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The Kelly Criterion

An empricial study of the growth optimal Kelly portfolio, backtested on the Oslo Stock Exchange

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Abstract

This study analyzes the performance of the growth optimal Kelly portfolio on the Norwegian stock market from February 2003 through December 2022. To measure the strategy's alpha, we employ the Capital Asset Pricing Model, Fama French's three-factor model and Carhart's four-factor model. The Kelly portfolio generates a higher annual growth rate than the benchmark, and consequently a higher ending wealth level. Our results indicate that the strategy generates an annualized alpha of 16.8%, significant on a 1% level. However, the models show very poor explanatory power, prohibiting us from drawing a meaningful conclusion. Furthermore, when accounting for transaction costs, the portfolio no longer achieves a higher wealth level than the benchmark, and the corresponding alpha is only significant on a 10% level, indicating that the strategy is unable to generate risk-adjusted excess returns in the real world.

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1. Introduction

Contradicting the highly influential efficient market hypothesis, many investors and financial practitioners believe there exists opportunities to exploit and beat the highly informed market through active management. With active management, strategies actively try to outperform a chosen benchmark and generate alpha. As such, many investing- and trading strategies have been generated with the purpose of achieving alpha. Such strategies can generally be subcategorized into qualitative and quantitative strategies for asset allocation. With regards to quantitative asset allocation Markowitz' mean variance optimal portfolio is acknowledged as the most widely used approach, seeking to generate the optimal portfolios for a given level of risk or return. However, Markowitz has been shown to perform poorly in numerous tests of the strategy, as performed by Ang (2014) and DeMiguel et al. (2007). In relation, many competing strategies have emerged, one such is the Kelly growth optimal portfolio.

The Kelly growth optimal portfolio seeks to maximize the expected growth rate of capital, thereby maximizing the expected value of the logarithm of wealth. The strategy has its foundation in John Kelly's paper "*A New Interpretation of Information Rate*". Kelly (1956) states that for repeated bets, a bettor should act as to maximize the expected growth rate of capital to maximize his expected wealth at the end. This contradicts the previous standard notion of purely maximizing expected return in each period, as Kelly proves that following this strategy for repeated bets implicates a probability of ruin close to 1 for the bettor. This is due to the effects of overbetting. Following, the Kelly criterion calculates the optimal amount to bet, given the probabilities and payouts, to maximize the expected growth rate. The criterion has since been modified for application in the financial markets. A Kelly growth optimal portfolio for the stock market should have several desirable features under the correct market conditions, whereas obtaining the highest level of wealth for the investor, while also reaching a specific wealth target in less time than essentially any other strategy. However, for the stock market, there is essentially an infinite number of possible outcomes, generating uncertainty for the Kelly criterion. With known probabilities and payoffs, the Kelly criterion will dominate for certain in the long run, although this is not necessarily the case for the securities market. Still, we want to test whether a growth optimal Kelly portfolio is able to beat the market. Our research question is as follows:

"Does the growth optimal Kelly portfolio offer alpha?"

Our Kelly portfolio is constructed on the basis of Ed Thorp's (2006) approach outlined in "*The Kelly Criterion in Blackjack Sports Betting and the Stock Market"*. The formula calculates the optimal investment fraction in a set of assets based on the expected excess returns of the assets, as well as the inverse of the variance-covariance matrix. Thorp's approach is identical to the one outlined by Robert Merton (1969) in "*Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case"* when using logarithmic utility. Merton constructs the optimal portfolio in a continuous setting for a constant relative or absolute risk aversion. The Kelly portfolio maximizes expected utility for an investor with log utility, as one maximizes the growth rate by having a log utility. Hence the two approaches coincide.

We implement the Kelly portfolio using monthly rebalancing and a 12-month rolling estimation window. Furthermore, as our Kelly portfolio takes on quite aggressive positions, we limit the stock selection to the ten stocks on the Oslo Stock Exchange with the highest NOK-volume to ensure sufficient liquidity. In addition, we allow for short sales, but apply a position size constraint of 25%. The portfolio is constructed using monthly data from Bloomberg and Compustat. Additionally, we construct a Markowitz mean-variance portfolio to rival Kelly, while we use the OSEBX as our benchmark.

To test whether the portfolio offers alpha, we regress our excess portfolio returns using the Capital Asset Pricing Model (CAPM), the Fama French three-factor model (FF3F) and Carhart's four-factor model (C4F).

The Kelly portfolio shows great results initially, outperforming both the benchmark and the Markowitz portfolio. The strategy also generates alpha on a 5% level in the CAPM, and on a 1% level in FF3F and C4F. Although, given the portfolio's high turnover, Kelly no longer outperforms the benchmark after accounting for transaction costs, and the alphas are consequently reduced considerably, only generating alpha significant on a 10% level in FF3F and C4F. However, Kelly exhibits a low market beta, and our factor models show very low R^2. Hence, the alpha measures appear inflated and prevent us from drawing any meaningful conclusions regarding the alpha.

We structure the thesis in five main sections. Section 2 presents relevant theoretical background on the Kelly criterion, portfolio theory, and the factor models we implement. Further, section 3 describes our data collection process and how we construct the portfolios, as well as empirical methodology. In section 4 we analyze the results and test for robustness. Finally, in section 5 we conclude our findings.

2. Theoretical Background

2.1 Kelly Criterion Background

As per MacLean et al. (2010), the Kelly criterion was first properly introduced in 1956 by John Larry Kelly Jr. in his paper "*A New Interpretation of Information Rate*". In his paper, Kelly (1956) introduced a gambler who seeks to maximize the value of his capital. Being exposed to a single favorable bet, the gambler could bet his entire fortune to maximize the expected value of his capital as the probability for success exceeded the probability for failure. However, should the game involve repeated bets, with a probability p of losing, the gambler would likely end up broke with this strategy, with a probability approaching 1 as he continued indefinitely.

Kelly then suggested that for repeated bets where one can reinvest one's winnings, one should rather seek to maximize the expected logarithm of capital. The logarithm is additive in repeated bets and to which the law of large numbers applies. By seeking to maximize the expected logarithm of capital, the gambler would maximize his wealth over a long period of time, representing the growth optimal strategy. Should the gambler not be able to reinvest, then he would be better of maximizing the expected value of capital on each bet as initially proposed.

Ed Thorp is known as the first to properly utilize the Kelly criterion in practical settings. After discovering Kelly's paper, Thorp started using the concept for gambling, and in 1962 in his book "*Beat the Dealer*", he introduced the "Kelly gambling system" (Thorp, 1962). This system would later be popularized as the Kelly criterion. Thorp initially utilized the criteria in blackjack and sports betting. However, he later started his own hedge fund applying the Kelly criterion in the financial markets, generating large profits trading warrants in a statistical arbitrage strategy (Thorp, 2006). The criterion has since become more popular in finance and portfolio theory, where an investor seeks to create an optimal portfolio maximizing the expected logarithm of wealth, and thus representing the growth optimal portfolio that leads to the highest ending wealth over time.

2.2 Kelly Criterion Illustration

A fitting situation to illustrate the Kelly criterion is through a coin toss scenario. With a coin toss, we have a simple binomial case where the probabilities for each outcome are known, as well as the subsequent payouts. We base the illustration on Thorp´s (2006) example in "*The Kelly Criterion in Blackjack Sports Betting and the Stock Market*".

Thorp´s coin tossing game assumes we have an infinitely wealthy opponent willing to wager even money bets made on repeated independent trials of coin tossing of a biased coin. The biased coin makes this a favorable game, with a win probability $p > \frac{1}{2}$ and the probability of loss $q = 1 - p$. As this is a favorable game, it fits the Kelly criterion. We have X_0 as our initial capital and want to maximize the expected value $E(X_n)$ after n trials. Our problem is to decide the optimal fraction B_k to bet on the k^{th} trial. If the k^{th} trial is a win, we let $T_k = 1$, and $T_k = -1$ with a loss. Then, $X_k = X_{k-1} + T_k B_k$ for $k = 1,2,3,$ and $X_n = X_0 + \sum_{k=1}^{n} T_k B_k$.

The expected value of the game is expressed as:

$$E(X_n) = X_0 + \sum_{k=1}^{n} E(B_k T_k) = X_0 + \sum_{k=1}^{n} (p - q) E(B_k) \tag{2.1}$$

The game has positive expected value, and we wish to maximize our expected end capital $E(X_n)$. Logically we would then bet the entirety of our capital in each trial to maximize the expected value $E(B_k)$ at each trial, and hence maximize the expected gain in the game. However, as the win probability $p < 1$, and the probability of ruin during the game is $1 - p^n$, we have that $\lim_{n \to \infty} [1 - p^n] = 1$. As seen from Kelly's paper, when gambling with such a strategy, financial ruin is practically certain, naturally making it unappealing when seeking to maximize end wealth.

Oppositely, one can bet to minimize probability of ruin according to Feller (1966), where we minimize the probability of ruin by making a minimum bet on each trial. However, this strategy also minimizes the expected gain, making this strategy undesirable as well.

Clearly, the optimal strategy is somewhere in between these two extremes, where we don´t assure ruin or minimize the expected gain. The strategy proposed by Kelly (1956) serves as an asymptotically optimal strategy.

As the probabilities and payoffs in the game are constant, we realize that the optimal strategy to maximize wealth will likely wager a fixed fraction f of your capital. An assumption of infinitely divisible capital is required to make this strategy work. The mathematical derivation of the problem is outlined as by Thorp in the appendix in section A1.1.

Thorp shows that the optimal fraction to bet is then $f = f^* = p - q$. This is the difference in probabilities for win and loss and represents the edge the gambler has in the situation.

This optimal fraction is illustrated in the following figure:

Figure 2.1: Illustration of Optimal Bet-Fraction

The figure illustrates the optimal bet-fraction f^* that on a repeated favorable bet, maximizes the expected growth rate of capital, and results in the highest ending wealth (Thorp, 2006).

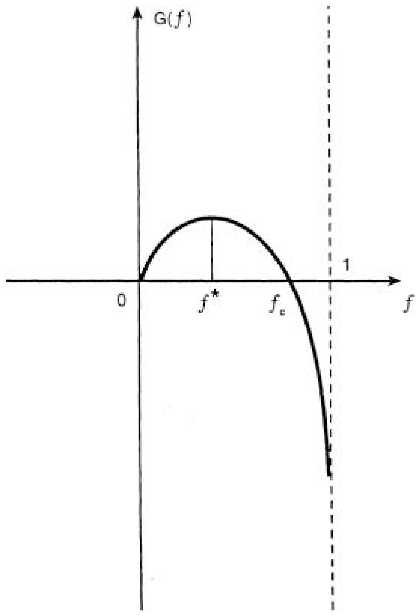


Figure 2.1 illustrates the optimal fixed fraction f^* to bet. This fraction maximizes the expected growth rate of capital and will over time lead to a higher level of wealth than essentially any other strategy. Also, the expected time for the current capital X_n to reach any fixed preassigned goal is asymptotically least with a strategy that maximizes $E \log X_n$ as per Thorp (2006). The figure also highlights the dangers of overbetting. With a fraction f approaching 1, $G(f)$ is rapidly turning negative. So, the probability of going bust even on a favorable bet by overbetting is considerable.

2.3 Kelly Portfolio Theory

When applying the Kelly criterion to a portfolio of assets, we wish to create a strategy that identifies the growth optimal portfolio. Roll (1973) observes that by maximizing the expected value of the logarithm of wealth one will achieve the highest expected growth rate of capital, resulting in the highest expected ending wealth for the investor. Also, Breiman (1961) has shown that the time to reach asymptotically large wealth levels is minimized by the Kelly strategy.

Kelly Criterion for a Portfolio of Securities

Based on the desirable features of the Kelly criterion, we wish to implement a growth optimal trading strategy for the stock market. As we are applying the strategy to a portfolio of multiple securities in the stock market, we need a formula to calculate the optimal portfolio weights to maximize the expected growth rate of the Kelly portfolio. Such a formula is derived by Thorp (2006). The mathematical derivation in section A1.2 in the appendix yields the following results:

$$f^* = \frac{m - r}{s^2} \tag{2.2}$$

$$g_\infty(f) = r + f(m - r) - \frac{s^2 f^2}{2} \tag{2.3}$$

$$g_\infty(f^*) = \frac{(m - r)^2}{2s^2} + r \tag{2.4}$$

Equation 2.2 for fraction f^*, illustrates how much to invest in a single security, where m is the expected return, r is the risk-free rate, and s^2 is the variance of the security.

We note that equation 2.4 for the maximum growth rate can be rewritten as $g_\infty(f^*) = \frac{S^2}{2} + r$, where S equals the Sharpe ratio. Hence, both the optimal allocation and the maximum growth rate of the portfolio depends heavily on the Sharpe ratio.

Equation 2.2 illustrates the bet size in one single risky asset. However, for this paper's problem, we are dealing with a portfolio of securities. Thus, we have to factor in the correlation between the securities in our portfolio when determining the optimal bet size. To account for opportunity costs and the correlation between the securities in our portfolio, we need the joint properties of all current and possible new investments in our portfolio as stated by Thorp (2008). Thorp (2006) further modifies the formula for a portfolio of multiple securities, as outlined below.

We first consider an unconstrained case with a riskless security, portfolio fraction f_0 and n securities with portfolio fractions $f_1, \dots f_n$. We suppose the rate of return on the riskless security is r, and assume for simplicity that r is also the rate for borrowing, lending and the rate paid on short sale proceeds. We let $C = (s_{ij})$ be the covariance matrix for the i^{th} and j^{th}

security with $i, j = 1, \dots, n$. $M = (m_1, \dots, m_n)^T$ is the row vector such that m_i is the drift rate for the i^{th} security for $i = 1, \dots, n$. Our portfolio then satisfies:

$$f_0 + \cdots + f_n = 1 \tag{2.5}$$

$$\begin{aligned} m = f_0 r + f_1 m_1 + \cdots + f_n m_n &= r + f_1(m_1 - r) + \cdots + f_n(m_n - r) \\ &= r + F^T(M - R) \end{aligned} \tag{2.6}$$

$$s^2 = F^T C F \tag{2.7}$$

Here, $F^T = (f_1, \dots, f_n)$, and R is the column vector $(r, r, \dots, r)^T$ of length n.

Our previous formulas for one single security apply to $g_\infty(f_1, \dots, f_n) = m - \frac{s^2}{2}$. We now have a standard quadratic maximization problem. By using the equations above that our portfolio satisfies and solving the simultaneous equations $\frac{\partial g_\infty}{\partial f_i} = 0$ for $i = 1, \dots, n$, we get:

$$F^* = C^{-1}[M - R] \tag{2.8}$$

$$g_\infty(f_1^*, \dots, f_n^*) = r + \frac{(F^*)^T C F^*}{2} \tag{2.9}$$

The vector of optimal fractions in equation 2.8 is calculated by multiplying the inverse of the variance-covariance matrix with the excess return vector. To achieve a unique solution, we require C^{-1} to exist, so $\det C \neq 0$. If all the securities in our portfolio are uncorrelated, then $f_i^* = (m_i - r)/s_i^2$, which when $n = 1$ is identical to equation 2.2 for one single security. Equation 2.9 represents the maximum growth rate of the optimal portfolio.

Merton's Lifetime Portfolio Selection

Stutzer (2003) states that the growth optimal portfolio is the portfolio maximizing the expected log utility of wealth. Hence, we can also apply the framework by Merton (1969) where he develops optimal portfolios over an investor's lifetime in continuous time for investors with constant relative or absolute risk aversion. Merton finds that the optimal fraction w^* to invest in a single risky asset can be written as:

$$w^* = \frac{(a - r)}{\sigma^2 \delta} \tag{2.10}$$

Where a is the expected return, r is the risk-free rate, σ^2 is the variance and δ is Pratt's relative risk aversion measure.

When extending the model to several risky assets, Merton derives the formula for the optimal fractions w^* in the case of many assets:

$$w_\infty^*(t) = \frac{1}{(1-\gamma)} \Omega^{-1}(a - \hat{r}) \tag{2.11}$$

Here, γ is a measure of risk aversion, defined as $1 - \gamma \equiv \delta$. Ω^{-1} is the inverse of the variance-covariance matrix, and a and r still represent the expected return and risk-free rate.

With logarithmic utility, we have that $U(w) = \log(w)$. Arrow (1964) and Pratt (1963) defines the Arrow-Pratt coefficient of relative risk aversion as:

$$\delta = \frac{wU''(w)}{U'(w)} \tag{2.12}$$

From equation 2.12, we have that $\delta = 1$ for log utility, and consequently, $\gamma = 0$. As a result, when deriving the optimal portfolio for an investor maximizing the expected log utility of wealth we have:

$$w^* = \frac{(a-r)}{\sigma^2} \tag{2.13}$$

For a single risky asset, and:

$$w_\infty^*(t) = \Omega^{-1}(a - \hat{r}) \tag{2.14}$$

When extending the portfolio to several risky assets. We note that the results are identical to those derived by Thorp (2006) but with different mathematical notations. Thus, our approach is also robust in a more general utility setting.

Alternative Kelly Approaches

MacLean, Thorp and Ziemba (2010) express that to achieve the perfect growth optimal strategy, we require accurately known probabilities distributions and payoffs, which we do not have for the securities market. Merton (1969) in his model assumes that the risky assets follow geometric Brownian motion. Hence, there is uncertainty involved in the real world, and not one perfect approach, but rather different alternatives related to the Kelly criterion's objective of maximizing the expected growth rate of a portfolio of securities. As such, other approaches than the one we pursue in this paper exists as possible alternatives to achieve a growth optimal Kelly portfolio.

One alternative approach is to apply the standard criterion from Thorp (2006) qualitatively. With qualitative investing, the investor disregards the correlation between the securities, solely focusing on stock picking instead. As a result, such an investor can use the formula for one single security that we have introduced earlier:

$$f^* = \frac{m - r}{s^2} \tag{2.15}$$

With m being the expected return of the asset, r the risk-free rate, and s^2 the variance of the asset. The formula will then propose a fraction f^* to bet on each investment, ignoring potential covariance between the stocks in the portfolio. Disregarding the covariance can be dangerous when designing an optimal portfolio. However, for a concentrated portfolio, this can be a good qualitative approach to favor the most attractive investments individually, while also hindering overbetting. Ziemba (2005) shows that Warren Buffet invests in a manner that resembles such a strategy.

An alternative quantitative approach is Luenberger's Portfolio of Maximum Growth Rate presented in his book "*Investment Science*". Luenberger (1998) states that we obtain the optimal growth portfolio by maximizing the growth rate v. This is accomplished by finding the weights $w_1, \dots, w_n$ that solve:

$$\max \sum_{i=1}^{n} w_i \mu_i - \frac{1}{2} \sum_{i=1}^{n} w_i \sigma_{ij} w_j \tag{2.16}$$

$$subject\ to \sum_{i=1}^{n} w_i = 1 \tag{2.17}$$

Using Luenberger's approach we seek to maximize the growth rate in each period to generate the highest growth rate. The formula looks to weigh in on the securities with the highest expected returns, while also penalizing for variance. However, such a portfolio will be highly concentrated to only a select securities with the highest historical returns. Also, Merton and

Samuelson (1974) show that such a log mean-variance approach is not consistent with expected utility theory.

Kelly Drawbacks

Despite its many favorable properties, there are some drawback and weaknesses with the Kelly criterion. First, in the setting of portfolio management with several securities, we use historical returns for expected returns, variances and covariances. Chopra and Ziemba (1993) have shown that such quantities can have huge estimation errors. As a result, we will not know the true probabilities and the constructed Kelly portfolio will not fully represent the optimal growth portfolio. This is not in essence a drawback with the Kelly criterion, but rather how we implement it given the circumstances. Furthermore, equity prices do not have a given set of outcomes with assigned probabilities. Meaning, that for our chosen market, the Kelly criterion should not be completely accurate in practice.

Even if we could model the distributions correctly, there are further drawbacks. Thorp (2008) states that as time t tends to infinity the Kelly bettor's fortune will, with probability tending to 1, permanently surpass that of a bettor following any essentially different strategy. So, for the Kelly portfolio to dominate for certain, we would need to invest for the long run. Many investors might not have this patience, or liquidity to run a trading strategy for a sufficient time period.

Furthermore, Hausch and Ziemba (1985) and Clark and Ziemba (1987) have demonstrated that the Kelly criterion often instruct the investor to bet large portions of his capital on what is perceived as favorable bets. Consequently, the strategy can be quite risky, and may in periods result in drawdowns that are uncomfortably large for the average investor. Furthermore, Thorp (2008) expresses that while the growth optimal Kelly strategy will dominate in the long run it can be extremely volatile at times.

Fractional Kelly

A solution to the possibly uncomfortable volatility of the full Kelly strategy is to implement fractional Kelly strategies. Thorp (2008) explains fractional Kelly strategies, where the investor to reduce volatility and the risk of large drawdowns, bet a fraction c of the original Kelly weights, with $0 < c < 1$. Using a fractional Kelly strategy will result in a lower expected growth rate of the portfolio, but also lower volatility. However, MacLean, Thorp, Zhao and Ziemba (2010) show that the wealth accumulated from the full Kelly strategy does not stochastically dominate the fractional Kelly wealth, where the downside is often more favorable with a fraction of less than one invested. Following, there is a tradeoff of risk and return with the fraction in the Kelly portfolio. With cases of large uncertainty from intrinsic volatility of estimation errors, the investor can gain security by reducing the Kelly investment fraction. As we are in this paper dealing with historical returns as estimates for expected returns and covariances, we naturally have considerable estimation errors. Consequently, in such an instance, implementing a fractional Kelly may be desirable to reduce the risk of overbetting due to uncertainty.

Markowitz Mean-Variance

Thorp (1971) acknowledges the Markowitz mean-variance approach as the most widely used guide to portfolio selection. Moreover, Thorp (1969) also states that the Kelly criterion should replace the Markowitz criterion as the guide to portfolio selection. Thus, we consider the Markowitz mean-variance portfolio a rival strategy to the Kelly growth optimal portfolio. Harry Markowitz first introduced the theory in 1952 in his paper "*Portfolio Selection*". Markowitz (1952) states that the investor should seek to maximize expected return and minimize variance, this will yield the best risk-adjusted return. Markowitz creates a set of efficient portfolios that produce the highest expected return for a given level of risk, or the lowest level of risk for a given expected return. Through diversification one can up to a point lower the risk in the portfolio without sacrificing expected return. The efficient portfolios create the efficient frontier in the mean-variance space.

On the efficient frontier lies the tangent portfolio. This is the given portfolio with the highest Sharpe ratio. With the existence of a risk-free rate, an investor can combine the tangent portfolio with a riskless bond to achieve the highest risk adjusted return. The subsequent allocation to the risk-free rate and the tangent portfolio will depend on the risk aversion of the investor, where the use of leverage is allowed.

When implementing the Markowitz mean-variance approach, we can add several constraints regarding position sizes, overall weight, and short positions. This can serve as risk management of the portfolio, or to fit a specified investment mandate.

Markowitz' model assumes perfect capital markets. Furthermore, it requires inputs on expected returns, variance and covariances. Quality inputs of such parameters can be hard to obtain, often forcing us to use historical data. Chopra and Ziemba (1993) has shown that the use of historical data can cause sizeable estimation errors in both means, variances and covariances. Mean-variance optimization is very sensitive to errors in such estimates, with errors in means having considerably the most degrading effect, followed by errors in variances and covariances respectively. Following, Ang (2014) has shown the Markowitz portfolio to perform poorly when tested and compared to alternative strategies. Also, DeMiguel et al. (2007) finds that the equally weighted $1/N$ portfolio outperforms Markowitz.

Closely related to Markowitz mean-variance theory is the CAPM. In the CAPM world, every investor holds the market portfolio as the market portfolio is the tangent portfolio. Holding the market portfolio eliminates the idiosyncratic risk, leaving the investor with the undiversifiable systematic risk. From Tobin (1958), an investor will hold a combination of the tangent market portfolio and the risk-free rate in relation to their risk aversion.

Kelly vs Markowitz

MacLean et al. (2010) explains how the Kelly growth optimal investment strategy differs from the standard Markowitz mean-variance approach in the sense that it is a multi-period model. Whereas Markowitz seeks to maximize the expected returns for a given level of risk, or vice versa, for a single period, the Kelly approach is interested in geometric or compound rates of return. The growth optimal strategy maximizes $E\,[\log X]$ to achieve the portfolio with the highest growth over a prolonged period of time. Furthermore, the Kelly strategy is always geometric mean-variance efficient and has the highest growth rate of all such strategies. Contrastingly, Thorp (1971) shows that the Kelly strategy is not necessarily arithmetic mean-variance efficient.

A Kelly investor's focus is achieving the highest end wealth, paying less attention to the risk adjusted return compared to a Markowitz investor. In Kelly portfolio theory the geometric average dominates the arithmetic average as this is the rate of which the portfolio will compound over time.

2.4 Efficient Market Hypothesis and Active Portfolio Management

Active vs Passive Portfolio Management

Both Kelly and Markowitz are actively managed portfolio strategies. Active management refers to the approach of actively trying to outperform a benchmark. Passive management refers to the approach of trying to replicate the market index or benchmark return while keeping transaction costs at a minimum. Active management necessitates more frequent trading, leading to higher transaction costs relative to passive management.

For actively managed portfolios, the manager seeks to generate alpha. Consequently, we analyze whether this is generated by the Kelly portfolio with regards to the Norwegian stock market. Alpha is defined as the portfolio performance in excess of a benchmark, adjusted for risk (Jensen, 1967). There are several empirical studies like for instance Malkiel (2003) and Fama and French (2010) that suggest that on average active fund managers are unable to outperform their benchmarks when accounting for transaction costs. On the other hand, Berk and van Binsbergen (2014) find that while fund managers may generate alpha, the value does not necessarily benefit the fund investor, but rather the managers themselves.

Efficient Market Hypothesis

The achievability of alpha with active management depends on how efficient the market is. The Efficient Market Hypothesis (EMH) states that new information on individual stocks or market conditions is immediately reflected in stock prices. The EMH comes in three forms; weak-, semi-strong-, and strong form, as proposed by Fama (1970). The weak form implies that stock prices reflect all past information. The semi-strong form implies that stock prices reflect all publicly available information. The strong form implies that stock prices reflect all information, both public and private. Our Kelly portfolio relies on historical data for asset allocation in the search for alpha.

The challenge for the Kelly portfolio and active management is to outperform a benchmark over a prolonged period, to demonstrate persistent skill. Bodie, Kane and Marcus (2009) review various studies and suggest that markets are generally close to efficient, this implies it can be difficult for the Kelly portfolio to consistently outperform the market. However, there are numerous market anomalies like for instance the size effect by Banz (1981) and the book-to-market effect by Stattman (1980) which are questioning the EMH, even though Fama and French (1993) argue that these effects can be explained as risk premiums.

2.5 Factor Models

We seek to evaluate the performance of Kelly and Markowitz relative to a benchmark. Therefore, we utilize the Capital Asset Pricing Model, Fama and French's (1992) three-factor model, and Carhart's (1997) four-factor model to measure the alpha of the portfolios.

Capital Asset Pricing Model

The Capital Asset Pricing Model was introduced in the 1960's by William Sharpe (1964), John Lintner (1965) and Jan Mossin (1966).

In a CAPM world, investors are only compensated for the systematic risk which is the risk that cannot be mitigated by diversifying. Hence, the CAPM functions as a single-factor model for evaluating alpha, where the single factor is the excess return of the market portfolio, known as the market risk factor. When measuring alpha with CAPM, we obtain the alpha described by Jensen (1967), widely known as Jensen's alpha. Jensen's Alpha can be expressed as:

$$\alpha = R_i - \left(r_f + \beta\left(R_M - r_f\right)\right) \tag{2.18}$$

Consequently, we run the following time-series regression of the CAPM to estimate Jensen's alpha:

$$R_{it} - R_{F_t} = \alpha_i + \beta_i(R_{Mt} - R_{Ft}) + \epsilon_{it} \tag{2.19}$$

Where $R_{it} - R_{F_t}$ represents the excess return of asset or portfolio i at time t. α denotes the excess return of the portfolio or asset that cannot be explained by the included factor, which is the excess return of the market portfolio. β represents the sensitivity of the asset's return relative to the return of the overall market portfolio. ϵ_{it} is the error term and reflects the variability in the data that is not explained by the regression model.

There are several empirical contradictions to the CAPM suggesting that it does not capture all appropriate risk factors to explain variation in the cross-sectional returns. Fama and French (1992) argues that the β in CAPM is inadequate in explaining the cross-section of returns. As a result, we also employ multifactor models such as the Fama French three-factor model and Carhart four-factor model.

Multifactor Models

Fama and French (1992) point out the size effect discovered by Banz (1981), who observed that market equity or market capitalization contributes to explaining the cross-sectional variation of returns. Another notable finding is from Stattman (1980) and Rosenberg, Reid and Lanstein (1985), who found a positive relationship between average returns on US stocks and the book-to-market equity ratio. Fama and French consolidated these discoveries and further developed the CAPM single-factor model by adding the size- (SMB) and value factor (HML) to construct the Fama French three-factor model. By including size- and value risk, the model considers the fact that small cap stocks tend to outperform large cap stocks and that high book-to-market firms (value stocks) tend to outperform low book-to-market firms (growth stocks).

According to Carhart (1997), Fama and French's three-factor model falls short in explaining the fluctuations in returns of momentum-sorted portfolios. As a result, Carhart proposes a four-factor model by extending the three-factor model with an additional momentum factor (UMD) to capture the effect of 1-year momentum. The momentum factor was identified by Jagadeesh and Titman (1993) and aims to capture the tendency that outperforming stocks continue their outperformance relative to their peers, while underperforming stocks tends to persist in underperformance.

Carhart's four-factor model builds upon the three-factor model by extending its variables with the momentum factor. When estimating alpha we run the following time-series regression:

$$R_{it} - R_{F_t} = \alpha_i + \beta_i(R_{Mt} - R_{Ft}) + s_i SMB + h_i HML + u_i \text{UMD} + \epsilon_{it} \tag{2.20}$$

The model extends the CAPM single-factor model in equation 2.19 with the SMB, HML and UMD factor. SMB (small minus big) is recognized as the size factor or size premium and is given as the return spread of small and large stocks measured by the market capitalization. s_i is the coefficient that measures the portfolio's sensitivity or exposure to the size factor, indicating whether the portfolio is tilted towards small or large firms. HML (high minus low) is recognized as the value factor or value premium and is given as the difference in returns between high- and low book-to-market firms. h_i measures the portfolio's sensitivity or exposure to the value factor, indicating whether the portfolio is tilted towards high or low book-to-market firms. UMD (up minus down) is recognized as the momentum factor and is given as the difference in returns between the top- and bottom-performing firms. The coefficient u_i represents the portfolio's sensitivity to the momentum factor, indicating whether the portfolio is tilted towards positive or negative momentum firms.

Limitations of the Factor Models

Fama and French (1992) argue that the β in the single-factor CAPM is inadequate in explaining the cross-section of returns. Therefore, they extend the model by introducing the size and value factor. Carhart (1997) further extends the three-factor model with an additional momentum factor to also capture the momentum effect. However, the factor models have only achieved partial success in empirical tests. Gregory et al. (2013) and Fama and French (2012) exhibit that the model falls short in explaining variation in returns for portfolios based on size and momentum.

The primary issue of the factor models is the exclusion of other relevant factors that may explain the cross-section of returns. This issue may lead to a potentially misleading alpha. To address this limitation, Fama and French (2015) added a profitability and an investment factor to their three-factor model, resulting in the Fama and French five-factor model.

However, an important finding is that simply adding more variables does not necessarily result in a model more adequate in explaining cross-section of returns. Blitz et al. (2016) found that adding more variables has drawbacks, mainly because the included factors interact which makes it more challenging to summarize the cross section of returns. Further, the Fama French (2015) five-factor model still ignores the momentum factor, although several researchers like Chui, Titman and Wei (2010) and Asness, Moskowitz and Pedersen (2013) conclude that momentum premia are present all over the world.

While the Fama French models have shown evidence of failure, they remain widely used in empirical asset pricing, likely due to their superiority over common alternatives such as the CAPM.

2.6 Risk-Adjusted Performance Measures

Sharpe Ratio

The Sharpe ratio is a reward-to-volatility measure proposed by William F. Sharpe (1964) in his study on the CAPM. The Sharpe ratio is a measure of the excess return of the portfolio relative to the standard deviation or volatility of the return. The Sharpe can be expressed as:

$$Sharpe\ ratio = \frac{R_i - r_f}{\sigma_i} \tag{2.21}$$

In our analysis, we consider the Sharpe ratio as a risk-adjusted performance measure for the Kelly portfolio, highlighting whether the excess return the strategy obtains is a result of a higher level of risk.

Sortino Ratio

The Sortino ratio is a modification of the Sharpe ratio and is a result of Sortino and Price (1994) who criticized the Sharpe ratio for penalizing large positive returns as the standard deviation increases. The standard deviation is defined as the upside risk plus the downside risk. As a result, the Sharpe ratio penalizes both positive and negative volatility. Based on this, Sortino and Price suggest the Sortino ratio, which only penalizes the returns as the downside deviation increases. Following, the Sortino closely resembles a symmetric downside-risk Sharpe ratio as proposed by Ziemba (2005). Such a measure can be appropriate for evaluating high return, volatile portfolios such as the Kelly portfolio. The Sortino ratio can be expressed as:

$$Sortino\ ratio = \frac{R_i - MAR}{\sigma_{i_d}} \tag{2.22}$$

Where R_i is the portfolio return, MAR is the minimum acceptable rate of return, and σ_{i_d} is the downside deviation of the portfolio return in excess of the MAR. The downside deviation is the square root of the sum of the squared differences of the portfolio return and the minimum acceptable return when the portfolio returns fall short of the minimum acceptable return. The downside deviation can be expressed as:

$$\sigma_{i_d} = \sqrt{\left(\frac{\sum_{i=1}^{N}[MIN(R_i - MAR; 0)]^2}{N-1}\right)} \tag{2.23}$$

The Kelly portfolio can be very volatile, with a lot of variation related to its upwards fluctuations. As such, it can be heavily penalized by the Sharpe ratio. Therefore, the Sortino ratio acts as an appropriate statistic to evaluate the performance of our Kelly portfolio.

Information Ratio

The information ratio was introduced by Treynor and Black (1973). In our analysis, the information ratio functions as a risk-adjusted performance measure to evaluate the Kelly and Markowitz returns relative to the benchmark. The information ratio can be expressed as:

$$Information\ Ratio = \frac{R_i - R_b}{\sigma(R_i - R_b)} \tag{2.24}$$

Where R_i and R_b are the return of the portfolio and benchmark, respectively. The denominator is recognized as the tracking error and is the standard deviation of the excess portfolio returns. The tracking error is the additional risk of the portfolio relative to the risk of the benchmark, and it reveals to what extent the portfolio tracks the benchmark. A low tracking error implies that the portfolio closely follows the benchmark, whereas a large tracking error indicates that the portfolio differs from the benchmark (Kinnel, 2021).

A positive information ratio indicates that the portfolio outperforms the benchmark given its level of risk.

3. Methodology

3.1 Data

Data Sample

For our analysis we wish to use securities from the Norwegian stock market with a sample period from 2003 to 2023. To ensure that we are able to properly execute the desired trades in our Kelly portfolio, we require stocks with a satisfactory amount of liquidity. Therefore, we limit our sample to the OBX index on the Oslo Stock Exchange (OSE). The OBX index comprises the twenty-five most liquid companies listed on the primary index of the Oslo Stock Exchange, with a semi-annual rebalancing.

Data Collection

To obtain the constituents of the OBX index, we utilize the Compustat Capital IQ database from the Wharton Research Data Services (n.d.) website. We obtain a list of the tickers of stocks included in the OBX index from 2003 to 2023.

We employ the tickers obtained from Compustat and procure the monthly stock prices for the OBX index through Bloomberg. We also retrieve monthly observations of the 10-year Norwegian government bond rate from Bloomberg as our risk-free rate. Bloomberg includes delisted firms, hence our sample is explicitly free of survivorship bias. Survivorship bias refers to the exclusion of firms that go bankrupt from a sample prior to their actual bankruptcy. Additionally, the prices provided by Bloomberg are adjusted for corporate actions such as stock splits, dividends, and other events such as rights offerings and spin-offs. We compute the return of the stock prices, using simple returns.

Bloomberg does not have sufficient historical data for the trading volumes. As a result, we use the Compustat database to collect daily trading volumes of the stocks. We use daily observations as we assume all trading by the Kelly strategy is executed in a single trading day. Compustat provides all financial and accounting information reported by the subsequent firms, thereby mitigating the look-ahead bias of the sample. Utilizing information not accessible during the designated timeframe will result in look-ahead bias and diminish the reliability of the analysis.

Securities in the sample may be delisted during the one-year sample period. If a company is delisted due to bankruptcy, we set a negative return of -100%. It is noteworthy that none of the firms comprising the sample set went bankrupt during the sample period, hence a possible bias from this assumption, will not affect the alpha of the Kelly portfolio in any direction. However, Tandberg ASA, included in the 2010 sample, was delisted in March 2010 following its acquisition by Cisco Systems Inc (DN, 2010). In this case, we set the return for the subsequent months to zero, assuming that the acquisition was executed at Tandberg ASA's share price at the time of delisting. Consequently, the sample set now comprises only nine stocks until the next rebalancing.

In order to explain the cross-section of returns for Kelly and Markowitz with the CAPM, Fama French’s three-factor model and Carhart’s four-factor model, we obtain monthly asset pricing factors for the Oslo Stock Exchange. We retrieve the factors from Bernt Arne Ødegaard’s (2023) website. Ødegaard has employed Fama and French’s methodology to compute the pricing factors using stocks listed on the Oslo Stock Exchange.

In terms of data collection, it is important to note that the selection of database is a crucial factor influencing the outcomes and conclusions of the analysis. The Bloomberg Terminal is one of the most popular databases and deemed to be a dependable source of financial data. When it comes to Compustat, Liu (2020) found that Compustat data had a high degree of accuracy. Furthermore, Fama and French used Compustat in their influential papers on the three- and five-factor models (Fama and French, 1992, 1993, 2015). Regarding the pricing factors for the Oslo Stock Exchange, Ødegaard has published these factors consistently for several years, and is a recognized economist. Therefore, we deem Ødegaard as a reliable source.

Adjusting for Transaction Costs

As we are analyzing actively managed portfolios, transaction costs are anticipated to have a significant impact on returns. Furthermore, transaction costs play a considerable role in our analysis of the portfolios’ ability to generate alpha. Transactions costs can, as per Demsetz (1968) definition, refer to the cost incurred while exchanging ownership titles. For the stock market, Demsetz states that brokerage fees and bid-ask spreads account for most of the transaction costs. Therefore, we have chosen to concentrate on the bid-ask spread and brokerage fees as a proxy for transaction costs in our analysis. We assume that a trade on average requires to cross half of the bid-ask spread, in line with Hyde (2016). Further, we apply a brokerage fee of 0.049%, as offered by Nordnet (n.d.).

Bloomberg lacks data for bid ask-spreads for certain years in our time period. However, Compustat has sufficient high and low prices for our sample period. Hence, we apply the spread estimator of Corwin and Schultz (2012) to estimate bid-ask spreads from daily high and low prices. The respective daily high and low prices are retrieved from Compustat. The derivation of the Corwin and Schultz estimator is highlighted in the appendix in section A1.3.

As we operate with monthly data, we want to estimate the monthly bid-ask spreads for the stocks in our sample. In compliance with Corwin and Schultz (2012) we set all the negative spread estimates equal to zero prior to computing the monthly average. They argue this will yield more accurate spread estimates, as opposed to including or excluding the negative spreads. Furthermore, we notice that on certain days, the closing price of certain stocks equals its high and low price, which may indicate that the stock is illiquid that day. As a result, we get disproportionately low daily spread estimates as the high and low price are equal, which will lower our monthly average spread estimates. To address this downward bias, we exclude all daily observations where the closing price is equivalent to the high and low prices.

There are several different well-known spread estimators, such as the Roll (1984) covariance estimator and the Hasbrouck (2005) estimator. However, we implement the more recent estimator proposed by Corwin and Schultz (2012). They proclaim that the estimator is more accurate than Roll's and more user-friendly relative to Hasbrouck's, which requires an iterative process. Moreover, Abdi and Ranaldo (2017) find that the Corwin-Schultz estimator exhibits a strong correlation with the effective spread levels of US stocks. Also, it is important to note that as an estimator, there are several underlying assumptions that can potentially bias the spread estimates. To mitigate this, we adjust for overnight price changes and negative spread values. Furthermore, we adjust for the same price for all daily trades, in compliance with Corwin and Schultz (2012).

3.2 Assumptions and Decisions

When constructing our portfolios and backtesting the results, we need to make quite a few assumptions and decisions for our implementation. These are expressed and explained below.

Time Horizon

The time horizon for our analysis is set to the start of 2003 to the end of 2022. This is a period where we find that we have sufficient data, while the analysis is still relatively “long term”. However, we note that by choosing 2003 as our starting year, the portfolios are excluded from the dot-com bubble and the following crash at the end of the 1990s and start of the 2000s. Thus, the performance of the selected portfolios may be affected by this choice.

Rebalancing

When constructing optimal portfolios, one calculates the optimal weights for a given time period. This time period is not expressed by the calculations, but rather subject to the selected rebalancing period. As our quantitative Kelly strategy is more of a trading strategy than an investing strategy by nature, we wish to operate with frequent rebalancing, while still prohibiting unnecessarily large transaction costs. Consequently, we use a monthly rebalancing period for both the growth optimal portfolio and the optimal mean-variance portfolio. After a one month holding period, the new optimal weights will subsequently be calculated, and the portfolios reweighted.

Historical Returns

The basis of the construction of both the Kelly portfolio and the Markowitz portfolio are expected returns as well as variances and covariances. We estimate our model inputs using historical returns and historical volatility. Chopra and Ziemba (1993) highlights how this can be a faulty approach, as the estimation errors in the returns, variances and correlations can be substantial. However, this approach is still the most widely used in financial theory.

To estimate expected returns and the variance covariance matrices, we use a rolling estimation window with the previous 12 months as our lookback period. This follows from the fact that a short rolling estimation window may precipitate some unwanted problems. First of all, a short rolling estimation window will lead to more pro-cyclical portfolios that bet in the direction the securities and the market is heading. Secondly, such portfolios will favor securities with a strong historical performance in the lookback period. Such securities will likely have higher prices given the previous price appreciation, and consequently lower expected future returns.

Using a shorter estimation period could be advantageous in periods of large market drawdowns such as the financial crisis, as it could react quickly and go short. However, such strategies may also be limited in the sense that large short positions are difficult to execute in a bear market.

By using a longer period, we are able to achieve a better estimate for the probability that a stock will appreciate or depreciate, which enables the Kelly criterion to likely bet more accurately over a longer time period. Nonetheless, using an excessively long estimation window can also introduce the problem of mean reversion. Hence, we find the rolling estimation window of 12 months as appropriate for our analysis.

Borrowing and Lending

We assume that the investor has the ability to borrow and lend at the risk-free rate. This is a standard assumption in asset pricing theory, and also one that Thorp (2006) assumes when deriving formula 2.8 for the growth optimal portfolio for a portfolio of securities. However, the mean-variance portfolio in our analysis is assumed to be fully invested in the market at all times. As a result, the Markowitz portfolio constructed is unaffected by this assumption. On the other hand, the Kelly portfolio takes on extremely large positions when unconstrained. Also, the relative position sizes between securities and time periods can vary substantially. If we assume a 100% fully invested Kelly portfolio with a long/short strategy, some individual weights will be too big from a risk management standpoint, while also restricting available stocks due to liquidity requirements. We find it better to use a sufficiently large fractional Kelly strategy to achieve “normal” portfolio weights, and rather borrow at the risk-free rate when using leverage or lend when short or underweighted, to achieve a 100% net weight.

Short Sales

As indicated by Thorp (2006), being able to sell a security short is a desirable feature of the stock market for the Kelly criterion. To take advantage of this opportunity we allow for short sales in the Kelly portfolio. Following, we do the same with the Markowitz portfolio to make the portfolios more comparable.

Position Sizes

Both the Kelly portfolio and the Markowitz portfolio can take on quite substantial position sizes when unconstrained. We want to avoid the extremely large high-risk positions, so we choose to constrain the sizes to a maximum of 25% of the total capital. Such positions are still quite large, but our Kelly portfolio is highly concentrated with the possibility of going both long and short, thereby offsetting some of the portfolio's net weight. The Kelly strategy can be unhinged at times with extreme position sizes, see table A3.1 in the appendix. To keep it aggressive but to still manage downside risk, we must use such a quantity constraint. Naturally, the size of this constraint is an important variable, as the choice of the position size constraint will affect the weights and subsequent performance of the portfolio considerably. Avoiding the extreme positions is also important for liquidity measures and to make it implementable as a strategy. Having to go several thousand percent of the assets under management long or short can make the stock availability diminish drastically. Furthermore, it severely affects the possibility of running the Kelly strategy in a larger scale. Factoring in the availability of leverage and short positions for such volumes, such an unrestricted portfolio would be unrealistic to implement properly in the real world.

Liquidity Requirements and Available Securities

As the portfolio weights in the Kelly portfolio can be quite aggressive, we have chosen to limit our sample to the ten most liquid stocks on the OBX index. An alternative would be to include all securities on the OSE above a certain minimum liquidity requirement, however, this would force us to make assumptions regarding the capital deployed in our strategy. For our approach to determine the ten most liquid stocks, we use the prior 1-year average of daily NOK-volume as a proxy for future liquidity. Subsequently, we sort the stocks on the prior 1-year average NOK-volume and update this selection at the start of each year. The NOK-volume is computed as:

$$NOK\ Volume = Daily\ Trading\ Volume \cdot Closing\ Price \tag{4.7}$$

Having a fixed number of securities in the portfolio over time can be desirable. When calculating optimal portfolios, obtaining high quality covariance estimates can be difficult. By having a low fixed number of securities in our portfolios, we find that the inverse covariance matrix is more precise, and able to generate more sensible portfolio weights.

However, by choosing a smaller subset of securities available, and imposing quantity constraints, we limit the optimal portfolios to local maximums and not global maximums, hence, the performance of the portfolios should be affected negatively. On the other hand, they should be more realistic to implement, as well as more desirable from a risk management perspective. Furthermore, Ødegaard (2021) states that most of the relevant diversification on the Oslo Stock Exchange is achieved with ten stocks or more in the portfolio. Also, Bessembinder (2018) documents that a small subset of stocks generates most of the excess return on most exchanges.

An unwanted consequence of our liquidity requirement and the resulting concentrated portfolio is that no companies go bankrupt while present in our portfolio. Hence, an element of survivorship bias may exist implicitly.

Kelly Portfolio

With the choice of having the ten stocks with the highest NOK-volume, we need to update our set of securities in the portfolios with a given frequency. We choose the start of each year, selecting the ten stocks with the highest average daily NOK-volume in the previous 12 months as an estimate of future liquidity. We generate matrices with historical returns for each year in our time horizon, allowing for a lookback period of 12 months. To solve for the weights of the Kelly portfolio, we use VBA in Excel. The constructed VBA formula in A6.1 in the appendix is based on Ed Thorp's formula as in equation 2.8, with the excess return of the last 12 months for each security as our input range.

Again, the initial weights calculated from this formula are ludicrously aggressive. Such a portfolio will not be possible to implement in real life, given the amount of borrowing and traded volume we have. Also, the transaction costs it would incur would be monumental. Furthermore, this strategy would ultimately exceed the limits when backtesting in R Studio. Consequently, we divide the weights by a factor of 50 to achieve more "normal" weights. Typically, fractional Kelly strategies use a fraction of $c = \frac{1}{2}$ or $c = \frac{1}{4}$ as per MacLean et al. (2010), so dividing by 50 is relatively extreme. This also introduces considerable uncertainty regarding the appropriate fraction for the growth optimal portfolio. To illustrate, we show the performance of three Kelly portfolios with identical quantity constraints and fractions $c = \frac{1}{25}$, $c = \frac{1}{50}$ and $c = \frac{1}{75}$ later in the paper. Another such performance sensitive parameter is the quantity constraint, specifying the upper and lower bounds for permitted proportions allocated to each asset in the portfolio. For our quantity constraint, we set the lower bound to 25% short, and the upper bound to 25% long. Similarly, this variable will also be sensitized later in the paper.

Once we have calculated the appropriate constrained weights in Excel, we use the Portfolio Analytics package in R to backtest the performance of the portfolio with the empirical returns gathered from Bloomberg (Peterson, 2018).

Markowitz Portfolio

As we consider Kelly and Markowitz rival strategies, we want to apply the Markowitz portfolio in a setting as similar as possible to the Kelly portfolio. Consequently, we use the same set of securities, with the same quantity constraint of 25% long or short. This way, deviations in performance will not be explained by our choice of a limited securities sample. The only major difference is the portfolio weights, where the mean-variance portfolio has an additional full investment constraint. This is because, we find that the Portfolio Analytics package assumes full investment by nature when calculating the Markowitz portfolio, even when the full investment constraint is relaxed.

When solving for the Markowitz weights, we solve for each subsequent year with its corresponding historical return matrix. As the Portfolio Analytics package in R has built in functions to calculate the optimal mean-variance portfolio, we use this as opposed to VBA (Peterson, 2018). Firstly, we design a portfolio with objectives to maximize the expected return and minimize the risk. Secondly, we add constraints for box size. The package then solves for the optimal portfolios with monthly rebalancing and a 12-month rolling estimation window. The optimal portfolio function runs with a search size of 5000 and 50 iterations as this is what our computers could handle. The resulting optimal portfolio weights for each corresponding year are then extracted in a complete Markowitz weights matrix, and then backtested with the empirical returns for the time period.

Market Portfolio

The select inclusion of the ten most liquid securities in our optimal portfolios is a choice made to ensure that the Kelly strategy is implementable, and that we can construct a comparable Markowitz portfolio to compare performances. This selection does not extend to the market portfolio as we wish to compare the performance with the simple strategy of holding the market portfolio as from the CAPM theory. Consequently, we use the OSEBX, the standard benchmark index of the Oslo Stock Exchange as our market portfolio and benchmark. Oppositely, we can construct a market weighted portfolio of the ten securities in our portfolio selection. However, this contradicts Næs, Skjeltorp and Ødegaard’s (2009) suggestion of using a broad market index.

4. Results

4.1 Initial Results

Figure 4.1 illustrates the holding period returns for the growth optimal Kelly strategy, the mean-variance Markowitz strategy, as well as the OSEBX. The plots have been indexed at 1 and illustrate the end wealth for the respective strategies. As expected, the graph highlights the outperformance of the Kelly portfolio with regards to both the Markowitz portfolio and the benchmark, achieving an ending wealth level of 16.39 against 8.44 and 10.84 respectively. The growth optimal portfolio is designed to maximize end of period wealth, so at least before adjusting for transaction costs the wealth obtained by the Kelly strategy is expected to be the highest. Furthermore, we note how Markowitz fails to generate higher holding period returns than our benchmark. This is in line with several studies highlighting the poor performance of this strategy.

Kelly underperforms in the years leading up to the financial crisis in 2008 but is able to avoid the drawdowns of Markowitz and the benchmark, and rather generate positive returns this period. This is due to the relaxation of the full investment constraint, where the Kelly portfolio is allowed to go net short in periods when the market does not appear attractive.

In light of this possibility, the Kelly portfolio still visually appears to follow the market portfolio relatively closely. Although, we note the strong returns and high volatility in the end of our period. This is due to the large bets made by the Kelly criterion. While the bets result in great gains, a higher risk for large drawdowns also follows.

Figure 4.1: Holding Period Returns

The figure plots the holding period returns for Kelly, Markowitz, and the benchmark from February 2003 through December 2022. We calculate the holding period return by multiplying the monthly portfolio returns. By reinvesting the capital gains for each month, we achieve the compounded returns. The returns represent the ending wealth we obtain through the different strategies.

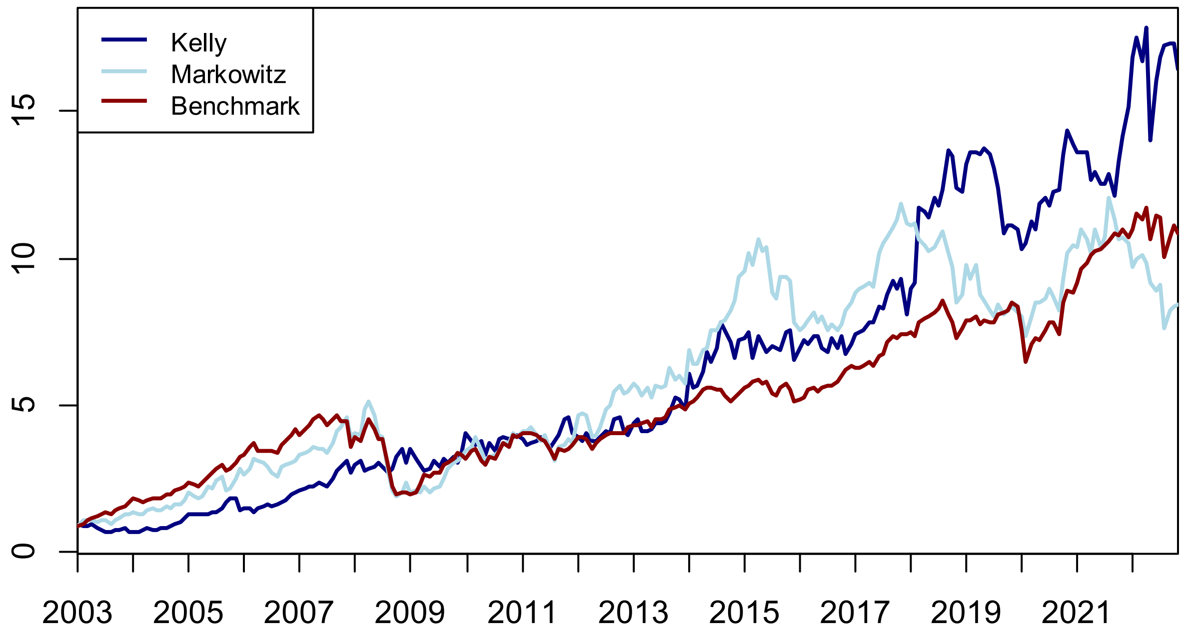


Table 4.1 presents a selection of key statistics for our strategies and the market portfolio. At first glance, we observe the nearly identical Sharpe ratios for Kelly and our benchmark. The Sharpe ratio of Markowitz is lower than Kelly and the market. The annualized arithmetic mean of Kelly is higher than Markowitz, while the standard deviation is marginally lower, resulting in a lower Sharpe ratio for Markowitz. The benchmark has lower arithmetic mean and volatility than Kelly but has the same return per unit of risk. An important metric is the geometric mean of the portfolios. Here, Kelly outperforms both the market and Markowitz, and consequently generates the highest ending wealth for the investor. We also note that while the arithmetic mean of the benchmark is almost similar to our Markowitz portfolio, its geometric mean is higher, resulting in a higher wealth by holding the market portfolio. While the Sharpe ratio is the most commonly used measure for portfolio performance, it punishes a portfolio for volatility related to both the upside and the downside. Oppositely, the Sortino ratio expresses return per unit of downside risk. Kelly's Sortino ratio is higher than Markowitz and the benchmark, thus indicating that a larger portion of the Kelly portfolio's standard deviation stems from volatility related to its upwards movements. Ziemba (2005) proposes a modified Sharpe ratio closely resembling the Sortino ratio to properly analyze the performance of portfolios. He found that the funds in his research generating the highest

wealth, such as Warren Buffett's Berkshire Hathaway, performed rather poorly compared to low volatility, low growth funds when applying the Sharpe ratio. However, by using the modified Sharpe targeting downside risk, the higher growth funds scored relatively higher.

Finally, the information ratio highlights how both Markowitz and Kelly generate higher returns than the market benchmark. IR is often used to measure a portfolio manager's skill, evaluating to what extent and with what consistency the strategy employed beats the benchmark. Kelly's information ratio is also in this instance higher than Markowitz, which only marginally beats the benchmark.

The key financials for our portfolios are largely as expected. The Kelly portfolio seeks to maximize the geometric mean to achieve the highest growth rate and does just so. Furthermore, the Kelly portfolio is more volatile than the market portfolio. However, as a lot of the volatility is related to upside risk, the Sortino ratio is, as expected, high.

Kelly's Sharpe ratio is nearly identical to the benchmark's, so we should under perfect market conditions be able to replicate the return of the Kelly portfolio by borrowing at the risk-free rate and investing with leverage in the market portfolio as per Tobin's separation theorem (1958). This is illustrated in figure A2.1 in the appendix.

Table 4.1: Key Statistics

The table reports annualized performance measures for Kelly, Markowitz, and benchmark. The performance measures are calculated utilizing monthly figures. The Sharpe, Sortino, IR and standard deviation are annualized by multiplying with $\sqrt{12}$. Arithmetic and geometric mean are annualized by multiplying with 12.

	Kelly	**Markowitz**	**Benchmark**
Sharpe Ratio	0.58	0.43	0.58
Sortino Ratio (MAR=rf)	0.95	0.62	0.68
Information Ratio	0.11	0.02	-
Arithmetic Mean	17.1%	14.3%	13.9%
Geometric Mean	14.1%	10.8%	12.0%
Standard Deviation	24.6%	26.4%	19.2%

After analyzing the return and various metrics of the Kelly strategy we wish to investigate the extent to which the performance can be attributed to risk factors in multifactor models. We analyze using linear regression in table 4.2 below:

Table 4.2: Regression Results on Gross Returns

The table reports the regression results for the Kelly and Markowitz portfolio before adjusting for transaction costs. The portfolios' excess returns are regressed on the CAPM, FF3F and C4F model. The regression results are estimated utilizing monthly excess returns from February 2003 through December 2022.

	Dependent variable:					
	Kelly			Markowitz		
	CAPM	FF3F	C4F	CAPM	FF3F	C4F
Alpha	0.011^{**}	0.014^{***}	0.014^{***}	0.001	0.002	-0.003
	(0.005)	(0.005)	(0.005)	(0.004)	(0.004)	(0.004)
Market	0.115	0.114	0.116	0.974^{***}	0.969^{***}	1.010^{***}
	(0.083)	(0.083)	(0.084)	(0.063)	(0.064)	(0.061)
SMB		-0.185^{*}	-0.183^{*}		-0.117	-0.069
		(0.096)	(0.097)		(0.073)	(0.070)
HML		-0.012	-0.011		0.023	0.033
		(0.080)	(0.080)		(0.061)	(0.058)
UMD			0.013			0.269^{***}
			(0.076)			(0.055)
Observations	239	239	239	239	239	239
R^2	0.008	0.024	0.025	0.503	0.508	0.554
Adjusted R^2	0.004	0.012	0.008	0.501	0.502	0.546
Residual Std. Error	0.071	0.071	0.071	0.054	0.054	0.051
Significance level				$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$		

Kelly generates a significant positive alpha across all three factor models. This aligns with our expectations as the performance measures in table 4.1 demonstrate that Kelly outperforms the benchmark. For the CAPM, the alpha is significant on a 5% level, while for the three- and four-factor models it is significant on a 1% level. The highest alpha of 1.4% is observed under the three- and four-factor models. This implies that on average, 1.4% of the portfolio's excess return cannot be explained by the exposure to the included risk factors. It should be noted that this is a monthly alpha, which translates into a remarkably high annualized alpha of 16.8%.

For the Markowitz portfolio, the alpha is statistically indistinguishable from zero for all models. This is expected as we observe that Markowitz does not outperform the benchmark as per the performance measures in table 4.1 and is in line with several empirical studies suggesting a poor performance of the Markowitz portfolio.

Kelly exhibits a market beta that is statistically indistinguishable from zero, which implies that the portfolio return cannot be explained by the market exposure. Markowitz's market beta on the other hand, displays a value around one for all models and is significant on a 1% level. This implies that the Markowitz portfolio is largely correlated with the market portfolio. The stark difference in betas is somewhat surprising based on the plots in figure 4.1. Also, as the securities in our sample represent a large proportion of the OSEBX, we would expect a higher beta for the Kelly portfolio. However, the Kelly portfolio has a tendency to appreciate during bear markets, resulting in some negative correlation with the market, lowering the estimated beta.

The size factor, SMB, is significantly negative for Kelly on a 10% level. A negative exposure to the size factor implies that the portfolio returns are influenced by the performance of larger firms. This is expected as our sample is annually sorted to include firms with the highest NOK-volume, and we observe that these firms also tend to have high market capitalization.

Furthermore, our regression results indicate that Kelly has a negative exposure to the HML factor. This is somewhat unexpected given that our stock sample mainly consists of typical value stocks. However, the exposure is statistically indistinguishable from zero. On the other hand, Markowitz has a positive but insignificant exposure to the HML. The opposing signs for the portfolios' exposure are surprising as both portfolios consist of the same securities.

The momentum factor, UMD, is significant for Markowitz on a 1% level, suggesting a tilt towards positive momentum stocks. For the Kelly portfolio, the momentum factor is statistically indistinguishable from zero.

The models' R-squared (R^2) values are consistently low for the Kelly portfolio, indicating low explanatory power. Furthermore, the inclusion of additional risk factors does not result in a significant increase in the adjusted R^2 values, suggesting the factors added have low explanatory power over the portfolio's excess returns. The highest R^2 is a mere 2.5% for the Kelly portfolio with the four-factor model, indicating that only 2.5% of the variation in excess returns can be explained by the risk factors in the model. This suggests that there are other excluded or unobservable factors that are driving the returns. As a result, our alpha values are too high and misleading.

If we address the CAPM model for the Kelly portfolio, we initially note the low beta. As the market exposure is the only risk factor in the model, it struggles to explain the excess returns of the strategy. Consequently, the explanatory power is very low and the alpha as a measure of unexplained variation of excess returns is inflated. When including the significant SMB factor in FF3F, the explanatory power increases. However, as Kelly has negative exposure to both new factors and the beta is practically unchanged, the alpha measure increases. By including an additional momentum factor in Carhart, the adjusted R^2 decreases, as the additional factor is insignificant.

As there are excluded factors driving the returns of the Kelly portfolio, supplementing, or replacing factors with others may be necessary to achieve better results. Black et al. (1972) argue that low beta assets outperform high beta assets, known as the low-beta anomaly. The anomaly contradicts the CAPM, where an investor can only achieve higher returns through higher systematic risk. Given its low beta, the Kelly portfolio may be subject to this anomaly, and a subsequent factor may be necessary to better explain the portfolio returns. Fama and French (2015), argue that by adding a profitability factor (RMW) and an investment factor (CMA), the extended five-factor model captures the returns associated with a low market beta. However, our findings in table A4.1 in the appendix, suggest that this is not the case for our Kelly portfolio. As such, other relevant factors are still excluded, and our alpha measures are likely still inaccurate.

Still, our Durbin-Watson and Breusch-Pagan test results in table A4.2 for our regression models, show no sign of autocorrelation or heteroskedasticity on a 5% level.

4.2 Transaction Costs Adjustment

Our results, albeit with a poor explanatory factor, indicate that Kelly generates alpha, whereas Markowitz fails to do so. However, both strategies have been implemented with relatively relaxed portfolio constraints allowing for large positions. Furthermore, we use monthly rebalancing, and as a result, both strategies incur quite substantial trading costs. To get more realistic real-world estimates of the strategies' alpha we want to account for these trading costs. To test the robustness of the positive alphas through accounting for transaction costs, we estimate each portfolio's turnover and the average bid-ask spreads for each stock in the corresponding months, in line with Corwin and Schultz (2012).

We find the portfolio turnover by taking the absolute value of the delta for each rebalancing period. This will then yield the percentage size of the strategies' capital traded for each security at each rebalancing period. The consequent turnover is then multiplied with half of the monthly average bid-ask spread for each security and the assumed brokerage fees of 0.049%. We adjust the returns of the Kelly and Markowitz portfolios by subtracting the overall monthly costs incurred.

Table 4.3: Regression Results Net of Transaction Costs

The table reports the regression results for the Kelly and Markowitz portfolio adjusted for transaction costs when crossing half of the bid-ask spread. The bid-ask spreads are estimated using Corwin and Schultz's (2012) estimator. The portfolios' excess adjusted returns are regressed on the CAPM, FF3F and C4F model. The regression results are estimated utilizing monthly excess adjusted returns from February 2003 through December 2022.

	Dependent variable:					
	Kelly			Markowitz		
	CAPM	FF3F	C4F	CAPM	FF3F	C4F
Alpha	0.006	0.009*	0.009*	-0.004	-0.002	-0.007*
	(0.005)	(0.005)	(0.005)	(0.004)	(0.004)	(0.004)
Market	0.115	0.115	0.117	0.982***	0.977***	1.018***
	(0.082)	(0.083)	(0.084)	(0.063)	(0.064)	(0.061)
SMB		-0.179*	-0.176*		-0.117	-0.069
		(0.095)	(0.096)		(0.073)	(0.070)
HML		-0.014	-0.014		0.022	0.032
		(0.079)	(0.080)		(0.061)	(0.058)
UMD			0.014			0.272***
			(0.075)			(0.055)
Observations	239	239	239	239	239	239
R^2	0.008	0.024	0.024	0.506	0.512	0.558
Adjusted R^2	0.004	0.011	0.007	0.504	0.506	0.550

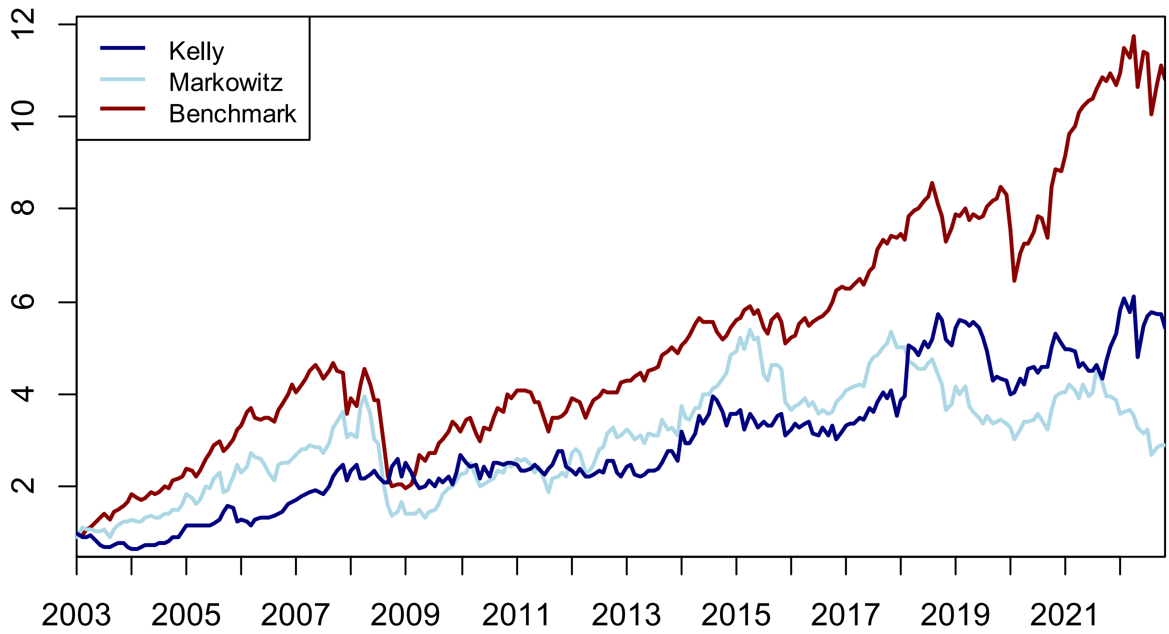
Residual Std. Error	0.070	0.070	0.070	0.054	0.054	0.051
Significance level						*p<0.1; **p<0.05; ***p<0.01

Table 4.3 shows that after accounting for transaction costs, the alphas are only significant for the three- and four-factor model for Kelly on a 10% level. Regarding Markowitz, the alpha is now statistically negative under the four-factor model, implying that the portfolio destroys value given the included risk factors. Consequently, in our case, adjusting for transaction costs considerably affects the performance of our portfolios, as expected. However, we still note the extremely low R^2, indicating that our models do not provide satisfactory explanatory power.

To further illustrate, figure 4.2 graphically depicts the effect of transaction costs on portfolio performance.

Figure 4.2: Holding Period Returns Net of Transaction Costs

The figure plots the holding period returns for Kelly, Markowitz, and the benchmark from February 2003 through December 2022 net of transaction costs. After subtracting the monthly transaction costs, the monthly portfolio returns are multiplied. By reinvesting the capital gains for each month, we achieve the compounded returns. The returns represent the ending wealth we obtain through the different strategies.



As predicted, transaction costs significantly harm the holding period returns for both Kelly and Markowitz, with the Kelly portfolio now considerably underperforming the market. We note that costs with holding the market portfolio have been ignored, however, this is expected to have little impact. The underperformance is also reflected in the key statistics in table 4.4, where both strategies underperform the benchmark on every measure. In light of the poor performance of the Kelly portfolio after transaction costs, our alpha measures in table 4.3 seems high.

Table 4.4: Key Statistics Net of Transaction Costs

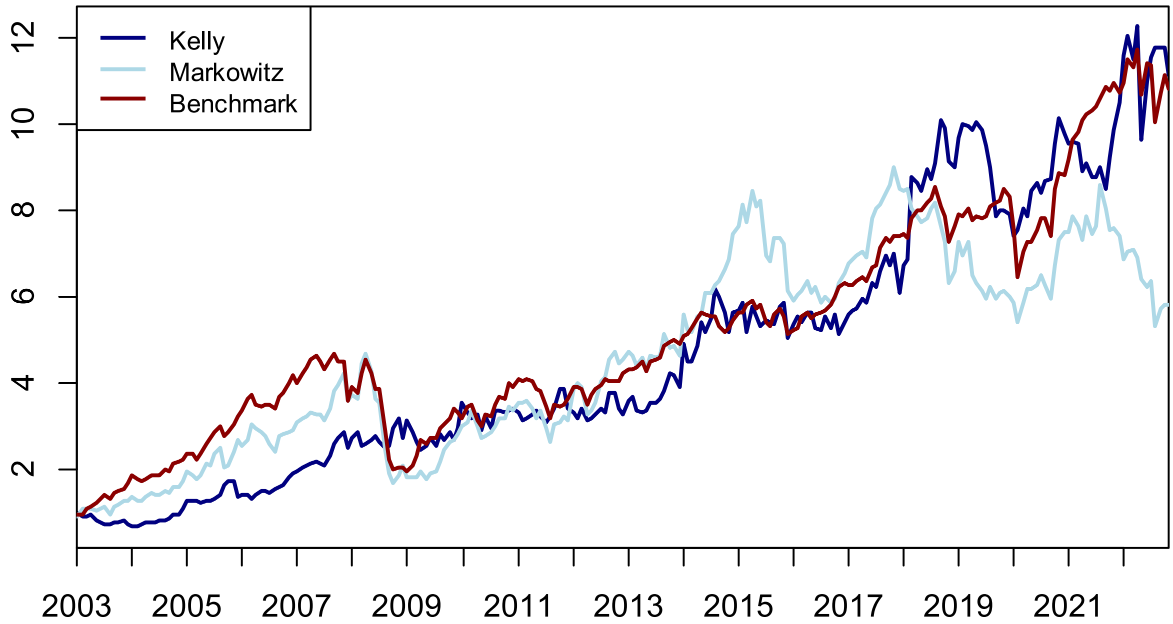
The table reports annualized performance measures for Kelly, Markowitz, and the benchmark when adjusting for transaction costs. The performance measures are calculated utilizing monthly figures. The Sharpe, Sortino, IR and standard deviation are annualized by multiplying with $\sqrt{12}$. Arithmetic and geometric mean are annualized by multiplying with 12.

	Kelly	**Markowitz**	**Benchmark**
Sharpe Ratio	0.36	0.23	0.58
Sortino Ratio (MAR=rf)	0.56	0.33	0.68
Information Ratio	-0.08	-0.27	-
Arithmetic Mean	11.5%	8.9%	13.9%
Geometric Mean	8.5%	5.4%	12.0%
Standard Deviation	24.3%	26.5%	19.2%

When estimating the spreads in line with Corwin and Schultz (2012), by using high and low prices, we achieve an average bid-ask spread for the securities in our sample of 0.87%. However, when evaluating the limited selection of bid-ask spreads we were able to obtain from Bloomberg, we note a considerable difference for the majority of the stocks in our portfolios compared to our estimated spreads. The observed spreads for the most liquid stock usually appear in a range of $[0.05\%, 0.15\%]$. Though, as this represents more recent data, we note that increases in liquidity in recent years as per Næs et al. (2008) naturally should result in narrower spreads that may not be representable for the entirety of our time period. On the other hand, given the significant difference in observed and estimated spreads, we wish to test the sensitivity of our portfolios to this parameter. As such, we illustrate by dividing the estimated spread by an arbitrary factor of 8, to achieve spreads closer to our observed interval, while acknowledging the relative difference in spreads for the securities in our portfolios.

Figure 4.3: Holding Period Returns Net of Transaction Costs, Using a Lower Spread Estimate

The figure plots the holding period returns for Kelly, Markowitz, and the benchmark from February 2003 through December 2022 net of transaction costs. Here, a lower spread estimate is utilized, resulting in lower transaction costs. After subtracting the monthly transaction costs, the monthly portfolio returns are multiplied. By reinvesting the capital gains for each month, we achieve the compounded returns. The returns represent the ending wealth we obtain through the different strategies.



When using a lower spread estimate, the holding period returns improve drastically, as shown in figure 4.3. The growth optimal portfolio now generates marginally higher wealth than the market portfolio. The key statistics in table 4.5 also highlights the increase in performance associated with lower transaction costs, with our key portfolio financials now resembling those for our gross returns. Focusing on our regression results net of transactions costs in table 4.6, we note that naturally, by using a lower spread estimate, our results show that transaction costs now have a less degrading effect on the alpha values. Nevertheless, the explanatory power of the models is still very low.

Table 4.5: Key Statistics Net of Transaction Costs, Using a Lower Spread Estimate

The table reports annualized performance measures for Kelly, Markowitz, and the benchmark when adjusting for transaction costs, using a lower spread estimate. The performance measures are calculated utilizing monthly figures. The Sharpe, Sortino, IR and standard deviation are annualized by multiplying with $\sqrt{12}$. Arithmetic and geometric mean are annualized by multiplying with 12.

	Kelly	**Markowitz**	**Benchmark**
Sharpe Ratio	0.50	0.36	0.58
Sortino Ratio (MAR=rf)	0.81	0.52	0.68
Information Ratio	0.04	-0.08	-
Arithmetic Mean	15.1%	12.4%	13.9%
Geometric Mean	12.2%	8.9%	12.0%
Standard Deviation	24.5%	26.4%	19.2%

Table 4.6: Regression Results Net of Transaction Costs, Using a Lower Spread Estimate

The table reports the regression results for the Kelly and Markowitz portfolio when adjusting for transaction costs, using a lower spread estimate. The portfolios' excess adjusted returns are regressed on the CAPM, FF3F and C4F model. The regression results are estimated utilizing monthly excess adjusted returns from February 2003 through December 2022.

	Dependent variable:					
	Kelly			Markowitz		
	CAPM	FF3F	C4F	CAPM	FF3F	C4F
Alpha	0.009**	0.012**	0.012**	-0.001	0.001	-0.004
	(0.005)	(0.005)	(0.005)	(0.004)	(0.004)	(0.004)
Market	0.114	0.114	0.116	0.977***	0.972***	1.013***
	(0.083)	(0.083)	(0.084)	(0.063)	(0.064)	(0.061)
SMB		-0.183*	-0.181*		-0.117	-0.069
		(0.095)	(0.096)		(0.073)	(0.070)
HML		-0.013	-0.013		0.023	0.033

		(0.080)	(0.080)		(0.061)	(0.058)
UMD			0.013			0.270***
			(0.076)			(0.055)
Observations	239	239	239	239	239	239
R^2	0.008	0.024	0.024	0.504	0.510	0.555
Adjusted R^2	0.004	0.012	0.008	0.502	0.503	0.548
Residual Std. Error	0.071	0.070	0.070	0.054	0.054	0.051
Significance level						*p<0.1; **p<0.05; ***p<0.01

4.3 Robustness

After analyzing the effect of transaction costs on portfolio performance, we wish to further analyze the robustness of other sensitive parameters for the Kelly portfolio. Consequently, our choice of fractional Kelly, as well as the quantity constraint is sensitized without transaction costs in the figures 4.4 and 4.5 below:

Figure 4.4: Sensitivity of the Kelly Portfolio's Quantity Constraint

The figure plots the holding period returns of three Kelly portfolios from February 2003 through December 2022 using different quantity constraints. The initial constraint of maximum 25% of the capital invested in a single security is sensitized, allowing for 40% and 10%. The returns of the portfolios are multiplied, representing the compounded returns and the ending wealth we achieve with the different constraints.

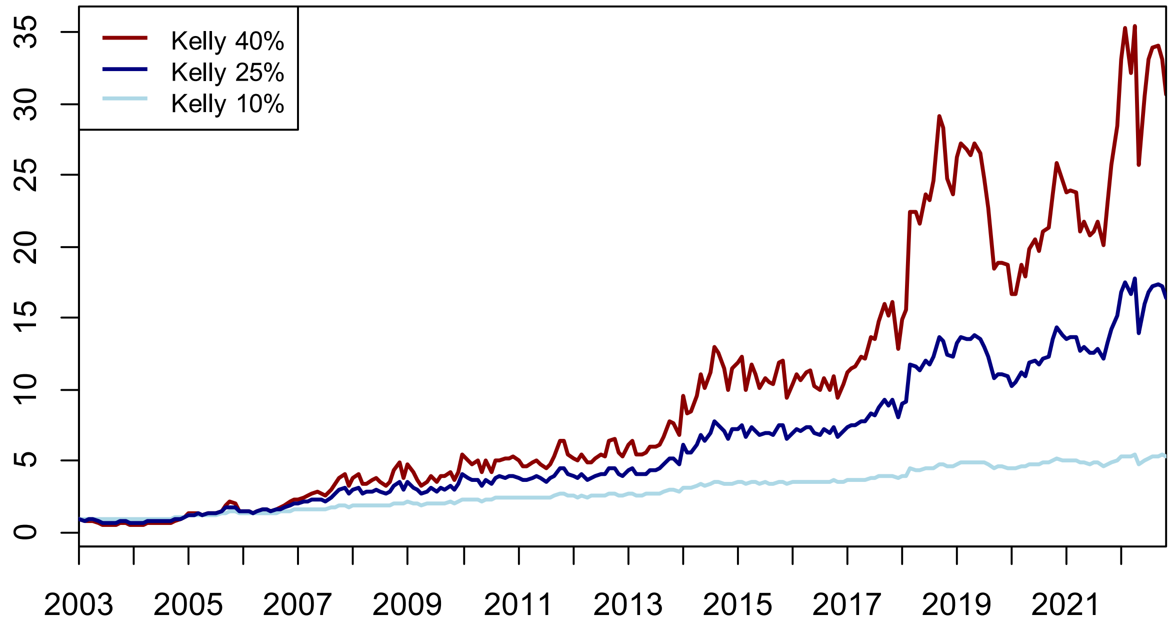


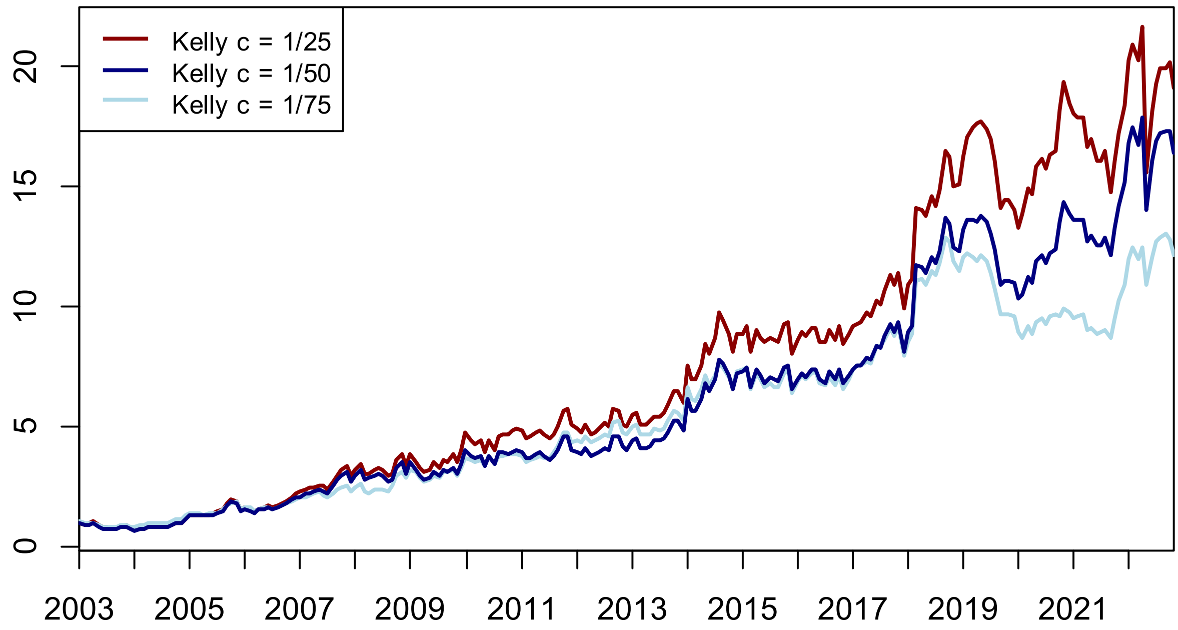
Figure 4.4 shows the sensitivity of our proposed quantity constraint for the Kelly portfolio.

With an initial constraint of 25% investments long or short in single securities, we now allow for 10% and 40%. The portfolio with a 40% constraint resembles our original Kelly portfolio, but has greater volatility and higher return. Every movement of our regular Kelly is exaggerated. However, the positive returns dominate, and as a result, the ending wealth achieved is at a significantly higher level than before, although with a higher risk of ruin. A natural discussion follows whether our growth optimal portfolio instead should be subject to a constraint of 40%, but making such a choice would subject us to hindsight bias. Furthermore, this strategy has larger turnover, hence it would be more difficult and unrealistic to implement in real life. Still, we note that the Kelly portfolio with our chosen constraints is not the optimal portfolio with the highest growth rate. Hence, further outperformance of the benchmark is certainly possible. However, the questions remain as to whether more aggressive portfolios are implementable, as well as how degrading the transaction costs are for performance.

On the other hand, the more conservative portfolio with a 10% constraint behaves quite differently than our original Kelly, exhibiting significantly lower volatility, but also very low final wealth. For a particularly risk-averse investor, such a portfolio can appear more attractive. Ultimately, the Kelly portfolio shows great sensitivity to the position size constraint, with not only the return achieved seemingly very reliant on our selection, but also the risk level and underlying characteristics of the portfolio.

Figure 4.5: Sensitivity of the Fractional Kelly

The figure plots the holding period returns of three Kelly portfolios from February 2003 through December 2022 using different Kelly fractions. The initial fraction of $c = \frac{1}{50}$ is sensitized with Kelly fractions of $c = \frac{1}{25}$ and $c = \frac{1}{75}$. The returns of the portfolios are multiplied, representing the compounded returns and ending wealth we achieve with the different fractional Kelly portfolios. All portfolios have the same position size constraint of maximum 25%.



Above, figure 4.5 displays the sensitivity of our choice of fractional Kelly. Our Kelly fraction of $c = 50$ is sensitized with $c = 25$ and $c = 75$. In this instance, the effects are visible but not as prominent as in figure 4.4. By using a lower Kelly fraction, the portfolio will have higher weights before being subject to the quantity constraint, making it a more aggressive strategy. Oppositely, when using a higher fraction, we get lower wealth but slightly lower volatility. The similarity of the three portfolios highlights the importance of using an appropriate position size constraint, as this clearly has a higher effect on our portfolios than the actual Kelly fraction. This is in line with expectations, as most Kelly portfolio weights will have to be constrained regardless, given the aggressive nature of our Kelly strategy.

4.4 Implementability

Our analysis of the growth optimal Kelly portfolio indicates that the transaction costs the strategy incurs are quite substantial. Following, this leads to a discussion of how implementable our strategy is in the real world. An implicit assumption made in the implementation of our strategy is the closing price being the obtainable price when purchasing or selling a security. Should a large fund run this strategy at a large scale, such an assumption may not hold particularly well. Larger funds are known to move prices when trading, consequently incurring even larger transaction costs than already accounted for. An additional problem arising for a larger fund is regarding available volume. Figure A2.2 shows that for certain periods, this might be a real problem. To account for the large trades observed in the Kelly strategy, we have chosen the ten stocks with the highest NOK-volume on the OBX, and further added a position size constraint of 25% to limit turnover. Still, Atea ASA, the security with the 10th highest average daily NOK-volume in 2003 had only 19M in average daily NOK-volume. The largest trade possible with the strategy is either from 25% short to 25% long or vice versa. This results in a trade of 50% of the capital deployed long or short. If all trading is executed in the last day of the month as assumed, the strategy could only have a maximum capital base of a mere 38M to avoid restrictions on portfolio weights suggested by the Kelly criterion. While trades of this size are not required particularly often, one does not want to be restricted by available volume when implementing a trading strategy.

Moreover, another implicit assumption that becomes apparent is the possibility of capturing the entire daily NOK-volume of a stock, should the strategy require you to do so. An additional buffer would likely be required to make the execution of the trade realistic. However, dependent on the size of the portfolio, the trades would likely be executed in the span of several trading days to ensure sufficient NOK-volume. This introduces further uncertainty regarding obtainable prices, making our estimates less precise.

One of the aspects related to the strong performance of the growth optimal strategy is its ability to bet big both ways. However, large short positions are harder to come by than long positions, creating an additional availability issue. Whereas both the market and the Markowitz portfolio suffer significant drawdowns during the financial crisis, the Kelly portfolio exhibit stellar performance. This is largely due to the Kelly portfolio being net short this period, as shown in figure A2.3 in the appendix. In reality, obtaining large short positions in bear markets can be extremely difficult. Consequently, the strategy at times of considerable market distress may be forced to take larger positions in the riskless security as an alternative to the ideal scenario of going short, at times when securities simply may not be available to short. Furthermore, short positions impose larger transaction costs. An investor usually pays a borrowing fee when lending the security, as well as margin interest on the margin account opened. These associated short costs also vary with the market climate, where short costs naturally increase at times of extreme market distress. Additional costs when shorting a security has been disregarded in our analysis, indicating that our transaction cost estimate could be too low, compared to the real-life costs of implementing the growth optimal Kelly portfolio.

5. Conclusion

In this study, we test whether the growth optimal Kelly portfolio is able to beat the benchmark and generate alpha in the Norwegian stock market from February 2003 through December 2022. We find that the Kelly portfolio yields a compound average growth rate of 14.1%, resulting in a final wealth of 16.39 (indexed at 1). This outperforms the OSEBX, who achieves an ending wealth of 10.84 and an annual growth rate of 12%. The Markowitz portfolio underperforms both Kelly and the benchmark. We also find that Kelly and the benchmark achieve nearly identical Sharpe ratios of 0.58, but that Kelly achieves a higher Sortino ratio of 0.95. The Kelly portfolio generates an annual alpha of 16.8% in the three- and four-factor models of Fama French and Carhart. The alpha is significant on a 1% level. However, the beta of the portfolio is low, and our models struggle to explain the excess returns generated by the Kelly portfolio, resulting in a very low R^2. This leads us to believe that our factor models are not sufficient in explaining the returns of our portfolio, and that the alpha measures are inflated.

Moreover, we find that the risk-adjusted excess returns may not be achievable in the real world. When accounting for transaction costs, Kelly underperforms the benchmark with regards to ending wealth, and the alpha is only significant on a 10% level. However, our results are very sensitive to the level of transaction costs, where a lower estimate yields significant improvement in performance.

Our Kelly portfolio is based on Thorp (2006) and Merton (1969). Both assume no transaction costs when deriving the optimal portfolio. Perhaps to be able to achieve risk-adjusted excess returns in the real world, the transaction costs would need to be internalized in the model when calculating the optimal portfolio, as opposed to accounted for afterwards. Extensions of Merton's model with transaction costs are studied by, amongst others, Davis and Norman (1990) and Morton and Pliska (1995). However, such extensions are not analyzed in this study.

Our Kelly portfolio for the stock market exhibits strong performance without transaction costs. However, we know the stock market lacks known probability distributions and outcomes that are required for an optimal performance of the Kelly criterion. Extending the analysis to a setting with more limited outcomes, such as exploring options strategies like the split strike conversion, holds promise in determining whether such a portfolio would outperform our approach.

Ultimately, although the Kelly criterion applied in our setting generates strong returns, our regression models have low explanatory power, making it difficult for us to draw a conclusion regarding the alpha. Furthermore, the alpha the strategy seems to generate, appears to diminish when applying the portfolio to real-world conditions.

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Appendix

A1 Mathematical Derivations

A1.1: Mathematical Derivation of the Coin Toss Problem (Thorp, 2006)

With a fixed fraction strategy, we bet $B_i = fX_{i-1}$, where $0 \leq f \leq 1$. We note number of successes and failures S and F, our capital after n trials is then $X_n = X_0(1+f)^S(1-f)^F$, where $S + F = n$. With f in the interval $0 < f < 1$, $P(X_n = 0) = 0$. Hence, gambler´s ruin is technically avoided.

Since:

$$e^{n \log\left[\frac{X_n}{X_0}\right]^{\frac{1}{n}}} = \frac{X_n}{X_0} \tag{A1.1}$$

We have the exponential rate of increase per trial equal to:

$$G_n(f) = \log\left[\frac{X_n}{X_0}\right]^{\frac{1}{n}} = \frac{S}{n}\log(1+f) + \frac{F}{n}\log(1-f) \tag{A1.2}$$

This measures the exponential rate of increase per trial. Kelly chose to maximize the expected value of the growth rate coefficient $g(f)$, where:

$$\begin{aligned} g(f) = E\left\{\left[\frac{X_n}{X_0}\right]^{\frac{1}{n}}\right\} &= E\left\{\frac{S}{n}\log(1+f) + \frac{F}{n}\log(1-f)\right\} \\ &= p\log(1+f) + q\log(1-f) \end{aligned} \tag{A1.3}$$

Since $g(f) = \frac{1}{n}E(\log X_n) - \frac{1}{n}\log X_0$, for n fixed, maximizing $g(f)$ is equivalent to maximizing $E\log X_n$.

To find the optimal fraction f^* to wager in each bet, we maximize $g(f)$ by setting $g'(f) = 0$ This yields:

$$g'(f) = \frac{p}{1+f} - \frac{q}{1-f} = \frac{p-q-f}{(1+f)(1-f)} = 0 \tag{A1.4}$$

A1.2: Mathematical Derivation of the Kelly Portfolio (Thorp, 2006)

First, when we apply the Kelly criterion to the stock market, we meet new challenges. Whereas in our coin tossing example, there are only a select possible outcomes, in the stock market there are practically an infinite number of outcomes. As a result, we use continuous probability distributions, instead of discrete probability distributions. We need to find the f that maximizes $g(f) = E\ln(1 + fX) = \int \ln(1 + fx)\,dP(x)$ where $P(x)$ is a probability measure describing the outcomes. The problem is to find an optimum portfolio among n securities. Here, x and f are n-dimensional vectors and fx is their scalar product. We also have constraints for the maximization problem. We require $1 + fx > 0$ so ln $(\cdot)$ is defined, and $\sum f_i = 1$ or a $c > 0$ to normalize to a unit investment. Additional constraints such as no short selling, limited quantities invested in the i^{th} security, or leverage limits may also be added. The maximization problem is generally solvable as $g(f)$ is concave. However, a liquidity issue might arise that prohibits us from betting the full optimal f^*, forcing us to under bet.

One technique to use is continuous approximation. We let X be a random variable with $P(X = m + s) = P(X = m - s) = 0.5$. Then $E(X) = m$, $Var(X) = s^2$. If we have initial capital V_0, betting fraction f, and return per unit of X, then the result is:

$$V(f) = V_0(1 + (1 - f)r + fX) = V_0\big(1 + r + f(X - r)\big) \tag{A1.5}$$

where r is here the rate of return on remaining capital that we invest in the risk-free rate.

Then:

$$\begin{aligned} g(f) = E\big(G(f)\big) = E\left(\ln\left(\frac{V(f)}{V_0}\right)\right) &= E\ln\big(1 + r + f(X - r)\big) \\ = 0.5\ln\big(1 + r + f(m - r + s)\big) &+ 0.5\ln\big(1 + r + f(m - r - s)\big) \end{aligned} \tag{A1.6}$$

We now subdivide the time interval into n equal independent steps, keeping the same drift and the same total variance. Accordingly, we replace m, s^2 and r with m/n, s^2/n, and r/n. We have n independent X_i, $i = 1, \dots, n$, with:

$$P\left(X_i = \frac{m}{n} + sn^{-\frac{1}{2}}\right) = P\left(X_i = \frac{m}{n} - sn^{-\frac{1}{2}}\right) = 0.5 \tag{A1.7}$$

Then:

$$\frac{V_n(f)}{V_0} = \prod_{i=1}^{n}(1 + (1 - f)r + fX_i) \tag{A1.8}$$

By taking the $E(\log(\cdot))$ of both sides, we get $g(f)$. Expanding the result in a power series leads to:

$$g(f) = r + f(m - r) - \frac{s^2 f^2}{2} + O\left(n^{-\frac{1}{2}}\right) \tag{A1.9}$$

where $O\left(n^{-\frac{1}{2}}\right)$ has the property $n^{\frac{1}{2}}O\left(n^{-\frac{1}{2}}\right)$ is bounded as $n \to \infty$. Letting $n \to \infty$, we have:

$$g_\infty(f) \equiv r + f(m - r) - \frac{s^2 f^2}{2} \tag{A1.10}$$

The limit $V \equiv V_\infty(f)$ of $V_n(f)$ as $n \to \infty$ corresponds to a log normal diffusion process. This is a well-known model for securities prices as claimed by Thorp (2006). The security has instantaneous drift rate m, variance s^2, and cash invested in the risk-free rate earns at the instantaneous rate r. The $g_\infty(f)$ of above is then the instantaneous growth rate of capital with fraction f invested. Any bounded random variable X with mean $E(X) = m$ and variance $Var(X) = s^2$, will lead to the same result.

Now f no longer needs to be less than or equal to 1, and the problems with $\log(\cdot)$ being undefined for negative arguments have disappeared. $f < 0$, is unproblematic, and simply corresponds to short selling the security.

Any investor who follows the policy f must adjust his investments "instantaneously". Thorp (2006) states that in practice this means adjusting in tiny increments whenever there is a small change in V. This is known from option theory and does not prevent the practical implementation of the theory (Black and Scholes, 1973). However, this is generally not implementable in the real world. As a compromise, we will use monthly rebalancing of the investments in the portfolio.

$g_\infty(f)$ is exactly parabolic and easy to study. Lognormality of $\frac{V(f)}{V_0}$ means $\log\left(\frac{V(f)}{V_0}\right)$ is $N(M, S^2)$ distributed with mean $M = g_\infty(f)t$ and variance $S^2 = Var\big(G_\infty(f)\big)t$ for any time t. Using this, we can determine the expected capital growth. Additionally, we can determine the time t_k required for $V(f)$ to be at least k standard deviations above $V(0)$. First, we can show by our previous methods that $Var(G_\infty(f)) = s^2 f^2$, hence $Sdev(G_\infty) = sf$. Solving $t_k g_\infty$, from which we find t_k.

A1.3: Derivation of the Corwin-Schultz Estimator

Corwin and Schultz (2012) state that the estimator is based on two assumptions. First, they assume that the low price is almost always a seller-initiated trade and that the high price is almost always a buyer-initiated trade. Hence, the price range between the high and low prices reflects the variance and bid-ask spread of the stock. Secondly, they assume that variance and spreads remain constant over consecutive two-day periods. Consequently, the variance of the high-low price range is twice as large for a two-day period than for a single day, while the spread is unaffected. This enables us to estimate bid-ask spread as a function of the high and low prices over both one- and two-day periods. The Corwin and Schultz spread estimator can be expressed as:

$$S = \frac{2(e^\alpha - 1)}{1 + e^\alpha} \qquad (A1.11)$$

Where:

$$\alpha = \frac{\sqrt{2\beta} - \sqrt{\beta}}{3 - 2\sqrt{2}} - \sqrt{\frac{\gamma}{3 - 2\sqrt{2}}} \qquad (A1.12)$$

$$\beta = \left[\ln\left(\frac{H_t^A}{L_t^A}\right)\right]^2 + \left[\ln\left(\frac{H_{t+1}^A}{L_{t+1}^A}\right)\right]^2 \qquad (A1.13)$$

$$\gamma = \left[\ln\left(\frac{MAX[H_t^A; H_{t+1}^A]}{MIN[L_t^A; L_{t+1}^A]}\right)\right]^2 \qquad (A1.14)$$

H_t^A and L_t^A represent the high and low prices respectively, adjusted for overnight returns. According to Corwin and Schultz (2012), we adjust for overnight returns by checking if the close price on day t is outside the range of the high or low prices for day t+1. As a result, the high and low prices adjusted for overnight returns can be computed as:

$$H_{t+1}^A = H_{t+1} + MAX[0; C_t - H_{t+1}] - MAX[0; L_{t+1} - C_t] \qquad (A1.15)$$

$$L_{t+1}^A = L_{t+1} + MAX[0; C_t - H_{t+1}] - MAX[0; L_{t+1} - C_t] \qquad (A1.16)$$

Where C_t is the closing price, and H_t and L_t are the high and low prices.

A2 Portfolio Plots

Figure A2.1: Mean-Variance Plot

The figure plots the Kelly portfolio, the Markowitz portfolio, and the benchmark in a mean-variance space. In addition, the five most frequent stocks in our selection are plotted. The benchmark has marginally lower return than Markowitz, but lower risk, while Kelly has both higher return and lower risk. As the benchmark and Kelly has similar Sharpe ratio, one should in perfect market conditions be able to replicate the higher return of the Kelly portfolio using leverage. This is illustrated by them both being on the capital allocation line.

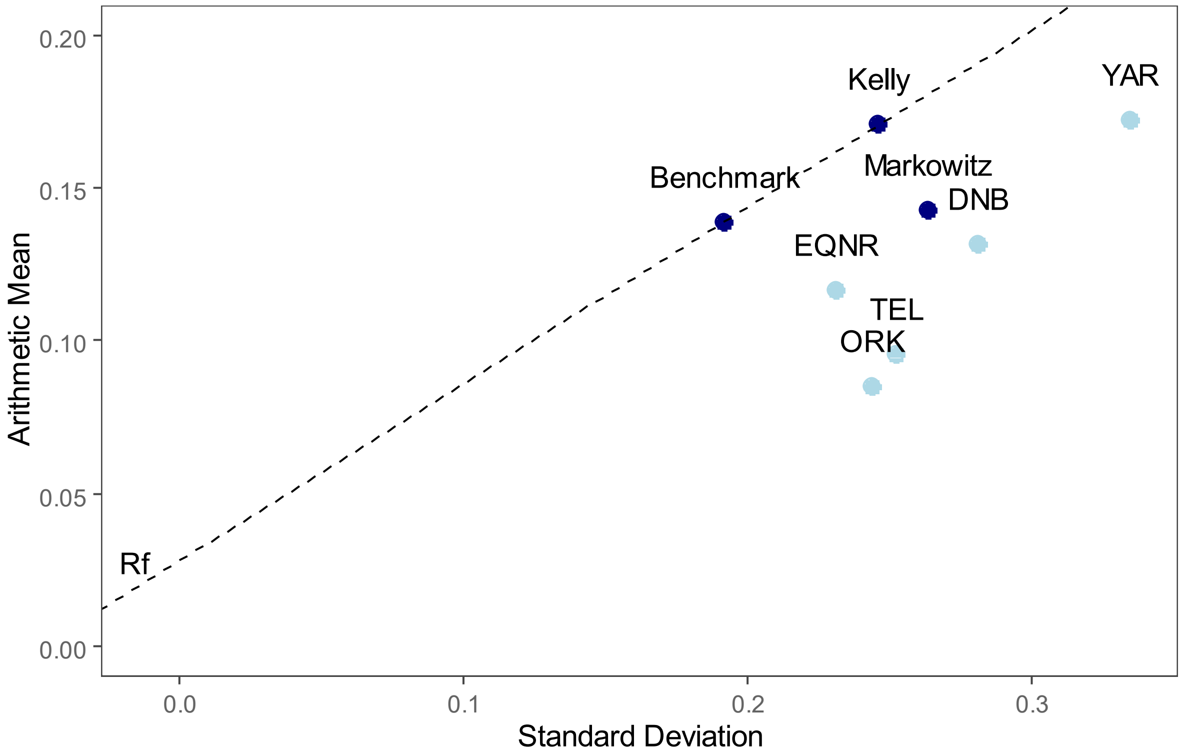


Figure A2.2: NOK-Volume vs Kelly Turnover

The figure plots the total NOK-volume versus the turnover of the Kelly portfolio. The NOK-volume is computed as the sum of the NOK-volumes on the last trading day each month for the ten stocks in our sample that particular year. The turnover is calculated by taking the absolute value of the delta for each rebalancing period.

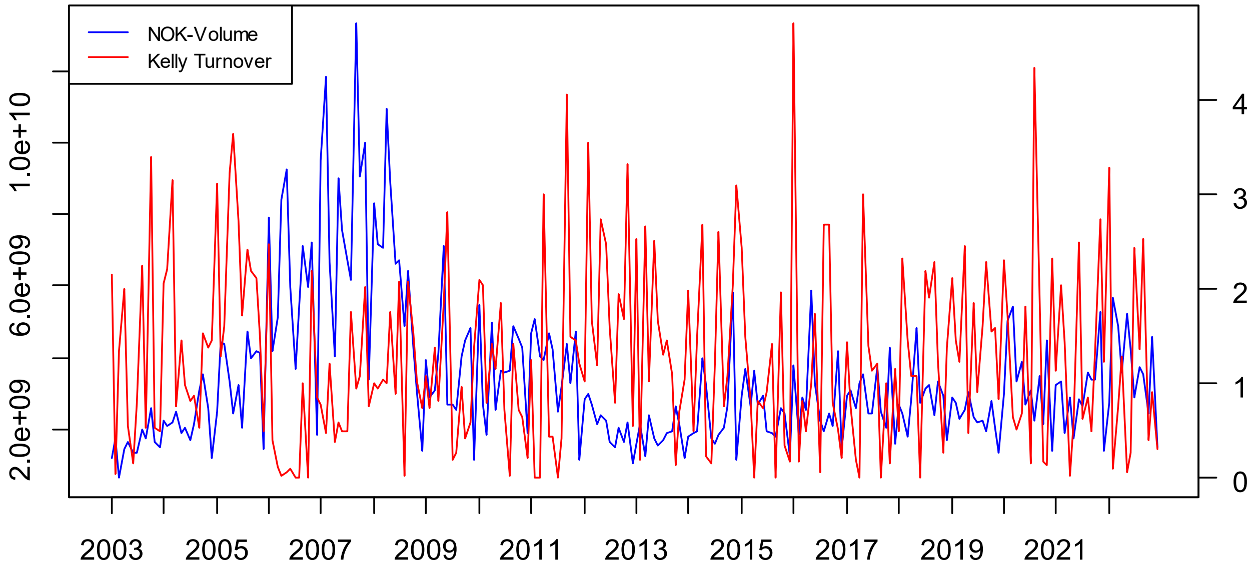
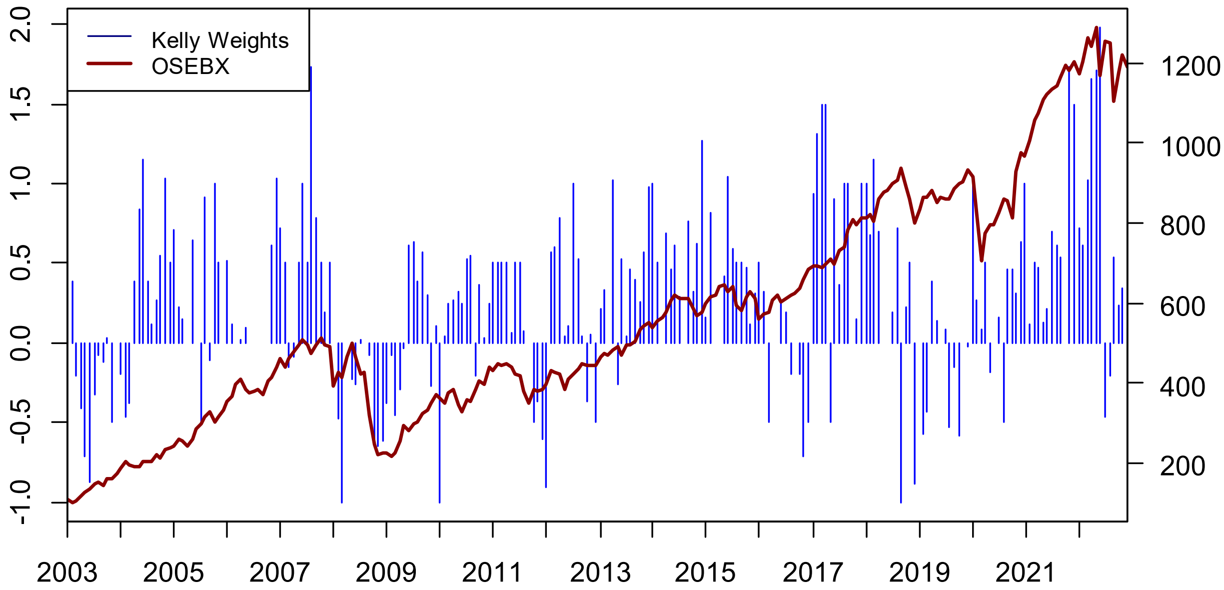


Figure A2.3: Net Kelly Portfolio Weights vs OSEBX

The figure plots the net weights of the Kelly portfolio as a bar plot in reference to the left y-axis, and a line graph of the OSEBX in reference to the right y-axis. The net weights of the Kelly portfolio is calculated by summing up the long and short positions for the portfolio for each month after rebalancing.



A3 Portfolio Statistics

Table A3.1: Unconstrained Kelly Portfolio Weights

The table illustrates the unconstrained portfolio weights of 2003 as suggested by the Kelly criterion.

Dates	NHY	DNB	ORK	STB	EQNR	TAA	TOM	TEL	OPC	GNO
2003-01	5673%	1653%	-8885%	2274%	2625%	2086%	-182%	-3011%	-625%	-11890%
2003-02	5567%	1421%	-10155%	2770%	3752%	2067%	65%	-2946%	-650%	-14096%
2003-03	2583%	1759%	-3579%	316%	-2118%	363%	-329%	665%	-774%	-2751%
2003-04	-13050%	3611%	621%	1593%	-152%	-2226%	-2785%	8803%	-1464%	-2062%
2003-05	-9368%	5562%	2701%	181%	-3247%	-2342%	-2765%	9860%	-1696%	-3180%
2003-06	-5457%	7862%	2884%	-638%	-6356%	-3068%	-1991%	9673%	-1659%	-5782%
2003-07	4672%	4062%	2286%	-1573%	-7531%	-1971%	-840%	5404%	-758%	-4503%
2003-08	853%	-1575%	-889%	579%	-1224%	33%	-600%	1457%	-394%	1298%
2003-09	951%	-1106%	-707%	1277%	-1839%	161%	-1108%	1327%	-859%	796%
2003-10	-4981%	16213%	10637%	7135%	15961%	-1073%	-6359%	-5846%	2189%	-17335%
2003-11	-18232%	69382%	64207%	-4554%	42274%	-3208%	-11763%	-25631%	3624%	-65074%
2003-12	-10593%	31487%	27221%	4158%	26811%	-1268%	-8788%	-12656%	2893%	-30624%

Table A3.2: Constrained Kelly Portfolio Weights

The table illustrates the constrained portfolio weights of 2003 for the Kelly portfolio. The unconstrained weights as suggested by the Kelly criterion are subject to a Kelly fraction of $\boldsymbol{c} = \frac{\mathbf{1}}{\mathbf{50}}$ and a 25% position size constraint.

Dates	NHY	DNB	ORK	STB	EQNR	TAA	TOM	TEL	OPC	GNO
2003-01	25.0%	25.0%	-25.0%	25.0%	25.0%	25.0%	-3.6%	-25.0%	-12.5%	-25.0%
2003-02	25.0%	25.0%	-25.0%	25.0%	25.0%	25.0%	1.3%	-25.0%	-13.0%	-25.0%
2003-03	25.0%	25.0%	-25.0%	6.3%	-25.0%	7.3%	-6.6%	13.3%	-15.5%	-25.0%
2003-04	-25.0%	25.0%	12.4%	25.0%	-3.1%	-25.0%	-25.0%	25.0%	-25.0%	-25.0%

Dates	NHY	DNB	ORK	STB	EQNR	TAA	TOM	TEL	OPC	GNO
2003-05	-25.0%	25.0%	25.0%	3.6%	-25.0%	-25.0%	-25.0%	25.0%	-25.0%	-25.0%
2003-06	-25.0%	25.0%	25.0%	-12.8%	-25.0%	-25.0%	-25.0%	25.0%	-25.0%	-25.0%
2003-07	25.0%	25.0%	25.0%	-25.0%	-25.0%	-25.0%	-16.8%	25.0%	-15.2%	-25.0%
2003-08	17.1%	-25.0%	-17.8%	11.6%	-24.5%	0.7%	-12.0%	25.0%	-7.9%	25.0%
2003-09	19.0%	-22.1%	-14.1%	25.0%	-25.0%	3.2%	-22.2%	25.0%	-17.2%	15.9%
2003-10	-25.0%	25.0%	25.0%	25.0%	25.0%	-21.5%	-25.0%	-25.0%	25.0%	-25.0%
2003-11	-25.0%	25.0%	25.0%	-25.0%	25.0%	-25.0%	-25.0%	-25.0%	25.0%	-25.0%
2003-12	-25.0%	25.0%	25.0%	25.0%	25.0%	-25.0%	-25.0%	-25.0%	25.0%	-25.0%

Table A3.3: Holding Period Returns

The table presents the annual holding period returns for the Benchmark, Kelly, and Markowitz from 2003 to 2023.

Holding Period	Benchmark	Kelly	Markowitz
2003	55.8%	-17.5%	31.1%
2004	38.4%	21.2%	24.0%
2005	40.5%	81.9%	56.2%
2006	32.4%	7.1%	19.4%
2007	11.5%	58.6%	51.8%
2008	-54.1%	14.3%	-55.0%
2009	64.8%	-13.4%	50.4%
2010	18.3%	28.5%	30.1%
2011	-12.5%	16.4%	-4.7%
2012	15.4%	-9.6%	40.2%
2013	23.6%	26.1%	10.8%

Holding Period	**Benchmark**	**Kelly**	**Markowitz**
2014	5.0%	26.3%	43.3%
2015	5.9%	14.9%	7.2%
2016	12.1%	-10.6%	-10.7%
2017	19.1%	37.9%	44.1%
2018	-1.8%	33.0%	-28.5%
2019	16.5%	-10.6%	-1.3%
2020	4.6%	29.2%	21.5%
2021	23.4%	-1.1%	5.3%
2022	-1.0%	15.7%	-21.1%

A4 Regression Results and Tests

Table A4.1: Regression Results Using the Fama French Five-Factor Model

The table presents the regression results for the Kelly portfolio before adjusting for transaction costs. The portfolios' excess returns are regressed on the Fama and French (2015) five-factor model using European pricing factors retrieved from Kenneth French's (2023) website. The regression results are estimated utilizing monthly excess returns from February 2003 through December 2022.

	Dependent variable:
	Kelly FF5F
Alpha	0.013^{***} (0.005)
Market	0.218^{**} (0.107)
SMB	-0.452^{*} (0.263)
HML	-0.527 (0.327)

RMW	-0.109
	(0.472)
CMA	0.567
	(0.428)
Observations	239
R^2	0.035
Adjusted R^2	0.014
Residual Std. Error	0.070
Significance level	$^{*}p<0.01$; $^{**}p<0.05$; $^{***}p<0.01$

Table A4.2: Breusch-Pagan and Durbin-Watson Test

The table presents the results of the Breusch-Pagan (BP) and Durbin-Watson (DW) test on the regressions of the factor models for the Kelly and Markowitz portfolio. We use BP to test for heteroscedasticity, where a BP statistic close to zero indicates no presence of heteroscedasticity. We use DW to test for autocorrelation in the residuals. The DW statistic ranges from 0 to 4, where a value of 2 indicates no autocorrelation. The p-values are above 5% for both BP and DW, hence we cannot reject the null hypotheses on a 5% significance level, indicating that there are no signs of autocorrelation or heteroscedasticity in the regression models.

	Breusch-Pagan test		**Durbin-Watson test**	
	BP statistic	**p-value**	**DW statistic**	**p-value**
Kelly CAPM	0.199	0.656	2.172	0.162
Kelly FF3F	5.356	0.148	2.186	0.154
Kelly C4F	5.419	0.247	2.186	0.152
Markowitz CAPM	2.868	0.090	2.026	0.842
Markowitz FF3F	6.802	0.078	2.013	0.918
Markowitz C4F	8.225	0.084	2.009	0.990

A5 Stock Sample

Table A5.1: Frequency of Firms in Sample

The table highlights the companies that are present in our sample between 2003 and 2023, when OBX is sorted annually based on NOK-volume. The "frequency" column indicates the number of years each firm is present in the sample. A star ("*") denotes that the company was delisted during the sample period.

Ticker	Company Name	Frequency
DNB	DNB Bank ASA	20
EQNR	Equinor ASA	20
NHY	Norsk Hydro ASA	20
TEL	Telenor ASA	20
YAR	Yara International ASA	17
ORK	Orkla ASA	16
MOWI	Mowi ASA	14
PGS	PGS ASA	11
AKA	Akastor ASA	8
STB	Storebrand ASA	8
RECSI	Rec Silicon ASA	7
TAA*	Tandberg AS	6
NAS	Norwegian Air Shuttle ASA	5
DNO	DNO ASA	4
TGS	TGS ASA	4
TOM	Tomra Systems A/S	4
AKERBP	Aker BP ASA	3
NEL	NEL ASA	3
ATEA	Atea ASA	1
FAST	Fast Search and Transfer AS	1

Ticker	Company Name	Frequency
GNO	Gjensidige NOR ASA	1
KAHOT	Kahoot ASA	1
NOD	Nordic Semiconductor	1
OPC	Opticom ASA	1
SALM	Salmar ASA	1
SCATC	Scatec ASA	1
SCHA	Schibsted ASA Ser A	1
TAT	Tandberg Television ASA	1

Table A5.2: Key Statistics for the Ten Most Frequent Firms

The table presents annualized performance measures for the ten most frequent firms in our sample. The performance measures are calculated utilizing monthly figures. The Sharpe ratio (SR) and standard deviation (SD) are annualized by multiplying with $\sqrt{12}$. The arithmetic mean (AM) is annualized by multiplying with 12.

	DNB	EQNR	NHY	TEL	YAR	ORK	MOWI	PGS	AKA	STB
AM	13.2%	11.6%	12.9%	9.6%	17.2%	8.5%	14.7%	15.0%	14.2%	14.3%
SD	28.1%	23.1%	33.2%	25.2%	33.5%	24.4%	63.5%	63.5%	45.0%	37.4%
SR	0.37	0.38	0.31	0.27	0.43	0.23	0.19	0.19	0.25	0.31

Table A5.3: List of Firms in the Kelly and Markowitz Portfolios (2003 through 2012)

The table presents the companies that are present in our sample from 2003 through 2012, when OBX is sorted annually based on NOK-volume. The firms are listed alphabetically.

2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
DNB	ATEA	DNB	DNB	AKA	AKA	AKA	AKA	AKA	AKA
EQNR	DNB	EQNR	DNO	DNB	DNB	DNB	DNB	DNB	DNB

2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
GNO	EQNR	FAST	EQNR	DNO	EQNR	EQNR	EQNR	EQNR	EQNR
NHY	NHY	NHY	NHY	EQNR	MOWI	NHY	NHY	MOWI	MOWI
OPC	ORK	ORK	ORK	MOWI	NHY	ORK	ORK	NHY	NHY
ORK	STB	PGS	PGS	NHY	ORK	PGS	PGS	ORK	ORK
STB	TAT	STB	STB	ORK	PGS	RECSI	RECSI	PGS	PGS
TEL	TEL	TEL	TEL	PGS	RECSI	TEL	TEL	RECSI	RECSI
TOM	TOM	TOM	TAA	TEL	TEL	TAA	TAA	TEL	TEL
TAA	TAA	TAA	YAR	YAR	YAR	YAR	YAR	YAR	YAR

Table A5.4: List of Firms in the Kelly and Markowitz Portfolios (2013 through 2022)

The table presents the companies that are present in our sample from 2013 through 2023, when OBX is sorted annually based on NOK-volume. The firms are listed alphabetically.

2013	2014	2015	2016	2017	2018	2019	2020	2021	2022
AKA	AKA	DNB	DNB	DNB	DNB	AKERBP	AKERBP	AKERBP	DNB
DNB	DNB	DNO	EQNR	EQNR	DNO	DNB	DNB	DNB	EQNR
EQNR	EQNR	EQNR	MOWI	MOWI	EQNR	EQNR	EQNR	EQNR	KAHOT
MOWI	MOWI	MOWI	NAS	NAS	MOWI	MOWI	MOWI	MOWI	MOWI
NHY	NAS	NHY	NHY	NHY	NAS	NAS	NEL	NEL	NEL
PGS	NHY	PGS	ORK	ORK	NHY	NHY	NHY	NHY	NHY
STB	PGS	RECSI	SCHA	STB	ORK	ORK	ORK	ORK	NOD
TEL	RECSI	TEL	TEL	TEL	STB	STB	SALM	TEL	SCATC
TGS	TEL	TGS	TGS	TGS	TEL	TEL	TEL	TOM	TEL
YAR	YAR	YAR	YAR	YAR	YAR	YAR	YAR	YAR	YAR

A6 VBA Code

A6.1: VBA Code for the Kelly Portfolio Weights

```
Function GeneratePortfolioWeights(dataRange As Range) As Variant
    Dim returns() As Variant
    Dim numReturns As Integer
    Dim numAssets As Integer
    Dim covMatrix() As Variant
    Dim invCovMatrix() As Variant
    Dim avgReturns() As Variant
    Dim weights() As Variant
    Dim i As Integer, j As Integer

    ' Get the returns data from the input range
    returns = dataRange.Value
    numReturns = UBound(returns, 1)
    numAssets = UBound(returns, 2)

    ' Calculate the covariance matrix
    ReDim covMatrix(1 To numAssets, 1 To numAssets)
    For i = 1 To numAssets
        For j = 1 To numAssets
            covMatrix(i, j) =
WorksheetFunction.Covariance_P(Application.Transpose(Application.Index(returns, 0, i)),
Application.Transpose(Application.Index(returns, 0, j)))
        Next j
    Next i

    ' Calculate the inverse of the covariance matrix
    invCovMatrix = WorksheetFunction.MInverse(covMatrix)

    ' Calculate the average returns vector
    ReDim avgReturns(1 To numAssets)
    For i = 1 To numAssets
        avgReturns(i) = WorksheetFunction.Average(Application.Index(returns, 0, i))
    Next i

    ' Calculate the portfolio weights using matrix multiplication
    weights = WorksheetFunction.MMult(invCovMatrix, Application.Transpose(avgReturns))

    ' Transpose the weights vector before returning it
    GeneratePortfolioWeights = Application.Transpose(weights)
End Function
```