Unraveling Coordination Problems

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Abstract

The interplay between strategic beliefs and policy complicates policy design in coordination games. To untangle this relationship, we study policy design in the context of equilibrium selection. We characterize the unique subsidy scheme that selects a targeted strategy vector as the unique equilibrium of a coordination game. These subsidies are continuous in model parameters and do not make the targeted strategies strictly dominant. While discrimination is optimal in games with multiple equilibria (Segal, 2003; Winter, 2004), we construct a non-discriminatory subsidy scheme the cost of which converges to that of a least-cost discriminatory policy when agents are symmetric.

JEL Codes: D81, D82, D83, D86, H20.

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1 Introduction

This paper studies policy design in coordination games. Many decision problems are fundamentally coordination games, and the design of policies to solve these problems is a major economic issue. How should government stimulate the use of technologies that foster economic development (Bandiera and Rasul, 2006; Cai et al., 2015; Beaman et al., 2021)? How should principal set rewards to incentivize work in teams (Holmstrom, 1982; Winter, 2004; Halac et al., 2021)? How should firms raise capital from multiple investors (Sakovics and Steiner, 2012; Halac et al., 2020)? How should a rebel leader induce citizens to participate in a revolution (Edmond, 2013; Morris and Shadmehr, 2023)? How should incentives be set to shift social norms (Brekke et al., 2003; Ferraro et al., 2011; Lane et al., 2023)?

Policy design is rarely simple. To begin with, it is often impossible to assess from the onset what the payoff to a particular course of action will be. When individuals or firms choose whether to adopt a new technology, for example, it is hard to tell precisely how beneficial it will eventually be. Confronted with uncertainty about payoffs, a planner's goal need not necessarily be to make players adopt the technology no matter what. Rather, she might seek to design a policy that induces adoption whenever the (potential) benefits are sufficiently high. Such a change of perspective becomes especially pertinent when the players possess relevant private information superior to the planner's, a possibility easily imagined – industry likely has a clearer idea about a new technology's true potential than, say, Congress. In those cases, the planner might want players to act according to their own knowledge. How should the planner design a policy that induces players to use their private information in a way the planner wants them to?

Another complicating factor, this one specific to coordination games, is the complex interplay between policy and players' *strategic beliefs*, that is, their beliefs about the choices other players will make. In a coordination game, players have an incentive to act the way their peers do. If a new technology exhibits network effects, for example, then the payoff to adopting is increasing in overall adoption. Whether a policy creates the right incentives then depends upon a player's strategic beliefs which, in turn, are themselves affected by the policy. A theory of policy design should unravel this two-way interaction between policies and strategic beliefs, ideally from economic first principles. This requires an understanding of how strategic beliefs are formed in the first place.

This paper develops a theory of policy design in coordination games that deals with these two challenges. To illustrate our problem, consider again the example of technology adoption. Players must choose whether to adopt a network technology. The payoff to adopting the technology is increasing both in the number of other players that adopt it (e.g. network effects) as well as a fundamental state, x, which is hidden. One could think about the state x as the technology's efficiency-enhancing potential. Not knowing the true state x, a planner publicly announces subsidies on technology adoption. Players then receive more precise information about the technology's quality in the form of private but noisy signals, and choose whether to adopt. The problem of the planner is to find a subsidy scheme that induces players to adopt the technology whenever the state exceeds some critical threshold $\tilde{x}$ and not adopt otherwise. A subsidy scheme is optimal if it solves the planner's problem and makes coordination on such strategies the unique Bayesian Nash equilibrium of the game.

As a preliminary result, we show that social welfare – which can be any increasing function of players' payoffs – is maximal if and only if players adopt the technology whenever the state x is sufficiently high. Although we present a mostly positive analysis, solving the planner's problem for any critical state $\tilde{x}$, this result helps motivate her problem: provided she chooses the critical state $\tilde{x}$ well, an optimal subsidy scheme induces players to coordinate on strategies that maximize expected welfare. The result also has a practical implication: full adoption is not necessarily the efficient outcome of the game. Because efficiency depends upon a state the planner does not observe, she may end up incentivizing adoption when, from the point of view of social welfare, players should not adopt. In considering uncertainty about the ex post efficient outcome of the game, our analysis deviates from the typical approach in the literature on policy design in coordination games.[1]

The main result of this paper shows that there exists a unique subsidy scheme that solves the planner's problem. It also characterizes the optimal scheme. Subsidies pursuant to the scheme are (i) symmetric for identical players; (ii) continuous functions of model parameters; and (iii) do not make the targeted strategies strictly dominant for any of the players. These findings run counter to well-known results in the literature (*cf.* Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020).

A distinctive feature of our analysis is that we connect the problem of policy design to that of equilibrium selection. Coordination games frequently have more than one Nash equilibrium. In games with multiple equilibria, "the rational decision maker [...] is uncertain which equilibrium strategy other decision makers will use (Van Huyck et al., 1990, 1991)." This complicates policy design as, lacking a sharp prediction on players' strategic beliefs, the planner is assured of her policy's effectiveness only if it works against *all* strategic beliefs. We therefore argue that an understanding of strategic belief formation is critical when designing policy in coordination games. By selecting one out of multiple equilibria, equilibrium selection eliminates any uncertainties about the strategies other players will use. A sharp specification of strategic beliefs in turn permits a precise delineation of the effect policy has on equilibrium strategies, which we exploit to characterize the optimal subsidy scheme. Given our focus on policy design under uncertainty, we address equilibrium selection using a global games approach (Carlsson and Van Damme, 1993).[2]

Because equilibrium selection allows for a sharp specification of strategic beliefs, our analysis is the first to reveal an "unraveling effect" of subsidies in coordination games. Consider again the example of technology adoption. We show that, for a given subsidy scheme, there is a unique vector of player-specific threshold states such that in equilibrium a

[1]This type of uncertainty about efficiency, though not at the forefront of most analyses, seems historically relevant. For example, Cowan (1990) describes the history of nuclear power generation. Nowadays, light water nuclear reactors are the dominant technology. This situation can be traced back to Captain Hyman Rickover of the U.S. Navy, whose preference for light water drove the early development of this technology led to its eventual domination of the field. There now is compelling evidence that two competing technologies, both of which were known to Captain Rickover, are economically and technologically superior to light water nuclear reactors. Similarly, Cowan and Gunby (1996) discuss competing pest control strategies in agriculture. They show that today's heavy reliance on pesticides – a consequence of targeted policies in the 1930s and 1940s – is inefficient. Evidence indicates that a competing technology that already existed at the time, Integrated Pest Management, is technologically and economically superior to pesticides. This wasn't known, however, when policymakers first had to choose which type of pest control to pursue.

[2]Global games are incomplete information games in which players do not observe the true game they play but receive private and noisy signals of it.

player adopts the technology if and only if his signal exceeds his threshold state. Furthermore, a player's threshold state is continuously decreasing in (i) his own subsidy and (ii) the threshold states of other players.[3] Combining (i) and (ii) yields the unraveling effect: a (raise in the) subsidy to player i makes him more likely to adopt the technology. This makes adoption more attractive for player j, whose incentive to adopt therefore also increases. Knowing that j is more eager to adopt, player i's adoption incentive increases even further, and so on. These effects keep on compounding, ever reinforcing one another, demonstrating how even small subsidies can go a long way toward unraveling coordination problems.

Our main analysis builds upon the canonical model of contracting with externalities in which players' actions are contractible and the externalities they impose upon one another are deterministic (Segal, 1999, 2003; Bernstein and Winter, 2012). Typical examples would be group participation problems or network technology adoption. To generalize our analysis to the broader literature, we also study several extensions.

In one extension, we study a global game of regime change in which individual actions are contractible but externalities are binary and (partly) stochastic. The classic example is a joint investment problem. Players choose whether to invest in a project. Upon investing, a player incurs a certain cost; if the project succeeds, investing players earn a return that is increasing in the project's unobserved "quality". The project succeeds if and only if total investments exceed a stochastic (and unobserved) critical threshold. A planner offers investment subsidies to induce investment whenever the project's quality is sufficiently high. We find that an optimal subsidy scheme subsidizes all players and makes investment a best response to a player's belief that the project succeeds with probability 1/2. Also, investment subsidies are continuous in model parameters (*cf.* Sakovics and Steiner, 2012).

In another extension, cast in a moral-hazard-in-teams setup, we study games in which externalities are binary and stochastic while individual actions are *not* contractible. Agents can work or shirk toward a common project; working is costly and agents' work provision is their private knowledge. If the project succeeds, players earn a reward that consists of a fixed "bonus" and a share of profits. Profits are uncertain, but agents receive noisy information about it before choosing to work. Project success is stochastic; the probability of success is increasing in total work provision. A principal wants to design bonuses that induce work whenever (projected) profits are sufficiently high. We show that there is a unique bonus scheme that solves the principal's problem. In this scheme, identical agents receive identical rewards and working is a best response to uniform beliefs about work provision by the other agents (*cf.* Winter, 2004; Halac et al., 2021).

Lastly, we can relate our analysis to celebrated discrimination results on policy in coordination games. Assuming complete information about payoffs, Segal (2003) and Winter (2004) establish that a least-cost subsidy scheme is fundamentally discriminating: it rewards even identical players asymmetrically. Using the language of technology adoption, these authors seek to identify the least-cost subsidy scheme that makes adoption by all players the unique Nash equilibrium of the game. Observe that the assumption of certainty about payoffs would be equivalent, in the framing of our model, to assuming it is common knowledge

[3]The latter derives from players' coordination incentives. Without going into the details of the model, the argument goes like this: if a player's threshold state is lower, he is more likely to adopt. This makes adoption more attractive for other players, leading to a decrease in their threshold states.

that the state is $\bar{x}$. With this in mind, we first show that there exists an infinite number of non-discriminating subsidy schemes that induce coordination on adoption as the unique equilibrium *outcome* of the global game should nature draw state $\bar{x}$. Furthermore, equilibrium spending on subsidies in the least expensive such scheme is the same as that of the cheapest discriminatory policy when players are symmetric. This shows that the least-cost property of discriminatory policies disappears when the problem of policy design is connected to that of equilibrium selection; optimality of discrimination hinges critically on the multiplicity of equilibria in coordination games.

Related literature.—A closely related paper is Sakovics and Steiner (2012), who study policy design in a global game of regime change. Sakovics and Steiner (2012) find that an optimal policy fully subsidizes a subset of players, targeting those who matter most for project success and/or have least incentive to invest. The difference between their results and ours is a consequence of the distinct information structures considered. In Sakovics and Steiner (2012), payoffs conditional on project success are known to the planner when she offers her subsidies while players receive noisy signals about the critical threshold for project success. The same distinction also set this paper apart from the broader literature on policy design in global games of regime change, most notable among which are Goldstein and Pauzner (2005), Angeletos et al. (2006, 2007), Edmond (2013), and Basak and Zhou (2020).

Similarly related is Halac et al. (2020), who study a game of regime change that is not a global game. A firm seeks to raise capital from multiple investors to fund a project; the firm offers payments contingent on project success. Halac et al. (2020) identify conditions under which larger investors receive higher per-dollar returns on investment in an optimal policy.

In the literature on moral hazard in teams, two closely related papers are Winter (2004) and Halac et al. (2021). In a complete information setup, Winter (2004) shows that an optimal reward scheme is inherently discriminatory; no two agents are rewarded equally even when agents are identical. This seminal result differs sharply from our finding that optimal subsidies (or rewards) are symmetric for identical agents; the difference derives from our focus on equilibrium selection. Halac et al. (2021) extend the model in Winter (2004) by allowing contract offers to be private. Halac et al. (2021) demonstrate that, with private contract offers, symmetric agents are offered identical rewards; this property of an optimal reward scheme is similar to our optimal subsidies. Interestingly, the results in Halac et al. (2021) depend upon contract offers being private; in contrast, offers have to be common knowledge for our results.

In the literature on contracting with externalities, directly related are Segal (1999, 2003) and Bernstein and Winter (2012). Our focus on policy design under fundamental uncertainty and in connection to equilibrium selection sets our approach apart from theirs. A notable contribution to this literature is our result that least-cost subsidies are not necessarily discriminatory: we construct a symmetric subsidy scheme that (i) gives the same guarantees on outcomes (in relevant states) as the least-cost policies proposed by Segal (1999, 2003) and Bernstein and Winter (2012) while (ii) total equilibrium costs are the same. This suggests that the least-cost property of discrimination is an artifact of unresolved equilibrium multiplicity in the coordination games considered.

2 The Game

2.1 Complete Information

Consider a normal form game played by a finite set $\mathscr{N} = \{1, 2, ..., N\}$ of players, indexed i. Each player $i \in \mathscr{N}$ chooses an action $a_i \in A_i = \{0, 1\}$. A planner publicly announces a *subsidy scheme* $s = (s_i)$, $i \in \mathscr{N}$, where s_i is the subsidy she offers player i for playing 1; we will come to the planner's choice of s shortly. The payoff $\pi_i(a \mid x, s)$ to player i depends upon the action vector $a = (a_i) \in A_1 \times A_2 \cdot \ldots \cdot A_N$ played as well as a common state x and the subsidy s_i as follows:

$$\pi_i(a \mid x, s) = \begin{cases} x + w_i\left(\sum_{j \neq i} a_j\right) + s_i & \text{if} \quad a_i = 1 \text{ in } a, \\ c_i & \text{if} \quad a_i = 0 \text{ in } a. \end{cases} \tag{1}$$

In (1), the common state x represents an intrinsic benefit to playing 1 whereas c_i is player i's (opportunity) cost of playing 1. The externalities other players impose upon player i are given by $w_i\left(\sum_{j \neq i} a_j\right)$. We are interested in coordination problems and assume that $w_i(n)$ is increasing in n. We say that players $i, j \in \mathscr{N}$ are *symmetric* if $c_i = c_j$ and $w_i(n) = w_j(n)$ for all $n = 0, 1, ..., N-1$. Extensions and generalizations of the game described here are given in Section 5.

The foregoing describes a game of complete information $\Gamma(x, s)$. In $\Gamma(x, s)$, a player's *incentive* to choose 1 is defined as his gain from playing 1, rather than 0:

$$u_i(a_{-i} \mid x, s) = \pi_i(1, a_{-i} \mid x, s) - \pi_i(0, a_{-i} \mid x, s) = x + w_i\left(\sum_{j \neq i} a_j\right) + s_i - c_i. \tag{2}$$

All else equal, a player's incentive u_i to play 1 is strictly increasing in x. Denote $x_i^0 := c_i - w_i(0)$ and $x_i^N := c_i - w_i(N-1)$. One has $u_i(\overline{a}_{-i} \mid x_i^0) = u_i(\underline{a}_{-i} \mid x_i^N) = 0$. In other words, to each player i playing 1 is strictly dominant for all $x > \overline{x}_i^0 - s_i$; playing 0 is strictly dominant for $x < \underline{x}_i^N - s_i$. Define $x^N := \max\{x_i^N \mid i \in \mathscr{N}\}$, $x^0 := \min\{x_i^0 \mid i \in \mathscr{N}\}$, $\underline{x} = \min\{x_i^0 \mid i \in \mathscr{N}\}$, and $\overline{x} = \max\{x_i^N \mid i \in \mathscr{N}\}$. We assume that $[\underline{x}, \overline{x}]$ is nonempty.

Let $\bar{\pi}_i(a \mid x) = \pi_i(a \mid x, s) - a_i \cdot s_i$ denote a player's payoff in (a, x) net of subsidies. Social welfare is given by

$$W(\bar{\pi}_1(a \mid x), \bar{\pi}_2(a \mid x), ..., \bar{\pi}_N(a \mid x)), \tag{3}$$

where W is increasing and symmetric in its arguments. Our first result shows that there exists a unique vector of (player-specific) thresholds states x_i^* such that social welfare is maximized if and only if player i plays 1 when the state x exceeds x_i^*, and 0 otherwise.

Proposition 1. *There exists a unique $x^* = (x_i^*) \in \mathbb{R}^N$ such that if $(a_i^*(x)) = \arg\max_{a \in A} W(\cdot)$, then $a_i^*(x) = 1$ iff $x \geq x_i^*$. Furthermore, if players $i, j \in \mathscr{N}$ are symmetric, then $x_i^* = x_j^*$.*

2.2 Fundamental Uncertainty

To reflect the many uncertainties that exist in the real world, we assume that the state x is hidden. Instead, it is common knowledge among the players that x is drawn from a continuous (improper) prior density $g : \mathbb{R} \to \mathbb{R}$ and that each player i receives a private noisy signal x_i^ε of x, given by

$$x_i^\varepsilon = x + \varepsilon \cdot \eta_i. \tag{4}$$

It is not necessary that x can take values on the entire line; the analysis and results also apply to games in which g has positive support on a closed internal $\mathcal{X} = [\underline{X}, \overline{X}] \subset \mathbb{R}$.[4] Some authors refer to x_i^ε as the player's *type*. The random variable η_i is a noise term that is distributed i.i.d. on $[-1/2, 1/2]$ according to a continuously differentiable distribution F, and $\varepsilon > 0$ is a scaling factor.[5] This information structure describes a global game $\Gamma^\varepsilon(s)$, see Carlsson and Van Damme (1993). The game $\Gamma^\varepsilon(s)$ is common knowledge among the players.

The timing of $\Gamma^\varepsilon(s)$ is as follow. First, the planner publicly commits to her subsidies s. Second, nature draws a state x. Third, each player i receives his private signal x_i^ε of x. Fourth, all players simultaneously choose their actions. Lastly, payoffs are realized according to the true x and the actions chosen by all players. We note that players play once and then the game is over; see Angeletos et al. (2007) and Chassang (2010) for analyses of dynamic global games.

2.3 Concepts & Notation

Posterior densities. Let $x^\varepsilon = (x_i^\varepsilon)$ denote the vector of signals received by all players, and let x_{-i}^ε denote the vector of signals received by all players but i, i.e. $x_{-i}^\varepsilon = (x_j^\varepsilon)_{j \neq i}$. Note that player i observes x_i^ε but neither x nor x_{-i}^ε. We write $F_i^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon)$ for player i's posterior distribution on (x, x_{-i}^ε) conditional on his signal x_i^ε.

Strategies. A strategy p_i for player i in $\mathbb{R}$ is a function that assigns to any $x_i^\varepsilon \in [\underline{X} - \varepsilon, \overline{X} + \varepsilon]$ a probability $p_i(x_i^\varepsilon) \geq 0$ with which the player chooses action $a_i = 1$ when they observe x_i^ε. Write $p = (p_1, p_2, ..., p_N)$ for a strategy vector for all player, and $p_{-i} = (p_j)_{j \neq i}$ for the vector of strategies for all players but i. A strategy vector p is *symmetric* if for every $i, j \in \mathcal{N}$ and every signal x^ε one has $p_i(x^\varepsilon) = p_j(x^\varepsilon)$. Conditional on the strategy vector p_{-i} and a private signal x_i^ε, the expected incentive to play 1 for player i is given by:

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) := \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x) \, \mathrm{d}F_i^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon).$$

When no confusion can arise, we refer to the expected incentive $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon)$ simply as a player's incentive.

Increasing strategies. For $X \in \mathbb{R}$, let p_i^X denote the particular strategy such that $p_i^X(x_i^\varepsilon) = 0$ for all $x_i^\varepsilon < X$ and $p_i^X(x_i^\varepsilon) = 1$ for all $x_i^\varepsilon \geq X$. The strategy p_i^X is called an *increasing strategy with switching point* X. Let $p^X = (p_1^X, p_2^X, ..., p_N^X)$ denote the strategy vector of increasing strategies with switching point X, and $p_{-i}^X = (p_j^X)_{j \neq i}$. Generally, for a vector of real numbers $y = (y_i)$ let $p^y = (p_i^{y_i})$ be a (possibly asymmetric) increasing strategy vector, and $p_{-i}^y = (p_j^{y_j})_{j \neq i}$.

[4]If g has finite domain $\mathcal{X}$, we must in addition assume that $[\underline{x} - s_i - \varepsilon, \overline{x} + s_i + \varepsilon]$ for all $i \in \mathcal{N}$. Observe that this interval depends upon s; as our main results are concerned primarily with characterizing schemes s that satisfy certain properties, it smooths the exposition to assume g has domain $\mathbb{R}$.

[5]The assumption that the support of η_i is $[-1/2, 1/2]$ is without loss. If η_i were systematically biased, rational players would simply take that into account when forming their posteriors. Moreover, we could also allows the noise distribution to have support on the entire real line without great technical complications.

Strict dominance. The action $a_i = 1$ is strictly dominant at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i}. Similarly, the action $a_i = 0$ is strictly dominant (in the global game G^ε) at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i}. When $a_i = \alpha$ is strictly dominant, the action $a_i = 1 - \alpha$ is said to be strictly dominated.

Conditional dominance. Let L and R be real numbers. The action $a_i = 1$ is said to be dominant at x_i^ε conditional on R if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > R$, all $j \neq i$. Similarly, the action $a_i = 0$ is dominant at x_i^ε conditional on L if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > L$, all $j \neq i$. Note that $a_i = 1$ is strictly dominant at x_i^ε conditional on R if and only if $u_i^\varepsilon(p_{-i}^R \mid x_i^\varepsilon) > 0$. Similarly, if $a_i = 0$ is strictly dominant at x_i^ε conditional on L then it must hold that $u_i^\varepsilon(p_{-i}^L \mid x_i^\varepsilon) < 0$.

Iterated elimination of strictly dominated strategies. The solution concept in this paper is iterated elimination of strictly dominated strategies (IESDS). Eliminate all pure strategies that are strictly dominated, as rational players may be assumed never to pursue such strategies. Next, eliminate a player's pure strategies that are strictly dominated if all other players are known to play only strategies that survived the prior round of elimination; and so on. The set of strategies that survive infinite rounds of elimination are said to survive IESDS.

3 Optimal Subsidies

3.1 Unique Implementation

By Proposition 1, there exists a unique vector of thresholds x_i^* such that it is welfare-maximizing for player i to play 1 whenever $x > x_i^*$ and 0 otherwise. It thus makes sense to consider implementation problems in which the planner seeks to induce coordination on increasing strategies. In particular, if each player i plays the increasing strategy $p_i^{x_i^*}$, then as $\varepsilon \to 0$ ex post social welfare is maximized with probability 1. We henceforth restrict attention to such problems. Furthermore, to reduce notation we will throughout the main analysis assume that the planner is after subsidies that induce players to play *the same* increasing strategy; in Section 5.4, we relax that restriction.

Let $\tilde{x} \in \mathbb{R}$ be a *critical state*. The planner's problem is to find the subsidy scheme $\tilde{s}$ such that $p^{\tilde{x}}$ is the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(\tilde{s})$. We say that $\tilde{s}$ *implements* $p^{\tilde{x}}$. The focus on unique equilibrium implementation is in keeping with the broader literature on policy design in coordination games (*cf.* Segal, 1999, 2003; Segal and Whinston, 2000; Sakovics and Steiner, 2012; Bernstein and Winter, 2012; Halac et al., 2020, 2021, 2022).

The planner faces two constraints. First, she cannot condition her policy on the realization of x or players' signals thereof; this assumption is customary in the literature on policy design in global games (*cf.* Sakovics and Steiner, 2012; Leister et al., 2022). Possible interpretations are that the policy intervention takes place prior to the realization of any private information, or that players have an informational advantage (e.g. expertise) relative to the planner. Note that, with the exception of the state x, the planner knows all parameters of the game.[6]

Second, the planner cannot coordinate players on her preferred equilibrium in a multiple equilibria setting but has to rely on simple subsidies (or taxes) to create the appropriate

[6]Interesting work by Carroll (2015) and Dai and Toikka (2022) explores contract-theoretic problems in which the planner designing a policy only knows a subset of the actions available to each player.

incentives. The focus on simple instruments also means that policies cannot condition directly upon other players' actions. These are standard assumptions in the literature (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020).

Fix a critical state $\tilde{x} \in \mathcal{X}$. Given $\tilde{x}$, let $s^*(\tilde{x}) = (s_i^*(\tilde{x}))$ denote the subsidy scheme such that each $s_i^*(\tilde{x}) \in s^*(\tilde{x})$ is given by

$$s_i^*(\tilde{x}) = c_i - \tilde{x} - \sum_{n=0}^{N-1} \frac{w_i(n)}{N}. \tag{$*$}$$

Let us write $\mathcal{B}_r(y)$ for the open ball with radius r centered at y. Our main result is Theorem 1.

Theorem 1. *Let $\tilde{x} \in \mathbb{R}$. The following holds:*

(i) For all ε sufficiently small, there exists a unique $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$;

(ii) For all $r > 0$, there exists $\varepsilon(r)$ such that $\tilde{s}$ is contained in $\mathcal{B}_r(s^*(\tilde{x}))$ for all $\varepsilon \leq \varepsilon(r)$.

The optimal subsidy scheme $\tilde{s}$ admits a number of notable properties, some of which are best understood with the analysis in mind. We therefore defer a discussion of the properties of $\tilde{s}$ to Section 4.4.

We observe that Theorem 1 holds for all continuous densities f and g. Thus, the informational requirements imposed upon the planner are slim. Moreover, the condition that ε be sufficiently small is necessary to permit an analysis of $\Gamma^\varepsilon(s)$ "as if" the common prior g were uniform. The following corollary to Theorem 1 is immediate from our proof.

Corollary 1. *If the common prior g is uniform and the noise distribution f is symmetric, then Theorem 1 holds for all $\varepsilon > 0$.*

Uniform common priors are often assumed in the applied literature on global games (*cf.* Morris and Shin, 1998; Angeletos et al., 2006, 2007; Sakovics and Steiner, 2012). In Appendix A we show why $\Gamma^\varepsilon(s)$ behaves "as if" g were uniform when ε is small.

The analysis will reveal that Theorem 1 remains valid under a slightly more general definition of implementation. We show that $\tilde{s}$ is the unique subsidy scheme such that $p^{\tilde{x}}$ is the unique strategy vector that survives IESDS in $\Gamma^\varepsilon(\tilde{s})$. Implementation as a unique strategy vector that survives IESDS is more general than implementation as a unique Bayesian Nash equilibrium because the former implies the latter but the reverse implication is not necessarily true. In this sense, as in Sandholm (2002, 2005), we need not impose that players play an equilibrium of the game but could depart from more primitive assumptions on players' strategic sophistication by requiring that none play a strategy that is iteratively dominated. Equilibrium play would then be obtained as a result, rather than an assumption, of the analysis.

Lastly, observe that Theorem 1 is a positive result: given the planner's choice of $\tilde{x}$, Theorem 1 characterizes the unique subsidy scheme that implements $p^{\tilde{x}}$. Though Proposition 1 could motivate a focus on implementing increasing strategy equilibria, the planner in this paper is not bound to choose those increasing strategies optimally. All we do is show how, conditional on her choice of $\tilde{x}$, the planner can implement $p^{\tilde{x}}$.

4 Analysis

4.1 Monotonicities

Suppose that all of player i's opponents are known to play increasing strategies, say $p^y_{-i} = (p^y_j)_{j \neq i}$. Then his incentive u^ε_i to play 1 satisfies a two intuitive monotonicity properties.

Lemma 1. *Given is a vector of real numbers $y = (y_i)$ and the associated increasing strategy vector $p^y = (p^{y_i}_i)$. Then,*

(i) $u^\varepsilon_i(p^y_{-i} \mid x^\varepsilon_i)$ is monotone increasing in x^ε_i;

(ii) $u^\varepsilon_i(p^y_{-i} \mid x^\varepsilon_i)$ is monotone decreasing in y_j, all $j \in \mathscr{N} \setminus \{i\}$.

Part (i) of Lemma 1 says that a player's incentive to play 1 is increasing in his type x^ε_i when his opponents play increasing strategies. There are two sides to this. First, taking as given the vector of actions a_{-i}, a player's expected payoff to playing 1 is linearly increasing in x^ε_i; hence, his expected incentive is increasing in his signal x^ε_i. Second, as x^ε_i increases player i's posterior distribution on the hidden state x and, therefore, the signals of his opponents shifts to the right. If his opponents play increasing strategies, this also shifts his distribution of the aggregate action to the right which, because externalities are increasing in the aggregate action, further raises his incentive to play 1. Note that monotonicity of $u^\varepsilon_i(p^y_{-i} \mid x^\varepsilon_i)$ in x^ε_i depends upon p^y_{-i} being increasing; for generic p_{-i}, $u^\varepsilon_i(p_{-i} \mid x^\varepsilon_i)$ can be locally decreasing in x^ε_i.

Part (ii) of Lemma 1 says that the incentive to play 1 of a player i whose opponents play increasing strategies is decreasing in the switching point of each of these increasing strategies. For given signal x^ε_i, the probability player i attaches to the event that his opponent j receives a signal $x^\varepsilon_j > y_j$ and thus, in $p^{y_j}_j$, plays 1 is decreasing in y_j. Therefore player i's incentive to play 1 is decreasing in the switching y_j.

The analysis relies repeatedly upon Lemma 1 for much of the heavy lifting. While a focus on increasing strategies seems natural in $\Gamma^\varepsilon(s)$, the results in Lemma 1 are of true practical use only once the focus on increasing strategies has been properly defended. The next section provides such a justification; Lemma 2 pushes it to its ultimate conclusion.

4.2 Subsidies, Strategies, Selection

Recall that x^N_i and x^0_i demarcate strict dominance regions for player i: when $x < x^N_i$ [$x > x^0_i$], playing 0 [playing 1] is strictly dominant for player i in $\Gamma(x)$. A subsidy s_i to player i shifts these boundaries to $x^N_i - s_i$ and $x^0_i - s_i$, respectively. In the game of incomplete information Γ^ε, the boundaries for strict dominance in terms of a player's signals instead are $x^N_i - s_i - \varepsilon/2$ and $x^0_i - s_i + \varepsilon/2$, respectively. That is, for all $x^\varepsilon_i > x^0_i - s_i + \varepsilon/2$ player i knows that any true state x consistent with his signal satisfies $x > x^0_i - s_i$, in which case playing 1 is strictly dominant. To make the following arguments work, we must assume that $\overline{X} \geq \overline{x} - s_i + \varepsilon/2$ and $\underline{X} \leq \underline{x} - s_i - \varepsilon/2$ for all $i \in \mathscr{N}$, imposing a joint restriction on permissible values of $(\underline{X}, \overline{X}, s)$ given ε. This assumption is henceforth maintained.

Per the foregoing argument, given the assumption that $\overline{X} \geq \overline{x} - s_i + \varepsilon/2$, we know that $u_i^\varepsilon(p_{-i} \mid \overline{X}, s_i) > 0$ for all p_{-i}. In particular, therefore, one has

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid \overline{X}, s_i) > 0.$$

Let r_i^1 be the solution to

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid r_i^1, s_i) = 0.$$

To any player i, the action $a_i = 1$ is strictly dominant at all $x_i^\varepsilon > r_i^1$ conditional on $\overline{X}$; denote $r^1 := (r_i^1)$. It is clear that r_i^1 depends upon the subsidy s_i, but for brevity we leave this dependence out of the notation for now. From Lemma 1 follows that $r_i^1 < \overline{X}$ for all i.

Player i knows that no player j will pursue a strategy $p_j < p_j^{r_j^1}$ since such a strategy is iteratively strictly dominated. Now define $r^2 = (r_i^2)$ as the signal that solves

$$u_i^\varepsilon(p_{-i}^{r^1} \mid r_i^2, s_i) = 0,$$

for all i. Because $p_i^{\overline{X}}$ is strictly dominated for every i, the any strategy $p_i < p_i^{r_i^1}$ is iteratively strictly dominated for all i, which in turn implies that any $p_i < p_i^{r_i^2}$ is iteratively dominated. This argument can – and should – be repeated indefinitely. We obtain a sequence $\overline{X} = r_i^0, r_i^1, \ldots$, all i. For any k and r_i^k such that $u_i^\varepsilon(p^{r^k_{-i}} \mid r_i^k, s_i) > 0$, there exists r_i^{k+1} that solves $u_i^\varepsilon(p_{-i}^{r^k} \mid r_i^{k+1}, s_i) = 0$. Induction on k, using Lemma 1, reveals that $r_i^{k+1} < r_i^k$ for all $k \geq 0$. Moreover, we know that $r_i^k \geq \underline{X}$ for all k. It follows that the sequence (r_i^k) is monotone and bounded. Such a sequence must converge; let $r_i(s)$ denote its limit and define $r(s) := (r_i(s))$. By construction, $r(s)$ solves

$$u_i^\varepsilon\left(p_{-i}^{r(s)} \mid r_i(s), s_i\right) = 0.$$

A symmetric procedure should be carried out starting from low signals, eliminating ranges of x_i^ε for which playing 1 is strictly (iteratively) dominated. For every player i this yields an increasing and bounded sequence (l_i^k) whose limit is $l_i(s)$, and $l(s) := (l_i(s))$. The limit $l(s)$ solves $u_i^\varepsilon(p_{-i}^{l(s)} \mid l_i(s), s_i) = 0$ for all i.

It is clear from the foregoing construction that a strategy p_i survives IESDS if and only if $p_i^{r_i(s)}(x_i^\varepsilon) \leq p_i(x_i^\varepsilon) \leq p_i^{l_i(s)}(x_i^\varepsilon)$ for all x_i^ε. We are particularly interested in games in which the points $l_i(s)$ and $r_i(s)$ converge to a common limit $x(s) := (x_i(s))$ that, hence, is the (essentially) unique solution to

$$u_i^\varepsilon\left(p_{-i}^{x(s)} \mid x_i(s), s_i\right) = u_i^\varepsilon\left(p_{-i}^{x(s)} \mid x_i(s)\right) + s_i = 0 \tag{5}$$

for all $i \in \mathscr{N}$. To work in such an environment, we must assume ε to be sufficiently small.

Lemma 2. *For all $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that $r_i(s) - l_i(s) < \delta$ for all $\varepsilon \leq \varepsilon(\delta)$ and all $i \in \mathscr{N}$.*

We note that assuming $\varepsilon \to 0$ is sufficient but not, in general, necessary to obtain convergence to the common limit $x(s)$; for example, when g is uniform we have $l_i(s) = r_i(s)$ for all $\varepsilon > 0$. We also observe that the identifying condition (5), combined with Lemma 1, reveals that the unique threshold $x_i(s)$ is continuously decreasing in (i) player i's own subsidy s_i and (ii) the thresholds $x_j(s)$, all $j \neq i$, of his opponents.

Given a subsidy scheme s and small enough ε, there is a unique increasing strategy vector $p^{x(s)}$ that survives IESDS in $\Gamma^{\varepsilon}(s)$. We next establish that the relation between $x(s)$ and s is one-to-one: given any $\hat{x}$, there is a unique subsidy scheme $\hat{s}$ such that $p^{\hat{x}}$ is the unique strategy vector that survives IESDS in $\Gamma^{\varepsilon}(\hat{s})$.

Lemma 3. *Let $\hat{x} = (\hat{x}_i)$ and ε sufficiently small. There is a unique subsidy scheme $\hat{s} = (\hat{s}_i)$ such that $x(\hat{s}) = \hat{x}$.*

Clearly, it follows from Lemma 3 that – for all ε sufficiently small – there is a unique subsidy vector $\tilde{s}$ such that $x_i(\tilde{s}) = \tilde{x}$ for all i.

4.3 Implementation and Characterization

Recall that a strategy vector $p = (p_1, p_2, ..., p_N)$ is a Bayesian Nash Equilibrium (BNE) of $\Gamma^{\varepsilon}(s)$ if for any p_i and x_i^{ε} it holds that:

$$p_i(x_i^{\varepsilon}) \in \underset{a_i \in \{0,1\}}{\arg\max}\, \pi_i^{\varepsilon}(a_i, p_{-i} \mid x_i^{\varepsilon}, s_i), \tag{6}$$

where $\pi_i^{\varepsilon}(a_i, p_{-i} \mid x_i^{\varepsilon}) := \int \pi_i(a_i, p_{-i}(x_{-i}^{\varepsilon}) \mid x) \, \mathrm{d}F_i^{\varepsilon}(x, x_{-i}^{\varepsilon} \mid x_i^{\varepsilon})$. It follows immediately that $p^{x(s)}$ is a BNE of $\Gamma^{\varepsilon}(s)$. Lemma 4 strengthens this result and establishes that $p^{x(s)}$ is the *only* BNE of $\Gamma^{\varepsilon}(s)$.

Lemma 4. *Given is s and ε sufficiently small. The essentially unique Bayesian Nash equilibrium of $\Gamma^{\varepsilon}(s)$ is $p^{x(s)}$. In particular, if p a BNE of $\Gamma^{\varepsilon}(s)$ then any $p_i \in p$ satisfies $p_i(x_i^{\varepsilon}) = p_i^{x_i(s)}(x_i^{\varepsilon})$ for all $x_i^{\varepsilon} \neq x_i(s)$ and all i.*

We know that for any subsidy scheme s and small enough ε the increasing strategy vector $p^{x(s)}$ is the unique BNE of $\Gamma^{\varepsilon}(s)$. From Lemma 2, we furthermore know that there is a unique subsidy scheme $\tilde{s}$ such that $x_i(\tilde{s}) = \tilde{x}$ for all i. It follows that the subsidy scheme $\tilde{s}$ that implements $p^{\tilde{x}}$ exists and is unique, provided we set ε sufficiently small. This proves part (i) of Theorem 1.

Before we proceed to characterize $\tilde{s}$, we recall that the unique switching point $x_i(s)$ is decreasing in both s_i and each $x_j(s)$, for all $i, j \in \mathscr{N}$. Because by Lemma 4 these monotonicities describe *equilibrium* effects of subsidies, we thus observe that subsidies have a compounded "unraveling" effect in coordination games. Consider an increase in the subsidy offered to player i. The higher subsidy raises his incentive to play 1 and lowers his equilibrium switching point $x_i(s)$. The drop in i's switching point in turn raises player j's incentive to play that action, shifting his switching point $x_j(s)$ down as well. The downward shift in $x_j(s)$ in turn raises player i's incentive to play 1 even more, further reducing his switching point $x_i(s)$. And so on. Because subsidies are common knowledge, these effects keep on compounding, ever reinforcing one another. Accounting for the total equilibrium effect of subsidies therefore shows that even fairly small subsidies (given $\tilde{x}$) can unravel a coordination game and solve the planner's problem.[7] But how small is small? To answer that question, we must characterize $\tilde{s}$. We rely on the following result.

[7]This feature of $\tilde{s}$ is a key counterpoint to several well-known results in the literature on policy design in coordination problems that stress optimality of subsidizing at least some players to strict dominance (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020).

Lemma 5. *For all* $\delta > 0$ *there exists* $\varepsilon(\delta) > 0$ *such that*

$$\left| u_i^{\varepsilon}\left(p_{-i}^{X} \mid X, s_i\right) - \left[X + \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - c_i + s_i \right] \right| < \delta \tag{7}$$

for $\varepsilon \leq \varepsilon(\delta)$ *and all* X *such that* $\underline{X} + \varepsilon \leq X \leq \overline{X} - \varepsilon$.

If his opponents all play the same increasing strategy p_j^X, then upon observing the threshold signal $x_i^{\varepsilon} = X$ player i's belief over the aggregate action $\sum_{j \neq i} a_j$ is uniform. Convergence to uniform strategic beliefs is a common property in global games; see Lemma 1 in Sakovics and Steiner (2012) for a reference in the context of policy design.[8]

Recall that, if $x(s)$ is the vector of switching points such that $p^{x(s)}$ is the unique BNE of $\Gamma^{\varepsilon}(s)$, then $x_i(s)$ solves (5) for all i. Imposing now that $\tilde{s}$ be such that $x_i(\tilde{s}) = \tilde{x}$ for all $i \in \mathscr{N}$, one obtains

$$u_i^{\varepsilon}\left(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i\right) = 0 \tag{8}$$

as the N identifying conditions for the subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$. Using the result in Lemma 5 when $X = \tilde{x}$ and solving (8) for $\tilde{s}_i$ establishes that for all $r > 0$ there exists $\varepsilon(r) > 0$ such that

$$|\tilde{s}_i - s_i^*(\tilde{x})| < r$$

for all $\varepsilon \leq \varepsilon(r)$ and all $i \in \mathscr{N}$. This proves part (ii) of Theorem 1.

4.4 Discussion

Our results characterize the subsidy scheme $\tilde{s}$ a planner must commit to when seeking to implement $p^{\tilde{x}}$ among rational players. Let us discuss several properties of this policy.

First, optimal subsidies are modest relative to the planner's goal: $\tilde{s}_i$ does not make $p_i^{\tilde{x}}$ strictly dominant for any player i. Subsidization up to strict dominance is unnecessary due to the sharp specification of players' strategic beliefs that our analysis reveals. In the unique equilibrium $p^{\tilde{x}}$, a player i whose signal exceeds the critical state $\tilde{x}$ cannot believe that all of his opponents will play 0 with probability 1; it is the impossibility of ruling out such extreme beliefs in games with multiple equilibria that requires subsidization of at least one player to strict dominance. Specifically, we note that the scheme $\tilde{s}$ is pinned down by the player's strategic beliefs in the critical state ($x_i^{\varepsilon} = \tilde{x}$) only. Furthermore, we also establish that these beliefs converge to a uniform distribution on the aggregate action for all continuous priors g and noise distributions f, making the optimal subsidy scheme $\tilde{s}$ independent of the precise prior and noise distributions assumed.

Second, symmetric players are offered identical subsidies. This symmetry deviates from a number of other notable proposals including a divide-and-conquer policy (*cf.* Segal, 2003;

[8]Note that Sakovics and Steiner (2012) assume a uniform prior. On the one hand, this makes our result more general. On the other hand, Sakovics and Steiner (2012) establish uniformity of strategic beliefs for all $\varepsilon > 0$. We observe that, in line with Sakovics and Steiner (2012), Lemma 5 applies for all $\varepsilon > 0$ when g is uniform.

Bernstein and Winter, 2012) and the incentive schemes studied in Winter (2004) and Halac et al. (2020).[9]

Third, subsidies target all players and are globally continuous in model parameters. While conditional on policy treatment the optimal subsidies in Sakovics and Steiner (2012) are continuous in the relevant model parameters as well, changes in one player's parameters could affect whether or not said player is targeted, causing a discrete jump in subsidies received. Similarly, subsidies are continuous conditional on a player's position in the policy ranking in a divide and conquer mechanism (Segal, 2003; Bernstein and Winter, 2012); however, a player's position in the optimal ranking may be affected by a change in its parameters, which can lead to discrete jumps in subsidy entitlement.

Fourth, the subsidy scheme $\tilde{s}$ is unique. In the complete information environments considered by Segal (2003), Winter (2004), and Bernstein and Winter (2012) the optimal policy is not unique when (some) players are symmetric. In the incomplete information environments considered by Sakovics and Steiner (2012) and Halac et al. (2021), the optimal policy is unique. Note, however, that the results in Sakovics and Steiner (2012) and Halac et al. (2021) establish uniqueness of the policy that minimizes the expected cost of implementing a given equilibrium; in their models, there still exist other, more expensive policies that implement the same equilibrium. In contrast, Theorem 1 establishes that only one policy can implement a given equilibrium of the game studied here.

Sixth, subsidies are decreasing in $\tilde{x}$, the threshold for coordination on 1 targeted by the planner. All else equal, a player's incentive to play 1 is increasing in his signal x_i^ε. Hence, for higher signals a player needs less subsidy to induce him to play 1. One can interpret $\tilde{x}$ as an inverse measure of the planner's ambition: the higher is $\tilde{x}$, the lower is the prior probability that coordination on 1 will be achieved. In this interpretation, being ambitious is costly: assuming coordination on 1 is indeed achieved, total spending on subsidies is increasing in the planner's ambition (decreasing in $\tilde{x}$). The same is true in Sakovics and Steiner (2012).

Seventh, subsidies are decreasing in spillovers, i.e. $\partial \tilde{s}_i / \partial w_i(n) < 0$. When observing the threshold signal $\tilde{x}$, a player i's belief over the aggregate action $\sum_{j \neq i} a_j$ is uniform; in particular, therefore, he assigns strictly positive probability to the event that $\sum_{j \neq i} a_j = n$ for all $n = 0, 1, ..., N-1$. If $w_i(n)$ increases, the *expected* spillover a player expects to enjoy upon playing 1 is hence greater. This raises his incentive to play 1 and, for given $\tilde{x}$, the subsidy required to make him willing to do so is smaller. Given a ranking of players, subsides for each player (except the first-ranked) are also decreasing in spillovers in a divide-and-conquer policy (Segal, 2003; Bernstein and Winter, 2012). The optimal subsidies in Sakovics and Steiner (2012) are not generally decreasing in spillovers, except insofar as players who benefit less from project success are more likely to be targeted.

5 Extensions

Throughout this section, we continue to assume that x is an unobserved random variable about which players receive private signals x_i^ε in the way described in Section 2. To simplify

[9] Onuchic and Ray (2023) also show that "identical agents" may be compensated asymmetrically in equilibrium; however, though identical in the payoff-relevant sense their players may still vary in payoff-irrelevant "identifies".

statements of result, we assume that $\varepsilon \to 0$. When for $z \in \mathbb{R}$ we write $\tilde{s}_i \to z$ this is to be read as $|\tilde{s}_i - z| < r$ for all $r > 0$.

5.1 Regime Change

Consider a global game of regime change in which the externality w_i is partly stochastic (Morris and Shin, 1998; Angeletos et al., 2006, 2007; Goldstein and Pauzner, 2005; Sakovics and Steiner, 2012).[10] There is a project in which N investors can invest (play $a_i = 1$). The cost of investment to investor i is $c_i > 0$. If the project succeeds, i who invested earns benefit $x + b_i$, where $b_i > c_i$. Our assumption that the benefit of project success is partly unknown seems plausible and reflects any kind of (fundamental) uncertainty pertaining to the cost or benefit of investment (Abel, 1983; Pindyck, 1993); note, however, that the literature on games of regime change typically assumes common knowledge of payoffs given project success and normalizes x to 0.[11] The project succeeds only if aggregate investments reach or exceed a critical mass; specifically, there exists $I \in \{1, ..., N\}$ such that the project succeeds if and only if $\sum_{i \in \mathscr{N}} a_i \geq I$. Investors do not observe I but it is common knowledge that I is distributed uniformly on $\{1, ..., N\}$.[12] We normalize the payoff to not investing to 0. A planner offers each investor i an investment subsidy s_i. We are interested in the subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$ for some $\tilde{x} \in \mathbb{R}$. To compare our results with those in the literature, the policy that implements p^0 (i.e. $\tilde{x} = 0$) is of particular interest.[13]

Proposition 2. *Consider a global game of regime change. The subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$ is given by*

$$\tilde{s}_i \to c_i - \frac{b_i + \tilde{x}}{2} \tag{9}$$

for every $i \in \mathscr{N}$.

In the subsidy scheme $\tilde{s}$, all investors are subsidized and subsidies are a fraction of their investment costs. The latter is explained through the contagious effect of policies: if investor i receives an investment subsidy, he is more likely to invest. Anticipating the increased likelihood that i invests, project success becomes more likely and this attracts investment by investor j. The greater likelihood that j invests in turn makes investment even more interesting for i, and so on.

While our investment problem bears close resemblance to the model considered in the applied global games literature, Sakovics and Steiner (2012) in particular, it differs in two

[10] The model in Halac et al. (2020) is also a game of regime change, but not a global game.

[11] If $x = 0$, the assumption that $b_i > c_i$ for all i instantly implies that coordinated investment is the efficient outcome of a (global) game of regime change. This fundamentally distinguishes our model, in which the ex post efficient outcome is not known with certainty, from the literature.

[12] The assumption of a uniform prior on the critical mass I is standard in the applied global games literature and here maintained for reasons of comparability (*cf.* Morris and Shin, 1998; Angeletos et al., 2006, 2007; Goldstein and Pauzner, 2005; Sakovics and Steiner, 2012).

[13] More precisely, we are interested in $\tilde{x} \nearrow 0$. Recall that, conditional on project success or failure, the literature assumes complete information about payoffs and normalizes the state to $x = 0$. To make our results comparable to those in the literature, we must hence consider those policies that offer the same kinds of guarantees about outcomes as those considered by other authors. When $\varepsilon \to 0$, as assumed, all $\tilde{x} < 0$ offer such guarantees for $x = 0$.

fundamental ways. First, Sakovics and Steiner (2012) do not model prior uncertainty about the efficient outcome of the game; coordinated investment is always the efficient equilibrium of their game. Instead, fundamental uncertainty in their model pertains exclusively to the critical investment threshold I, and it is about I that players receive their noisy signals. Second, conditional on the regime in place, there is certainty about payoffs in Sakovics and Steiner (2012); we instead work with uncertain payoffs even conditional on the regime.[14] It is interesting that adding an additional layer of uncertainty to the game leads to vastly different policy implications. The main result in Sakovics and Steiner (2012) is that an optimal policy fully subsidizes (i.e. $s_i = c_i$) a subset of players and does not subsidize the others. In contrast, we find that an optimal policy subsidizes all players partially; choosing a critical state $\tilde{x} = 0$ (as Sakovics and Steiner (2012) implicitly assume $x = 0$) dictates offering each player a subsidy less than half his investment cost.

5.2 Incentives in Teams

Consider the problem of a principal offering rewards to incentive work by agents in teams (Holmstrom, 1982; Winter, 2004; Fischer and Huddart, 2008; Halac et al., 2021, 2022; Dai and Toikka, 2022). There is an organizational project that involves N tasks each performed by one agent $i \in \mathscr{N}$. Each agent i decides whether to work ($a_i = 1$) towards completing his task or shirk ($a_i = 0$). The cost of working to agent i is given by $c_i > 0$. Success of the project depends upon the decisions of all agents through a production technology $q : \{0, 1, ..., N\} \to [0, 1]$, where $q(n)$ is the probability of success given that n agents work. Following Winter (2004) and Halac et al. (2021), we assume that q is strictly increasing and strictly supermodular, i.e. $q(n+1) > q(n)$ ($n = 0, 1, ..., N-1$) and $q(k+1) - q(k)$ is increasing in k. If the project succeeds, each agent gets a (common) direct payoff x, which one may interpret as a kind of profit-sharing arrangement; uncertainty about x then reflects uncertainty about profits, which seems realistic.[15] The payoff to shirking is normalized to 0.

A principal offers contracts that specify rewards $s = (s_i)$ to agents contingent on project success; if the project fails, all agents receive zero. We assume that agents' work effort a_i is their private knowledge – any rewards the principal offers can condition only upon project success. The principal seeks the reward scheme $\tilde{s}$ that implements $p^{\tilde{x}}$.

Proposition 3. *There exists a unique reward scheme $(\tilde{s}_i)$ that implements $p^{\tilde{x}}$ in the principal agent problem. For each $i \in \mathscr{N}$, the reward $\tilde{s}_i$ is given by*

$$\tilde{s}_i \to \frac{c_i}{\bar{q}} - \tilde{x}, \tag{10}$$

where $\bar{q} := \sum_{n=0}^{N-1} \frac{q(n+1)-q(n)}{N}$.

[14] This distinction applies more generally to the literature on global games of regime change, see Morris and Shin (1998), Angeletos et al. (2007), Goldstein and Pauzner (2005), Basak and Zhou (2020), and Edmond (2013). Similarly, Kets et al. (2022) (in section 3.3.1, and their Theorem 3.4) also assume that joint investment is the efficient outcome of their game.

[15] More generally, x can be any kind of uncertain fundamental that determines agents' payoffs, see also Halac et al. (2022) for a model of contracting under fundamental uncertainty.

For $x = 0$, the payoffs in our model exactly replicate those of the canonical problem studied by Winter (2004). A direct comparison between results is nevertheless not meaningful. Observe that our question is fundamentally different from Winter's: whereas we characterize the unique reward scheme that implements a particular equilibrium, Winter characterizes the least-cost reward scheme that induces work by all agents in his game of complete information. To make comparisons between results in such different frameworks, additional work is needed; we take up this task in Section 5.5.

It is interesting to compare Proposition 3 to a result in Halac et al. (2021, Theorem 2 and Corollary 1 in particular). These authors consider the problem of a planner who offers agents rewards in a *ranking scheme*. In a ranking scheme, agents first are ranked; conditional on his ranking, agent i is then offered a reward that makes him indifferent between working and shirking provided all agents who are ranked below [above] him work [shirk]. Moreover, contract offers a private so that agents face uncertainty about their ranking. For the case of symmetric agents, Halac et al. (2021) establish that an optimal ranking scheme induces uniform beliefs about each agent's ranking. One can interpret Proposition 3 along similar lines: if an agent is ranked n-th and believes that all agents ranked below [above] him work [shirk], the reward that is necessary to make him work for all $x_i^\varepsilon > \tilde{x}$ is $s_i(n) = (c_i - \tilde{x})/(q(n+1) - q(n))$. Thus, if an agent has uniform beliefs about his own ranking the necessary reward becomes $\sum_{n=0}^{N-1} s_i(n)/N$, which is exactly the optimal reward $\tilde{s}_i$ given in Proposition 3. Note that, in our analysis, the uniform belief over n also applies when agents are asymmetric.

Some readers suggested an alternative specification in which the cost of effort is uncertain, i.e. working costs agent i $c_i - x$. For example, the cost of work may be higher for more "complicated" tasks, but it is not always possible to assess the level of complication before embarking on a project. In this specification, the principal chooses a critical state $\hat{x}$ such that agent i works whenever the cost of work is at most $c_i - \hat{x}$. For this case, too, there exists a unique reward scheme $\hat{s} = (\hat{s}_i)$ that solves the principal's problem. The scheme $\hat{s}$ is given by $\hat{s}_i \to (c_i - \hat{x})/\bar{q}$ for each $i \in \mathscr{N}$.

5.3 Heterogeneous Externalities

Externalities are rarely symmetrical. In practice, externalities are hardly so symmetrical. Large stores attract more customers to shopping malls than smaller shops (Bernstein and Winter, 2012). "Team players" consistently cause their co-workers to over-perform (Weidmann and Deming, 2021). An inventor's premature death causes a large and long-lasting decline in their co-inventor's earnings and citation-weighted patents (Jaravel et al., 2018). Workers respond more to the presence of coworkers with whom they frequently interact (Mas and Moretti, 2009). At academic conferences, sessions featuring Nobel laureates attract bigger audiences (private observation).

In this section, we allow that the externality $w_i(a_{-i})$ depends upon the specific vector a_{-i} rather than only the number $\sum_{j \neq i} a_j$. We maintain a focus on games with strategic complementarities and assume that if $a'_{-i} \geq a_{-i}$, then $w_i(a'_{-i}) \geq w_i(a_{-i})$. This externality structure encompasses the games in Bernstein and Winter (2012) and Halac et al. (2021), where externalities are allowed to depend upon the subset $M \subseteq \mathscr{N}$ of players who play 1. It also nests the approach in Sakovics and Steiner (2012), where externalities depend upon the *weighed* aggregate action, and that in Galeotti et al. (2020); Leister et al. (2022), who study coordination games on (directed) graphs.

Let us write A^n_{-i} for the set of all (unique) action vectors a_{-i} in which precisely n players j play $a_j = 1$, i.e. $A^n_{-i} := \{a_{-i} \mid \sum_{a_j \in a_{-i}} a_j = n\}$. Observe that A^n_{-i} contains exactly $\binom{N-1}{n}$ elements. For all i, define

$$w_i^n := \frac{\sum_{a_{-i} \in A^n_{-i}} w_i(a_{-i})}{\binom{N-1}{n}}.$$

In words, w_i^n is the expected externality imposed upon player i who expects that n opponents play 1 and believes that any such outcome is equally likely.

Proposition 4. *Let $\tilde{x} \in \mathbb{R}$. There exists a unique subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$ in the game $\Gamma^{\varepsilon}(s)$ with heterogeneous externalities. The subsidy $\tilde{s}_i$ pursuant to the scheme is given by*

$$\tilde{s}_i \to c_i - \tilde{x} - \sum_{n=0}^{N-1} \frac{w_i^n}{N} \tag{11}$$

for all $i \in \mathscr{N}$.

We observe that Proposition 4 doubles down on the uniform strategic beliefs of the game with homogeneous externalities. In a game with heterogeneous externalities, players have uniform beliefs about the total number of opponents $n \in \{0, 1, ..., N-1\}$ that play 1. Moreover, conditional on the number n of opponents that play 1 a threshold type player also has uniform beliefs about the identity of the n opponents who play 1. Note that in the game with heterogeneous externalities, too, optimal subsidies are unique, symmetric for identical players, continuous functions of model parameters, and do not make playing 1 strictly dominant for any player (*cf.* Bernstein and Winter, 2012).

5.4 Asymmetric Targets

Let $\tilde{x}^M = (\tilde{x}_1, \tilde{x}_1, ..., \tilde{x}_M) \in \mathbb{R}^M$, $M \leq N$, be a vector of critical states such that $\tilde{x}_1 < \tilde{x}_1 < ... < \tilde{x}_M$. Partition the player set $\mathscr{N}$ into M subsets $\mathscr{N}_m$, $m = 1, 2, ..., M$, which we refer to as "groups". We write n_m for the the number of players in $\mathscr{N}_m$, i.e. $n_m = |\mathscr{N}_m|$, and $N_m = \sum_{k=1}^{m-1} n_k$ (define $N_1 = 0$). Let $\tilde{p}$ denote the vector of increasing strategies in which player $i \in \mathscr{N}_m$ plays $p_i^{\tilde{x}_m}$. The planner wants to implement $\tilde{p}$; that is, she wants to find the subsidy scheme $\tilde{s}$ such that $\tilde{p}$ is the unique BNE of $\Gamma^{\varepsilon}(\tilde{s})$.

Proposition 5. *Let $\tilde{x}^M \in \mathbb{R}^M$. There exists a unique subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $\tilde{p}$. For each $m = 1, 2, ..., M$ and all $i \in \mathscr{N}_m$, the subsidy $\tilde{s}_i$ is given by*

$$\tilde{s}_i \to \begin{cases} c_i - \tilde{x}_m - \sum_{n=0}^{n_m - 1} \frac{w_i(N_m + n)}{n_m} & \text{if } n_m \geq 2, \\ c_i - \tilde{x}_m - w_i(N_m) & \text{if } n_m = 1. \end{cases} \tag{12}$$

Proposition 5 emphasizes the importance of strategic beliefs for policy design in coordination games. When ε is small, a player $i \in \mathscr{N}_m$ faces strategic uncertainty only with respect to opponents in his own group. In particular, he knows that players in groups $\mathscr{N}_1$ up to and including $\mathscr{N}_{m-1}$ must all have observed signals that exceed there respective critical states and will, in equilibrium, play 1. Similarly, all players in groups $\mathscr{N}_{m+1}$ up to and including $\mathscr{N}_M$ must have received a signal below their critical state; these players will play 0. Hence, the only remaining strategic uncertainty pertains to opponents $j \in \mathscr{N}_m \setminus \{i\}$; in equilibrium, player i's strategic beliefs over their aggregate action converges to a uniform distribution.

Recall from Proposition 1 that there exists a unique $(x_i^*) \in \mathbb{R}^N$ such that an action vector $a(x) = (a_i(x))$ strictly maximizes social welfare $W(\cdot \mid x)$ if and only if, for each $i \in \mathscr{N}$, $a_i(x) = 1$ for all $x > x_i^*$ and $a_i(x) = 0$ for all $x < x_i^*$. Let $p^* = (p_i^{x_i^*})$ denote the vector of increasing strategies in which each player i has switching point x_i^*.

Corollary 2. *There is a unique subsidy scheme $s^* = (s_i^*)$ that implements p^*. The scheme s^* maximizes social welfare with probability 1. Specifically, in the unique equilibrium of $\Gamma^\varepsilon(s^*)$ players' actions maximize $W(\cdot \mid x)$ for almost all x.*

The term almost all is used in its measure theoretic interpretation: the set of states x for which s^* does not implement the first best has Lebesgue measure zero (as we assumed that $\varepsilon \to 0$). Furthermore, in those states there is no other policy that improves upon s^* as any inefficiencies derive exclusively from the noise in players' signals. Thus, a welfare-maximizing planner could credibly commit to the scheme s^*.

5.5 Ranking Policies

Consider the game of complete information $\Gamma(\bar{x}, s)$ in which a planner offers subsidy scheme s with the aim of making $(1, 1, ..., 1)$ the unique Nash equilibrium of $\Gamma(\bar{x}, s)$. Seminal results due to Segal (2003) and Winter (2004) establish that the least-cost policy that solves the planner's problem in $\Gamma(\bar{x}, s)$ is a *ranking policy* (see also Bernstein and Winter, 2012; Halac et al., 2021). A ranking policy is a tuple $\langle \sigma, s^R \rangle$ that consists of a ranking σ and an associated subsidy scheme s^R. A ranking, which is a permutation $\sigma(\mathscr{N}) = \{i_1, i_2, ..., i_N\}$ of the player set $\mathscr{N}$, assigns a rank to each player. Given $\sigma(\mathscr{N})$, let s_{i_n} denote the subsidy offered to player i_n that makes him indifferent between playing 0 and 1 in the belief that all players who precede him in the ranking play 1 while all others play 0. Observe that, if all subsidies s_{i_n} are raised to $s_{i_n} + \delta$, for any $\delta > 0$, then playing 1 is strictly dominant for the first-ranked player and iteratively strictly dominant for all others, thus yielding the desired equilibrium. The ranking-policy scheme s^R is given by $(s_{i_n} + \delta)$ upon letting $\delta \to 0$. Importantly, note that a ranking policy is fundamentally discriminating: symmetric players receive asymmetric subsidies (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012).

The question arises how costly a policy s must be to offer similar guarantees on outcomes in $\Gamma^\varepsilon(s)$. Specifically, suppose a scheme s is such that the unique BNE *outcome* of $\Gamma^\varepsilon(s)$ is $(1, 1, ..., 1)$ if nature happens to draw state $\bar{x}$ as $\varepsilon \to 0$. What is the equilibrium cost of s in $\bar{x}$? There is no unique answer to this question as there exist infinitely many s such that the unique BNE outcome of $\Gamma^\varepsilon(s)$ is $(1, 1, ..., 1)$ in state $\bar{x}$, one example being s^R. We now construct a non-discriminatory scheme, $\bar{s}$, that induces $(1, 1, ..., 1)$ in state $\bar{x}$ as $\varepsilon \to 0$. Write $\bar{s}' = (\bar{s}_i')$ for the subsidy scheme that implements $p^{\bar{x}}$, i.e. $p^{\bar{x}}$ is the unique BNE of $\Gamma^\varepsilon(\bar{s}')$. For $\delta > 0$, we define $\bar{s}' + \delta = (\bar{s}_i' + \delta)$ as the subsidy scheme obtained by adding δ to the subsidies in $\bar{s}'$. We write $\bar{s}$ for $\bar{s}' + \delta$ upon letting $\delta \to 0$.

To compare costs across policies (and games), let us write $K(s \mid \bar{x})$ for total equilibrium spending on subsidies in$\Gamma(\bar{x}, s^R)$:

$$K(s \mid \bar{x}) = \sum_{n=1}^{N} s_i. \tag{13}$$

Similarly, let $K^\varepsilon(s \mid \bar{x})$ denote (expected) equilibrium spending on subsidies in $\Gamma^\varepsilon(s)$ if nature draws state $\bar{x}$:

$$K^\varepsilon(s \mid \bar{x}) = \int \left[\sum_{i \in n} s_i \cdot p_i^{x_i(s)}(x_i^\varepsilon) \right] \mathrm{d}F(x^\varepsilon \mid \bar{x}). \tag{14}$$

Theorem 2. *Let $\bar{x} \in \mathbb{R}$. If players are symmetric, then*

(i) $K^\varepsilon(\bar{s} \mid \bar{x}) \to K(s^R \mid \bar{x})$ *as* $\varepsilon \to 0$*;*

(ii) $\bar{s}_i = \bar{s}_j$ *for all* $i, j \in \mathscr{N}$*.*

For the case of symmetric players, Proposition 2 says that the cost of inducing coordination on $(1, 1, ..., 1)$ in the global game through scheme $\bar{s}$ should the state be $\bar{x}$ is equal to the total cost of inducing that outcome under common knowledge that the state is $\bar{x}$ through s^R. Notably, however, the scheme $\bar{s}$ is non-discriminatory: all players receive exactly the same subsidy in $\bar{s}$.

A notable insight from Proposition 2 is that the least-cost property of discriminatory policies in coordination games is fundamentally an artifact of equilibrium multiplicity under complete information. Upon connecting the problems of policy design to that of equilibrium selection, we are able to construct a non-discriminatory policy that achieves the same outcome as an optimal ranking scheme and costs the same. This finding illustrates the importance of equilibrium selection for the study of policy design in coordination games.

5.6 Continuous Actions

Assume that $a_i \in [0, 1]$ for all i.[16] Given a vector of actions $a \in [0, 1]^N$, a state x, and a subsidy scheme s, let the payoff to player i be denoted $\hat{\pi}_i$, given by

$$\begin{aligned} \hat{\pi}_i(a \mid x, s) &= a_i \cdot \pi_i(1, a_{-i} \mid x, s) + (1 - a_i) \cdot \pi_i(0, a_{-i} \mid x, s) \\ &= a_i \cdot \left[x + w_i \left(\sum_{j \neq i} a_j \right) + s_i \right] + (1 - a_i) \cdot c_i. \end{aligned} \tag{15}$$

In the context of an investment problem, one might interpret the action a_i as the proportion of investor i's budget invested in a project. The per-dollar return on investment in the project (counting subsidies) is $x + w_i(a_{-i}) + s_i$ (as in Halac et al. (2020), subsidies are per dollar invested). The investor's outside option yields a (certain) per-dollar return of c_i. Given (15), a player's marginal incentive to increase his action is defined as $\hat{u}_i(a_{-i} \mid x, s) := \partial\hat{\pi}_i(a \mid$

[16] It is straightforward to extend the analysis to games in which, similar to Halac et al. (2020), players have heterogeneous budgets I_i so that $a_i \in [0, I_i]$ for each $i \in \mathscr{N}$. We discuss $I_i = 1$ for all i to reduce notation.

$x, s)/\partial a_i = x + w_i(a_{-i}) + s_i - c_i$, which is (2). We write $\hat{\Gamma}(x, s)$ for the continuous-action game of complete information; $\hat{\Gamma}^\varepsilon(s)$ denotes the global game obtained by embedding $\hat{\Gamma}(x, s)$ in the information structure described in Section 2.

We restrict attention to the problem of a planner who wants players to coordinate on playing $a_i = 1$ for all $x > \tilde{x}$ and $a_i = 0$ for all $x < \tilde{x}$. As before, we write $p^{\tilde{x}} = (p_i^{\tilde{x}})$ for the vector of strategies such that each player i plays 1 [plays 0] for all $x_i^\varepsilon > \tilde{x}$ [$x_i^\varepsilon < \tilde{x}$].[17] Proposition 6 establishes that our main result, Theorem 1, applies as given to the continuous action global game $\hat{\Gamma}^\varepsilon(s)$.

Proposition 6. *Let $\tilde{x} \in \mathcal{X}$ and consider the game $\hat{\Gamma}^\varepsilon(s)$. There exists a unique subsidy scheme $\tilde{\hat{s}} = (\tilde{\hat{s}}_i)$ that implements $p^{\tilde{x}}$, and $\tilde{\hat{s}}_i \to s^*(\tilde{x})$. Here, $s_i^*(\tilde{x})$ is given by $(*)$.*

An analysis of continuous action games where payoffs are non-linear in own actions lies beyond the scope of this paper.

6 Concluding Remarks

This paper presents a number of results on policy design in coordination games. Strategic uncertainty complicates policy design in coordination games. To deal with this complication, the planner in this paper connects the problem of policy design to that of equilibrium selection using a global games approach. Our main result characterizes the subsidy scheme that induces coordination on a given equilibrium of the game as its unique equilibrium. We show that optimal subsidies are unique and admit a number of properties that run counter to well-known results on policy design in coordination games. In particular, we show that optimal subsidies are symmetric for identical players, continuous functions of model parameters, and do not make the targeted strategies strictly dominant for any single player.

Two core features of the game considered here help explain the differences between our optimal policy and the policies previously proposed in the literature. First, as stated above, the planner in this paper connects the problem of policy design to that of equilibrium selection. Equilibrium selection allows the planner to make very precise inferences about player's actual, rather than hypothetical, strategic beliefs and to design her policy in response to those. Second, the planner must commit to her policy before knowing which strategy vector will be the ex post efficient outcome of the game. This kind of fundamental uncertainty leads to a degree of policy restraint as overly aggressive intervention may itself become a source of ex post coordination failure.

[17] In this continuous action game, an alternative problem would be that of a planner who seeks to induce actions $a_i \notin \{0, 1\}$ for all or some x's. This would be similar to the problem studied in Halac et al. (2020). We observe that, given the assumed linearity of payoffs in a_i, implementation of any $a_i \notin \{0, 1\}$ requires the planner to offer a non-constant return *schedule* (r_i, a_i), specifying a per-dollar return [subsidy] $r_i(a_i)$ for any feasible choice of action $a_i \in [0, 1]$. Proposition 3 and Corollary 2 in Halac et al. (2020) identify intuitive conditions under which a subset of agents is targeted to invest their entire endowment (play $a_i = 1$) whereas the remaining agents are targeted not to invest at all (play $a_i = 0$). Thus, in their model, the focus on implementation of equilibria in which players do not play actions $a_i \notin \{0, 1\}$ follows endogenously from the planner's (or firm's) optimization problem under well-specified conditions. A focus on implementation of equilibria other than $p^{\tilde{x}}$ lies beyond the scope of our paper.

The analysis also highlights an unraveling effect of policy in coordination games. A subsidy raises a player i's incentive to play the subsidized action. The raised incentive of player i also indirectly increases player j's incentive to play that action. This, in turn, makes the subsidized action even more attractive for player i, and so on. Under common knowledge of the policy, this positive feedback loop compounds indefinitely and allows seemingly modest policies to unravel coordination problems.

A Properties of Γ^ε When ε Is Small

As stated when introducing Corollary 1, the proofs in this Appendix rely upon our ability to analyze the problem "as if" the common prior g were uniform when ε is sufficiently small. Here, we make this claim more precise.

Let us write ϕ^ε for the density of $\varepsilon \cdot \eta_i$. Although in general $\phi^\varepsilon(z)$ can, for any z, become arbitrarily large if we pick ε very small, it remains true that $\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x$ is (proportional to) a density and, consequently, that for any continuous function $h : \mathcal{X} \to \mathbb{R}$ the quantity $\int h(x)\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x$ is bounded. In particular, therefore, we know that $\int g(x)\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x$ is bounded.

Conditional on his signal x_i^ε the density of player i on the vector of signals x_{-i}^ε received by his opponents is

$$f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon) = \frac{\int g(x)\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x}{\iint g(x)\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x_{-i}^\varepsilon\mathrm{d}x} \tag{16}$$

for all $x_i^\varepsilon \in [\underline{X} - \varepsilon/2, \overline{X} + \varepsilon/2]$ and all $x_j^\varepsilon \in [x_i^\varepsilon - \varepsilon, x_i^\varepsilon + \varepsilon]$ while $f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon) = 0$ otherwise. Under a uniform prior g the density $f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ simplifies to $\overline{f_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon) := \int \phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x$.

Proposition 7. *For all $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that $|f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon) - \overline{f_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon)| < \delta$ for all $\varepsilon \leq \varepsilon(\delta)$ and all $(x_i^\varepsilon, x_{-i}^\varepsilon) \in \mathbb{R}^N$.*

An immediate implication of Proposition 7 is that the cumulative distribution function $F(z_{-i} \mid x_i^\varepsilon) := \int^{z_{-i}} f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)\mathrm{d}x_{-i}^\varepsilon$ can also, for sufficiently small ε, be approximated arbitrarily closely by the distribution $\overline{F_i^\varepsilon}$ obtained under a uniform prior g. Moreover, the probability distribution $\overline{F_i^\varepsilon}$ admits a highly useful property: its shape is independent of x_i^ε. To be more precise, and abusing notation, let us write Δ for both a real number $\Delta \in \mathbb{R}$ and the vector of real numbers $(\Delta, \Delta, ..., \Delta) \in \mathbb{R}^{N-1}$ such that $z_{-i} + \Delta = (z_j + \Delta)_{j\neq i}$.

Proposition 8. *For all Δ and all $(z_i, z_{-i}) \in \mathbb{R}^N$, we have $\overline{F_i^\varepsilon}(z_{-i} + \Delta \mid z_i + \Delta) = \overline{F_i^\varepsilon}(z_{-i} \mid z_i)$.*

B Proofs

Let h^ε denote a function that is (implicitly) parametrized by ε, and let H be defined on the same domain as h^ε. Throughout this Appendix, when we write $h^\varepsilon(z) \to H(z)$ we mean that for all $\delta > 0$ there exists $\varepsilon(\delta) > 0$ such that $|h^\varepsilon(z) - H(z)| < \delta$ for all $\varepsilon \leq \varepsilon(\delta)$ and all z in the domain of h^ε and H. Thus, $h^\varepsilon(z) \to H(z)$ should be read as saying that $h^\varepsilon(z)$ can be brought arbitrarily close to $H(z)$ provided we choose ε sufficiently small. Whenever such a claim is made without further explanation, it is implied that this follows Proposition 7. We emphasize that the symbol "$\to$" should *not* be read as a limit as ε goes to zero; since $\varepsilon > 0$ by assumption, that limit is not defined.

B.1 Proofs of Results in Appendix A

PROOF OF PROPOSITION 7

Proof. Given x_i^ε, for all $x \in [x_i^\varepsilon - \varepsilon/2, x_i^\varepsilon + \varepsilon/2]$, let us define $g_-^\varepsilon(x_i^\varepsilon) = \min_x g(x)$ and $g_+^\varepsilon(x_i^\varepsilon) = \max_x g(x)$. Clearly, $g_-^\varepsilon(x_i^\varepsilon) \leq g(x) \leq g_+^\varepsilon(x_i^\varepsilon)$ in the relevant domain. Therefore

$$\frac{\int g_-^\varepsilon(x_i^\varepsilon)\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x}{\iint g_+^\varepsilon(x_i^\varepsilon)\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x_{-i}^\varepsilon\mathrm{d}x} \leq f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)$$
$$\leq \frac{\int g_+^\varepsilon(x_i^\varepsilon)\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x}{\iint g_-^\varepsilon(x_i^\varepsilon)\phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x_{-i}^\varepsilon\mathrm{d}x}$$

for all $(x_i^\varepsilon, x_{-i}^\varepsilon) \in \mathbb{R}^N$ and all $\varepsilon > 0$. Because $g_-^\varepsilon(x_i^\varepsilon)$ and $g_+^\varepsilon(x_i^\varepsilon)$ are constants relative to the variable of integration, we can factor them out of the integral. Noting that $\iint \phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x_{-i}^\varepsilon\mathrm{d}x = 1$, the above then becomes

$$\frac{g_-^\varepsilon(x_i^\varepsilon)}{g_+^\varepsilon(x_i^\varepsilon)}\int \phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x \leq f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon) \leq \frac{g_+^\varepsilon(x_i^\varepsilon)}{g_-^\varepsilon(x_i^\varepsilon)}\int \phi^\varepsilon(x_i^\varepsilon - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x,$$

or

$$\frac{g_-^\varepsilon(x_i^\varepsilon)}{g_+^\varepsilon(x_i^\varepsilon)}\overline{f_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon) \leq f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon) \leq \frac{g_+^\varepsilon(x_i^\varepsilon)}{g_-^\varepsilon(x_i^\varepsilon)}\overline{f_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon).$$

From the uniform continuity of g (i.e. g is continuous on a compact set, which by the Heine-Cantor theorem implies g is uniformly continuous) follows that for any $k > 0$ there exists $\varepsilon(k) > 0$ such that $g_+^\varepsilon(x_i^\varepsilon) - g_-^\varepsilon(x_i^\varepsilon) < k$ for all $\varepsilon \leq \varepsilon(k)$ and all x_i^ε. It follows immediately (by the squeeze theorem) that for all $\delta > 0$ there exists $\varepsilon(\delta) > 0$ such that $|\overline{f_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon) - < f_i^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)| < \delta$ for all $\varepsilon \leq \varepsilon(\delta)$ and all $(x_i^\varepsilon, x_{-i}^\varepsilon) \in \mathbb{R}^N$. □

PROOF OF PROPOSITION 8

Proof. Fix $(z_i, z_{-i}) \in \mathbb{R}^N$ and Δ. We have

$$\begin{aligned}
\overline{F_i^\varepsilon}(z_{-i} \mid z_i) &= \int_{z_i-\varepsilon}^{z_{-i}} \overline{f_i^\varepsilon}(x_{-i}^\varepsilon \mid z_i)\mathrm{d}x_{-i}^\varepsilon \\
&= \int_{z_i-\varepsilon}^{z_{-i}} \left[\int_{z_i-\varepsilon/2}^{z_i+\varepsilon/2} \phi^\varepsilon(z_i - x)\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon - x)\mathrm{d}x\right]\mathrm{d}x_{-i}^\varepsilon \\
&= \int_{z_i-\varepsilon}^{z_{-i}} \left[\int_{z_i-\varepsilon/2}^{z_i+\varepsilon/2} \phi^\varepsilon(z_i + \Delta - (x+\Delta))\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon + \Delta - (x+\Delta))\mathrm{d}x\right]\mathrm{d}x_{-i}^\varepsilon \\
&= \int_{z_i-\varepsilon}^{z_{-i}} \left[\int_{z_i+\Delta-\varepsilon/2}^{z_i+\Delta+\varepsilon/2} \phi^\varepsilon(z_i + \Delta - x')\prod_{j\neq i}\phi^\varepsilon(x_j^\varepsilon + \Delta - x')\mathrm{d}x'\right]\mathrm{d}x_{-i}^\varepsilon \\
&= \int_{z_i-\varepsilon}^{z_{-i}} \overline{f_i^\varepsilon}(x_{-i}^\varepsilon + \Delta \mid z_i + \Delta)\mathrm{d}x_{-i}^\varepsilon
\end{aligned}$$

$$= \int_{z_i+\Delta-\varepsilon}^{z_{-i}+\Delta} \overline{f_i^\varepsilon}(x_{-i}^\varepsilon \mid z_i + \Delta) \mathrm{d}x_{-i}^\varepsilon$$
$$= \overline{F_i^\varepsilon}(z_{-i} + \Delta \mid z_i + \Delta),$$

as claimed. □

B.2 Proofs of Results in Section 4

PROOF Of PROPOSITION 1

Proof. First we relabel the players so that $x_{i_1}^* \leq x_{i_2}^* \leq \ldots \leq x_{i_N}^*$. For each $i \in \mathscr{N}$ there exists a unique $\underline{x}_i$ such that $\underline{x}_i + w_i(0) = c_j$. From the definition of $\underline{x}_i$ follows that $W((1, \mathbf{0}_{-i}) \mid x) > W((0, \mathbf{0}_{-i}) \mid x)$ iff $x > \underline{x}_i$. Set $x_{i_1}^* = \min\{\underline{x}_i \mid i \in \mathscr{N}\}$ and label as i_1 that player i for whom $\underline{x}_i = x_{i_1}^*$. From the construction of $x_{i_1}^*$ follows that to maximize welfare player i_1 must play 1 for all $x \geq x_{i_1}^*$ (and play 0 otherwise).

Suppose now the result is true for players $i_1, i_2, ..., i_n$, i.e. assume there are $x_{i_1}^* \leq x_{i_2}^* \leq ... \leq x_{i_n}^*$ such that it is welfare-maximizing for player $i \in \{i_1, i_2, ..., i_n\}$ to play 1 iff $x > x_i^*$. Given this hypothesis, we observe that for each $j \in \mathscr{N} \setminus \{i_1, ..., i_n\}$ there exists a unique $\underline{x}_j(n)$ such that

$$\underline{x}_j(n) + w_j(n) + \sum_{m=1}^{n} [w_{i_m}(n+1) - w_{i_m}(n)] = c_j.$$

Choose $x_{i_{n+1}}^* = \min\{\underline{x}_j(n) \mid j \in \mathscr{N} \setminus \{i_1, i_2, ..., i_n\}\}$. Clearly, $a_j = 1$ is welfare-maximizing for all $x > \underline{x}_j(n)$ when players $i_1, ..., i_n$ play 1 (and the others play 0). Furthermore, our inductive hypothesis implies that $\underline{x}_j(n) \geq x_{i_n}^*$ (otherwise $j \in \{i_1, i_2, ..., i_n\}$, but $j \in \mathscr{N} \setminus \{i_1, ..., i_n\}$ by assumption). As we proved our inductive hypothesis for $n = 1$, the result follows from induction on n. □

PROOF OF LEMMA 1

Proof. First, observe that

$$\begin{aligned} u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) &= \int u_i \left(p_{-i}\left(x_{-i}^\varepsilon\right) \mid x\right) \, \mathrm{d}F_i^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= \int w_i \left(p_{-i}\left(x_{-i}^\varepsilon\right)\right) + x \, \mathrm{d}F_i^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) - c_i \\ &\to \int w_i \left(p_{-i}\left(x_{-i}^\varepsilon\right)\right) \, \mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon) + x_i^\varepsilon - c_i, \end{aligned}$$

for any strategy vector p_{-i}.

To prove part (i), it suffices to show that $\int w_i \left(p_{-i}^y\left(x_{-i}^\varepsilon\right)\right) \mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is increasing in x_i^ε. First we introduce a random variable $v_i(x_{-i}) = w_i(p_{-i}^y(x_{-i}^\varepsilon))$ and observe that, since $w_i(p_{-i}^y(x_{-i}^\varepsilon))$ is increasing in $p_{-i}^y(x_{-i}^\varepsilon)$ and $p_{-i}^y(x_{-i}^\varepsilon)$ is increasing in x_{-i}^ε, v_i is increasing in x_{-i}^ε. Next, we note that the distribution $\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is first-order stochastic dominant over the distribution $F_i^\varepsilon(x_{-i}^\varepsilon \mid \hat{x}_i^\varepsilon)$ iff $x_i^\varepsilon > \hat{x}_i^\varepsilon$; this follows from Bayes' theorem upon application of the two facts that (a) each ε_j (and indeed ε_i) is drawn independently of x, and (b) player i's conditional distribution on x given x_i^ε first-order stochastic dominates his conditional distribution on x given $\hat{x}_i^\varepsilon$ iff $x_i^\varepsilon > \hat{x}_i^\varepsilon$. Hence, because v_i is increasing we have $\int v_i(x_{-i}^\varepsilon)\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon) > \int v_i(x_{-i}^\varepsilon)\mathrm{d}F_i^\varepsilon(x_{-i}^\varepsilon \mid \hat{x}_i^\varepsilon)$ and the result follows.

To prove part (ii), we reiterate the observation from the proof of part (i) that the distribution $\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is first-order stochastic dominant over the distribution $\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid \hat{x}_i^\varepsilon)$ iff $x_i^\varepsilon > \hat{x}_i^\varepsilon$. Next, we note that $p_{-i}^y(x_{-i}^\varepsilon)$ is (weakly) decreasing in $y_j \in y$, all $j \neq i$ (and, therefore, the random variable $v_i(x_{-i}^\varepsilon)$ we introduced in the proof of part (i) is also decreasing in y_j). Therefore $\int w_i\left(p_{-i}^y\left(x_{-i}^\varepsilon\right)\right)\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is decreasing in y_j and the result follows. □

PROOF OF LEMMA 2

Proof. We omit the argument s to reduce notation. By construction, $l_i \leq r_i$. Define $\Delta_i := r_i - l_i$, so $\Delta_i \geq 0$. We first establish a useful claim.

Claim 1. *If $\Delta_i = \Delta$ for all $i \in \mathscr{N}$, then $\Delta = 0$.*

Proof of the claim. If $\Delta_i = \Delta$ for all $i \in \mathscr{N}$, we have

$$\begin{aligned}
u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i, s_i) &\to r_i + \int w_i(p_{-i}^{r_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid r_i) + s_i - c_i \\
&= l_i + \Delta + w_i(p_{-i}^{l_{-i}+\Delta}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid l_i + \Delta) + s_i - c_i \\
&= l_i + \Delta + \int w_i(p_{-i}^{l_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid l_i) + s_i - c_i \\
&\to \Delta + u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i, s_i).
\end{aligned}$$

By construction, $u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i, s_i) = u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i, s_i)$, and it follows that $\Delta = 0$. □

Now let $\Delta_i \neq \Delta_j$ for at least one pair of players $i, j \in \mathscr{N}$ and suppose (without loss) that player i is such that $\Delta_i = \max\{\Delta_j \mid j \in \mathscr{N}\}$. Because $\Delta_i \geq \Delta_j$ for all $j \neq i$ with a strict inequality for at least one j, we have

$$\begin{aligned}
&u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i, s_i) - u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i, s_i) \\
&\to r_i - l_i + \int w_i(p_{-i}^{r_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid r_i) - \int w_i(p_{-i}^{l_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid l_i) \\
&= \Delta_i + \int w_i(p_{-i}^{r_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid r_i) - \int w_i(p_{-i}^{l_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid l_i) \\
&> \int w_i(p_{-i}^{r_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid r_i) - \int w_i(p_{-i}^{l_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid l_i) \\
&> \int w_i(p_{-i}^{l_{-i}+\Delta_i}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid l_i + \Delta_i) - \int w_i(p_{-i}^{l_{-i}}(x_{-i}^\varepsilon))\mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid l_i) \\
&= 0,
\end{aligned}$$

where the first inequality follows from $\Delta_i > 0$ and the final equality is a consequence of Proposition 8. Hence, for player i we have $u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i, s_i) > u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i, s_i)$, contradicting that $u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i, s_i) = u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i, s_i)$ by construction. Hence, there cannot be a player i such that $\Delta_i \geq \Delta_j$ for all $j \neq i$ with a strict inequality for at least one j. Therefore $\Delta_i = \Delta$ for all $i \in \mathscr{N}$. By the claim at the start of the this proof, this implies $\Delta = 0$. □

PROOF OF LEMMA 3

Proof. Suppose, in contrast, that there are two distinct vectors of subsidies $\hat{s}_1 = (\hat{s}_{1i})$ and $\hat{s}_2 = (\hat{s}_{2i})$ that both implement $p^{\hat{x}}$ such that $\hat{s}_1 \neq \hat{s}_2$. Per Lemmas 2 and 4, $\hat{s}_1$ and $\hat{s}_2$ must solve $x(\hat{s}_1) = x(\hat{s}_2) = \hat{x}$. By (5), this means that $\hat{s}_{1i}$ and $\hat{s}_{2i}$ are both solutions to

$$u_i^{\varepsilon}\left(p_{-i}^{\hat{x}} \mid \hat{x}_i, \hat{s}_{1i}\right) = u_i^{\varepsilon}\left(p_{-i}^{\hat{x}} \mid \hat{x}_i, \hat{s}_{2i}\right) = 0, \tag{17}$$

for each $i \in \mathscr{N}$. We thus have

$$u_i^{\varepsilon}\left(p_{-i}^{\hat{x}} \mid \hat{x}_i\right) + s_{1i} = u_i^{\varepsilon}\left(p_{-i}^{\hat{x}} \mid \hat{x}_i\right) + s_{2i}, \tag{18}$$

which implies

$$\hat{s}_{1i} = \hat{s}_{2i} \tag{19}$$

for all $i \in \mathscr{N}$. This contradicts our assumption that $\hat{s}_1 \neq \hat{s}_2$. □

PROOF OF LEMMA 4

Proof. Let $p = (p_i)$ be a BNE of $\Gamma^{\varepsilon}(s)$. For any player i, define

$$\underline{\underline{x}}_i = \inf\{x_i^{\varepsilon} \mid p_i(x_i^{\varepsilon}) > 0\}, \tag{20}$$

and

$$\overline{\overline{x}}_i = \sup\{x_i^{\varepsilon} \mid p_i(x_i^{\varepsilon}) < 1\}. \tag{21}$$

Observe that $\underline{\underline{x}}_i \leq \overline{\overline{x}}_i$. Now define

$$\underline{\underline{x}} = \min\{\underline{\underline{x}}_i\}, \tag{22}$$

and

$$\overline{\overline{x}} = \max\{\overline{\overline{x}}_i\}. \tag{23}$$

By construction, $\overline{\overline{x}} \geq \overline{\overline{x}}_i \geq \underline{\underline{x}}_i \geq \underline{\underline{x}}$. Observe that p is a BNE of $\Gamma^{\varepsilon}(s)$ only if, for each i, it holds that $u_i^{\varepsilon}(p_{-i}(x_{-i}^{\varepsilon}) \mid \underline{\underline{x}}_i) \geq 0$. Consider then the expected incentive $u_i^{\varepsilon}(p_{-i}^{\underline{\underline{x}}}(x_{-i}^{\varepsilon}) \mid \underline{\underline{x}}_i)$. It follows from the definition of $\underline{\underline{x}}$ that $p^{\underline{\underline{x}}}(x^{\varepsilon}) \geq p(x^{\varepsilon})$ for all x^{ε}. The implication is that, for each i, $u_i^{\varepsilon}(p_{-i}^{\underline{\underline{x}}}(x_i - i^{\varepsilon}) \mid \underline{\underline{x}}_i) \geq u_i^{\varepsilon}(p_{-i}(x_{-i}^{\varepsilon}) \mid \underline{\underline{x}}_i) \geq 0$. From Proposition 5 then follows that $\underline{\underline{x}} \geq x$.

Similarly, if p is a BNE of $\Gamma^{\varepsilon}(s)$ then, for each i, it must hold that $u_i^{\varepsilon}(p_{-i}(x_{-i}^{\varepsilon}) \mid \overline{x}_i) \leq 0$. Consider the expected incentive $u_i^{\varepsilon}(p_{-i}^{\overline{x}}(x_{-i}^{\varepsilon}) \mid \overline{\overline{x}}_i)$. It follows from the definition of $\overline{\overline{x}}$ that $p^{\overline{\overline{x}}}(x^{\varepsilon}) \leq p(x^{\varepsilon})$ for all x^{ε}. For each i it therefore holds that $u_i^{\varepsilon}(p_{-i}^{\overline{\overline{x}}}(x_{-i}^{\varepsilon}) \mid \overline{\overline{x}}_i) \leq u_i^{\varepsilon}(p_{-i}(x_i - i^{\varepsilon}) \mid \overline{\overline{x}}_i) \leq 0$. Hence $\overline{\overline{x}} \leq x$.

Since $\underline{\underline{x}} \leq \overline{\overline{x}}$ while also $\underline{\underline{x}} \geq x$ and $\overline{\overline{x}} \leq x$ it must hold that $\underline{\underline{x}} = \overline{\overline{x}} = x$. Moreover, since $p^{\underline{\underline{x}}} \geq p$ while also $p^{\overline{\overline{x}}} \leq p$, given $\underline{\underline{x}} = \overline{\overline{x}} = x$, it follows that $p_i(s_i^{\varepsilon}) = p_i^{x}(x_i^{\varepsilon})$ for all $x_i^{\varepsilon} \neq x$ and all i (recall that for each player i one has $u_i^{\varepsilon}(p_{-i}^{x} \mid x) = 0$, explaining the singleton exeption at $x_i^{\varepsilon} = x$). Thus, if $p = (p_i)$ is a BNE of $\Gamma^{\varepsilon}(s)$ then it must hold that $p_i(x_i^{\varepsilon}) = p_i^{x}(x_i^{\varepsilon})$ for all $x_i^{\varepsilon} \neq x$ and all i, as we needed to prove. □

PROOF OF LEMMA 5

Proof. Let $\Omega^\varepsilon(n \mid X, x_i^\varepsilon)$ denote the probability that a player i who observes signal x_i^ε attaches to the event that n other players j receive a signal $x_j^\varepsilon \geq X$:

$$\Omega^\varepsilon(n \mid X, x_i^\varepsilon) = \frac{\int g(x) f \phi^\varepsilon\left(x_i^\varepsilon - x\right) \binom{N-1}{n} \left[\Phi\left(X-x\right)\right]^{N-n-1} \left[1-\Phi\left(X-x\right)\right]^n \mathrm{d}x}{\int g(x)\phi^\varepsilon\left(x_i^\varepsilon - x\right)\mathrm{d}x}, \tag{24}$$

where $\Phi^\varepsilon(z) := \int_{\varepsilon/2}^{z} \phi^\varepsilon(\lambda)\mathrm{d}\lambda$ is the c.d.f. of ϕ^ε. When g is uniform, this simplifies to:

$$\overline{\Omega^\varepsilon}(n \mid X, x_i^\varepsilon) = \binom{N-1}{n} \int \phi^\varepsilon\left(x_i^\varepsilon - x\right) \left[\Phi^\varepsilon\left(X-x\right)\right]^{N-n-1} \left[1-\Phi^\varepsilon\left(X-x\right)\right]^n \mathrm{d}x \tag{25}$$

Clearly, if player i's opponents play p_{-i}^X then $\Omega^\varepsilon(n \mid X, x_i^\varepsilon)$ is also i's conditional distribution on $\sum_{j\neq i} a_j = n$. Therefore

$$\begin{aligned} u_i^\varepsilon(p_{-i}^X \mid x_i^\varepsilon, s_i) &= \int x\phi^\varepsilon(x_i^\varepsilon - x)\mathrm{d}x + \sum_{n=0}^{N-1} w_i(n)\Omega^\varepsilon(n \mid X, x_i^\varepsilon) - c_i + s_i \\ &\to x_i^\varepsilon + \sum_{n=0}^{N-1} w_i(n)\overline{\Omega^\varepsilon}(n \mid X, x_i^\varepsilon) - c_i + s_i \end{aligned}$$

as $\varepsilon \to 0$. To prove the Lemma, we need only evaluate $\overline{\Omega^\varepsilon}(n \mid X, x_i^\varepsilon)$ at $x_i^\varepsilon = X$. Define $y := X - x$, so we may write $\int \phi^\varepsilon\left(y\right)\left[\Phi^\varepsilon\left(y\right)\right]^{N-n-1}\left[1-\Phi^\varepsilon\left(y\right)\right]^n \mathrm{d}y$.[18] Repeatedly carrying out the integration by parts, we obtain

$$\begin{aligned} \frac{1}{\binom{N-1}{n}} \cdot \overline{\Omega^\varepsilon}(n \mid X, X) &= \int \phi^\varepsilon(y)\left[\Phi^\varepsilon(y)\right]^{N-n-1}\left[1-\Phi^\varepsilon(y)\right]^n \mathrm{d}y \\ &= \frac{n}{N-n}\int \phi^\varepsilon(y)\left[\Phi^\varepsilon(y)\right]^{N-n}\left[1-\Phi^\varepsilon(y)\right]^{n-1} \mathrm{d}y = \\ &= \frac{n\cdot(n-1)}{(N-n)\cdot(N-n+1)}\int \phi^\varepsilon(y)\left[\Phi^\varepsilon(y)\right]^{N-n+1}\left[1-\Phi^\varepsilon(y)\right]^{n-2} \mathrm{d}y \\ &\vdots \\ &= \frac{n\cdot(n-1)\cdot(n-2)\cdots 1}{(N-n)\cdot(N-n+1)\cdots(N-1)}\int \phi^\varepsilon(y)[\Phi^\varepsilon(y)]^{N-1}dy \\ &= \frac{n!\,(N-n-1)!}{(N-1)!}\frac{1}{N}[\Phi^\varepsilon(y)]_{-\infty}^{\infty} \\ &= \frac{1}{N}\frac{1}{\binom{N-1}{n}}, \end{aligned}$$

[18]To evaluate this integral, recall that for two functions u and v of y intergration by parts gives

$$\int_a^b u(y)v'(y)dy = [u(y)v(y)]_a^b - \int_a^b u'(y)v(y)dy.$$

A convenient choice of u and v will prove to be $v'(y) := \phi^\varepsilon(y)[\Phi^\varepsilon(y)]^{N-n-1}$ and $u(y) := [1-\Phi^\varepsilon(y)]^n$. We thus have $u'(y) = -n[1-\Phi^\varepsilon(y)]^{n-1}\phi^\varepsilon(y)$ and $v(y) = \frac{1}{N-n}[\Phi^\varepsilon(y)]^{N-n}$.

which shows that $\overline{\Omega^{\varepsilon}}(n \mid X, X) = 1/N$ for all $n = 0, 1, ..., N-1$. Therefore

$$u_i^{\varepsilon}(p_{-i}^X \mid X, s_i) \to X + \sum_{n=0}^{N-1} w_i(n)\overline{\Omega^{\varepsilon}}(n \mid X, X) - c_i + s_i = X + \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - c_i + s_i,$$

as given. □

B.3 Proofs of Results in Section 5

PROOF OF PROPOSITION 2

Proof. Let $u_i^{\varepsilon}(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}, I)$ denote player i's expected incentive to invest given that his opponents play $p_{-i}^{\tilde{x}}$, his investment subsidy is $\tilde{s}_i$, and the critical threshold for investment is known to be I. Assuming i knows I (and that ε is sufficiently small), we have

$$u_i^{\varepsilon}(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}, I) = \tilde{s}_i + \frac{N-I}{N}(\tilde{x} + b_i) - c_i,$$

which follows from Lemma 5 which establishes that a player's strategic belief over aggregate investments (given $p_{-i}^{\tilde{x}}$ and $x_i^{\varepsilon} = \tilde{x}$) is uniform.

We assumed, however, that players do not know I, only that it is is uniformly distributed on $\{1, 2, ..., N\}$. Therefore player i's expected investment incentive in $(p_{-i}^{\tilde{x}}, \tilde{x}, \tilde{s}_i)$ is given by

$$u_i^{\varepsilon}(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}) = \frac{1}{N}\sum_{I=1}^{N} u_i^{\varepsilon}(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}, I) = \tilde{s}_i - c_i + \frac{1}{N}\sum_{I=1}^{N}\frac{N-I}{N}(\tilde{x} + b_i).$$

Noting that $\sum_{I=1}^{N} I/N = N/2$, we have

$$\frac{1}{N}\sum_{I=1}^{N}\frac{N-I}{N} = \frac{1}{N}\frac{N^2}{N} - \frac{1}{N}\frac{N}{2} = \frac{1}{2},$$

so

$$u_i^{\varepsilon}(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}) = \tilde{s}_i + \frac{\tilde{x} + b_i}{2} - c_i. \tag{26}$$

Finally, since $\tilde{s}_i$ is pinned down by the indifference condition $u_i^{\varepsilon}(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}, \bar{I}) = 0$, solving for $\tilde{s}_i$ yields the result. □

PROOF OF PROPOSITION 3

Proof. Given a_{-i} and the reward scheme v, the payoff to agent i is given by:

$$\pi_i(a_i, a_{-i} \mid x, v_i) = \begin{cases} x + v_i - c_i & \text{if the project succeeds and } a_i = 1 \\ x + v_i & \text{if the project succeeds and } a_i = 0 \\ -c_i & \text{if the project does not succeed and } a_i = 1 \\ 0 & \text{if the project does not succeed and } a_i = 0 \end{cases} \tag{27}$$

Because project success is stochastic through the technology q, we define

$$\pi_i(a_i, a_{-i} \mid x, v_i, q) = \begin{cases} q\left(\sum_{j\neq i} a_j + 1\right) \cdot (v_i + x) - c_i & \text{if } a_i = 1, \\ q\left(\sum_{j\neq i} a_j\right) \cdot (v_i + x) & \text{if } a_i = 0, \end{cases} \tag{28}$$

and an agent's incentive to work is:

$$u_i(a_{-i} \mid x, v_i, q) = \left(q\left(\sum_{j\neq i} a_j + 1\right) - q\left(\sum_{j\neq i} a_j\right)\right) \cdot (v_i + x) - c_i. \tag{29}$$

Define

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, v_i, q) = \int \left(q\left(\sum_{j\neq i} p_j(x_j^\varepsilon) + 1\right) - q\left(\sum_{j\neq i} p_j(x_j^\varepsilon)\right)\right) \cdot (v_i + x) - c_i \,\mathrm{d}F_i^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \tag{30}$$

Note that if p_{-i} is an increasing strategy vector, then by Lemma 1 the expected incentive $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, v_i, q)$ is increasing in x_i^ε since the technology q is increasing and supermodular. From the proof of Theorem 1 then follows that for any reward scheme v there exists a unique $x(v) = (x_i(v))$ such that $p^{x(v)}$ is the unique Bayesian Nash equilibrium of the game (provided ε is sufficiently small). Furthermore, it also follows that for any $\tilde{x}$, there is a unique reward scheme $\tilde{v} = (\tilde{v}_i)$ such that $x_i(\tilde{v}) = \tilde{x}$ for all $i \in \mathscr{N}$. We now proceed to characterizing $\tilde{v}$.

Next, recall that the reward scheme $\tilde{v}$ implements $p^{\tilde{x}}$ iff for all i the indifference condition

$$u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{v}_i) = \int (\tilde{v}_i + x)\left[q\left(\sum_{j\neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right) + 1\right) - q\left(\sum_{j\neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right)\right)\right] \mathrm{d}F_i^\varepsilon(x_{-i}^\varepsilon \mid \tilde{x}) - c_i = 0 \tag{31}$$

is met. Using (as was assumed throughout this section) that $\varepsilon \to 0$, we have

$$\begin{aligned} &\int (\tilde{v}_i + x)\left[q\left(\sum_{j\neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right) + 1\right) - q\left(\sum_{j\neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right)\right)\right] \mathrm{d}F_i^\varepsilon(x_{-i}^\varepsilon \mid \tilde{x}) \\ \to &(\tilde{v}_i + \tilde{x}) \int \left[q\left(\sum_{j\neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right) + 1\right) - q\left(\sum_{j\neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right)\right)\right] \mathrm{d}\overline{F_i^\varepsilon}(x_{-i}^\varepsilon \mid \tilde{x}). \end{aligned}$$

To see this, note first that the convergence $F_i^\varepsilon \to \overline{F_i^\varepsilon}$ follows from Proposition 7. Furthermore, writing $Q(\tilde{x}) := q\left(\sum_{j\neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right) + 1\right) - q\left(\sum_{j\neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right)\right)$, we have

$$\int (\tilde{v}_i + \tilde{x} - \varepsilon) Q(\tilde{x}) \,\mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}) \leq \int (\tilde{v}_i + x) Q(\tilde{x}) \,\mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}) \leq \int (\tilde{v}_i + \tilde{x} + \varepsilon) Q(\tilde{x}) \,\mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}),$$

which, noting that $\tilde{x}$, ε and $\tilde{v}_i$ are constants relative to the variables of integration, gives

$$(\tilde{v}_i + \tilde{x} - \varepsilon) \int Q(\tilde{x}) \,\mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}) \leq \int (\tilde{v}_i + x) Q(\tilde{x}) \,\mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}) \leq (\tilde{v}_i + \tilde{x} + \varepsilon) \int Q(\tilde{x}) \,\mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}).$$

When $\varepsilon \to 0$, the upper and lower bounds given above both converge to the same level $(\tilde{v}_i + \tilde{x}) \int Q(\tilde{x})\, \mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x})$ (since obviously $\tilde{x} + \varepsilon - (\tilde{x} - \varepsilon) = 2\varepsilon \to 0$ as $\varepsilon \to 0$), and so it follows that

$$\int (\tilde{v}_i + x) Q(\tilde{x})\, \mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}) \to (\tilde{v}_i + \tilde{x}) \int Q(\tilde{x})\, \mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}).$$

Invoking Lemma 5, we have

$$\int Q(\tilde{x})\, \mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}) = \int q\left(\sum_{j \neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right) + 1\right) - q\left(\sum_{j \neq i} p_j^{\tilde{x}}\left(x_j^\varepsilon\right)\right) \mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid \tilde{x}) \quad (32)$$

$$= \sum_{n=0}^{N-1} \frac{q(n+1) - q(n)}{N} := \overline{q}. \quad (33)$$

Therefore, $\tilde{v}_i$ solves

$$\overline{q} \cdot (\tilde{v}_i + \tilde{x}) - c_i = 0,$$

as given. □

PROOF OF PROPOSITION 4

Proof. Recall from the proof of Lemma 5 that $\Omega^\varepsilon(n \mid X, x_i^\varepsilon)$ denotes the probability that n players j receive a signal $x_j^\varepsilon \geq X$ while $N - n - 1$ receive a signal $x_j^\varepsilon < X$. Moreover, given that n players j receive a signal x_j^ε, the probability that any given subset of players $\{j_1, j_2, ..., j_n\} \subseteq \mathscr{N} \setminus \{i\}$ receive signals above X is the same (e.g. uniform) across such subsets; as there are exactly $\binom{N-1}{n}$ (unique) subsets $\{j_1, j_2, ..., j_n\} \subseteq \mathscr{N} \setminus \{i\}$, this (conditional) probability is simply $1/\binom{N-1}{n}$. Given the strategy vector p_{-i}^X played, and conditional on exactly n players j receiving a signal $x_j^\varepsilon > X$, the expected spillover on player i is hence $\sum_{a_{-i} \in A_{-i}^n} w_i(a_{-i})/\binom{N-1}{n}$, where we recall that $A_{-i}^n := \{a_{-i} \mid \sum_{a_j \in a_{-i}} a_j = n\}$. Putting all this together, we get

$$\begin{aligned} u_i^\varepsilon(p_{-i}^X \mid x_i^\varepsilon, s_i) &= \frac{\int x g(x) \phi^\varepsilon(x_i^\varepsilon - x) \mathrm{d}x}{\int g(x) \phi^\varepsilon(x_i^\varepsilon - x) \mathrm{d}x} + \sum_{n=0}^{N-1} \frac{\sum_{a_{-i} \in A_{-i}^n} w_i(a_{-i})}{\binom{N-1}{n}} \Omega^\varepsilon(n \mid X, x_i^\varepsilon) - c_i + s_i \\ &= \frac{\int x g(x) \phi^\varepsilon(x_i^\varepsilon - x) \mathrm{d}x}{\int g(x) \phi^\varepsilon(x_i^\varepsilon - x) \mathrm{d}x} + \sum_{n=0}^{N-1} w_i^n \Omega^\varepsilon(n \mid X, x_i^\varepsilon) - c_i + s_i \\ &\to x_i^\varepsilon + \sum_{n=0}^{N-1} w_i^n \overline{\Omega^\varepsilon}(n \mid X, x_i^\varepsilon) - c_i + s_i. \end{aligned}$$

Furthermore, we need only concern ourselves with the event that $x_i^\varepsilon = X$, in which case we have

$$u_i^\varepsilon(p_{-i}^X \mid X, s_i) \to X + \sum_{n=0}^{N-1} \frac{w_i^n}{N} - c_i + s_i,$$

where we use the result that $\overline{\Omega^\varepsilon}(n \mid X, X) = 1/N$ for all $n = 0, 1, ..., N-1$ established in the proof of Lemma 5. Finally, solving $u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i) = 0$ for $\tilde{s}_i$ yields the result. □

PROOF OF PROPOSITION 5

Proof. For each m, let $\Omega_m^\varepsilon(n \mid \tilde{x}_m, x_i^\varepsilon)$ denote the probability that n players $j \neq i$ in $\mathscr{N}_m$ receive a signal $x_j^\varepsilon \geq \tilde{x}_m$. Now recall that $\tilde{x}_1 < \tilde{x}_2 < \ldots < \tilde{x}_M$, all inequalities strict. Assume that ε is small enough so that $\tilde{x}_{m+1} - \tilde{x}_m > 2\varepsilon$ for all m (recall that $\varepsilon \to 0$, so this assumption will be satisfied). Conditional on his signal x_i^ε, player i knows that $x_j^\varepsilon \in [x_i^\varepsilon - \varepsilon, x_i^\varepsilon + \varepsilon]$ for each $j \neq i$. Hence, for each $k \in \{1, 2, ..., M\}$ we have $\Omega_k^\varepsilon(n_k \mid \tilde{x}_k, \tilde{x}_m) = 1$ for all $k = 1, 2, ..., m-1$ and $\Omega_k^\varepsilon(n_k \mid \tilde{x}_k, \tilde{x}_m) = 0$ for all $k = m+1, m+2, ..., M$. Moreover, for player $i \in \mathscr{N}_m$ we know that, following the same procedure used to prove Lemma 2,

$$\overline{\Omega_m^\varepsilon}(n \mid \tilde{x}_m, \tilde{x}_m) = \frac{1}{n_m},$$

for $n = 1, 2, ..., n_m - 1$. Let p^* denote the vector of strategies such that each player i is assigned strategy $p_i^{\tilde{x}_m}$ if $i \in \mathscr{N}_m$. Then, for each player $i \in \mathscr{N}_m$ and all $m \in \{1, 2, ..., M\}$, we have

$$u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_m, s_i) = \frac{\int x g(x) \phi^\varepsilon(\tilde{x}_m - x)\mathrm{d}x}{\int g(x)\phi^\varepsilon(\tilde{x}_m - x)\mathrm{d}x} + w_i\left(N_m + n\right)\Omega_m^\varepsilon(n \mid \tilde{x}_m, \tilde{x}_m) - c_i + s_i,$$

which, choosing ε sufficiently small, tends to:

$$u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_m, s_i) \to \tilde{x}_m + \sum_{n=0}^{n_m - 1} \frac{w_i(N_m + n)}{n_k} - c_i + s_i.$$

Solving $u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_m, \tilde{s}_i) = 0$ for s_i yields the result. □

PROOF OF THEOREM 2

Proof. (i) Given common knowledge of $\bar{x}$, the optimal ranking policy s^R offers subsidy $s_{i_n}^R = c - \bar{x} - w(n-1)$ to the n-th ranked player (see Segal (2003) and Winter (2004)). Hence,

$$K(s^R \mid \bar{x}) = \sum_{n=1}^{N} s_{i_n}^R = N \cdot (c - \bar{x}) + \sum_{m=0}^{N-1} w(m).$$

Next, we note that

$$K^\varepsilon(\bar{s} \mid \bar{x}) = \sum_{i \in \mathscr{N}} \bar{s}_i$$

as ε since $x_i(\bar{s}) < \bar{x}$ for all i; hence, as $\varepsilon \to 0$ we have $\bar{x} - \varepsilon > x_i(\bar{s})$ for all $i \in \mathscr{N}$ and the unique equilibrium outcome of $\Gamma^\varepsilon(\bar{s})$ is $(1, 1, ..., 1)$. Furthermore, we know from Theorem 1 that $\bar{s}_i \to s^*(\bar{x})$ as $\varepsilon \to 0$. Using $(*)$, we thus have

$$K^\varepsilon(\bar{s} \mid \bar{x}) = \sum_{i \in \mathscr{N}} \bar{s}_i \to \sum_{i \in \mathscr{N}} \left\{ c - \bar{x} - \sum_{n=0}^{N-1} \frac{w(n)}{N} \right\} = N \cdot (c - \bar{x}) - \sum_{n=0}^{N-1} w(n) = K(s^R \mid \bar{x})$$

as $\varepsilon \to 0$.

(ii) By Theorem 1, each $\bar{s}_i \to s_i^*(\bar{x})$ for all $i \in \mathscr{N}$. Furthermore, we can see from the definition of s_i^* in $(*)$ that $s_i^* = s_j^*$ for any two symmetric players $i, j \in \mathscr{N}$. Hence, because players assumed to be symmetric, we have $\bar{s}_i = \bar{s}_j$ for all $i, j \in \mathscr{N}$. □

PROOF OF PROPOSITION 6

Proof. Recall that $\hat{u}_i(a_{-i} \mid x, s) := \partial\hat{\pi}_i(a \mid x, s)/\partial a_i = x + w_i(a_{-i}) + s_i - c_i$. We observe that $\hat{u}_i(a_{-i} \mid x, s)$, a player's incentive to increase/decrease his action a_i, is independent of his own action and either strictly positive or strictly negative. The same clearly applies to $\hat{u}_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s)$, which we define as

$$\begin{aligned}\hat{u}_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) &:= \int \hat{u}_i(p_{-i}(x_{-i}^\varepsilon) \mid x, s)\,\mathrm{d}F_i^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &\to \int \hat{u}_i(p_{-i}(x_{-i}^\varepsilon) \mid x, s)\,\mathrm{d}\overline{F_i^\varepsilon}(x, x_{-i}^\varepsilon \mid x_i^\varepsilon).\end{aligned}$$

Hence, a player maximizes his (expected) payoff by playing $a_i = 1$ whenever $\hat{u}_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) > 0$ and playing $a_i = 0$ when $\hat{u}_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) < 0$. This means the exact same reasoning upon which the proof of Theorem 1 is based can be applied. □

References

Abel, A. B. (1983). Optimal investment under uncertainty. *American Economic Review*, 73(1):228–233.

Angeletos, G.-M., Hellwig, C., and Pavan, A. (2006). Signaling in a global game: Coordination and policy traps. *Journal of Political Economy*, 114(3):452–484.

Angeletos, G.-M., Hellwig, C., and Pavan, A. (2007). Dynamic global games of regime change: Learning, multiplicity, and the timing of attacks. *Econometrica*, 75(3):711–756.

Bandiera, O. and Rasul, I. (2006). Social networks and technology adoption in northern mozambique. *The economic journal*, 116(514):869–902.

Basak, D. and Zhou, Z. (2020). Diffusing coordination risk. *American Economic Review*, 110(1):271–97.

Beaman, L., BenYishay, A., Magruder, J., and Mobarak, A. M. (2021). Can network theory-based targeting increase technology adoption? *American Economic Review*, 111(6):1918–1943.

Bernstein, S. and Winter, E. (2012). Contracting with heterogeneous externalities. *American Economic Journal: Microeconomics*, 4(2):50–76.

Brekke, K. A., Kverndokk, S., and Nyborg, K. (2003). An economic model of moral motivation. *Journal of public economics*, 87(9-10):1967–1983.

Cai, J., Janvry, A. D., and Sadoulet, E. (2015). Social networks and the decision to insure. *American Economic Journal: Applied Economics*, 7(2):81–108.

Carlsson, H. and Van Damme, E. (1993). Global games and equilibrium selection. *Econometrica*, pages 989–1018.

Carroll, G. (2015). Robustness and linear contracts. *American Economic Review*, 105(2):536–563.

Chassang, S. (2010). Fear of miscoordination and the robustness of cooperation in dynamic global games with exit. *Econometrica*, 78(3):973–1006.

Cowan, R. (1990). Nuclear power reactors: a study in technological lock-in. *Journal of Economic History*, 50(3):541–567.

Cowan, R. and Gunby, P. (1996). Sprayed to death: path dependence, lock-in and pest control strategies. *Economic Journal*, 106(436):521–542.

Dai, T. and Toikka, J. (2022). Robust incentives for teams. *Econometrica*, 90(4):1583–1613.

Edmond, C. (2013). Information manipulation, coordination, and regime change. *Review of Economic studies*, 80(4):1422–1458.

Ferraro, P. J., Miranda, J. J., and Price, M. K. (2011). The persistence of treatment effects with norm-based policy instruments: evidence from a randomized environmental policy experiment. *American Economic Review*, 101(3):318–322.

Fischer, P. and Huddart, S. (2008). Optimal contracting with endogenous social norms. *American Economic Review*, 98(4):1459–1475.

Galeotti, A., Golub, B., and Goyal, S. (2020). Targeting interventions in networks. *Econometrica*, 88(6):2445–2471.

Goldstein, I. and Pauzner, A. (2005). Demand–deposit contracts and the probability of bank runs. *Journal of Finance*, 60(3):1293–1327.

Halac, M., Kremer, I., and Winter, E. (2020). Raising capital from heterogeneous investors. *American Economic Review*, 110(3):889–921.

Halac, M., Lipnowski, E., and Rappoport, D. (2021). Rank uncertainty in organizations. *American Economic Review*, 111(3):757–86.

Halac, M., Lipnowski, E., and Rappoport, D. (2022). Addressing strategic uncertainty with incentives and information. In *AEA Papers and Proceedings*, volume 112, pages 431–437.

Holmstrom, B. (1982). Moral hazard in teams. *The Bell journal of economics*, pages 324–340.

Jaravel, X., Petkova, N., and Bell, A. (2018). Team-specific capital and innovation. *American Economic Review*, 108(4-5):1034–1073.

Kets, W., Kager, W., and Sandroni, A. (2022). The value of a coordination game. *Journal of Economic Theory*, 201:105419.

Lane, T., Nosenzo, D., and Sonderegger, S. (2023). Law and norms: Empirical evidence. *American Economic Review*, 113(5):1255–1293.

Leister, C. M., Zenou, Y., and Zhou, J. (2022). Social connectedness and local contagion. *Review of Economic Studies*, 89(1):372–410.

Mas, A. and Moretti, E. (2009). Peers at work. *American Economic Review*, 99(1):112–145.

Morris, S. and Shadmehr, M. (2023). Inspiring regime change. *Journal of the European Economic Association*, page jvad023.

Morris, S. and Shin, H. S. (1998). Unique equilibrium in a model of self-fulfilling currency attacks. *American Economic Review*, pages 587–597.

Onuchic, P. and Ray, D. (2023). Signaling and discrimination in collaborative projects. *American Economic Review*, 113(1):210–52.

Pindyck, R. S. (1993). Investments of uncertain cost. *Journal of Financial Economics*, 34(1):53–76.

Sakovics, J. and Steiner, J. (2012). Who matters in coordination problems? *American Economic Review*, 102(7):3439–61.

Sandholm, W. H. (2002). Evolutionary implementation and congestion pricing. *Review of Economic Studies*, 69(3):667–689.

Sandholm, W. H. (2005). Negative externalities and evolutionary implementation. *Review of Economic Studies*, 72(3):885–915.

Segal, I. (1999). Contracting with externalities. *Quarterly Journal of Economics*, 114(2):337–388.

Segal, I. (2003). Coordination and discrimination in contracting with externalities: Divide and conquer? *Journal of Economic Theory*, 113(2):147–181.

Segal, I. R. and Whinston, M. D. (2000). Naked exclusion: comment. *American Economic Review*, 90(1):296–309.

Van Huyck, J. B., Battalio, R. C., and Beil, R. O. (1990). Tacit coordination games, strategic uncertainty, and coordination failure. *The American Economic Review*, 80(1):234–248.

Van Huyck, J. B., Battalio, R. C., and Beil, R. O. (1991). Strategic uncertainty, equilibrium selection, and coordination failure in average opinion games. *The Quarterly Journal of Economics*, 106(3):885–910.

Weidmann, B. and Deming, D. J. (2021). Team players: How social skills improve team performance. *Econometrica*, 89(6):2637–2657.

Winter, E. (2004). Incentives and discrimination. *American Economic Review*, 94(3):764–773.

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