paradox, to divide the ball of volume V (where $P(V) = V$) into five parts, all non-measurable, and put the parts together again to two balls each of which has the same volume V as the original. This can be done where two of these five parts can be put together (without stretching or twisting, by rotation and translation only) to a new unit ball and the three others can be put together to another unit ball. The point is that none of these five parts are Borel-measurable, and one can not attach any kind of volume to them.[1]

With probability allowed on non-measurable sets this would imply that

$$P(V) = \frac{\pi}{6} = \sum_{i=1}^{5} P(V_i) = \frac{2\pi}{6},$$

or $V = V + V$. Since $P(V) > 0$ this gives $1 = 2$, a contradiction.

The existence of these five sets is constructed by the axiom of choice, which says the following:

[1] For an account of these prodigies, see Wagon (1985).

Axiom of Choice: *Given an arbitrary family $\mathcal{C}$ of non-empty sets, there exists a new set which consists of exactly one element from each of the sets in $\mathcal{C}$.*

This sounds innocuous (and if the family $\mathcal{C}$ is countable, there is no problem). The problem is that the family may be uncountable, and then it is not obvious. The axiom of choice tells us something about the existence of certain sets.

Kurt Gödel (1940) has shown that this axiom is independent of the other axioms of mathematics. If one accepts the axiom of choice, which most mathematicians do, then there is no finitely additive measure on $\mathbb{R}^n$ for $n \geq 3$ which is translation- and rotation-invariant and which have finite measure on the unit-ball.

If you do not accept the axiom of choice, there is no Banach-Tarski paradox, and no Hahn-Banach Theorem. Also, most classifications of events into classes have proofs based on this axiom, like "if A occurs, then A^c does not occur", etc.

Provided the starting point is the volume-measure m, then the largest possible family of sets on which m can be defined is the set of Lebesgue-measurable sets. This family of sets contains the Borel-sets, in addition to some other subsets of $\mathbb{R}^3$ (but not all).

For any subset E of $\mathbb{R}^3$ the outer measure of E is defined by $m^*(E) := \inf\{m(A); A$ is Borel-measurable, E⊆A$\}$, and the inner measure of E, $m_*(E)$, by $m_*(E) := \sup\{m(B); B$ is Borel-measurable, B⊆E$\}$. Then we have equality, i.e., $m^*(E) = m_*(E)$ if and only if E is Lebesgue-measurable.

In dynamical situations we have in ordinary probability theory stochastic processes $[X_t : t \in T]$, a collection of random variables (acts) on a probability space $(\Omega, \mathcal{F}, P)$, where T is a set of time points. A process is usually described in terms of distributions it induces on Euclidian space. For each k-tuple $(t_1, t_2, \ldots, t_k)$ of distinct elements of T the random vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_k})$ has over $\mathbb{R}^k$ some distribution

$$P[(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) \in B], \quad B \in \mathcal{B}^k,$$

the finite-dimensional distributions of the stochastic process $[X_t : t \in T]$. This system does not necessarily completely determine the properties of the process, but under certain conditions it does (such as the Kolmogorov's existence theorem). The probability measure is thus defined on Borel-sets $\mathcal{B}^k$ in Euclidian k-space, for all k, and the subjective probability theory should obviously be able to handle basic situations like this.

Moreover, as information accrues, there is typically an associated sequence of increasing σ-fields $\mathcal{F}_t$, interpreted as information at time t, where $\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s \leq t$, for example in martingale theory, and the theory of stochastic processes in general. In this theory one is confronted by conditional probability, and conditional expectation with respect to such a filtration of σ-fields. Provided the subjective theory is extended to dynamic models, the theory should be able to handle standard situations like this. The development of ordinary probability theory should also benefit the subjective interpretation of probability.

There are of course also reasons why the finite additivity approach on all subsets has been developed. De Finetti, a pioneer in the field, defended this choice by various arguments. In particular, Dubins and Savage (1965) in their book on gambling, explains this choice by reference to simplicity. About the standard approach they write: "If this tradition were followed in this book, tedious technical measurability difficulties would beset the theory from the outset." If, on the other hand, the result of this approach is met with hard paradoxes, the choice should be easy.

4 A modification of Savage's theory to countable additive probability P on a σ-algebra.

In this section we point out how to modify the theory of Savage and de Finetti to account for the objections of the previous section. For completeness, let us recall some properties of a countable additive probability measure: Given is a probability space $(S, \mathcal{A}, P)$, where S is the set of states, $\mathcal{A}$ is a σ-algebra of measurable events, and P is a *countable additive* probability measure defined on $\mathcal{A}$. This means that if $\{A_n\}$ is any denumerable sequence of disjoint events, then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

It follows immediately from the axioms of probability that P is also *finitely additive*, that is, for any natural number n the following holds: $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$ for any finite set of pairwise disjoint events A_i. Furthermore if $A_n \subset A_{n+1}, n = 1, 2, \dots$, then $P(\bigcup_{n=1}^{\infty} A_n) = \lim_n P(A_n)$.

4.1 Countable additive probability on the state space.

In order to reformulate Savage's theory to avoid the paradoxes, consider the following properties of the qualitative probability $\succ$: First it satisfies the four basic axioms of de Finetti: weak order, nontriviality, nonnegativity, and additivity.

Villegas (1964) identifies a key assumption on $\succ$ for countably additivity. Modified to accomodate a Boolean algebra that is not also a sigma-algebra, the postulate is:

P8: *Monotone continuity*: For all $A, B, A_1, A_2, \cdots$ in $\mathcal{A}$, if $A_1 \subseteq A_2, \ldots$, $A = \bigcup_{i=1}^{\infty} A_i$ and $B \succeq A_i$ for all i, then $B \succeq A$.

Thus, if the nondecreasing sequence A_i converges to a limit event A, then the judgement that B is at least as probable as A_i for all i cannot be reversed in the limit.

Villegas (1964) proves that if $\mathcal{A}$ is a σ-algebra and if P_0 is a finitely additive probability measure that agrees with $\succ$, then P_0 is countably additive if and only if $\succ$ is monotonely continuous. This remains true even when $\mathcal{A}$ is not a σ-algebra (see Chateauneuf and Jaffray (1984)).

We give a short demonstration of the the main elements in the proof: Assume that P_0 on $\mathcal{A}_0$ that agrees with $\succ$ is countable additive. Consider a sequence of events $A_n \uparrow A$ and assume that $B \succeq A_n$ for all n. Since this is equivalent to $P_0(A_n) \leq P_0(B)$ for all n, by countable additivity $P_0(A) = P_0(\cup_{n=1}^{\infty} A_n) = \lim_n P_0(A_n) \leq P_0(B)$, which is equivalent to $A \preceq B$. Thus $\succ$ is monotonely continuous.

The reverse is built on the following result: If P_0 is a finitely additive probability measure on an algebra $\mathcal{A}_0$, and if $A_n \downarrow \emptyset$ for sets $A_n \in \mathcal{A}_0$ implies $P_0(A_n) \downarrow 0$, then P_0 is countably additive (see Billingsley (1995) p. 25). This property is called continuity from above at the empty set.

Assume from now on that $\succ$ is monotonely continuous. We then have the following situation: We are given an algebra $\mathcal{A}_0$ and a finitely additively probability measure P_0 defined on this algebra which is also *countably* additive by this postulate.

We can then use the Hahn-Kolmogorov theorem, which is a version of Carathéodory's extension theorem, a deep result in measure theory. It states that if P_0 is a finitely additive probability measure on an algebra $\mathcal{A}_0$ which is also countably additive, then P_0 extends uniquely to a countably additive probability measure P on the σ-algebra $\mathcal{A}$ generated by the algebra $\mathcal{A}_0$ (see, for example, Ash (1999) or Kallenberg (2002)).

Since the countably additive probability measure P whose existence is assured by this result satisfies

$$P(A) = \sup\{P_0(A_0) : A_0 \subset A, A_0 \in \mathcal{A}_0\}$$

for any $A \in \mathcal{A}$, it follows that, if the given probability measure P_0 is atomless as in Savage's theory, then P is also atomless (see, for example, Sirkorski (1960)).

Next, from Villegas (1964) we have the following two results: (i) If a subjective probability algebra is fine and tight, it can be extended to a subjective probability sigma-algebra; (ii) If a subjective probability sigma-algebra is atomless, it is fine and tight.

From the above it follows that in Savage's theory there exists a *unique* countably additive extension P, defined on the sigma-algebra $\mathcal{A}$ generated by $\mathcal{A}_0$, that agrees with $\succ$, where the latter is atomless, fine and tight. We summarize as follows:

Theorem 1 *Suppose in the theory of Savage that $\succ$ is monotonely continuous. Then the model can be extended from the probability algebra $(\mathcal{A}_0, P_0)$ to the probability σ-algebra $(\mathcal{A}, P)$, where the relation (1) holds. Here P is a uniquely determined quantitative probability measure defined on the σ-algebra $\mathcal{A}$ of measurable events generated by $\mathcal{A}_0$. The corresponding subjective probability $\succ$ is atomless, fine and tight, where the relationship (3) holds with P defined on $\mathcal{A}$.*

In the next section we turn to the structure of the range space in the situation where the space Z can be Euclidian of dimension $n \geq 3$, where the real problems lie. In the latter case the σ-field $\mathcal{B}$ in Z is assumed to be the Borel σ-field (generated by the open sets in Z).

4.2 Countable additive probability measures on the range spaces.

We now demonstrate that when the probability measure P given by Theorem 1 is countably additive, the probability measures P_f, $f \in H$ in the range space are countably additive as well.

Towards this end, let $\mathcal{P} = \{P_f : f \in H\}$ be the set of all probability measures induced by P. Assume that the functions $f \in H$ are measurable, i.e., $f^{-1}(B) = A \in \mathcal{A}$ for any $B \in \mathcal{B}$. Since P is countably additive, so is P_f for all $f \in H$: Let $\{B_i\}$ be a sequence of disjoint, Borel-measurable sets, and let $A_i = f^{-1}(B_i)$. Then by the mere property of f being a function, $\{A_i\}$ is a sequence of disjoint $\mathcal{A}$-measurable sets. Accordingly, $\sum_{i=1}^{\infty} P_f(B_i) = \sum_{i=1}^{\infty} P\big(f^{-1}(B_i)\big) = \sum_{i=1}^{\infty} P(A_i) = P(\bigcup_{i=1}^{\infty} A_i)$, where the latter equality follows from the countable additivity of P. On the other hand, $P_f(\bigcup_{i=1}^{\infty} B_i) = Pf^{-1}(\bigcup_{i=1}^{\infty} B_i) = P(\bigcup_{i=1}^{\infty} f^{-1}(B_i)) = P(\bigcup_{i=1}^{\infty} A_i)$. Together this shows that P_f is countably additive for $f \in H$. We summarize as follows:

Corollary 1 *Under the conditions of Theorem 1, assume that the acts $f \in H$ are measurable. Then the probability measures P_f induced by P are countably additive and defined on a σ-algebra of measurable events $\mathcal{B}$ in Z, where the relationship (2) holds.*

With this result we can proceed as in Savage's theory, where we obtain the representation (2), except that now the measures P_f and P_g are countable additive and defined on the Borel sets in the respective range spaces when Z is Euclidian. Thus the Banach-Tarski paradox is avoided as well as the problem that finite additivity on Euclidian n-spaces is not possible for $n \geq 3$.

Also Savage suggests that something like this can be be done. He writes about the construction of subjective probability measure: "First, there is no technical obstacle to work with a limited domain of definition, and except for expository complications, it might have been mildly preferable to have done so throughout. Second, it is a little better not to assume countable additivity as a postulate, but rather as a special hypothesis in certain contexts. A different and much more extensive treatment of these questions has been given by de Finetti."

Fishburn (1986) remarks that (he) "would not hesitate to invoke it (countable additivity) when its denial would create mathematical complexities .."

Wakker (1989) also claims that the finite additivity of the resulting probability measure in Savage's theory is essentially a technical requirement, that can be removed by an additional assumption.

The central point is to make sure that the we consider countably additive probability measures on the measurable sets $B \in \mathcal{B}$ in Euclidian n-spaces. When $n \geq 3$ it is not enough with finite additivity. As explained above, this can be achieved by first extending to countable additive probability measure P on the σ-algebra $\mathcal{A}$ in the state space. This in turn leads to an associated extension to the σ-algebra of events $\mathcal{B}$ in Euclidian space $E^{(n)}$. The connection to the σ-algebra $\mathcal{A}$ in S is such that $f^{-1}(\mathcal{B}) \subset \mathcal{A}$ for all $f \in H$, where P is defined on the measurable subsets $A \in \mathcal{A}$, whith $\mathcal{A} = \sigma\{\cup_{f \in H} f^{-1}(\mathcal{B})\}$.[2] With these extensions we avoid the paradox and other complications mentioned, and the theory is better founded.

There are other obvious advantages with this reformulation. One is that we can now use ordinary probability theory, and still retain the subjective interpretation if we so wish. This represents an improvement for many reasons, in particular regarding limit theorems, but also extensions to dynamical systems under uncertainty, containing increasing filtrations of sigma algebras $\{\mathcal{F}_t\}_{t=0}^T$ where $\mathcal{F}_t \subset \mathcal{F}_s$ when $s \geq t$, with all the applications that are developed in this direction, and which continue to move modern probability theory forward.

5 Applications to Pareto optimality and pricing of contingent claims.

Based on gambling theory and horse race lotteries one may be led to think that an independent bookmaker is needed in economic models in addition to the traditional auctioneer, provided we prefer the subjective interpretation of probability. First the bookmaker coordinates the final probabilities relevant for pricing such that the axioms of probability are met (coherence), then the auctioneer determines the marginal utility part of the pricing problem, which in theory is solved by a sup convolution type procedure. Thus we require that not only relative prices, but also probabilities are intenalized. However, even when these two parts can be separated into a representative marginal utility times a consensus probability measure, still a complex interdependence between probability and preferences may remain.

1. Syndicates. To illustrate, consider the theory of syndicates (Borch (1962), Wilson (1968)). With homogeneous probability beliefs, and under conditions where agents have affine risk tolerances with identical cautiousness, there is unanimity on the management of risk in the syndicate. The attitude towards the aggregate risk of each member of the pool is identical

[2] In Savage's model, similarly $\cup_{f \in H} f^{-1}(\mathcal{B}_0) \subseteq \mathcal{A}_0$, for $\mathcal{B}_0$ the algebra of all the subsets of $E^{(n)}$.

and equal to the one of the central planner *after* pooling has taken place, despite the fact that the members have different preference to start with. A Pareto optimum is characterised by the existence of non-negative agent weights λ_i and a real function u such that $\lambda_i u_i'(y_i(x)) = u'(x)$ for all agents i. Here the u_i are the utility functions of the agents and x signifies the aggregate risk, whereas $y_i(x)$ is member i's position after pooling of risk.

The aggregate risk $x = g(\alpha, z)$, where g is a real function of the state z and a decision variable α. The state is represented by a probability distribution, for example given by a density function $f(z)$. In this situation the pricing kernel turns out to have the form $u'(x)f(z)$ in the homogeneous case. The function u is interpreted as the utility function of the central planner, or the syndicate, and in this case it is assumed to be of HARA type with risk tolerance equal to the sum of the agent's risk tolerances. In this situation one might separate the two tasks of the bookmaker and the auctioneer, and probability does not explicitly depend on preference parameters.

With heterogenous probability beliefs this is different. Assuming that the pool members hold probability beliefs represented by density functions $f_i(z)$, mutually absolutely continuous with respect to each other, conditions exist when the pricing kernel still has the simple product form $u'(x)f(z)$ for some density function f, called a surrogate probability assessment or consensus probability density, and function $u'(x)$ which is called the surrogate marginal utility function (see Wilson (1968) for details), or marginal utility function of the representative agent. However, the consensus probability function will typically not be determined from the individual f_i's only, but also depends on the pool members' utility functions as well.

In an example, where the $f_i(z)$ are all normal probability densities with means m_i and variances v_i, the function $f(z)$ is a normal probability density with associated mean m and variance v, where both m and v depend on all the m_i and v_i, and also the individual risk tolerances ρ_i of the agents as well as the risk tolerance ρ of the pool. Moreover, the Pareto optimum exists in this situation only in the case where the pool members have negative exponential utility functions with constant risk tolerances ρ_i, and the syndicate is also now unanimous. Also, probability calculations with more than 3 members in the syndicate will take place using Euclidian-n space ($n \geq 3$), where the associated joint probability measure can not be defined on all subsets of $\mathbb{R}^n$ for reasons explained.

With different probability beliefs, not only is it harder to obtain a Pareto optimum (and consequently an equilibrium), but when it exists and when surrogate marginal utility and probability exist, the "separation" $u'(x)f(z)$ can be rather complex in that the surrogate probability depends on preference parameters of the members in the syndicate. If in this situation the the market clearing mechanism is carried out by a bookmaker and an auctioneer, these must indeed cooperate.

Wilson remarks that the behavioral significance of surrogate functions for a syndicate can be characterized by the observation that Savage's fourth postulate (Savage (1954), p. 31) is equivalent to the existence of surrogate functions. With densities representing probability beliefs, it is also clear that Postulate 8 suggested above with its associated extension is relevant for a syndicate, in order to avoid the paradoxes.

In this type of model the agents know their own probability distribution as well as all those of the other agents. Accordingly the model can not be used directly to analyze asymmetric information. However, close variants of the model have been utilized for this topic as well. An example is the model of Holmstrøm (1979), who analyses the situation of moral hazard with two agents, a principal and an agent. Only the agent can make the decision, which can not be observed by the principal, which is where the asymmetric information comes in.

The above model, on the other hand, leads to betting between the members (see e.g., Aase (2022)), which may appear as an annoying side-issue. For models of a neo-classical market economy homogeneous beliefs may therefore be the preferred choice.

2. Contingent claims evaluation. Our last demonstration makes the assumption about homogeneous beliefs, but in a considerably more advanced market model from a probabilistic point of view, where also subjective probability can play a role. Consider the theory of contingent claims analysis, where a claim X with maturity at time T in the future is to be priced at any time $t \in [0, T)$. We are given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $\mathcal{F}_t$ is an increasing filtration of σ-fields, and where $\mathcal{F} = \mathcal{F}_T$. There is an underlying stock market, with price processes given by Ito-diffusions, jump-diffusions or more generally by semi-martingales, governed by the probability measure P. The contingent claim is a real function of one or more of the underlying assets. If there exists an equivalent probability measure Q, under som additional technical conditions there are no arbitrage possibilities in such a market, and the price of the contingent claim at any time t is given

by the expression

$$X_t = \frac{1}{\zeta_t} \int_{\Omega} \zeta_T(\omega) X_T(\omega) dP_t(\omega),$$

where P_t is the conditional probability measure at time t, given $\mathcal{F}_t$, ξ_t is the Arrow-Debreu state price at time t (in units of probability), $\zeta_t = \xi_t e^{-\int_0^t r(s)ds}$, where $r(t)$ is the spot risk-free interest rate at time t (see for example Aase (2002)).

We can define the equivalent martingale measure Q in this setting as follows: $\xi_T = dQ/dP$, where dQ/dP is the Radon-Nikodym derivative of Q with respect to P. The state price ξ_t at time t can be recovered as $\xi_t = E(dQ/dP|\mathcal{F}_t)$. In this theory the discounted price process can be expressed as a conditional expectation under Q, in which case the price $X(t)$ can alternatively be written:

$$X(t) = E^Q(e^{-\int_t^T r(s)ds} X_T|\mathcal{F}_t).$$

The measure Q is a technical, albeit convenient construction, and represents normalized prices rather than probabilities. The construction demonstrates the close connection between probability and price.

In the range spaces where the calculation of $X(t)$ is performed regardless of which formula is being used, probability measures on Euclidian-n spaces will result. Such measures must accordingly be countably additive and defined on the Borel-sets only, according to our earlier observations.

If the probability measure P is subjective, the underlying probability theory must account for this by satisfying Kolomgorov's axioms in order to avoid paradoxes. In this literature P is referred to as the "physical measure", assumed given exogenously. Here, the agents in the model first determine the probability measures P_t at any time t (possibly coordinated by a bookmaker) to be coherent and free of paradoxes and next prices at each time t (possibly cleared by an auctioneer or market maker) to be free of arbitrage possibilities. For an equilibrium, market clearing is also required, among other things.

6 Conclusions.

Subjective probability is traditionally assumed to be finitely additive and defined on all the subsets of an underlying state space S. Functions from this space into Euclidian n-space create a new probability space for each such function. We point out that the associated probability measures, induced by the subjective probability, on these new spaces can not be finitely additive and defined on all the subsets of Euclidian n-space, for $n \geq 3$. This creates intolerable paradoxes as explained in the paper.

We considered the decision theory of Savage (1954) and suggested that this theory can be reformulated by an extension to take this objection into account. In the paper we demonstrated such an extension, which first involved the additional assumption of monotone continuity of the subjective probability. With this in place the finitely additive probability measure in Savages approach that agrees with $\succ$ on the algebra $\mathcal{A}_0$, will also be countably additive. This paved the way for application of the Hahn-Kolmogorov theorem, a version of the Carathéodory's extension theorem. From this theorem we were allowed to extend to a unique countable additive quantitative probability measure on the sigma-algebra generated by the algebra in Savage's model, which agrees with the corresponding subjective probability on the sigma-algebra. This structure carries over to the range spaces, in which case the Banach-Tarski paradox and other difficulties were avoided. The nature of this extension was carried out in such a way that the basic remaining structure of Savage's approach remains in tact.

Lastly we related the subjective probability theory to the theory of syndicates as well as contingent claims evaluations, and indicated how probability in general can be rather naturally internalized in economic models, indicating that the subjective interpretation can be applied more widely.

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