Optimal risk sharing with translation invariant recursive utility for jump-diffusions

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Abstract

We consider optimal risk sharing where agents have preferences represented by translation invariant recursive utility. The dynamics in continuous time is driven by diffusion processes and a random jump measure. The model has some appealing features compared to the scale invariant version. Economic effects of sudden events, like catastrophes or pandemics, can be interpreted and separated from ordinary shocks to the economy. Unlike the scale invariant version, this model allows for a treatment of heterogeneous preferences, and consequently optimal risk sharing at a general and basic level. A new endogenous variable, a traded security, enters via the preference structure, affecting the key relations between agents. We also implement a stock market in this setting, and derive a consumption based capital asset model. A catastrophe-insurance forward contract is analyzed as an application of our general model, where the jump part is priced and plays the essential role.

KEYWORDS: optimal risk sharing, recursive utility, translation invariance, jump dynamics, CCAPM, the stochastic maximum principle, the mutuality principle, catastrophe forward contracts.
JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

We consider a version of Lucas' (1978) exchange economy in continuous time, where the uncertainty is modelled by jump-diffusions. The preferences of the agents are represented by translation invariant recursive utility functions.

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This version of recursive utility allows the agents to have different preferences, which means that we can study optimal risk sharing. Conditions under which the classical result that risk tolerant agents benefit from risk taking on the average, and compensate more risk averse agents by side-payments, is recovered, but our model can explain a variety of other interesting patterns, which cannot be accounted for by the standard model. For the one-period version of a syndicated market, see for example Borch (1960a,b,1962,1990), Wilson (1968) and Aase (2022).

The jump part of the model adds to the analysis. It shows that given enough time, the large shocks to the economy are transitory, that is, the economic effects of catastrophic events gradually diminish with time.

The consumption based CAPM is derived in the setting of our general model, and contains two additional terms compared to the standard expected utility model without jumps. These are caused by a new variable, endogenously derived, stemming from the recursive preferences, and the jump component. The new variable is an annuity, a marketed asset, which is different from the scale invariant version, where the corresponding variable is agent wealth.

Confronted with data, we indicate how the model has the potential to explain the standard empirical puzzles, like the equity premium puzzle (Mehra and Prescott (1985)), in connection with this kind of equilibrium models. The puzzle has been verified by others, e.g., Hansen and Singleton (1983).

The basic framework for the scale invariant version of recursive utility was developed by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994), which elaborate the foundational work by Kreps and Porteus (1978) and Epstein and Zin (1989-91) of recursive utility in dynamic models. We base the current paper on the method developed in Aase (2016), where the stochastic maximum principle was used to solve the relevant optimization problems. The current paper is an extension from Aase (2024b) to include jump dynamics. Aase (2024a) contains the discrete time version.

The inclusion of a jump part implies that the model can be used as a theoretical underpinning for financial contracts where jump dynamics is essential. This we have illustrated in the paper's last part, where a catastrophe insurance forward contract has been analyzed, and priced, with climate change as a background.

The paper is organized as follows: Section 2 introduces the basics of the jump-diffusion dynamics. Section 3 highlights some of the problems of the conventional model. In Section 4 we present the developments of the recursive model. The first order conditions using the stochastic maximum principle is outlined in Section 5. In Section 6 the financial market is established, and Section 7 analyzes the single agent economy. In Section 8 the formula for risk premiums and the short rate are presented. Aggregation is discussed in Section 9. Section 10 addresses some market implications. Section 11 treats the mutuality principle, and optimal risk sharing. A few related issues are discussed in Section 12, a catastrophe insurance contract is analyzed in Section 13, and Section 14 concludes. The paper contains two appendices with more technical material.

2 Jump-diffusions

The stochastic processes in this paper will be jump-diffusions. A jump-diffusion process (sometimes called an Itô-Lévy process) is of the form

$$dX(t) = \mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}^l} \gamma(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T, \tag{1}$$

where $\mu : [0,T] \to \mathbb{R}^N$, $\sigma : [0,T] \to \mathbb{R}^{N\times d}$ amd $\gamma : [0,T] \times \mathbb{R}^l \to \mathbb{R}^{N\times l}$ are predictable processes such that the integrals exist. Here $B(t)$ is a d-dimensional Brownian motion and

$$\begin{aligned}\tilde{N}(dt,dz)' &= (\tilde{N}_1(dt,dz_1), \cdots, \tilde{N}_l(dt,dz_l)) \\ &= (N_1(dt,dz_1) - \nu_1(dz_1)dt, \cdots, N_l(dt,dz_l) - \nu_l(dz_l)dt) \end{aligned} \tag{2}$$

where prime means transpose, $z = (z_1, z_2, \cdots, z_l)$ and B_t and $\tilde{N}(dt,dz)$ are independent martingales. Here $\{N_j\}$ are independent Poisson random measures with Lévy measures ν_j coming from l independent (one-dimensional) Lévy prodesses $\eta_1, \cdots \eta_l$ satisfying $E[\eta_i^2(t)] < \infty$ for all $t \in [0,T]$, $i = 1, 2, \cdots, l$.

We let $\mathcal{F}_t^B$ be the σ-algebra generated by $B(s); s \le t$ and we let $\mathcal{F}_t^N$ be the σ-algebra generated by $N(ds,dz); s \le t$. Finally we set $\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{F}_t^N$, the σ-algebra generated by $\mathcal{F}_t^B$ and $\mathcal{F}_t^N$.

Both risky assets, consumption dynamics and indexes will be of this form in what follows. For example could (1) be the model of a stock market consisting of N risky assets. If there exists a risk-less asset, its dynamics is given by

$$dX_0(t) = r_t X_0(t)dt, \quad X_0 = x_0 > 0$$

where r_t is the short rate and x_0 is a constant.

Consider an arbitrary security with strictly positive price process

$$dX_n(t) = \mu_n(t)dt + \sigma_n(t)dB(t) + \int_{\mathbb{R}^l} \gamma_n(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T.$$

Here $\sigma_n(t)$ and $B(t)$ are d-dimensional vectors, and $\gamma_n(t,z)$ and $\tilde{N}(dt,dz)$ are both l-dimensional vectors. The cumulative-return process of this security is an Itô-Lévy process R_i defined by $R_i(0) = 0$ and

$$dR_n(t) = \mu_{R_n}(t)dt + \sigma_{R_n}(t)dB(t) + \int_{\mathbb{R}^l} \gamma_{R_n}(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T,$$

where $\mu_{R_n}(t) = \mu_n(t)/X_n(t)$ is the rate of return on the n-th risky asset, $\sigma_{R_n}(t) = \sigma_n(t)/X_n(t)$ is the return volatility of the continuous part, and $\gamma_{R_n}(t,z) = \gamma_n(t,z)/X_n(t)$ is the corresponding quantity associated to the jumps of the n-th risky asset, $n = 1, 2, \dots N$.

Consumption processes $c(t)$ are also one-dimensional and would typically have dynamics

$$dc(t) = \mu_c(t)dt + \sigma_c(t)dB(t) + \int_{\mathbb{R}^l} \gamma_c(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T, \qquad (3)$$

where $\sigma_c(t)$ and $B(t)$ are d-dimensional vectors, and $\gamma_c(t,z)$ and $\tilde{N}(dt,dz)$ are both l-dimensional vectors. With this interpretation the term $\mu_c(t)/c(t^-)$ is the growth rate of consumption at time t.

Smooth functions of jump-diffusions are again of this type, a result of Itô's lemma for such processes. We have also a version of Girsanov's theorm for jump-diffusions, important for the pricing of derivatives in such markets.

3 The problems with the conventional model

The conventional asset pricing model in financial economics, the consumption-based capital asset pricing model (CCAPM) of Breeden (1979), assumes a representative agent with a utility function of consumption that is the expectation of a sum, or a time integral, of future discounted utility functions. The model has been criticized for several reasons. First, it does not perform well empirically. Second, the usual specification of utility can not separate the risk aversion from the elasticity of intertemporal substitution, while it would clearly be advantageous to disentangle these two conceptually different aspects of preference.

In the conventional model the utility $U(c)$ of a consumption stream c_t is given by $U(c) = E\{\int_0^T u(c_t,t)\,dt\}$, where the felicity index u has the separable form $u(c_t,t) = \frac{1}{\alpha}(1 - e^{-\alpha c_t})e^{-\delta t}$. The parameter α is the representative agent's absolute risk aversion and δ is the utility discount rate, or the impatience rate. T is the time horizon. These parameters are assumed to satisfy $\alpha \ge 0$, $\delta \ge 0$, and $T < \infty$.

When jumps are included the risk premium $(\mu_R - r)$ of any risky security labeled R (for "risky") is given by

$$\mu_R(t) - r_t = \alpha\, \sigma_{c,R}(t) + \alpha \int_{\mathcal{Z}} \gamma_c(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta). \qquad (4)$$

Here r_t is the equilibrium real interest rate at time t, and the term $\sigma_{c,R}(t) = \sum_{i=1}^{d} \sigma_{c,i}(t)\sigma_{R,i}(t)$ is the covariance rate between returns of the risky asset and the aggregate consumption at time t, a measurable and adaptive process satisfying standard conditions. The dimension of the Brownian motion is $d > 1$. Underlying the jump dynamics we have $\{N_j\}$, $j = 1, 2, \cdots, l$ independent Poisson random measures with Lévy measures ν_j coming from l independent (1-dimensional) Lévy processes. The possible time inhomogeneity in the jump processes is expressed through the terms denoted $\gamma_{R,j}(t, \zeta_j)$ for the risky asset under consideration, and $\gamma_{c,j}(t, \zeta_j)$ for the aggregate consumption process, both measuring the jump sizes. The jump frequencies at time t are embedded in the Lévy measures. The "mark space" $\mathcal{Z} = \mathbb{R}^l$ in this paper, where $\mathbb{R} = (-\infty, \infty)$. Thus the last term in (4) is short-hand notation for the following

$$\int_{\mathcal{Z}} \gamma_c(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) = \sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{c,j}(t,\zeta_j))\gamma_{R,j}(t,\zeta_j)\nu(d\zeta_j).$$

This is a continuous-time version of the consumption-based CAPM, allowing for jumps at random time points. Similarly the expression for the risk-free, real interest rate is

$$r_t = \delta + \alpha\, \mu_c(t) - \frac{1}{2}\, \alpha^2 \sigma_c'(t)\sigma_c(t) - \frac{1}{2}\, \alpha^2 \int_{\mathcal{Z}} \gamma_c(t,\zeta)\, \gamma_c(t,\zeta)\nu(d\zeta). \qquad (5)$$

In the risk premium (4), the last term stems from the the discontinuous dynamics and is the jump interpretation as the previous term, while in (5) the last term is the jump analog of the previous term. The consumption process is assumed to have the dynamics given by (3).

If the consumption process were as volatile as the stock market index, the jump dynamics could potentially contribute to a better explanation of empirical regularities than the continuous model can alone. However, because of the relatively small sizes of the potential jumps in the consumption process, it is unlikely that the last terms in these two relationships move these quantities enough in the more plaussible direction. As with the continuous model, the problem stems from the low covariance rate between consumption and the market index and the low consumption variance.

Notice that the instantaneous correlation coefficient between returns and the consumption is given by

$$\kappa_{c,R}(t) = \frac{\sigma_{c,R}(t)}{||\sigma_R(t)|| \cdot ||\sigma_c(t)||} = \frac{\sum_{i=1}^{d} \sigma_{R,i}(t)\sigma_{c,i}(t)}{\sqrt{\sum_{i=1}^{d} \sigma_{R,i}(t)^2}\sqrt{\sum_{i=1}^{d} \sigma_{c,i}(t)^2}},$$

and similarly for other correlations given in this model. Here $-1 \leq \kappa_{c,R}(t) \leq 1$ for all t. With this convention we can equally well write $\sigma_R'(t)\sigma_c(t)$ for $\sigma_{c,R}(t)$, and the former does *not* imply that the instantaneous correlation coefficient between returns and the consumption growth rate is equal to one. Prime means transpose.

The process $\mu_c(t)$ is interpreted as the annual drift rate of aggregate consumption while $(\sigma_c'(t)\sigma_c(t))$ is the annual consumption variance, both at time t, again dictated by the Itô-isometry. Both these quantities are measurable and adaptive stochastic processes, satisfying usual conditions.

In the data below we have consumption growth rates and the associated volatility of the growth rate, which we obtain by simply dividing these quantities by c_t (and c_t^2). Also the relative risk aversion is needed in empirical work, which is obtained from from the absolute risk aversion α via $(\alpha c_t) := \tilde{\alpha}_t$. To prepare for calibration to data, we assume that the consumption process has the following dynamics

$$\frac{dc(t)}{c_t} = \tilde{\mu}_c(t)dt + \tilde{\sigma}_c(t)dB(t) + \int_{\mathbb{R}^l} \tilde{\gamma}_c(t,\zeta)\tilde{N}(dt,d\zeta), \quad 0 \leq t \leq T. \qquad (6)$$

This is consistent with the representation in (3) provided

$$\tilde{\mu}_c(t) = \mu_c(t)/c_t, \;\; \tilde{\sigma}_c(t) = \sigma_c(t)/c_t, \text{ and } \tilde{\gamma}_c(t,\zeta) = \gamma_c(t,\zeta)/c_t.$$

The results in (4) and (5) can then be rewritten in terms of consumption growth-rate characteristics and relative risk aversion as follows:

$$\mu_R(t) - r_t = \tilde{\alpha}_t\tilde{\sigma}_c(t)\sigma_R(t) + \tilde{\alpha}_t \int_{\mathcal{Z}} \tilde{\gamma}_c(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta). \qquad (7)$$

and

$$r_t \;=\; \delta \,+\, \tilde{\alpha}_t\,\tilde{\mu}_c(t) \,-\, \frac{1}{2}\,\tilde{\alpha}_t^2\tilde{\sigma}_c(t)'\tilde{\sigma}_c(t) \,-\, \frac{1}{2}\,\tilde{\alpha}_t^2 \int_{\mathcal{Z}} \tilde{\gamma}_c(t,\zeta)\tilde{\gamma}_c(t,\zeta)\nu(d\zeta). \quad (8)$$

In calibrations the return processes as well as the consumption growth rate process in this paper are also assumed to be such that statistical estimation makes sense (ergodic processes).

Similarly the term $\sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{R,j}(t,\zeta_j)(\gamma_{c,j}(t,\zeta_j)/c_t)\nu(d\zeta_j)$ is the covariance rate at time t between returns of the risky asset and aggregate consumption stemming from the discontinuous dynamics. We use the short-hand notation $\int_{\mathcal{Z}} \gamma_R(t,\zeta)\tilde{\gamma}_c(t,\zeta)\nu(d\zeta)$ for this term as well.

In Table 1 we present the key summary statistics of the data in Mehra and Prescott (1985), of the real annual return data related to the S&P-500, denoted by M, as well as for the annualized consumption data, denoted c, and the government bills, denoted b [1].

	Expectat.	Standard dev.	Covariances
Consumption growth	1.83%	3.57%	$\text{cov}(M,c) = .002226$
Return S&P-500	6.98%	16.54%	$\text{cov}(M,b) = .001401$
Government bills	0.80%	5.67%	$\text{cov}(c,b) = -.000158$
Equity premium	6.18%	16.67%	

Table 1: Key US-data for the time period 1889-1978. Discrete-time compounding.

Here we have, for example, estimated the covariance between aggregate consumption and the stock index directly from the data set to be .00223. This gives the estimate .3770 for the correlation coefficient [2].

Since our development is in continuous time, we have carried out standard adjustments for continuous-time compounding, from discrete-time compounding. The results of these operations are presented in Table 2. This gives, e.g., the estimate $\hat{\kappa}_{Mc} = .4033$ for the instantaneous correlation coefficient $\kappa(t)$. The overall changes are in principle small, and do not influence our comparisons to any significant degree, but are still important.

	Expectation	Standard dev.	Covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 2: Key US-data for the time period 1889-1978. Continuous-time compounding.

[1]There are of course newer data by now, but these retain the same basic features. If our model can explain the data in Table 1, it can explain any of the newer sets as well.

[2]The full data set was provided by Professor Rajnish Mehra.

Interpreting the risky asset R as the value weighted market portfolio M corresponding to the S&P-500 index, equations (7) and (8) are two equations in two unknowns that can provide estimates of the two preference parameters by the "method of moments". The result is $\tilde{\alpha}_t = 26.3$ and $\delta = -0.028$, i.e., a relative risk aversion of about 26 and an impatience rate of minus 2.8%. Both these are of course implausible.

Here we have not separated the data in continuous and jump parts, so this would be the result of the continuous model.

This is the same puzzle we obtain when we use the CRRA power felicity index $u(c_t, t) = \frac{1}{1-\gamma} c_t^{1-\gamma} e^{-\delta t}$, in which case the relative risk aversion $\gamma = 26.3$ and $\delta = -0.015$. As far as the equity premium puzzle is concerned, these models are more or less equivalent. The models obviously do not explain the data.

The jump terms might mitigate these numbers somewhat, since the jump model can, under certain assumptions, produce a larger equity premium than the continuous model can alone. It is an empirical question to estimate these quantities (e.g., Ait Sahalia and Jacod (2009-11)), but see below.

3.1 Deviations from normality in the standard model

In the conventional model we may use jump dynamics to study the effects of deviations from normality. This we have done by using the pure jump model alone to fit the data summarized in Table 1, and its logarithmic version. In doing so we have fixed the frequency of "jumps" to one per year on the average. The advantage with this approach is that we do not have to separate the jump dynamics from the continuous part in the data. We have modelled the simultaneous jumps in the Lévy-measure $\nu(d\zeta_1, d\zeta_2)$ by a joint Normal Inverse Gaussian (NIG)-distribution. This distribution measures heavy tails, kurtosis, skewness, etc, often found in financial stock market data. It fits fat-tailed and skewed data very well and is analytically tractable. This distribution was brought to the attention of workers in empirical finance by Barndorff-Nielsen (1997).

The result of this analysis weakened the puzzle somewhat, using the above model when calibrated to the data (for details, see Aase and Lillestøl (2015)).

By maximum likelihood estimators for the NIG-parameters, we obtain the same estimates of the moments as given in Table 1, from which we obtain the following calibrated values: $(\gamma, \delta) = (22.2, 0.0083)$. Moreover, by varying the NIG-estimates, one by one, within the bounds given by sampling errors, and using resampling techniques, the puzzle was further weakened to $(17.7, 0.058)$, assuming a felicity index of the CRRA-type.

As a comparison, under joint normality we get $(\gamma, \delta) = (24.3, -.044)$.

Jumps alone move the risk premium down somewhat relative to the diffusion model, deviations from normality accounts for the rest.

The result of this is encouraging for the task we now set out to do, namely to include jumps in the recursive model.

4 Recursive Stochastic Differentiable Utility

In this section we give a brief introduction to recursive, stochastic differential utility in the continuous-time model including jumps, along the lines of Øksendal and Sulem (2014). The starting point for this theory for the continuous model is Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994). Our approach based on Øksendal and Sulem (2014) includes jump dynamics.

We are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0,T], P)$ satisfying the 'usual' conditions, and a standard model for the stock market with Lévy-process driven uncertainty, N risky securities and one riskless asset (Section 6 provides more details). Consumption processes are chosen from the space L of square integrable progressively measurable processes with values in R_+. The agent has utility function U, to be specified below, and an endowment process $e \in L$.

The stochastic differential utility $U : L \to \mathbb{R}$ is defined as follows by three primitive functions: $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $A : \mathbb{R} \to \mathbb{R}$ and $A_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The function $f(t, c_t, U_t, \omega)$ corresponds to a felicity index at time t, A is associated with a measure of absolute risk aversion related to the continuous dynamics, while A_0 is connected to a similar measure related to jump size risk. Both the latter two terms may also depend on t. In addition to current consumption c_t, the function f also depends on utility U_t.

The utility process U for a given consumption process c that we consider, satisfying $U_T = 0$, is given by the representation

$$U_t = E_t\Big\{ \int_t^T \big(f(s, c_s, U_s) - \frac{1}{2} A(U_s)\, Z(s)' Z(s) \\ - \frac{1}{2} \int_{\mathcal{Z}} A_0(U_s, \zeta) K'(s,\zeta) K(s,\zeta) \nu(d\zeta) \big) ds \Big\}, \quad t \in [0,T] \quad (9)$$

where $E_t(\cdot)$ denotes conditional expectation given $\mathcal{F}_t$, and $Z(t)$ as well as $K(t, \cdot)$ are square-integrable progressively measurable processes, to be determined in our analysis. The term $Z(t)'Z(t)dt = d[U^c, U^c]_t$ where $[U^c, U^c]_t$ is the quadratic variation of the continuous part of V, and

$$\int_{\mathcal{Z}} K'(t,\zeta)K(t,\zeta)N(d\zeta,dt) = d[U^{di},U^{di}]_t$$

is the quadratic variation of the discontinuous part of U. The Brownian motion B_t has dimension d, and $K(t,\cdot)$ is an l dimensional vector. We think of U_t as the utility for c at time t, conditional on current information $\mathcal{F}_t$. The term $A(U_t)$ is penalizing for risk in the continuous model, while the term $A_0(U_t,\cdot)$ has the same property for jump size risk.

Recall the *timeless* situation with a mean zero risk X having variance σ^2, where the certainty equivalent m is defined by $Eu(w+X) := u(w-m)$ for a constant wealth w. Then the Arrow-Pratt approximation to m, valid for "small" risks, is given by $m \approx \frac{1}{2}A(w)\sigma^2$, where $A(\cdot)$ is the absolute risk aversion associated with u. This approximation is often good also when risks are not necessarily "small". In continuous-time models the approximation is exact.

If, for each consumption process c_t, there is a well-defined utility process U, the stochastic differential utility U is defined by $U(c) = U_0$, the initial utility. The triplet (f, A, A_0) generating U is called an aggregator.

Since $U_T = 0$ and $\int Z(t)dB_t$ and $\int\int_{\mathcal{Z}} K(t,\zeta)\tilde{N}(dt,d\zeta)$ are assumed to be martingales, (9) has the stochastic differential equation representation

$$dU_t = \Big(- f(t,c_t,U_t) + \frac{1}{2}A(U_t)\, Z(t)'Z(t) + \\ \frac{1}{2}\int_{\mathcal{Z}} A_0(U_t,\zeta)K'(t,\zeta)K(s,\zeta)\nu(d\zeta)\Big)\, dt + Z(t)\, dB_t + \int_{\mathcal{Z}} K(t,\zeta)\tilde{N}(dt,d\zeta). \tag{10}$$

Here $\tilde{N}(dt,d\zeta) = N(dt,d\zeta) - \nu(d\zeta)dt$ is an l-dimensional compensated Poisson random measure of the underlying l-dimensional Lévy process, and $B(t)$ is an independent d dimensional, standard Brownian motion.

If terminal utility different from zero is of interest, like for applications to life insurance, then U_T may be different from zero. We may think of A and A_0 as associated with functions $h, h_0 : \mathbb{R} \to \mathbb{R}$ such that $A(u) = -\frac{h''(u)}{h'(u)}$, where h is two times continuously differentiable, and similarly for h_0. U is monotonic and risk averse if $A(\cdot) \geq 0$, $A_0(\cdot,\cdot) \geq 0$ and f is jointly concave and increasing in consumption.

The preference ordering represented by recursive utility is usually assumed to satisfy A1: Dynamic consistency (in the sense of Johnsen and Donaldson (1985)), A2: Independence of past consumption, and A3: State independence of time preference (see Skiadas (2009)).

The version of RU with scale invariance and jump/diffusions has been examined by Aase (2020) and Schroder and Skiadas (2008).

The version we consider has the Kreps-Porteus CES utility representation in discrete time, which here corresponds to the aggregator with the specification

$$f(c,u) = \frac{\delta}{\theta}\frac{e^{-\theta u} - e^{-\theta c}}{e^{-\theta u}}, \quad A(u) = \alpha \text{ and } A_0(u,\zeta) = \alpha_0, \ \forall u, \zeta \in R \qquad (11)$$

This is the continuous-time limit of recursive exponential utility, examined in discrete time by Skiadas (2009), Aase (2024a) and in continuous time with Brownian uncertainty only by Aase (2024b).

The above version will be demonstrated below to be translation invariant, while the standard version of recursive utility, corresponding to the Epstein-Zin model in discrete time, is known to be scale invariant.

If $A(u) = A_0(u,\zeta) = 0$ for all u, ζ, this means that the recursive utility agent is risk neutral, which here requires that the parameter $\alpha = \alpha_0 = 0$. Also, when the parameter $\theta \to 0$, then $f(c,u) \to \delta(c-u)$.

The parameters are assumed to satisfy $\delta \geq 0, \theta \geq 0, \alpha \geq 0$. For fixed δ and θ, increasing α increases absolute risk aversion. The parameter δ is the impatience rate, while θ is the relative degree of resistance to intertemporal substitution of consumption, inversely related to the elasticity of the intertemporal substitution in consumption in a mutliperiod setting, as we will come back to below.

At this stage, when $\alpha = \theta$ and $\alpha_0 = 0$, the standard separable, additive EU representation results, where

$$U(c) = E\Big(\int_0^T e^{-\delta t}\frac{1}{\alpha}(1 - e^{-\alpha c_t})\, dt\Big).$$

General existence results for our application do not yet exist, although several partial results are available in the BSDE literature.[3]

It is instructive to recall the that the conventional additive and separable

[3]For example, Duffie and Epstein (1992b) applies a general filtration but requires that there is an ordinally equivalent version in which the aggregator is not a function of the volatility terms. Also required is a Lipschitz restriction on the aggregator. Pardoux (1997) showed an existence result for the case of Poisson random measures with a Lipschitz restriction. For the aggregator of the Kreps and Porteus type, a Lipschitz condition in the above reference related to the drift term of the BSDE is not satisfied, however, existence and uniqueness has then been proven in Duffie and Lions (1992) for diffusion processes. One may conjecture that this can be extended to jump-diffusions, since it is the drift term that poses the problem.

utility has aggregator

$$f(c,u) = u(c) - \delta u, \quad A = 0, \quad A_0 = 0. \tag{12}$$

in the present framework (an ordinally equivalent one). As can be seen, even if $A = A_0 = 0$, the agent of the conventional model is not risk neutral.

4.1 Translation invariance

The following result will be made use of in sections 7.3-4. For a given consumption process c_t we let $(U_t^{(c)}, Z_t^{(c)}, K_t(\zeta)^{(c)})$ be the solution of the BSDE

$$\begin{cases} dU_t^{(c)} = \Big(- f(t, c_t, U_t^{(c)}) + \frac{1}{2} A(U_t^{(c)})\, Z(t)'^{(c)} Z(t)^{(c)} + \\ \frac{1}{2} \int_{\mathcal{Z}} A_0(U_t^{(c)}, \zeta) K'(t,\zeta)^{(c)} K(t,\zeta)^{(c)} \nu(d\zeta) \Big)\, dt + Z(t)^{(c)}\, dB_t \\ + \int_{\mathcal{Z}} K(t,\zeta)^{(c)} \tilde{N}(dt, d\zeta); \qquad 0 \le t \le T \\ U_T^{(c)} = 0 \end{cases} \tag{13}$$

Theorem 1 *Assume that, for all $\eta \in \mathbb{R}$,*
(i) $f(t,\ c + \eta,\ u + \eta) = f(t,\ c,\ u)$; $\forall\ t,\ c,\ u,\ \omega$
(ii) $A(u + \eta) = A(u)$; $\forall\ u$
(iii) $A_0(u + \eta) = A_0(u)$; $\forall\ u$
Then

$$U_t^{(c+\eta)} = U_t^{(c)} + \eta,\ Z_t^{(c+\eta)} = Z_t^{(c)} \text{ and } K_t^{(c+\eta)}(\zeta) = K_t^{(c)}(\zeta);\ \forall \zeta,\ t \in [0,T]. \tag{14}$$

Proof: Assuming uniqueness of the solution of the BSDEs of the type (13), all we need to do is to verify that the triple $(\lambda U_t^{(c)}, \lambda Z_t^{(c)}, \lambda K_t(\cdot)^{(c)})$ is a solution of the BSDE (13) with c_t replaced by $c_t + \eta$, i.e. that

$$\begin{cases} d(U_t^{(c+\eta)}) = \Big(- f(t, c_t + \eta, U_t^{(c+\eta)}) + \frac{1}{2} A(U_t^{(c+\eta)})\, Z(t)'^{(c+\eta)} Z(t)^{(c+\eta)} + \\ \frac{1}{2} \int_{\mathcal{Z}} A_0(U_t^{(c+\eta)}, \zeta) K'(t,\zeta)^{(c+\eta)} K(t,\zeta)^{(c+\eta)} \nu(d\zeta) \Big)\, dt + Z'(t)^{(c+\eta)}\, dB_t \\ + \int_{\mathcal{Z}} K(t,\zeta)^{(c+\eta)} \tilde{N}(dt, d\zeta); \qquad 0 \le t \le T \\ U_T^{(c+\eta)} = 0 \end{cases} \tag{15}$$

By (i), (ii) and (iii) and the quadratic variation interpretations of $Z'Z$ and $K'Kd\nu$, the BSDE (15) can be written

$$\begin{cases} dU_t^{(c)} = \Big(- f(t, c_t, U_t^{(c)}) + \frac{1}{2}A(U_t^{(c)})\, Z(t)'^{(c)} Z(t)^{(c)} + \\ \frac{1}{2}\int_{\mathcal{Z}} A_0(U_t^{(c)}, \zeta) K'(t,\zeta)^{(c)} K(t,\zeta)^{(c)} \nu(d\zeta)\Big)\, dt + Z(t)^{(c)}\, dB_t \\ + \int_{\mathcal{Z}} K(t,\zeta)^{(c)} \tilde{N}(dt, d\zeta); \qquad 0 \le t \le T \\ U_T^{(c)} = 0 \end{cases} \tag{16}$$

But this is exactly the equation (13). Hence (16) holds and the proof is complete. □

Remarks 1) Note that the system need not be Markovian in general, since we allow

$$f(t, c, u, \omega);\ (t, \omega) \in [0, T] \times \Omega$$

to be an adapted process, for each fixed c, u.
2) Similarly, we can allow A_0 and A to depend on t as well[4].

Corollary 1 *Define $U(c) = U_0^{(c)}$. Then $U(c + \eta) = \lambda U(c) + \eta$ for all $\eta \in \mathbb{R}$.*

Notice that the aggregator in (11) satisfies the assumptions of the theorem.

4.2 Itô's Lemma for Jump-Diffusions

Also jump-diffusion processes have the pleasant property that smooth functions of such processes are again jump-diffusions, which is of great importance in applications. We will use the following one-dimensional extension of Itô's formula to jump-diffusions:. Consider a jump-diffusion model of the following form:

$$dX(t) = \mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}^l} \gamma(t, z)\tilde{N}(dt, dz), \quad 0 \le t \le T. \tag{17}$$

[4] not common in economics

Let $f \in C^{1,2}([0,T] \times \mathbb{R}^2; \mathbb{R})$ and define $Y(t) = f(t, X(t))$. Then $Y(t)$ is again a jump-diffusion with the representation

$$\begin{aligned} dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}\big(t, X(t)\big)\big(\mu(t)dt + \sigma(t)dB(t)\big) \\ + \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial x^2}\big(t, X(t)\big)dt \\ + \int_{\mathbb{R}} \Big\{ f\big(t, X(t^-) + \gamma(t)\big) - f(t, X(t^-)) - \frac{\partial f}{\partial x}\big(t, X(t^-)\big)\gamma(t) \Big\} \nu(dz)dt \\ + \int_{\mathbb{R}} \Big\{ f\big(t, X(t^-) + \gamma(t)\big) - f\big(t, X(t^-)\big) \Big\} \tilde{N}(dt, dz). \quad (18) \end{aligned}$$

As with ordinary diffusion processes, there is also a convenient extension of this result to the multidimensional case, see for example Gihman and Skorohod (1979) or Øksendal and Sulem (2018).

5 The First Order Conditions

In the following we solve the consumer's optimization problem using the stochastic maximum principle and forward/backward stochastic differential equations. We have the specification in (10) and (11) in mind, formulated in the previous section, where the $\tilde{f}$ to appear below is the drift term in (10). However, in principle the analysis is valid for any f, A and A_0 satisfying the stated conditions. The representative agent's problem is to solve

$$\sup_{c \in L} U(c)$$

subject to

$$E\Big\{ \int_0^T c_t \pi_t dt \Big\} \leq E\Big\{ \int_0^T e_t \pi_t dt \Big\},$$

where e is the endowment process of the agent. Here $U_t = U_t^c$, and $(U_t, Z(t), K(t, \cdot))$ is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dU_t = -g(t, c_t, U_t, Z(t), K(t, \zeta))\, dt + Z(t)\, dB_t + \int_{\mathcal{Z}} K(t, \zeta) \tilde{N}(dt, d\zeta) \\ U_T = 0, \end{cases} \quad (19)$$

where

$$g(t, c, u, z, k) = f(c, u) - \frac{1}{2}A(u)z'z - \frac{1}{2}\int_{\mathcal{Z}} A_0(u, \zeta)k'(t, \zeta)k(t, \zeta)\nu(d\zeta). \quad (20)$$

Here f, A and A_0 are given in (11).

For $\lambda > 0$ we define the Lagrangian

$$\mathcal{L}(c;\lambda) = U(c) - \lambda\, E\Big(\int_0^T \pi_t(c_t - e_t)dt\Big).$$

The volatility $Z(t)$ as well as the jump size quantity $K(t,\zeta)$ are both part of the solution, together with the dynamics of utility U. Market clearing combined with properties of recursive utility in Theorem 1 will be used to internalize the corresponding quantities for "prices", by connecting these to Z and K.

In order to set down the first order condition for the representative consumer's problem, we use Kuhn-Tucker and either directional derivatives in function space, or the stochastic maximum principle. Both these methods are rather robust. The problem is well posed since U is increasing and concave and the constraint is convex.

Below we utilize the stochastic maximum principle (see Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Øksendal and Sulem (2014), Hu and Peng (1995), or Peng (1990)): We then have a forward backward stochastic differential equation (FBSDE) system consisting of the simple FSDE $dX(t) = 0; X(0) = 0$ and the BSDE (19). The Hamiltonian for this problem is

$$H(t,c,u,z,k,y) = y_t\, g(t,c_t,u_t,z_t,k_t) - \lambda\, \pi_t(c_t - e_t), \tag{21}$$

Here y_t refers to the adjoint variable to be defined shortly. Let $\bigtriangledown_k g$ denote the Frechet derivative of g with respect to k, and $\frac{d\bigtriangledown_k g}{d\nu}(\zeta)$ denote its Radon-Nikodym derivative with respect to ν. The adjoint equation is then

$$\begin{cases} dY(t) = Y(t\text{-})\big\{\big(\frac{\partial f}{\partial u}(c_t,U_t) - \frac{1}{2}(\frac{\partial}{\partial u}A(U_t))\, Z'(t)Z(t) \\ -\frac{1}{2}\int_{\mathcal{Z}}(\frac{\partial}{\partial u}A_0(U_t,\zeta))\, K'(t,\zeta)K(t,\zeta)\nu(d\zeta)\big)dt \\ -\frac{1}{2}\frac{\partial}{\partial z}\big(A(U_t)Z_t'Z_t\big)dB_t + \int_{\mathcal{Z}} \frac{d\bigtriangledown_k g}{d\nu}(t,c_t,U_t,Z_t,K(t,\cdot))(\zeta)\tilde{N}(dt,d\zeta)\big\}, \\ Y(0) = 1. \end{cases}$$

With a general form of $A_0(u,\zeta)$ as in (9), we see that the Frechet derivative, $\bigtriangledown_k g$, is the linear operator

$$h \to (\bigtriangledown_k g)(h) = -\int_{\mathcal{Z}} A_0(u,\zeta)k'(\zeta)h(\zeta)\nu(d\zeta);\;\; h \in L^2(\nu).$$

Therefore, as a random measure we have that $\bigtriangledown_k g << \nu$, with Radon-

Nikodym derivative

$$\frac{d\,\nabla_k\, g}{d\nu}(\zeta) = -A_0(u,\zeta)k(\zeta).$$

Based on this, the adjoint equation can be written

$$\begin{cases} dY(t) = Y(t\text{-})\{\frac{\partial f}{\partial u}(c_t, U_t) - \alpha Z'(t)dB_t - \int_{\mathcal{Z}} \alpha_0 K(t,\zeta)\tilde{N}(dt,d\zeta)\}, \\ Y(0) = 1. \end{cases} \tag{22}$$

Defining $V(t) = \ln(Y(t))$ and using Itô's lemma, we can "solve" this stochastic differential equation, where the solution for Y is

$$Y(t) = \exp\Big(\int_0^t \big(\frac{\partial f}{\partial u}(c_s, U_s) - \frac{1}{2}\alpha^2 Z'(s)Z(s) \\ + \int_{\mathcal{Z}} \{\ln(1-\alpha_0 K(s,\zeta)) + \alpha_0 K(s,\zeta)\}\nu(d\zeta)\big)ds - \alpha \int_0^t Z'(s)dB_s \\ + \int_0^t \int_{\mathcal{Z}} \ln(1-\alpha_0 K(s,\zeta))\tilde{N}(ds,d\zeta)\Big). \tag{23}$$

The adjoint variable Y is seen to depend on primitives of the economy only. The interpretation of Y_t is a shadow price; the marginal value as of time zero of an additional "unit of utility" at time t.

Sufficient conditions for a unique, optimal solution using the stochastic maximum principle are the same as the corresponding conditions for the existence and uniqueness of a solution to the BSDE (19).

Maximizing the Hamiltonian with respect to c gives the first order equation

$$y\,\frac{\partial g}{\partial c}(t, c^*, u, z, k) - \lambda\,\pi = 0$$

or

$$\lambda\,\pi_t = Y(t)\frac{\partial g}{\partial c}(t, c_t^*, U(t), Z(t), K(t,\cdot)) \quad \text{a.s. for all } t \in [0,T]. \tag{24}$$

where c^* is optimal. The state price deflator π_t at time t formally depends, through the adjoint variable Y_t, on the entire optimal paths of the basic variables $(c_s^*, U_s, Z(s), K(s,\cdot))$ for $0 \le s \le t$, which means that marginal value at time t depends, in theory, on the consumption history.

For the representative agent equilibrium the optimal consumption process is the given aggregate consumption c in society, and for this consumption process the utility U_t is optimal at each time t.

We now have the first order conditions for translation invariant recursive utility in the jump/diffusion world. Before we proceed to a solution of the problem, we return to the financial market model.

6 The financial market

Having established the general recursive utility form of interest, in his section we further specify our model for the financial market. The dynamics of the various assets has been explained in Section 2. The model is much like the one used by Duffie and Epstein (1992a), except that we do not assume any unspecified factors in our model. In addition we include general jump-diffusion processes.

Let $\lambda_R(t) \in R^N$ denote the vector of expected rates of return of the N given risky securities in excess of the riskless instantaneous return r_t, and let $\sigma(t)$ denote the $N \times d$-matrix of diffusion coefficients of the risky asset prices, normalized by the asset prices, so that $\sigma(t)\sigma(t)'$ is the instantaneous covariance matrix for the continuous part of asset returns. The jumps in the various assets are captured by the $N \times l$-matrix $\gamma(t, \zeta)$ and a vector valued, compensated random measure

$$\tilde{N}(dt, d\zeta)' = (\tilde{N}_1(dt, d\zeta_1), \cdots, \tilde{N}_l(dt, d\zeta_l)) = \\ (N_1(dt, d\zeta_1) - \nu_1(d\zeta_1)dt, \cdots, N_l(dt, d\zeta_l) - \nu_l(d\zeta_l)dt),$$

where $\{N_j\}$ are independent Poisson random measures with Lévy measures ν_j coming from l independent (1-dimensional) Lévy processes.

The representative consumer's problem is, for each initial level w of wealth to solve

$$\sup_{(c,\varphi)} U(c) \tag{25}$$

subject to the intertemporal budget constraint

$$dW_t = \big(W_t(\varphi_t' \cdot \lambda_R(t) + r_t) - c_t\big)dt + W_t\varphi_t' \cdot \sigma(t)dB_t \\ + W_t\varphi_t' \cdot \int_{R^l} \gamma(t, \zeta)\tilde{N}(dt, d\zeta). \tag{26}$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ are the fractions of total wealth W_t held in the risky securities. The processes $\nu_R(t)$, $\sigma(t)$ and $\gamma(t)$ are progressively measurable processes.

Market clearing requires that $\varphi_t'\sigma(t) = (\delta_t^M)'\sigma(t) = \sigma_M(t)$ and $\varphi_t'\gamma(t, \cdot) = (\delta_t^M)'\gamma(t, \cdot) = \gamma_M(t, \cdot)$ in equilibrium, where $\sigma_M(t)$ is the volatility of the return on the market (wealth) portfolio, $\gamma_M(t, \cdot)$ is the corresponding jump size function, and δ_t^M are the fractions of the different securities, $j = 1, \cdots, N$ held in the value-weighted market portfolio. That is, the representative agent must hold the market portfolio in equilibrium, by construction.

The model is a pure exchange economy where the aggregate endowment process e_t in society is exogenously given, and prices are determined such that in equilibrium the single agent optimally consumes $c_t = e_t$ in every period. The main issue is then the determination of prices, including risk premiums and the interest rate, consistent with this behaviour.

7 The analysis of the single agent economy

We now turn our attention to pricing restrictions relative to the given optimal consumption plan. Recall the first order conditions are given in (24):

$$\lambda\, \pi_t = Y(t)\, \frac{\partial f}{\partial c}(c_t, U_t) \quad a.s. \text{ for all } \; t \in [0, T] \tag{27}$$

where the aggregator f is given in (11), which follows since $g_c = f_c$. The volatilities Z_t and $K(t, \cdot)$ and the utility U_t satisfy the equations (19)-(20). Aggregate consumption has dynamics given in (3) and the adjoint process Y has dynamics given in (23).

We seek the equilibrium connection between (U, Z, K) and the rest of the economy. From the FOC in equation (27) we derive the dynamics of the key quantity in this regard, the state price deflator (the Arrow-Debreu state prices in units of probability). Toward this end, by Ito's lemma, normalizing without loss of generality to $\lambda = 1$, we have

$$d\pi_t = f_c(c_t, U_t)\, dY_t + Y_t\, df_c(c_t, U_t) + d[Y, f_c(c, U)](t), \tag{28}$$

where $[X, Y](t)$ is the quadratic covariation of the processes X and Y given by

$$[X, Y](t) = \int_0^t \big(\sigma_X(s)\sigma_Y(s) + \int_{\mathcal{Z}} \gamma_X(s, \zeta)\gamma_Y(s, \zeta)\nu(d\zeta)\big) ds + \int_0^t \int_{\mathcal{Z}} \gamma_X(s, \zeta)\gamma_Y(s, \zeta)\tilde{N}(ds, d\zeta).$$

7.1 The dynamics of the state price deflator

By the dynamics of aggregate consumption, the adjoint variable and the backward equations for utility, the relation (28) can be written by use of

Ito's multi-dimensional formula for jump/diffusions (Gihman and Skorohod (1972))

$$
\begin{aligned}
d\pi_t = Y(t^-)\, f_c(c_t,U_t)\Big(f_u(c_t,U_t)dt - \alpha Z'(t)dB_t - \int_{\mathcal{Z}} \alpha_0 K(t,\zeta)\tilde{N}(dt,d\zeta)\Big)\\
+ Y(t^-)\, \frac{\partial f_c}{\partial c}(c_t,U_t)(\mu_c(t)dt + \sigma_c(t)dB_t)\\
+ Y(t^-)\, \frac{\partial f_c}{\partial u}(c_t,U_t)(-g(t,(c_t,U_t,Z_t,K(t,\cdot))dt + Z_t dB_t)\\
+Y(t^-)\Big(\frac{1}{2}\frac{\partial^2 f_c}{\partial c^2}(c_t,U_t)\,\sigma_c'(t)\sigma_c(t) + \frac{\partial^2 f_c}{\partial c\partial u}(c_t,U_t)\sigma_c'(t)Z_t + \frac{1}{2}\frac{\partial^2 f_c}{\partial u^2}(c_t,U_t)\, Z_t'Z_t\Big)dt\\
+ Y(t^-)\Big(\int_{\mathcal{Z}}\big\{f_c(c_{t-}+\gamma_c(t,\zeta),U_t+K(t,\zeta)) - f_c(c_t,U_t)\\
- \gamma_c(t,\zeta)\frac{\partial f_c}{\partial c}(c_t,U_t) - K(t,\zeta)\frac{\partial f_c}{\partial u}(c_t,U_t)\big\}\nu(d\zeta)dt\Big)\\
+ Y(t^-)\Big(\int_{\mathcal{Z}}\big\{f_c(c_{t-}+\gamma_c(t,\zeta),U_t+K(t,\zeta)) - f_c(c_t,U_t)\big\}\tilde{N}(dt,d\zeta)\Big)\\
+ Y(t^-)\Big((-\alpha Z_t)\big(\frac{\partial f_c}{\partial c}(c_t,U_t)\sigma_c(t) + \frac{\partial f_c}{\partial u}(c_t,U_t)Z_t\big)dt\Big)\\
Y(t^-)\int_{\mathcal{Z}}(-\alpha_0 K(t,\zeta))\{f_c(c_{t-}+\gamma_c(t,\zeta),U_t+K(t,\zeta)) - f_c(c_t,U_t)\}\nu(d\zeta)dt\\
Y(t^-)\int_{\mathcal{Z}}(-\alpha_0 K(t,\zeta))\{f_c(c_{t-}+\gamma_c(t,\zeta),U_t+K(t,\zeta)) - f_c(c_t,U_t)\}\tilde{N}(dt,d\zeta).
\end{aligned}
\tag{29}
$$

Here

$$f_c(c,u) := \frac{\partial f(c,u)}{\partial c} = \delta e^{-\theta(c-u)},\ f_u(c,u) := \frac{\partial f(c,u)}{\partial u} = -\delta e^{-\theta(c-u)}$$

$$\frac{\partial f_c(c,u)}{\partial c} = -\delta\theta e^{-\theta(c-u)},\ \frac{\partial f_c(c,u)}{\partial u} = \delta\theta e^{-\theta(c-u)},$$

$$\frac{\partial^2 f_c}{\partial c^2}(c,u) = \delta\theta^2 e^{-\theta(c-u)},\quad \frac{\partial^2 f_c}{\partial c\,\partial u}(c,u) = -\delta\theta^2 e^{-\theta(c-u)},$$

and

$$\frac{\partial^2 f_c}{\partial u^2}(c,u) = \delta\theta^2 e^{-\theta(c-u)}.$$

From the canonical representation of the state price deflator

$$d\pi_t = \mu_\pi(t)dt + \sigma_\pi(t)dB_t + \int_{\mathcal{Z}} \gamma_\pi(t,\zeta)\tilde{N}(dt,d\zeta),$$

from (29) we find the key characteristics of π. They are

$$\mu_\pi(t) = Y_t\Big(f_c(c_t,U_t)f_u(c_t,U_t) + \frac{\partial f_c}{\partial c}(c_t,U_t)\mu_c(t) \\ - \frac{\partial f_c}{\partial u}(c_t,U_t)\{f(c_t,U_t) - \frac{1}{2}\alpha Z_t' Z_t - \frac{\alpha_0}{2}\int_{\mathcal{Z}} K'(t,\zeta)K(t,\zeta)\nu(d\zeta)\} \\ + \frac{1}{2}\frac{\partial^2 f_c}{\partial c^2}(c_t,U_t)\sigma_c'(t)\sigma_c(t) + \frac{\partial^2 f_c}{\partial c\partial u}(c_t,U_t)\sigma_c'(t)Z_t + \frac{1}{2}\frac{\partial^2 f_c}{\partial u^2}(c_t,U_t)\, Z_t' Z_t \\ + \int_{\mathcal{Z}}\{f_c(c_{t-}+\gamma_c(t,\zeta),U_{t-}+K(t,\zeta)) - f_c(c_t,U_t) \\ - \gamma_c(t,\zeta)\frac{\partial f_c}{\partial c}(c_t,U_t) - K(t,\zeta)\frac{\partial f_c}{\partial u}(c_t,U_t)\}\nu(d\zeta) \\ (-\alpha Z_t)\{\sigma_c(t)\frac{\partial f_c}{\partial c}(c_t,U_t) + Z_t\frac{\partial f_c}{\partial u}(c_t,U_t)\} \\ + \int_{\mathcal{Z}}(-\alpha_0 K(t,\zeta))\{f_c(c_{t-}+\gamma_c(t,\zeta),U_{t-}+K(t,\zeta)) - f_c(c_t,U_t)\}\nu(d\zeta)\Big), \tag{30}$$

$$\sigma_\pi(t) = Y_t\Big(- f_c(c_t,U_t)\alpha Z_t + \frac{\partial f_c}{\partial c}(c_t,U_t)\sigma_c'(t) + \frac{\partial f_c}{\partial u}(c_t,U_t)Z_t\Big) \tag{31}$$

and

$$\gamma_\pi(t,\zeta) = Y_t\Big(f_c(c_t,U_t)(-\alpha_0 K(t,\zeta)) \\ + \{f_c(c_{t-}+\gamma_c(t,\zeta),U_{t-}+K(t,\zeta)) - f_c(c_t,U_t)\} \\ - \alpha_0 K(t,\zeta)\{f_c(c_{t-}+\gamma_c(t,\zeta),U_{t-}+K(t,\zeta)) - f_c(c_t,U_t)\}\Big). \tag{32}$$

These results will next be used to find risk premiums of risky securities and the risk-free interest rate, among other things.

7.2 The risk premiums

The risk premium of any risky security with return process R is given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t}\,\sigma_\pi'(t)\,\sigma_R(t) - \frac{1}{\pi_t}\int_{\mathcal{Z}}\gamma_\pi'(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) \tag{33}$$

where the jump term follows from Aase (2020). By the first order conditions $\pi_t = Y_t f_c(c_t,V_t)$. Using this and of the results (31) and (32) together with the various partial derivatives of the aggregator f we find that

$$\sigma_\pi(t) = \pi_t\big((\theta - \alpha)Z_t - \theta\sigma_c(t)\big) \tag{34}$$

and

$$\gamma_\pi(t,\zeta)) = \pi_t\big[- \alpha_0 K(t,\zeta) + \big(e^{-\theta(\gamma_c(t,\zeta)-K(t,\zeta))} - 1\big)\big(1 - \alpha_0 K(t,\zeta)\big)\big]. \tag{35}$$

From this it follows that the risk premium of any risky asset can be expressed as

$$\begin{aligned}\mu_R(t) - r_t &= \theta\sigma_c'(t)\sigma_R(t) + (\alpha - \theta)Z'(t)\sigma_R(t) \\ &+ \int_{\mathcal{Z}} \Big(\alpha_0 K(t,\zeta) + \big(1 - e^{-\theta(\gamma_c(t,\zeta)-K(t,\zeta))}\big)\big(1 - \alpha_0 K(t,\zeta)\big)\Big)\gamma_R(t,\zeta)\nu(d\zeta).\end{aligned} \tag{36}$$

It remains to connect the variables Z_t and $K(t,\cdot)$ to observables in the economy, which we do below. Before that we turn to the interest rate.

7.3 The equilibrium interest rate

The equilibrium short-term, real interest rate r_t is given by the formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \tag{37}$$

The real interest rate at time t can be thought of as the expected exponential rate of decline of the representative agent's marginal utility, which is π_t in equilibrium.

In order to find an expression for r_t in terms of the primitives of the model, we use (30). Using the expression for f and its various partial derivatives, we obtain the expression

$$\begin{aligned}\mu_\pi(t) = \pi(t)\Big[&- \delta - \theta\mu_c(t) + \frac{1}{2}\theta\alpha Z_t'Z_t + \frac{1}{2}\alpha_0\theta\int_{\mathcal{Z}} K'(t,\zeta)K(t,\zeta)\nu(d\zeta) \\ &+ \frac{1}{2}\theta^2\sigma_c'(t)\sigma_c(t) - \theta^2\sigma_c'(t)Z_t + \frac{1}{2}\theta^2 Z_t'Z_t - \alpha Z_t(-\theta\sigma_c(t) + \theta Z_t) \\ &+ \int_{\mathcal{Z}} \Big\{\big(e^{-\theta(\gamma_c(t,\zeta)-K(t,\zeta))} - 1\big) + \theta\gamma_c(t,\zeta) - \theta K(t,\zeta)\Big\}\nu(d\zeta) \\ &+ \int_{\mathcal{Z}} \big(- \alpha_0 K(t,\zeta)\big)\big(e^{-\theta(\gamma_c(t,\zeta)-K(t,\zeta))} - 1\big)\nu(d\zeta)\Big]\end{aligned} \tag{38}$$

From this we find the equilibrium short term interest rate:

$$
\begin{aligned}
r_t = \delta + \theta\mu_c(t) - \frac{1}{2}\,\theta^2\,\sigma_c'(t)\sigma_c(t) \\
+\,\theta(\theta-\alpha)\sigma_c'(t)Z_t - \frac{1}{2}\theta\,(\theta-\alpha)Z_t'Z_t \\
-\int_{\mathcal{Z}}\Big\{\frac{1}{2}\theta\alpha_0 K'(t,\zeta)K(t,\zeta) + \Big(\big(e^{-\theta(\gamma_c(t,\zeta)-K(t,\zeta))}-1\big)\Big(1-\alpha_0K(t,\zeta)\Big) \\
+\,\theta(\gamma_c(t,\zeta)-K(t,\zeta))\Big\}\,\nu(d\zeta), \quad (39)
\end{aligned}
$$

which is our basic result for the equilibrium short rate.

In Appendix 1 we show the following connections between the two variables $K(t,\zeta)$ and Z_t and observables in the market:

$$(I) \qquad Z_t = \sigma_c(t) - \frac{1}{\theta}\sigma_{R_a}(t)$$

and

$$(II) \qquad K(t,\zeta) = \gamma_c(t,\zeta)) - \frac{1}{\theta}\ln(1+\gamma_{R_a}(t,\zeta))$$

Using these two results in equation (36) for the risk premiums and in equation (39) for the short term interest rate, we formulate in the next section final versions of the risk premiums and the short rate in a form that may in principle be tested on real consumption and market data.

8 The final form of the risk premiums and the short rate.

We can summarize our results so far in the following theorem

Theorem 2 *For the transaltion invariant recursive model with jump dynamics included, in equilibrium the risk premium of any risky asset R is given by*

$$
\begin{aligned}
\mu_R(t) - r_t = \alpha\sigma_c(t)\sigma_R(t) + \frac{\theta-\alpha}{\theta}\,\sigma_{R_a}'(t)\sigma_R(t) + \\
\int_{\mathcal{Z}}\Big(1 - \frac{1}{1+\gamma_{R_a}(t,\zeta)}\big[1-\alpha_0\gamma_c(t,\zeta) + \frac{\alpha_0}{\theta}\ln(1+\gamma_{R_a}(t,\zeta))\big]\Big)\gamma_R(t,\zeta)\nu(d\zeta)
\end{aligned}
$$

and the real interest rate by

$$r_t = \delta + \theta\mu_c(t) - \frac{1}{2}\theta\alpha\sigma_c'(t)\sigma_c(t) - \frac{1}{2}\,\frac{\theta-\alpha}{\theta}\sigma_{R_a}'(t)\sigma_{R_a}(t)$$
$$-\int_{\mathcal{Z}}\Big\{\frac{1}{2}\theta\alpha_0\big[\gamma_c'(t,\zeta)\gamma_c(t,\zeta) - \frac{2}{\theta}\gamma_c'(t,\zeta)\ln(1+\gamma_{R_a}(t,\zeta)) + \frac{1}{\theta^2}\ln^2(1+\gamma_{R_a}(t,\zeta))\big]$$
$$-\Big(\frac{\gamma_{R_a}(t,\zeta)}{1+\gamma_{R_a}(t,\zeta)}\Big)\big(1-\alpha_0[\gamma_c(t,\zeta)-\frac{1}{\theta}\ln(1+\gamma_{R_a}(t,\zeta))]\big)+\ln(1+\gamma_{R_a}(t,\zeta))\Big\}\nu(d\zeta).$$

In order for these expressions to be testable with consumption growth rate data, the parameters α, α_0 and θ should be in relative forms rather than in absolute. This is achieved by multiplication by c_t, and division by c_t in the consumption characteristics. As before we denote the relative versions of the preference parameters by

$$\tilde{\alpha}_t = \alpha c_t, \;\; \tilde{\theta}_t = \theta c_t, \;\; \text{and} \;\; \tilde{\alpha}_{0t} = \alpha_0 c_t,$$

and the characteristics of the consumption growth rate in terms of the consumption characteristics by

$$\tilde{\mu}_c(t) = \mu_c(t)/c_t, \;\; \tilde{\sigma}_c(t) = \sigma_c(t)/c_t, \;\; \text{and} \;\tilde{\gamma}_c(t,\zeta) = \gamma_c(t,\zeta)/c_t.$$

Using this the results can alternatively be written

$$\mu_R(t) - r_t = \tilde{\alpha}_t\tilde{\sigma}_c'(t)\sigma_R(t) + \frac{\tilde{\theta}_t - \tilde{\alpha}_t}{\tilde{\theta}_t}\,\sigma_{R_a}'(t)\sigma_R(t) +$$
$$\int_{\mathcal{Z}}\Big(1 - \frac{1}{1+\gamma_{R_a}(t,\zeta)}\big[1 - \tilde{\alpha}_{0t}\tilde{\gamma}_c(t,\zeta) + \frac{\tilde{\alpha}_{0t}}{\tilde{\theta}_t}\ln(1+\gamma_{R_a}(t,\zeta))\big]\Big)\gamma_R(t,\zeta)\nu(d\zeta) \tag{40}$$

and

$$r_t = \delta + \tilde{\theta}_t\tilde{\mu}_c(t) - \frac{1}{2}\tilde{\theta}_t\tilde{\alpha}_t\tilde{\sigma}_c'(t)\tilde{\sigma}_c(t) - \frac{1}{2}\,\frac{\tilde{\theta}_t-\tilde{\alpha}_t}{\tilde{\theta}_t}\sigma_{R_a}'(t)\sigma_{R_a}(t)$$
$$-\int_{\mathcal{Z}}\Big\{\frac{1}{2}\tilde{\theta}_t\tilde{\alpha}_{0t}\big[\tilde{\gamma}_c(t,\zeta)\tilde{\gamma}_c(t,\zeta) - \frac{2}{\tilde{\theta}_t}\tilde{\gamma}_c(t,\zeta)\ln(1+\gamma_{R_a}(t,\zeta)) + \frac{1}{\tilde{\theta}_t^2}\ln^2(1+\gamma_{R_a}(t,\zeta))\big]$$
$$-\Big(\frac{\gamma_{R_a}(t,\zeta)}{1+\gamma_{R_a}(t,\zeta)}\Big)\big(1-\tilde{\alpha}_{0t}[\tilde{\gamma}_c(t,\zeta)-\frac{1}{\tilde{\theta}_t}\ln(1+\gamma_{R_a}(t,\zeta))]\big)+\ln(1+\gamma_{R_a}(t,\zeta))\Big\}\nu(d\zeta). \tag{41}$$

By setting $\alpha = \alpha_0 = \theta$ and $\gamma_{R_a}(t,\zeta) = 0, \ \forall \ (t,\zeta)$, we obtain the expected utility version presented in Section 3.

8.1 An approximation for the pure jump part

If we use Taylor series expansions on the jump parts, retaining only terms up to second order, we can simplify the jump expressions. The risk premiums can then be written as

$$\mu_R(t) - r_t \approx \tilde{\alpha}_t \tilde{\sigma}_c(t)' \sigma_R(t) + \frac{\tilde{\theta}_t - \tilde{\alpha}_t}{\tilde{\theta}_t} \sigma'_{R_a}(t)\sigma_R(t) +$$
$$\tilde{\alpha_0}_t \int_{\mathcal{Z}} \tilde{\gamma}_c(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) + \frac{\tilde{\theta}_t - \tilde{\alpha_0}_t}{\tilde{\theta}_t} \int_{\mathcal{Z}} \gamma'_{R_a}(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) \quad (42)$$

to the second order approximation. The corresponding expression for the short term interest rate is the following

$$r_t \approx \delta + \tilde{\theta}_t \tilde{\mu}_c(t) - \frac{1}{2}\tilde{\theta}_t \tilde{\alpha}_t \tilde{\sigma}'_c(t)\tilde{\sigma}_c(t) - \frac{1}{2}\frac{\tilde{\theta}_t - \tilde{\alpha}_t}{\tilde{\theta}_t}\sigma'_{R_a}(t)\sigma_{R_a}(t)$$
$$- \frac{1}{2}\tilde{\alpha_0}_t \tilde{\theta}_t \int_{\mathcal{Z}} \tilde{\gamma}_c(t,\zeta)\tilde{\gamma}_c(t,\zeta)\nu(d\zeta) - \frac{1}{2}\frac{\tilde{\theta}_t - \tilde{\alpha_0}_t}{\tilde{\theta}_t} \int_{\mathcal{Z}} \gamma'_{R_a}(t,\zeta)\gamma_{R_a}(t,\zeta)\nu(d\zeta) \quad (43)$$

Notice the complete symmetry between the continuous parts and the jump parts, which is not surprising when we take into account how the continuous time theory is developed. For the expected utility version, we noticed in Section 3 that this symmetry was exact, while it is here an approximation. For completeness, we have adjusted all the parameters to their relative forms, and amended the characteristics of the consumption dynamics accordingly.

9 Aggregation

In this section we treat equilibrium with several agents, all having recursive utilities of the translation invariant type, and possibly with *different* preference parameters. The first-order conditions for the problem with m agents is

$$\pi(t) = \lambda_1 \pi_1(t) = \lambda_2 \pi_2(t) = \cdots = \lambda_m \pi_m(t) \, a.s.$$

where the agent weights λ_i are all positive constants and $\pi_i(t)$ is the marginal utility of agent i, $i = 1, 2, \cdots, m$. Here π_t, the equilibrium state price deflator, equals the marginal utility of the representative agent in equilibrium.

The discrete-time version of this model aggregates (Aase (2021, 24a)), similarly does the version in continuous time. (Aase (2024b)). Below we show that this is also true for the continuous-time model including jump dynamics. What we mean by this will be clear below.

We focus on the risk premiums. The above condition can be rewritten as

$$\frac{d\pi(t)}{\pi(t)} = \frac{d\pi_1(t)}{\pi_1(t)} = \frac{d\pi_2(t)}{\pi_2(t)} = \cdots = \frac{d\pi_m(t)}{\pi_m(t)} \, a.s., \tag{44}$$

where the agent weights have cancelled.

From our previous results, this condition can be written

$$\frac{d\pi(t)}{\pi(t)} = \frac{\mu_{\pi_i}(t)}{\pi_i(t)}dt - (\alpha_i\sigma_{c_i} + \frac{\theta_i - \alpha_i}{\theta_i}\sigma'_{R_a}(t))dB_t$$
$$- \int_{\mathcal{Z}} \Big(1 - \frac{1}{1+\gamma_{R_a}(t,\zeta)}\big[1 - \alpha_{0i}\gamma_{c_i}(t,\zeta) + \frac{\alpha_{0i}}{\theta^i}\ln(1+\gamma_{R_a}(t,\zeta))\big]\Big)\tilde{N}(dt,d\zeta),$$
$$i = 1, 2, \cdots, m. \tag{45}$$

We here make the simplifying assumption that $\alpha_i = \alpha_{0i}$ for all i. Then first divide the above expression by α_i and sum over the agents. By Fubini's theorem this gives

$$\frac{d\pi(t)}{\pi(t)} = \Psi \sum_{i=1}^{m} \frac{1}{\alpha_i}\frac{\mu_{\pi_i}(t)}{\pi_i(t)}dt - \Psi\Big(\sigma_c(t) - (\frac{1}{\Theta} - \frac{1}{\Psi})\sigma_{R_a}(t)\Big)dB_t$$
$$- \int_{\mathcal{Z}} \Big(1 - \frac{1}{1+\gamma_{R_a}(t,\zeta)}\big[1 - \Psi\gamma_c(t,\zeta) + \frac{\Psi}{\Theta}\ln(1+\gamma_{R_a}(t,\zeta))\big]\Big)\tilde{N}(dt,d\zeta) \tag{46}$$

where $\frac{1}{\Psi} := \sum_{i=1}^{m}\frac{1}{\alpha_i}$, $\frac{1}{\Theta} := \sum_{i=1}^{m}\frac{1}{\theta_i}$, $\sigma_c(t) = \sum_{i=1}^{m}\sigma_{c_i}(t)$ and $\gamma_c(t,\zeta) = \sum_{i=1}^{m}\gamma_{c_i}(t,\zeta)$.

From the standard theory of syndicates (e.g., Borch (1962)), we know that in a Pareto optimum the aggregate risk tolerance is the sum of the individual risk tolerances. Here the individual risk tolerance corresponds to $1/\alpha_i$ for agent i, $i = 1, 2, \cdots, m$. From this it is both natural and common to define the aggregate risk tolerance by $1/\Psi := \sum_{i=1}^{m} 1/\alpha_i$, and interpret Ψ as the absolute risk aversion of the representative agent (the syndicate, the market or more generally society as a whole) related to the continuous part.

In a two-date, deterministic economy with consumption at both dates, we can interpret θ_i as the degree of resistance to intertemporal substitution in consumption. In analogy with the above, we can define the quantity $1/\Theta := \sum_{i=1}^{m} 1/\theta_i$, and interpret Θ as the degree of resistance to intertemporal substitution in consumption of the representative agent

The impatience parameters δ_i and the parameters α_i partly mix in the aggregate model, where we define the impatience rate δ of the representative agent by

$$\delta := \Psi \sum_{i=1}^{m} \frac{1}{\alpha_i} \delta_i. \tag{47}$$

Note that when all the impatience rates δ_i are equal to some constant, then this constant equals δ defined above.

Recall that the risk premium of any risky asset can be written

$$\mu_R(t) - r(t) = -\frac{\sigma'_\pi(t)}{\pi(t)} \sigma_R(t) - \frac{1}{\pi_t} \int_{\mathcal{Z}} \gamma'_\pi(t,\zeta) \gamma_R(t,\zeta) \nu(d\zeta).$$

In the present situation we then obtain the following formula

$$\mu_R(t) - r(t) = \Psi \sigma'_c(t) \sigma_R(t) + \big(1 - \frac{\Psi}{\Theta}\big) \sigma'_{R_a}(t) \sigma_R(t) + \int_{\mathcal{Z}} \Big(1 - \frac{1}{1 + \gamma_{R_a}(t,\zeta)} \big[1 - \Psi \gamma_c(t,\zeta) + \frac{\Psi}{\Theta} \ln\big(1 + \gamma_{R_a}(t,\zeta)\big)\big]\Big) \gamma_R(t,\zeta) \nu(d\zeta) \tag{48}$$

where the quantities Ψ and Θ are as defined above.

Let us consider the above expression in relative terms, where $\tilde{\Psi}_t = \Psi c_t$ and $\tilde{\Theta}_t = \Theta c_t$. Suppose we also allow α_{0i} to be different from α_i for some i's. Then we obtain the following

$$\begin{aligned} \mu_R(t) - r(t) = \tilde{\Psi}_t \tilde{\sigma}'_c(t) \sigma_R(t) + \big(1 - \frac{\tilde{\Psi}_t}{\tilde{\Theta}_t}\big) \sigma'_{R_a}(t) \sigma_R(t) \\ + \int_{\mathcal{Z}} \Big(1 - \frac{1}{1 + \gamma_{R_a}(t,\zeta)} \big[1 - \sum_{i=1}^{m} \big(\frac{\tilde{\Psi}_t \tilde{\alpha}^i_{0,t}}{\tilde{\alpha}_{i,t}} \tilde{\gamma}_{c_i}(t,\zeta) \\ + \frac{\tilde{\Psi}_t \tilde{\alpha}^i_{0,t}}{\tilde{\alpha}_{i,t} \theta_i} \ln(1 + \gamma_{R_a}(t,\zeta))\big)\big]\Big) \gamma_R(t,\zeta) \nu(d\zeta) \end{aligned} \tag{49}$$

where $\tilde{\sigma}_c(t) = \sigma_c(t)/c_t$ is the conditional volatility at time t of the aggregate consumption growth rate in society, with a corresponding interpretation of the term $\tilde{\gamma}_{c_i}(t,\zeta) = \gamma_{c_i}(t,\zeta)/c_i(t)$.

This version requires knowledge of the different individuals $\tilde{\gamma}_{c_i}(t,\zeta)$, $i = 1, 2, \ldots, m$, with the corresponding preference parameters, information which normally is not available for statistical agents. But see below.

The short term interest rate r_t with m agents is given by

$$r(t) = -\frac{\mu_\pi(t)}{\pi(t)} = -\Psi \sum_{i=1}^{m} \frac{\mu_{\pi_i}(t)}{\alpha_i \pi_i(t)}.$$

We next use the same simplification as in Section 8.1, and here we again require that $\alpha_{0i} = \alpha_i$ for all i. Then we obtain the following

$$r(t) = \delta + \Psi \sum_{i=1}^{m} \frac{\theta_i}{\alpha_i} \mu_{c_i}(t) - \frac{1}{2} \Psi \sum_{i=1}^{m} \theta_i \sigma_{c_i}(t)' \sigma_{c_i}(t) - \frac{1}{2}\,(1 - \frac{\Psi}{\Theta}) \sigma'_{R_a}(t) \sigma_{R_a}(t)$$
$$- \frac{1}{2} \Psi \sum_{i=1}^{m} \theta_i \int_{\mathcal{Z}} \gamma'_{c_i}(t,\zeta) \gamma_{c_i}(t,\zeta) \nu(d\zeta) - \frac{1}{2}(1 - \frac{\Psi}{\Theta}) \int_{\mathcal{Z}} \gamma'_{R_a}(t,\zeta) \gamma_{R_a}(t,\zeta) \nu(d\zeta), \tag{50}$$

with a similar version in terms of the relative preference parameters and the growth rate characteristics of aggregate consumption.

Also this formula requires knowledge of individual's consumption data and preferences.

In some situations we may have enough information to use a model of this kind. An example is related to the investigation of Mehra and Prescott (1985). Vissing-Jørgensen, A. (1999) claims that just a minor fraction, 8-9 per cent, of the US-population participated in the stock market. This information was used in Aase (2019) in a model with recursive utility of the scale invariant type, and diffusion driven uncertainty. With two agents in the economy allowed to have different preferences, one participated in the stock market, the other did not, the data could be "explained" with reasonable values of the different preference parameters of the two agents, under plausible assumptions regarding the division of the data between the two groups.

The present model, with the above expressions for risk premiums and the interest rate, could be used to more elegantly address these problems, since it works so well with agents having different preferences.

However, if we seek expressions for the risk premiums and the short rate with parameters only depending on the representative agent's preferences, there are still possibilities to do just that: From the requirements of the first order conditions in equation (44), we may utilize an extended form of "diffusion invariance" to overcome these problems, which we have not considered so far:

For jump-diffusions canonical representations are given by the Ito-Levy decomposition theorem and the Levy-Kintchine formula, see Theorem 1.7 and Theorem 1.10 in Øksendal and Sulem (2018). From the definition of a Levy process, we can define the jump measure $N(t)$ and subsequently the Levy measure ν, and vice versa: Given a drift μ volatility σ and a Levy measure ν, the Levy process is uniquely determined. Accordingly, the Levy triplet (μ, σ, ν) characterizes a Levy process.

These properties will allow us to find extra conditions, since the first order condition give restrictions on how different the agents are allowed to be for an equilibrium to exist.

Since we will only need the present versions in what follows, we defer this topic to a future investigation.

The "aggregation property" is not true for the scale invariant version of recursive utility, it is true for the EU-version with CARA felicity, but not true for the corresponding EU-version with CRRA or HARA felicity index.

10 Market implications.

In Cochrane (2015)[5] it is proposed that the US public debt should be made up of two securities. The first would have a fixed value of \$1 forever and a coupon payment that is continually set by the market in line with daily interest rates. The second would have a fixed coupon payment of \$1 forever and a price that is determined by the market. It is the latter security that is close to the annuity a, provided the coupon rate is in real terms. It has the character of long-term debt. When the stock market is down, then any holder of the long term debt may typically benefit, and vice versa when the stock market is up, indicating a negative, conditional correlation coefficient.

Let us consider the two results in (49) and (50), and for intuition we focus on the terms related to the continuous part. This is justified from the approximative results in (42) and (43). If the covariance rate $\sigma_{R_a,R}$ is negative, this implies that $\tilde{\Psi}_t > \tilde{\Theta}_t$, the representative agent has preference for early resolution of uncertainty, and the recursive model can account for a larger risk premium than the standard EU-model for reasonable values of the parameters. The risk-free interest rate can also be smaller in this situation than allowed by the EU-model. This is primarily due to the second term $\Theta\mu_c(t)$, since the parameter $\tilde{\Theta}_t < \tilde{\Psi}_t$, while for the EU-model the large value of $\tilde{\Psi}_t$ required to fit the data results in a negative value of the impatience rate δ, which is meaningless. The fourth term on the right-and side in (50) works in the wrong direction, but does not amount to much, since its magnitude is small.

[5] The paper is published with discussions by Darrel Duffie and John Campbell.

If, on the other hand, the conditional covariance between the annuity and any risky asset happens to be positive on the average, this implies that $\tilde{\Theta}_t > \tilde{\Psi}_t$, and the agent has preference for late resolution of uncertainty. The recursive model can in principle account for a larger risk premium than the standard EU-model also in this situation, but notice that the term $\frac{\tilde{\Theta}_t - \tilde{\Psi}_t}{\tilde{\Theta}_t}$ is bounded upward by 1. In this situation the second term in (50) can now be a problem, since it may be fairly large. The fourth term now works in the right direction, but agin does not help much since its magnitude is still small. It should be noted that preference for late resolution of uncertainty is not irrational. However, with the data considered in Table 2, the situation with $\tilde{\Theta}_t > \tilde{\Psi}_t$ does not fit with reasonable values of the parameters.

The scale invariant version can calibrate to both kinds of preferences, but the most convincing fit turns out to be for the type of agent with preference for early resolution of uncertainty (see Aase (2016)).

Below we illustrate by some calibrations to market data. This should of course not be taken literally, since we lack the necessary long-term data on the annuity and its relationship to the other securities. However, we can obtain a preliminary, and we claim not an unrealistic, picture from the statistical properties of the short-rate with a few amendments, explained below.

10.1 A numerical example.

As a numerical illustration, suppose we use the market and consumption data from Mehra and Prescott (1985), see Tables 1 and 2 in Section 3.

Data for our annuity are not directly available in this study, but we obtain relevant information from the data related to the risk-free interest rate. We employ all the data in the tables with one amendment: We assume the conditional covariance between the market return, represented by the S&P-500 index, and the return on the annuity is of the order of $-.001477$, the same magnitude but the opposite sign of $\sigma_{R,b}(t)$ in Table 2 below. As we have discussed above, we find this reasonable for the annuity. This implies a conditional, instantaneous correlation coefficient $\kappa_t = -.16$ between the return on the annuity and the return on the market portfolio. With this interpretations of Table 2, we obtain the following:

The recursive model calibrates (in the manner introduced by Kydland and Prescott (1982)) to the following parameter values: An estimate of the relative risk aversion $\tilde{\Psi}_t = 2.5$, the parameter $\tilde{\Theta}_t = .07$ and the impatience rate $\delta = .0009$. This corresponds to an EIS of 14.

The jump part might influence this result. Here we refer to the discussion in Section 3.1. An analysis would require the specification of a joint probability distribution of the three basic variables in this model, which could, for example, be multi-normal or of the NIG type in a theoretical model, but this must be tested on real data. Since we do not have the relevant data regarding the annuity, we can not go further into this here.

11 The Relation to the Mutuality Principle

In the one-period model with uncertainty, the mutuality principle says that the Pareto-efficient consumption of any agent i in state ω is $c_i(c(\omega))$, where $c(\omega)$ is the aggregate consumption in state ω. Here the real functions $c_i(\cdot)$ are increasing for all i. Hence the individual's consumption, after pooling, only depends on the state of the world via the aggregate consumption. The result is a direct consequence of Borch's (1962) analysis of equilibrium in a reinsurance market.

The principle roughly says that if all risk can be diversified, everyone is in the "same boat". When the aggregate consumption $c(\omega)$ is high in state ω, everyone benefits, and vice versa if $c(\omega)$ is low. In a one-period framework the result is of course interesting, and yields some basic insights on how risks affect society at large. However, in a multi-period model the principle is difficult to be integrated with the concept an economic equilibrium with separable, expected utility maximizing agents. The reason for this is that when uncertainty is revealed, then all agents are affected in the same "direction", which gives a situation with either sellers, or buyers, for the next period's trade. However, who compensates who also depends on the endowments. For an equilibrium to exist, both buyers and sellers should be present to clear markets.

In this section we demonstrate that, with recursive utility, this situation is different. The key observation is that in the economy there may be both early, and late resolvers, and agents in these two different groups may not react in the same monolithic fashion on uncertainty revelation (i.e., to market shocks).

Toward this end, let us recall the first order conditions:

$$\frac{d\pi(t)}{\pi(t)} = \frac{d\pi_i(t)}{\pi_i(t)} \quad a.s. \text{ for any agent } i, \; i = 1, 2, \cdots, m.$$

We now make the assumption that $\alpha_i = \alpha_{0i}$ for $i = 1, 2, \ldots, m$. Dividing both sides above by α_i and adding over the agents, by equations (45) and

(46) we obtain the following equations

$$\frac{d\pi(t)}{\pi(t)} = \big(-\delta - \Psi\sum_{i=1}^{m}\frac{\theta_i}{\alpha_i}\mu_{c_i}(t) + \frac{1}{2}\Psi\sum_{i=1}^{m}\theta_i\sigma'_{c_i}(t)\sigma_{c_i}(t) + \frac{1}{2}(1-\frac{\Psi}{\Theta})\sigma'_{R_a}(t)\sigma_{R_a}(t)$$
$$+\frac{1}{2}\Psi\sum_{i=1}^{m}\theta_i\int_{\mathcal{Z}}\gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta) + \frac{1}{2}(1-\frac{\Psi}{\Theta})\int_{\mathcal{Z}}\gamma'_{R_a}(t,\zeta)\gamma_{R_a i}(t,\zeta)\nu(d\zeta)\big)dt$$
$$+\big((\frac{\Psi}{\Theta}-1)\sigma'_{R_a}(t) - \Psi\sigma'_c(t)\big)dB(t)$$
$$-\int_{\mathcal{Z}}\Big(\frac{1}{\Psi} - \frac{1}{1+\gamma_{R_a}(t,\zeta)}\Big[\frac{1}{\Psi} - \gamma_c(t,\zeta) + \frac{1}{\Theta}\ln\big(1+\gamma_{R_a}(t,\zeta)\big)\Big]\Big)\tilde{N}(dt,d\zeta) =$$
$$\big(-\delta_i - \theta_i\mu_{c_i}(t) + \frac{1}{2}\theta_i\alpha_i\sigma'_{c_i}(t)\sigma_{c_i}(t) + \frac{1}{2}(1-\frac{\alpha_i}{\theta_i})\sigma'_{R_a}(t)\sigma_{R_a}(t)$$
$$+\frac{1}{2}\alpha_i\theta_i\int_{\mathcal{Z}}\gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta) + \frac{1}{2}(1-\frac{\alpha_i}{\theta_i})\int_{\mathcal{Z}}\gamma'_{R_a}(t,\zeta)\gamma_{R_a i}(t,\zeta)\nu(d\zeta)\big)dt$$
$$+\big((\frac{\alpha_i}{\theta_i}-1)\sigma'_{R_a}(t) - \alpha_i\sigma'_{c_i}(t)\big)dB(t)$$
$$-\int_{\mathcal{Z}}\Big(1 - \frac{1}{1+\gamma_{R_a}(t,\zeta)}\Big[1 - \alpha_i\gamma_{c_i}(t,\zeta) + \frac{\alpha_i}{\theta_i}\ln\big(1+\gamma_{R_a}(t,\zeta)\big)\Big]\Big)\tilde{N}(dt,d\zeta), \tag{51}$$

valid for $i = 1, 2, \cdots, m$,

By adding and subtracting the two terms $dc_i(t)$ and $\Psi dc(t)$ on both sides of the above equation and rearranging, and finally dividing the resulting equation by α_i, we obtain the following:

$$
\begin{aligned}
dc_i(t) = \frac{\Psi}{\alpha_i}dc(t) + \frac{1}{\alpha_i}\big(\frac{\alpha_i}{\theta_i} - \frac{\Psi}{\Theta}\big)\sigma'_{R_a}(t)dB(t) \\
+ \int_{\mathcal{Z}}\Big(\frac{1}{1+\gamma_{R_a}(t,\zeta)}\Big[\frac{1}{\alpha_i}\big(\frac{\alpha_i}{\theta_i} - \frac{\Psi}{\Theta}\big)\ln\big(1+\gamma_{R_a}(t,\zeta)\big) \\
- \big(\gamma_{c_i}(t,\zeta) - \frac{\Psi}{\alpha_i}\gamma_c(t,\zeta)\big)\Big] + \big(\gamma_{c_i}(t,\zeta)\big) - \frac{\Psi}{\alpha_i}\gamma_c(t,\zeta)\big)\Big)\tilde{N}(dt,d\zeta) \\
+ \Big(\big(\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\frac{\theta_j}{\alpha_j}\mu_{c_j}(t) - \frac{\theta_i}{\alpha_i}\mu_{c_i}(t)\big) + \big(\mu_{c_i}(t) - \frac{\Psi}{\alpha_i}\mu_c(t)\big) + \big(\frac{1}{2}\theta_i\sigma'_{c_i}(t)\sigma_{c_i}(t) \\
- \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\theta_j\sigma'_{c_j}(t)\sigma_{c_j}(t)\big) + \frac{1}{2}\frac{1}{\alpha_i}\big(\frac{\Psi}{\Theta} - \frac{\alpha_i}{\theta_i}\big)\sigma'_{R_a}(t)\sigma_{R_a}(t) \\
+ \big(\frac{1}{2}\theta_i\int_{\mathcal{Z}}\gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta) - \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\theta_j\int_{\mathcal{Z}}\gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta)\big) \\
+ \frac{1}{2}\frac{1}{\alpha_i}\big(\frac{\Psi}{\Theta} - \frac{\alpha_i}{\theta_i}\big)\int_{\mathcal{Z}}\gamma'_{R_a}(t,\zeta)\gamma_{R_a}(t,\zeta)\nu(d\zeta) + \frac{1}{\alpha_i}(\delta - \delta_i)\Big)dt. \quad (52)
\end{aligned}
$$

In the above have used the drift term simplification of the state price deflators in the absolute risk aversion version of equation (43).

The equation (52) tells us that after Pareto optimal pooling of risk at time t, any agent i holds a linear combination of three random variables at each time consisting of a fraction $\frac{\Psi}{\alpha_i} \in (0,1)$ of the aggregate consumption, a path continuous fraction of the annuity risk and a jump size risk term, where the latter contains both annuity and consumption risk. Both the latter terms have t-conditional expected zero. The annuity is seen to be in net zero supply.

In addition the agent holds an intercept of lower order of magnitude. This term is $\mathcal{F}_t$-measurable.

If we sum both side of this equation over the agents, we obtain $dc(t) = dc(t)$, where the terms related to the Brownian motion and related to the compensated jump term each sum to zero and so does the terms in the large parenthesis, the side-payments. This just says the obvious; no risk disappears in the exchange, it is just differently distributed among the agents according to their preferences in each period.

From the equation (52) we notice the following:

The mutuality principle does not hold for the translation invariant recursive utility model. The first term on the right-hand side affect all the agents in the same direction from a shock to aggregate consumption, positive or negative, but to varying degrees. This corresponds to the mutuality principle in the one-period model of a reinsurance syndicate. However, the next two random terms do not have this property. Depending on the individual preferences parameters and their relations to the population parameters, individuals are differently affected by random shocks to the economy. For example, the direction of the shocks stemming from these two random terms have different effects on the individual i depending on the sign of $\frac{1}{\alpha_i}\left(\frac{\alpha_i}{\theta_i} - \frac{\Psi}{\Theta}\right)$, which may be positive or negative.

Consider the jump stochastic integral w.r.t. $\tilde{N}(dt, d\zeta)$: The consumption parts of the integrand more of less cancel, in particular if the term $\gamma_{R_a}(t, \zeta)$ is small in absolute value. As a consequence of this, the agent has about the same type exposure to the annuity from the jump term as from the Brownian motion driven term, that is, if the coefficient $\frac{1}{\alpha_i}\left(\frac{\alpha_i}{\theta_i} - \frac{\Psi}{\Theta}\right) = \left(\frac{1}{\theta_i} - \frac{\Psi}{\alpha_i}\frac{1}{\Theta}\right) < 0$, the agent is diversified between aggregate consumption and the annuity, and when the inequality is reversed, the agent is exposed to the annuity from both types of uncertainty.

The overall nature of the jump dynamics is, however, that the effects of the jumps are mitigated with time, i.e., the consequences of large shocks to society tends to be transitory. Casual observations suggest that this is close to what we observe in real life. The results of catastrophic events gradually diminish with time. This is an important lesson for any insurance syndicate, as well as for society at large.

We conclude the following:

Theorem 3 *In the jump-diffusion risk exchange model where the agents have preferences represented by recursive utility of translation invariant type, the mutuality principle does not hold.*

This may be good news for existence of equilibrium. After shocks to the economy, there may be both "sellers" and "buyers" present, since agents are are not all moved in the same direction by such shocks.

11.1 Optimal risk sharing in the recursive model

Here we discuss the recursive utility model by considering optimal risk sharing that follow from equation (52). Below we address optimal risk sharing with EU.

We rewrite the basic equation (52 as follows:

$$
\begin{aligned}
c_i(t+dt) = {} & \frac{\Psi}{\alpha_i} c(t+dt) + \frac{1}{\alpha_i}\big(\frac{\alpha_i}{\theta_i} - \frac{\Psi}{\Theta}\big)\sigma'_{R_a}(t)dB(t) \\
& + \int_{\mathcal{Z}} \Big(\frac{1}{1+\gamma_{R_a}(t,\zeta)} \Big[\frac{1}{\alpha_i}\big(\frac{\alpha_i}{\theta_i} - \frac{\Psi}{\Theta}\big) \ln\big(1+\gamma_{R_a}(t,\zeta)\big) \\
& - (\gamma_{c_i}(t,\zeta) - \frac{\Psi}{\alpha_i}\gamma_c(t,\zeta))\Big] + (\gamma_{c_i}(t,\zeta)) - \frac{\Psi}{\alpha_i}\gamma_c(t,\zeta))\Big)\tilde{N}(dt,d\zeta) \\
& + (c_i(t) - \frac{\Psi}{\alpha_i}c(t)) + \Big(\big(\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\frac{\theta_j}{\alpha_j}\mu_{c_j}(t) - \frac{\theta_i}{\alpha_i}\mu_{c_i}(t)\big) + \big(\mu_{c_i}(t) - \frac{\Psi}{\alpha_i}\mu_c(t)\big) \\
& + \big(\frac{1}{2}\theta_i\sigma'_{c_i}(t)\sigma_{c_i}(t) - \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\theta_j\sigma'_{c_j}(t)\sigma_{c_j}(t)\big) + \frac{1}{2}\frac{1}{\alpha_i}\big(\frac{\Psi}{\Theta} - \frac{\alpha_i}{\theta_i}\big)\sigma'_{R_a}(t)\sigma_{R_a}(t) \\
& + \big(\frac{1}{2}\theta_i\int_{\mathcal{Z}}\gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta) - \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\theta_j\int_{\mathcal{Z}}\gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta)\big) \\
& + \frac{1}{2}\frac{1}{\alpha_i}\big(\frac{\Psi}{\Theta} - \frac{\alpha_i}{\theta_i}\big)\int_{\mathcal{Z}}\gamma'_{R_a}(t,\zeta)\gamma_{R_a}(t,\zeta)\nu(d\zeta) + \frac{1}{\alpha_i}(\delta - \delta_i)\Big)dt. \quad (53)
\end{aligned}
$$

The 4th term on the left-hand side, the one after the jump term, plays the following role: Without this an agent, with a small consumption in period t but with a large risk tolerance, could end up with a large consumption in the next period, but this could not possibly be consistent with the agent's budget constraint. This is the basic story in the one-period risk-sharing model of a syndicated market, due to Borch (1962).

This term together with the other inside the large parentheses we call the zero-sum side-payments. The latter all add new features compared to the basic one-period mo

Let us consider an agent i who is more risk tolerant than average. From the equation (52) such an agent is more exposed to aggregate consumption shocks than less risk tolerant agents. In the long run such agents will, more likely than not, obtain a larger than average share of the aggregate consumption.[6] In a Pareto optimum based on the one-period model with EU, such an agent will typically have a negative side-payment, compensating the other members of society. Below we show that this property holds in the present model as well, adjusted for some additional complexity.

We may rewrite the side-payments a bit using certainty equivalents: The quantity $ce(c_i)(t) := \mu_{c_i}(t) - \frac{1}{2}\alpha_i\sigma'_{c_i}(t)\sigma_{c_i}(t)$ is known as the certainty equiv-

[6]This is based on based on data for the last 150 years or so, where the annual consumption growth rate has been around 2 per cent.

alent to $c_i(t)$, primarily a positive quantity provided we rule out negative consumption. It has this interpretation under EU, and it has been shown that it also has the same interpretation for recursive utility (see, for example, Aase and Bjerksund (2021)). Using this, the new side-payments per time unit in equation (52) can be written

$$\text{New side-payments} :=$$
$$\Big(\big(\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\frac{\theta_j}{\alpha_j}ce(c_j)(t) - \frac{\theta_i}{\alpha_i}ce(c_i)(t)\big) + \big(\mu_{c_i}(t) - \frac{\Psi}{\alpha_i}\mu_c(t)\big) + \frac{1}{2}\frac{1}{\alpha_i}\Big(\frac{\Psi}{\Theta} - \frac{\alpha_i}{\theta_i}\Big)\sigma'_{R_a}(t)\sigma_{R_a}(t)$$
$$+ \big(\frac{1}{2}\theta_i\int_{\mathcal{Z}}\gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta) - \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\theta_j\int_{\mathcal{Z}}\gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta)\big)$$
$$+ \frac{1}{2}\frac{1}{\alpha_i}\big(\frac{\Psi}{\Theta} - \frac{\alpha_i}{\theta_i}\big)\int_{\mathcal{Z}}\gamma'_{R_a}(t,\zeta)\gamma_{R_a}(t,\zeta)\nu(d\zeta) + \frac{1}{\alpha_i}(\delta - \delta_i)\Big). \quad (54)$$

Consider an agent i is risk tolerant, early resolver and more impatient than average ($\delta < \delta_i$). In comparing the two first terms in the side-payments, $\sum_{j=1}^{m}\frac{\theta_j}{\alpha_j}ce(c_j)(t)$ is smaller than $\sum_{j=1}^{m}\mu_{c_j}(t)$ if the agents are early resolvers, since by definition $ce(c_j)(t) < \mu_{c_j}(t)$ for all j. In this situation $\mu_{c_i}(t) > \frac{\theta_i}{\alpha_i}ce(c_i)(t)$ but the first inequality dominates, since it involves a large number of similar terms. Similarly, the difference between the two terms in the fourth group related to the jumps are also negative.

Suppose for this agent i that the following inequality $\frac{\alpha_i}{\theta_i} > \frac{\Psi}{\Theta}$ holds. From the discussion in the last section, this means that the agent is not diversified with respect to the annuity from both sources of risk dynamics, that is, the risk tolerant agent assumes a risky position. The contribution from the drift terms of the annuity to the side-payments is negative. This penalises the risk tolerant agent for lack of diversification, making the side-payment negative.

On the other hand, suppose now that the agent i is risk averse, an early resolver, more patient than average and with the above inequality reversed, that is $\frac{\alpha_i}{\theta_i} < \frac{\Psi}{\Theta}$ holds. The agent is now diversified with respect to annuity risk from both sources of dynamics, that is, the risk averse agent diversify and assumes a less risky position than the above described agent. The contribution from the two drift terms of the annuity to the side-payments are now both positive, compensating the agent for being diversified, which increases the size of the side-payment. Such an agent will typically loose inn the long run, but will be compensated through side-payments. But recall, these are of a lower order of magnitude than the direct random shocks.

Since the side-payments sum to zero, if some agents have negative side-payments, other agents must necessarily receive positive compensations.

We summarize our informal discussion as follows:
(i) Suppose agent i is risk tolerant and impatient, where the inequality $\frac{\alpha_i}{\theta_i} > \frac{\Psi}{\Theta}$ is satisfied. This agent has a risky position from the presence of the annuity. The agent will typically pay a compensation to society, via a negative side-payment.
(ii) Suppose agent i is risk averse and patient, where the reverse inequality $\frac{\alpha_i}{\theta_i} < \frac{\Psi}{\Theta}$ holds. This agent is diversified with respect to annuity risk, and obtains a positive side-payment.

According to Eeckhoudt, Gollier and Schlesinger (2005) is preference for diversification intrinsically equivalent to risk aversion.

Many comparisons can be made in this model. The two different situations we have focused on above make sense and tell a basic story: This rather complex syndicate is still consistent with the original one period risk-sharing model of a syndicated market, due to Borch (1962). The basics are not lost in our generalization.

Let us illustrate why the situations in (i) and (ii) above are quite natural:

Example: Consider a situation with four agents, where $\alpha_1 = 110, \alpha_2 = 120, \alpha_3 = 130, \alpha_4 = 140$ and $\theta_1 = 15, \theta_2 = 30, \theta_3 = 60, \theta_4 = 100$. It follows that $\Psi = 31$ and $\Theta = 7.9$. For the most risk tolerant agent $\frac{110}{15} = 7.3 > \frac{31}{7.9} = 3.9$, which is consistent with (i). For the most risk averse agent $\frac{140}{100} = 1.4 < \frac{31}{7.9} = 3.9$, which is consistent with (ii) above. □

In this example both the market and all the agents have preference for early resolution of uncertainty. The most risk tolerant agent also has the lowest value of θ in the example, which means that agent 1 has low resistance to consumption substitution across time. This could, for example, mean that this agent invests now for future consumption. The most risk averse agent has a higher value of θ, and has thus more resistance to consumption substitution. For example would $\frac{140}{60} = 2.3 < 3.9$ still represent an agent in category (ii).

Individuals with positive side-payments are typically risk averse. This model also indicates that they may have large θ as well. Such agents tend to loose in the long run, but will be compensated (moderately) by the side-payments. Risk tolerant agents, on the other hand, may be risk-takers with a low resistance to consumption substitution across time. Assuming that such agents are also impatient, in the long run some agents of this type may end up with a much larger than average share of aggregate consumption, in which case the model prescribes a social compensation to society. However, many of these agents may also fail, and this could mean loosing everything; there is no limited liability in the setting of a syndicate. But this is what risk-taking is all about.

If we do not see this common picture in risk sharing, the EU-model can not explain possible deviations, while the recursive model can. This is one of the advantages with the latter model. For example, situations where some agents are late resolvers can obviously occur - there is nothing irrational about having preference for late resolution of uncertainty.

The above results are in agreement with the discrete time version and the continuous-time version with diffusion dynamics, in Aase (2024a,b).

11.2 The ordinary additive and separable expected utility model with diffusion and jump-driven dynamics.

Here we consider the case of expected utility in which case the basic equation (52) simplifies, where, for example, all the characteristics of the annuity drop out. The relationship reduces to the following when $\alpha_i = \theta_i$ for $i = 1, 2, \cdots, m$.

$$dc_i(t) = \frac{\Psi}{\alpha_i} dc(t) + \Big(\frac{1}{2}\alpha_i \sigma'_{c_i}(t)\sigma_{c_i}(t) - \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m} \alpha_j \sigma'_{c_j}(t)\sigma_{c_j}(t) + \big(\frac{1}{2}\alpha_i \int_{\mathcal{Z}} \gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta) - \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\alpha_j \int_{\mathcal{Z}} \gamma'_{c_i}(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta)\big) + \frac{1}{\alpha_i}(\delta - \delta_i)\Big)dt. \quad (55)$$

This illustrates that the mutuality principle holds in a dynamic framework for the standard model with continuous time and jump-diffusion dynamics. All the agents are seen to be affected in the same direction from an aggregate consumption shock, but to different degrees.

Agents who are more risk tolerant than average, supposing they are also more impatient than average, will be most affected, be it positively or negatively, from direct aggregate consumption shocks.

As above, this equation can be rewritten, where the term $(c_i(t) - \frac{\Psi}{\alpha_i}c(t))$ is added to the side-payments, securing consistency with the budget constraints:

$$
\begin{aligned}
c_i(t+dt) = \frac{\Psi}{\alpha_i}c(t+dt) + (c_i(t) - \frac{\Psi}{\alpha_i}c(t)) + \Big(\frac{1}{2}\alpha_i\sigma_{c_i}'(t)\sigma_{c_i}(t) \\
- \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\alpha_j\sigma_{c_j}'(t)\sigma_{c_j}(t) + \big(\frac{1}{2}\alpha_i\int_{\mathcal{Z}}\gamma_{c_i}'(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta) \\
- \frac{1}{2}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\alpha_j\int_{\mathcal{Z}}\gamma_{c_i}'(t,\zeta)\gamma_{c_i}(t,\zeta)\nu(d\zeta)\big) + \frac{1}{\alpha_i}(\delta-\delta_i)\Big)dt. \quad (56)
\end{aligned}
$$

Consider the side-payment of such an agent. There must be some agent i where $\alpha_i\sigma_{c_i}'(t)\sigma_{c_i}(t) < \frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\alpha_j\sigma_{c_j}'(t)\sigma_{c_j}(t)$ for α_i smaller than average, since $\sum_{i=1}^{m}\alpha_i\sigma_{c_i}'(t)\sigma_{c_i}(t) - \sum_{i=1}^{m}\frac{\Psi}{\alpha_i}\sum_{j=1}^{m}\alpha_j\sigma_{c_j}'(t)\sigma_{c_j}(t) = 0$.

A similar reasoning is valid for the two integrals with respect to the Levy measure $\nu(d\zeta)$. Finally $\frac{1}{\alpha_i}(\delta-\delta_i) < 0$ since the agent is assumed impatient.

That is, agent i then pays, according to the model, a compensation to society, consisting of more risk averse participants in the economy, represented by the side-payment which is now negative.

As noted above, agents in the category of our agent i tend to enjoy a larger share of aggregate consumption than the average consumer, at least in the long run.[7] According to theory, such agents often have negative side-payments.[8]

Agents with higher risk aversions than average are least affected from direct shocks to consumption, and may receive positive social compensations, provided they are patient enough. These compensations are, however, of a smaller order of magnitude than the shocks.

When it comes to existence of equilibrium, however, this is not good news for the standard model, since just after each shock to the economy, it is more or less out of equilibrium.

11.3 Discussion of the Pareto optimal risk sharing.

In this model we analyze basic risk sharing, not wealth redistributions, which is usually carried out via taxes. This may sometimes be suboptimal from an economic point of view.

Entrepreneurs typically take risk, can prosper from that, albeit sometimes loose, but will in general compensate others depending on preferences, provided they are still in business. As in the one-period model with EU,

[7]The latter claim is rooted in consumption statistics from about the last 150 years.

[8]But notice, optimal risk sharing must not be confused with wealth redistribution.

also here a side-payment prevents that a risk tolerant entrepreneur with low initial endowment of consumption ends up with a large share of the aggregate consumption in any period. In addition to this standard story, our model can also explain a variety of other interesting patterns, which cannot be accounted for by the standard EU-model. For example, it is not enough focus on the agents risk aversions only, the other parameters θ, which measures agent's resistance to consumption substitution, must also be taken into account, as well as the corresponding market quantities Ψ and Θ.

The major shocks tend to be transitory, where the effects diminish as time goes. Shocks generated by the Brownian motions B as well as shocks generated by the process jump process N are prices in our analysis, recall the form of the state price deflator $\pi(t)$ in Section 7, where the quantities Z_t and $K(t, \zeta)$ are determined in Appendix 1. This is of great importance to the economic model, and distinguishes our approach from almost any actuarial treatment of these issues.

Our risk sharing model is fairly general. For example, the agents can alternatively be interpreted as countries or regions, in which case it describes risk sharing arrangement between regions of the World, where the utility functions must be considered as broad objectives for the regions, which may differ from region to region. It can be useful to think through how risk sharing is carried out in real life, and compare it to the results of this model.

The non-monolithic character of the relationship (52) is good news for the possibility of existence of equilibrium in the presented model: After shocks to the economy the various classes of agents behave differently, meaning that supply can meet demand after the shocks. For the traditional separable and additive expected utility model this is, as pointed out above, not the case.

12 Some other model related issues

We have noticed that including all the features of the jump model can make some difference regarding the model's predictions. In particular, this becomes important for the recursive model. In this paper we have obtained exact formulas in the CCAPM-framework, which allows us to study deviations from the classical mean-square analysis. Similarly, our approach with jumps allows for deviations from normality, which both can be of importance in various applications, in particular in insurance.

The topic of separating the jump part from the continuous part of a data set is dealt with by, for example, Ait-Sahalia and Jacod (2009-11), a topic which is far from trivial.

In the construction a *security-spot market equilibrium* (for a definition, see Duffie (2001)) embedded in our basic risk sharing model, we can fix an Arrow-Debreu equilibrium of the type that we have considered in this paper, and examine a spanning condition on the gains process, or the price process adjusted for dividends, of the risky securities under which there exists a security-spot market equilibrium with the same consumption allocation. The result is that the dynamic spanning condition requires at least as many securities as there are independent sources of risk. In the pure diffusion case with a risk-less asset and d independent Brownian motions, precisely $d+1$ securities turns out to be both necessary and sufficient. With a jump component added the conventional wisdom seems to be that the model is generally incomplete. In the recent paper by Aase (2023) it is demonstrated why this is an hastened conclusion. The clue is to consider a full market model with as many assets as there are sources of risk. This means that when the jump components are market point processes with a finite common support, we need a finite number of risky asset in order to make the market complete.

On the other hand, results like the consumption based capital asset pricing theory and extensions, as in the present article, can be derived without the spanning condition. In order to study financial derivatives, it is usually required that the model is complete. The irony is that when this is the case, such contracts are also redundant, and thus do not contribute to economic efficiency.

Risk sharing in a stock market is linear, recall agents hold portfolios of shares in the various securities. In the present model we have shown that optimal risk sharing in general is linear with no jumps in the model. With jumps risk sharing is still linear provided the annuity has no jump component, i.e., if $\gamma_{R_a}(t,\zeta) = 0$ a.s. Referring to Rubinstein (1974), in such cases the stock market can be said to be "essentially complete", since there is no limitation in restricting attention to linear risk sharing represented by a stock market. Also recall, in a complete securities market, non-linear contracts are redundant. In contrast, nonlinear contracts when $\gamma_{R_a}(t,\zeta) \neq 0$ may be analyzed in our model.

We round off the paper with some practical applications of the presented theory. Here we employ the general framework worked out in the model of this paper, primarily the jump part, which is used to take a new look at a theory of catastrophe futures and forward contracts. It turns out that the general fundament of the paper will add to this theory and its applications. The connection to existing EU-based theory is pointed out, showing consistency between these two approaches.

First we connect to a situation where such catastrophe futures or forward contracts may be of particular interest.

12.1 Relation to Weitzman's "dismal theorem".

Weitzman (2009) expressed, with climate change as a prime example, worries about the implications of structural uncertainty for economics of low probability, high-impact catastrophes. He focused on the conditional expected value of the marginal rate of substitution, which could become large in such cases. As an illustration, he considered a perturbation of the volatility of consumption. However, such a perturbation has no "direction". A more concise model is one containing unpredictable jumps at random times, as in Aase (1999), and in the present paper, which instruments that might mitigate the effects of such catastrophes.

Weitzman's analyses have been discussed by several authors. As one way to react to a scenario as he describes, it has been suggested that society should be willing to exchange today's consumption to the future at a large rate, a form of ex post insurance. But, for many reasons, , this does not seem very likely to happen in such a scenario. To use the capacity of the world's financial markets seems more appropriate in situations of catastrophes, since institutions are already in place.[9]

The catastrophe forward and futures contracts described below may be a constructive device to tackle negative shocks of this type, by sharing the negative consequences in any particular region with the rest of the world. Futures and forward contracts transfer the risk to the financial markets, which presumably have the capacity to absorb shocks of this type.

However, other means must be made in order to reduce the likelihood of such events. In insurance this goes under the heading of risk management.

13 A catastrophe insurance forward contract

In this section we consider a catastrophe insurance futures/forward contract, where jump processes play a natural and central role.

In Aase (1999) an equilibrium model for a catastrophe insurance forward contract was analyzed using expected utility. The model for the claims against the insurers is of the type we consider; $V(t) = \int_0^t \int_{\mathbb{R}} \zeta N(ds, d\zeta)$. This can also be written $V(t) = \sum_{k=1}^{N(t)} Z_k$, where $N(t)$ is a counting process recording the number of claims by time t and $Z_1, Z_2, \dots$ are the consecutive claim sizes.

We give a short demonstration illustrating why the theory in the present paper can be used to recover the results in this article. In particular a closed form forward formula is derived.

[9]assuming of course that this does not mean the end of the world.

In order to describe the model, it may be useful to consider a reinsurance syndicate consisting of I members all having preferences represented by negative exponential utility functions with absolute risk aversions α_i. Assuming agent i has net reserves

$$X^i(t) = c^i(t) - V^i(t), \quad i = 1, 2, \ldots, I,$$

where $c^i(t)$ is consumption at time t. We assume the agents pool their risks in the usual way and reach a Pareto optimum $\hat{X}^i(t)$ at each time t. The syndicate's risk aversion is then α, where $\frac{1}{\alpha} = \sum_{i=1}^{I} \frac{1}{\alpha_i}$, and the Pareto optimal allocations are affine functions $\hat{X}^i(t) = \frac{\alpha}{\alpha_i} X^i(t) + b_i$, $i = 1, 2, \ldots, I$, where the constants b_i are zero-sum side-payments. Also $X(t) = \sum_{i=1}^{I} X^i(t) = \sum_{i=1}^{I} \hat{X}^i(t)$, since no risk or consumption disappears in the exchange.

The aggregate net reserves are

$$X(t) = c(t) - V(t),$$

where $c(t)$ is the aggregate consumption in the syndicate at time t, where $V(t)$ contains the jumps (in principal in consumption). The term $c(t)$ may, for example, be generated by a continuous Ito-diffusion process.

Here $V(t) = \sum_{k=1}^{N(t)} Z_k$ represents the aggregate claims by time t, reported to, say, a statistical agent associated to a catastrophe insurance exchange (like the Chicago Board of Trade). The idea with these derivatives on various loss indexes is to transfer some of the catastrophe risk from the reinsurance markets to the financial markets, where the capacity is presumably much larger.

The syndicate can benefit from derivatives on the loss ratio index for risk management purposes. The loss ratio index is defined by $\hat{V}(t) = V(t)/\Pi$, where Π, the total premiums for the next quarter, is considered prepaid and known before the event quarter starts. The contract value is \$25,000 times the loss ratio. Below we consider a forward contract, where the forward price is given by the formula

$$F(t) = \frac{E_t(e^{-\alpha(c(T)-V(T))}\hat{V}(T))}{E_t(e^{-\alpha(c(T)-V(T)})}.$$

If the short term interest rate $r(t)$ is conditionally independent of c and V given $\mathcal{F}_t$, this is also the formula for the *futures* price. With an assumption that the consumption term $c(T)$ has no jump components (other than those contained in $V(T)$), by the nature of the forward formula it vanishes and the formula simplifies to

$$F(t) = \frac{E_t(e^{\alpha V(T)}\hat{V}(T))}{E_t(e^{\alpha V(T)})}. \tag{57}$$

Assuming the process V is compound Poisson with frequency λ and the losses Y_i are iid Gamma-distributed with parameters μ and n, where $\mu > \alpha$, it is shown in Aase (1999) that

$$F(t) = \hat{V}(t) + \frac{n\mu^n \lambda(T-t)}{\Pi(\mu-\alpha)^{(n+1)}}. \tag{58}$$

This expression implies a risk premium stemming from the risk aversions of the members of the syndicate, which also implies that the parameter $\alpha > 0$. The forward price can be seen to increase with the absolute risk aversion in the syndicate, given that Π does not change.[10] Under risk neutrality $F(t) = \hat{V}(t) + E(Z)\frac{\lambda(T-t)}{\Pi}$, where $E(Z) = n/\mu$, as the case should be.

We now demonstrate that these results can be derived by our above, more general theory, as a special case when $\theta = \alpha_0$.

In this theory the forward price is given by

$$F(t) = \frac{E_t(\pi_T \hat{V}(T))}{E_t(\pi_T)}, \tag{59}$$

so we have to determine the state price deflator π_T. The insurance claims are assumed independent of consumption and there is no diffusion term.

Moving this story out of the insurance syndicate, and into the finance market of this paper, the component of the state price state price relevant for the latter market is

$$\gamma_\pi(t,\zeta) = \pi_{t-}\big[-\alpha_0 K(t,\zeta) + \big(e^{-\theta(\gamma_c(t,\zeta)-K(t,\zeta))}-1\big)\big(1-\alpha_0 K(t,\zeta)\big)\big],$$

given in equation (35), since the term $\mu_\pi(t)$ in the drift of π_t cancels in the forward formula due to independence between the insurance claims, which here are purely jumps, and the drift terms in π.

For comparisons with expected utility, recalling the representation (9), the terms $K(t,\zeta)$ and $Z(t)$ are seen to both drop out, which means that as far as the jump part is concerned, it is sufficient to consider

$$\gamma_\pi(t,\zeta) = \pi_{t-}\big(e^{-\theta\gamma_c(t,\zeta)}-1\big). \tag{60}$$

[10] According to professor Paul Embrechts of the ETH, this paper has been taken to the Swiss reinsurance market, where the application was a success. Also, Merton Miller recommended the author to establish such a market in Norway.

The term $\gamma_c(t,\zeta) = \zeta$, without loss of generality.

We now need Itô's lemma in this situation in order to find the relevant expression for the state price deflator: It says that the solution π_t of the equation

$$d\pi_t = \pi_{t-} \int_{\mathbb{R}} c(t,\zeta)\tilde{N}(dt, d\zeta)$$

for an appropriate function $c(t,\zeta)$, is given by

$$\pi_t = \exp\Big\{ \int_0^t \int_{\mathbb{R}} \big(\ln(1 + c(s,\zeta)) - c(s,\zeta)\big)\nu(d\zeta)ds + \int_0^t \int_{\mathbb{R}} \ln\big(1 + c(s,\zeta)\big)\tilde{N}(ds, d\zeta)\Big\}. \quad (61)$$

Recalling that with expected utility $\theta = \alpha_0$, by using equation (60) with $c(t,\zeta) = \big(e^{-\alpha_0\zeta} - 1\big)$, where we have set $\gamma_c(t,\zeta) = \zeta$, we find that

$$\pi_t = \exp\Big\{ \int_0^t \int_{\mathbb{R}} \big(1 - e^{-\alpha_0\zeta}\big)\nu(d\zeta)ds - \alpha_0 \int_0^t \int_{\mathbb{R}} \zeta N(ds, d\zeta)\Big\}. \quad (62)$$

It follows by our model assumptions that the forward price is given by the formula

$$F(t) = \frac{E_t\big\{\exp\big(\alpha_0 \int_0^T \int_{\mathbb{R}_-} \zeta N(dt, d\zeta)\big)\hat{V}(T)\big\}}{E_t\big\{\exp\big(\alpha_0 \int_0^T \int_{\mathbb{R}_-} \zeta N(dt, d\zeta)\big)\big\}}. \quad (63)$$

With the assumption about independent jump sizes, this expression can be seen to be the same formula as given in equation (57). This follows since we are concerned about the negative jumps, which are the ones contained in $\hat{V}(T)$, but with positive signs. The positive jumps in the state-price deflator are, by assumption, independent of the negative jumps, and will therefore cancel in the forward formula, by the nature of this the formula. Appendix 2 contains a proof. Without loss of generality we therefore restrict attention to the negative jumps only in the state price deflator.

With this assumption, we can write $V(T) = \int_0^T \int_{\mathbb{R}_+} \zeta N(dt, d\zeta)$, and by this we mean that the integral

$$-\int_0^T \int_{\mathbb{R}_-} \zeta N(dt, d\zeta) = \int_0^T \int_{\mathbb{R}_+} \zeta N(dt, d\zeta) = V(T).$$

The jumps in the two integrals are then the same.

The recursive utility version of the forward price must be based on the general expression for the state-price deflator, in particular the jump part $\gamma_\pi(t,\zeta)$, and is in principle more complex.

First we consider the general case. The objective is the same as above, to have a catastrophe forward or futures index related to negative shocks in aggregate consumption in society. Following the approach in (61) and (62), we have now one more variable, the annuity, with

$$c(z_1,z_2) = \frac{1}{1+\gamma_{R_a}(z_2)}\big(1-\alpha_0 z_1 + \frac{\alpha_0}{\theta}\ln(1+\gamma_{R_a}(z_2))\big) - 1, \tag{64}$$

which follows from the expressions for $\gamma_\pi(t,\zeta)$ in equation (35) and the formula for $K(t;z_1,z_2) = z_1 - \frac{1}{\theta}\ln(1+\gamma_{R_a}(t,z_2))$, we find the CAT-forward formula below. Since we can restrict attention to negative consumption jumps in the state-price deflator, the formula becomes

$$F(t) = E_t\big\{\exp\big(\int_0^T \int_{\mathcal{Z}} \ln\big(\frac{1}{1+\gamma_{R_a}(z_2)}\big[1-\alpha_0 z_1 + \frac{\alpha_0}{\theta}\ln(1+\gamma_{R_a}(z_2))\big]\big) \\ N(dt;dz_1,dz_2)\hat{V}(T)\big\}/E_t\big\{\exp\big(\int_0^T \int_{\mathcal{Z}} \ln\big(\frac{1}{1+\gamma_{R_a}(z_2)}\big[1-\alpha_0 z_1 + \\ \frac{\alpha_0}{\theta}\ln(1+\gamma_{R_a}(z_2))\big]\big)N(dt;dz_1,dz_2)\big)\big\} \tag{65}$$

where $\mathcal{Z}$ limits the region so that the integrands are well-defined, and $\hat{V}(T)$ is the same as above, it is a compound Poisson process containing the negative jumps in consumption, but with positive signs.

The formula depends on both α_0 and θ, and requires knowledge of the joint distribution of the jumps in the annuity process and in the aggregate consumption process. When the parameter θ increases, the resistance to consumption substitution increases, and demand for the forward contract should normally decrease.

To see the effect of the parameter θ, recall that aggregate consumption and the annuity are typically negatively correlated. With the annuity in the model, this instrument has the effect of increasing the forward price. When θ increases, this effect is weakened, i.e., $F(t)$ decreases. On the other hand, when resistance to consumption substitution decreases, the forward price $F(t)$ increases.

Provided we ignore the effect of the annuity on the cat forward contract altogether, this is achieved by setting $\gamma_{R_a} = 0$, and the general forward formula (65) takes the form

$$F(t) = \frac{E_t\big\{\exp\big(\int_0^T \int_{\mathbb{R}_+} \ln\big[1+\alpha_0\zeta\big]\big)N(dt,d\zeta)\big)\hat{V}(T)\big\}}{E_t\big\{\exp\big(\int_0^T \int_{\mathbb{R}_+} \ln\big[1+\alpha_0\zeta\big]\big)N(dt,d\zeta)\big)\big\}}. \tag{66}$$

This follows from equations (61) and (64).

As above, let us restrict attention to the negative jumps in the state price deflator. We then have the following result:

Theorem 4 *Assume that the jumps in $V(T)$ are iid with support on the positive reals, where the first two central moments, $E(Z_1) := \mu_1$ and $E(Z_1^2) := \mu_2$, both exist. We then obtain the following catastrophe forward price formula:*

$$F(t) = \hat{V}(t) + (\mu_1 + \alpha_0\mu_2)\frac{\lambda(T-t)}{\Pi}. \tag{67}$$

The proof is given in Appendix 2.

The term with the parameter θ disappeared by setting $\gamma_{R_a} = 0$. Since consumption substitution is made possible by the annuity, θ becomes irrelevant for the jump part in this situation.

This latter simplification does not recover the forward formula (58), but comes close in the situation with Gamma-distributed claim sizes. One reason is that setting $\gamma_{R_a}(t,\zeta) \equiv 0$ is not the same as setting $K(t,\zeta) \equiv 0$, where only the latter gives the EU-version. (Recall that when $\gamma_{R_a}(t,\zeta) \equiv 0$, then $K(t,\zeta) = \gamma_c(t,\zeta)$.)

Unlike for the formula (58), a restriction like $\alpha_0 < \mu$ is not required in the formula (67) if the claim-size distribution were Gamma (n,μ) (an assumption which is no longer required).

Recall that we have a more general model at our hands than the plain EU-version with jump dynamics, which is here formalized to deal with jump dynamics in a fairly general environment.

The difference between these two formulas turns out to be rather small when $\alpha < \mu$: By a Taylor series approximation, $\frac{n}{\mu}\frac{\mu^{n+1}}{(\mu-\alpha_0)^{n+1}} \approx \frac{n}{\mu}\frac{1}{(1-\alpha_0\frac{n+1}{\mu})} \approx \frac{n}{\mu}(1+\alpha_0\frac{n+1}{\mu}) = (\mu_1+\alpha_0\mu_2)$, where we have used the Binomial series for the function $(1-x)^m$.

As a numerical illustration of this, let us consider the parameters in Example 1 of Aase (1999) with $n = 10, \mu = 10^{-6}, \lambda = 10, (T-t) = 0.25, \alpha = \alpha_0 = 5\cdot 10^{-9}$, $\Pi = \$26417200$, the latter term in the above formula (the loss ratio) is 0.99840 while the corresponding term in formula (58) is 0.99999. As the parameter α decreases, the two expressions converge.

Suppose for the moment that the agents in the model are simply mutual insurance companies in a syndicate. If the number og agents increases, the parameter α decreases, as we have seen above.[11] For example, when $\alpha = \alpha_0 = 5 \cdot 10^{-10}$, the above loss ratio is 0.95155 while the corresponding loss ratio in formula (58) is 0.95157 (keeping Π fixed).

The risk premium, the difference between the market price and the expected payout under the contract, is here

$$F(t) - \{\hat{V}(t) + E(\hat{V}(T) - \hat{V}(t))|\mathcal{F}(t)\} = \frac{1}{\Pi}\lambda(T-t)\alpha_0\mu_2 > 0.$$

Under risk neutrality $\alpha_0 = 0$ in which case the risk premium is zero, as it should, and $F(t) = \hat{V}(t) + (EZ)\frac{\lambda(T-t)}{\Pi}$.

13.1 An interpretation of the market price of insurance risk.

Some readers may be used to the so-called equivalent martingale measure Q in pricing contexts. This also has a natural analogue here, which we now demonstrate.

Starting with the stochastic model we have chosen, the Radon-Nikodym derivative $\xi(T) = \frac{dQ}{dP}$ and its associated density process $\xi(t) = E(\xi(T) \mid \mathcal{F}_t)$ can alternatively be represented as (see, for example, Bremaud (1981), T10 Theorem, p241)[12]

$$\xi(t) = \left(\prod_{n\geq 1} \kappa v(Z_n) 1(\tau_n \leq t)\right) \exp\left\{\int_0^t \int_{\mathcal{Z}} (1-\kappa v(z))\lambda F(dz)ds + \int_0^t r(s)ds\right\}, \tag{68}$$

holding for any $t \in [0, T]$, where τ_n represent the time points of jumps. Here $F(dz)$ is the probability distribution of the claim sizes Z_i, κ is a positive constant and $v(z)$ is a function satisfying

$$\int_0^\infty v(z)F(dz) = 1\,, \qquad P\text{- a.s.} \tag{69}$$

The term $v(z)F(dz)$ is accordingly the probability distribution of the claim sizes Z_i under the probability measure Q, whereas $\kappa\lambda$ is the corresponding

[11] On the other hand, if we talk about competition between publicly owned insurance companies, the risk premium will, according to conventional wisdom, be traded down to zero, corresponding to $\alpha = 0$, in which case the two formulas become identical.

[12] In the literature this sometimes goes under the heading of likelihood ratios, or Radon-Nikodym derivatives

frequency of the Poisson process under Q. We interpret κ as the market price of frequency risk, and $v(z)$ as the market price of claim size risk. The function v is non-negative and strictly positive on the support of F.

The state price deflator for positive claim sizes, corresponding to negative jumps in consumption, when $\gamma_{R_a} = 0$, is

$$\pi_t = \exp\Big(\int_0^t \int_{\mathbb{R}_+} \ln(1+\alpha_0 z) N(dt, dz)\Big)$$

and using the connection between this quantity and the density process ξ_t, which is

$$\pi_t = \xi_t \exp\Big(-\int_0^t r(t)dt\Big)\pi_0,$$

we obtain the following ($\pi_0 = 1$):

$$\xi_t = \exp\Big(\int_0^t \int_{\mathbb{R}_+} \ln(1+\alpha_0 z) N(ds, dz) + \int_0^t r(s)ds\Big)$$
$$= \prod_{i=1}^{N_t} (1+\alpha_0 Z_i) e^{\int_0^t r(s)ds}. \quad (70)$$

We now assume, for simplicity, that the interest rate process $r(s)$ is independent of the claim sizes, for example, r has no jump component, in which case the forward price equals the futures price. By this assumption we can ignore r in what follows. The other term in the exponent of (68) can likewise be ignored since it is a non-random constant.

Comparing the above expression with (68), we obtain

$$\kappa v(z) = 1 + \alpha_0 z.$$

This means that the jump size distribution under Q, $F^Q(dz)$, is given by

$$F^Q(dz) = \frac{1}{\kappa}(1+\alpha_0 z)F(dz).$$

Since both F^Q and F are bona fide probability distributions, this implies that $1 = \int_0^\infty F^Q(dz) = \int_0^\infty \frac{1}{\kappa}(1+\alpha_0 z)F(dz) = \frac{1}{\kappa}(1+\alpha_0\mu_1)$, where $\mu_1 = E(Z)$, which gives that $\kappa = 1 + \alpha_0\mu_1$, so that $v(z) = \frac{1+\alpha_0 z}{1+\alpha_0\mu_1}$. By this, both the adjustment to frequency risk and jump size risk are determined.

Note that under Q the frequency of the claims is larger than under P since $\kappa > 1$, and the claim size distribution has a larger expected value under Q than under P. The latter claim follows, since $E^Q(Z_1) = (\mu_1 + \alpha_0\mu_2)/(1+$

$\alpha_0\mu_1)$, where $\mu_2 = E(Z^2)$, which can be written as

$$E^Q(Z_1) = \mu_1 + \frac{\alpha_0(\mu_2 - \mu_1^2)}{1 + \alpha_0\mu_1} > \mu_1 = E(Z_1),$$

since the variance of Z is strictly positive. Also it follows that $S(y) = \int_0^y (F^Q(z) - F(z))dz < 0$ for all $y > 0$. Together this means that Z_1 under Q is more risky than Z_1 under P in the sense of second degree stochastic monotonic dominance (see, for example, Huang and Litzenberger (1988)).

Alternatively, and more to the point here, this can be connected to the Monotone Likelihood Ratio (MLR) order.[13] Here Z_1^Q (Z_1 under Q) is larger than Z_1^P according to monotone likelihood ratio ordering if $\frac{f^Q(z)}{f(z)}$ is nondecreasing is z, see Lehmann (1955). Since $f^Q(z)dz = F^Q(dz)$ and $f(z)dz = F(dz)$, this condition holds by the above, where $\frac{f^Q(z)}{f(z)} = \frac{1}{\kappa}(1+\alpha_0 z)$, which is an increasing function of z. In the present context this can be interpreted as "claims are more risky" under Q than under P.

Under the probability measure Q we have the following pricing rule:[14]

$$F_t = E^Q(\hat{V}_T|\mathcal{F}_t) = \hat{V}_t + E^Q(\hat{V}_T - \hat{V}_t|\mathcal{F}_t), \tag{71}$$

which can readily be found from the above. First recall that

$$\int_0^\infty zF^Q(dz) = \int_0^\infty v(z)F(dz) = \frac{1}{\kappa}\big(\mu_1 + \alpha_0\mu_2\big)\big).$$

This means that

$$F_t = \hat{V}_t + \lambda\kappa(T-t)\frac{1}{\kappa}\big(\mu_1 + \alpha_0\mu_2\big)\frac{1}{\Pi},$$

which is our previous result, since κ cancels. That is, formula (67) is demonstrated by the method of an equivalent martingale measure.

14 Conclusions

We have studied optimal risk sharing an exchange economy in continuous time driven by jump-diffusions. An equilibrium model is worked out, where agents have preferences represented by translation invariant recursive utility. The model is demonstrated to have more degrees of freedom related to im-

[13]often used in the principal agent literature.

[14]Q is formally (mathematically) a probability measure, but expectation under Q is in reality a linear pricing functional.

portant economic issues, compared to the conventional model where agents have preferences represented by separable and additive expected utility. Our formal optimization approach is to use the stochastic maximum principle for forward/backward stochastic differential equations. When consumption substitution and risk aversion coincide, the model reduces to the standard expected utility version where absolute risk aversions are constants.

The general model has several appealing features compared to the scale invariant version. First, the model aggregates, which allows for a treatment of heterogeneous preferences. Second, the new endogenous variable entering into the intertemporal marginal rate of substitution is a traded security, an annuity, while in the scale invariant model the new variable is the agent's wealth process, which is not traded. In the syndicated version of the model, this annuity can have a diversifying effect on risk averse agents, and increase exposure to risk for risk tolerant agents, depending on preferences. Important is that the model can also explain deviations from this standard picture, which broadens the scope considerably in relation to the standard model. Third, the model allows us to take a new look at the mutuality principle, with consequences for the existence of equilibrium.

With respect to the consumption based CAPM, although empirical questions with translation invariance are somewhat different from the model with scale invariance, both models seem to favour a representative agent with preference for early resolution of uncertainty, and an EIS larger than 1.

The inclusion of a jump part implies that the model can be used as a theoretical underpinning for financial contracts where jump dynamics is essential. In the last part of the paper this is illustrated by the analysis of catastrophe insurance forward contracts, where all risk is priced, including jump-size risk.

Appendix 1

The connection between the quantities Z anf K and observables in the economy.

In this section we connect the processes Z_t and $K(t, \cdot)$ to the corresponding quantities of a marketed asset, an annuity, in the economy.

For the scale invariant version one uses that the wealth W_t of the agent at any time t is given by

$$W_t = \frac{1}{\pi_t} E_t\Big(\int_t^T \pi_s c_s \, ds \Big), \tag{72}$$

where c is the optimal consumption of the agent. Then one can use scale invariance to find a connection between Z and K and the corresponding quantities of wealth and of the aggregate consumption, where the preference parameters also enter (Aase (2016)).

Here we do something similar, except that we use the quasi-linearity of the utility function U_t. Suppose that the problem

$$\sup_{c \in L} U(c)$$

has a strictly positive solution c^*, and that U has a strictly positive continuous derivative at c^*. For any adapted stochastic process x_t we know that the directional derivative of the utility function U in the direction x, $\bigtriangledown U(c;x)$, is given by

$$\bigtriangledown U(c;x) = \lim_{\kappa \downarrow 0} \frac{U(c+\kappa x) - U(c)}{\kappa}.$$

Under these conditions there is no arbitrage and a state-price deflator is given by the Riesz representation π of $\bigtriangledown U(c^*)$:

$$\bigtriangledown U(c^*;x) = \lim_{\kappa \downarrow 0} \frac{U(c^*+\kappa x) - U(c^*)}{\kappa} = E\big(\int_0^T \pi_t x_t \, dt\big).$$

Similarly, at any time $t \in (0,T)$ we also know that

$$\bigtriangledown U_t(c^*;x) = \lim_{\kappa \downarrow 0} \frac{U_t(c^*+\kappa x) - U_t(c^*)}{\kappa} = E_t\Big(\int_t^T \pi_s^{(t)} \, c_s \, ds\Big), \qquad (73)$$

where the Riesz representation $\pi_s^{(t)}$ for $s \geq t$ is the state price deflator at time $s \geq t$, as of time t, where $\pi_s^{(t)} = \pi_s / Y_t$ for all $t \leq s \leq T$ (see Skiadas (2009), Prop 6.15 for the discrete-time model, Aase (2016) for the continuous-time version). Recalling that $\pi_t = Y_t f_c(c_t^*, U_t(c^*))$, or $\pi_t = Y_t f_c(t)$ for short, we have that $\pi_s^{(t)} = Y_s f_c(s)/Y_t$.

The terms "translation invariance" and "quasilinear with respect to 1" are synonymous when applied to recursive utility. By the latter we have that

$$\bigtriangledown U_t(c^*;1) = \lim_{\kappa \downarrow 0} \frac{U_t(c^*+\kappa 1) - U_t(c^*)}{\kappa} = \lim_{\kappa \downarrow 0} \frac{U_t(c^*) + \kappa - U_t(c^*)}{\kappa} = 1.$$

By (73) this means that

$$E_t\big(\int_t^T \pi_s^{(t)} 1 \, ds\big) = 1.$$

This can be written $f_c(t)E_t(\int_0^T \frac{\pi_s}{\pi_t} 1ds) = 1$, or

$$\frac{1}{\pi_t} E_t\Big(\int_t^T \pi_s \, 1 \, ds\Big) = \frac{1}{f_c(t)}. \tag{74}$$

Recalling that for any risky asset denoted X with market price process $V_X(t)$ and real dividend-rate process x_s, $s \geq t$, the following relationship holds

$$V_X(t) = \frac{1}{\pi_t} E_t\Big(\int_t^T \pi_s \, x_s \, ds\Big).$$

From this we see that the left-hand-side of equation (74) is the market price at time t, $V_a(t)$, of an annuity paying a real rate-process of 1 in (t, T). Accordingly is

$$V_a(t) = \frac{1}{f_c(t)}. \tag{75}$$

It turns out that this is an important relationship which will enable us to connect the volatility of utility, Z_t, to the volatilities of consumption and of the rate of return on the annuity. This we now set out to demonstrate.

Recall that $f_c(c_t, U_t) := \delta e^{-\theta(c_t - U_t)}$, writing c_t for c_t^* for simplicity of notation, which together with (75) gives that

$$\ln V_a(t) = -\ln\delta + \theta(c_t - U_t).$$

Assuming that the price process of the annuity is a geometric Itô-Lévy process with cumulative-return process $dR_a(t)$, where $dV_a(t) = V_a(t^-)dR_a(t)$, so that

$$dR_a(t) = \frac{dV_a(t)}{V_a(t^-)} = \mu_{R_a}(t)dt + \sigma_{R_a}(t)dB_t + \int_{\mathcal{Z}} \gamma_{R_a}(t, \zeta)\tilde{N}(dt, d\zeta), \tag{76}$$

we define the process X by $X_t = \frac{1}{\theta}\ln V_a(t)$. First, the above relationship can be written

$$dU_t = dc_t - dX_t. \tag{77}$$

Second, using Itô's lemma in (76), $V_a(t)$ can be written

$$V_a(t) = V_a(0)\exp\Big\{\int_0^t (\mu_{R_a}(s) - \frac{1}{2}\sigma'_{R_a}(s)\sigma_{R_a}(s))ds + \int_0^t \sigma_{R_a}(s)dB_s$$
$$+ \int_0^t \int_{\mathcal{Z}} \Big(\ln(1+\gamma_{R_a}(s,\zeta)) - \gamma_{R_a}(s,\zeta)\Big)\nu(d\zeta)$$
$$+ \int_0^t \int_{\mathcal{Z}} \ln(1+\gamma_{R_a}(s,\zeta))\tilde{N}(ds,d\zeta)\Big\}. \quad (78)$$

Third, we utilize the canonical representation of a jump-diffusion process in that the triplet (μ, σ, γ) uniquely determines the process (modulo the starting point).

Taking the logarithm in equation (78) and using the basic relationship (77), from the above steps we obtain the following characterisations of Z and K:

$$(I) \qquad Z_t = \sigma_c(t) - \frac{1}{\theta}\sigma_{R_a}(t)$$

and

$$(II) \qquad K(t,\zeta) = \gamma_c(t,\zeta)) - \frac{1}{\theta}\ln(1+\gamma_{R_a}(t,\zeta))$$

The result in equation (I) is the connection that links the volatility Z of utility to the two observable volatilities on the right-hand side and the preference parameter θ. Similarly is equation (II) the connection that links the jump term K of utility to the two corresponding, observable quantities on the right-hand side.

Appendix 2

Derivation of the forward formula in equation (67). When $\gamma_{R_a} = 0$, the general forward formula (65) takes the form

$$F(t) = \frac{E_t\big\{\exp\big(\int_0^T \int_{\mathcal{Z}} \ln\big[1-\alpha_0\zeta\big]\big)N(dt,d\zeta)\big)\hat{V}(T)\big\}}{E_t\big\{\exp\big(\int_0^T \int_{\mathcal{Z}} \ln\big[1-\alpha_0\zeta\big]\big)N(dt,d\zeta)\big)\big\}}. \quad (79)$$

This follows from equations (61) and (64). By our earlier convention, the negative jumps in the state-price deflator are the ones contained in $\hat{V}(T)$, but with positive signs.

Assuming that $\sum_{i=1}^{N_T} Z_i$ is a compound Poisson process, where the frequency of the Poisson process N_T is λ and the claim sizes Z_i are i.i.d., all independent of the Poisson process $N_t, t \geq 0$, let us start with the denominator. Recalling that the positive jumps in the state-price deflator cancel, by independence, in the forward formula, we have to compute the following:

$$E_t\{\exp\big(\int_0^T \int_{\mathbb{R}_-} \ln\big[1-\alpha_0\zeta\big]\big)N(dt,d\zeta)\big)\} = E_t\{\exp\big(\sum_{i=1}^{N_T} \ln(1+\alpha_0 Z_i)\big)\} =$$

$$E_t\{\prod_{i=1}^{N_T}(1+\alpha_0 Z_i)\} = \prod_{i=1}^{N_t}(1+\alpha_0 Z_i^{(obs)})E_t\{\prod_{i=N_t+1}^{N_T}(1+\alpha_0 Z_i)\} =$$

$$\prod_{i=1}^{N_t}(1+\alpha_0 Z_i^{(obs)})E_t\big(1+\alpha_0\mu_1\big)^{N_{T-t}} = \prod_{i=1}^{N_t}(1+\alpha_0 Z_i^{(obs)})\exp\big(\lambda(T-t)\alpha_0\mu_1\big).$$

Here $Z_i^{(obs)}$ are the observed values of Z_i by time t. We have used iterated expectation, the independence structure and the time homogeneity of the Poisson process to replace $(N_T - N_t)$ by N_{T-t}. The last expression follows from the moment generating function of the Poisson random variable given by $\varphi(\beta) = E(e^{\beta N_{T-t}}) = e^{\lambda(T-t)(e^\beta - 1)}$.

Moving to the numerator, we need to compute for $V(T) = \sum_{i=1}^{N_T} Z_i$ the following

$$E_t\{\exp\big(\int_0^T \int_{\mathbb{R}_-} \ln\big[1-\alpha_0\zeta\big]\big)N(dt,d\zeta)\big)V(T)\} =$$

$$\prod_{i=1}^{N_t}(1+\alpha_0 Z_i^{(obs)})E_t\Big\{\prod_{i=N_t+1}^{N_T}(1+\alpha_0 Z_i)\big(V_t + \sum_{i=N_t+1}^{N_T} Z_i\big)\Big\}.$$

Since

$$E_t\{\prod_{i=N_t+1}^{N_T}(1+\alpha_0 Z_i)V_t\} = V_t E_t\big(1+\alpha_0\mu_1\big)^{N_{T-t}} = V_t\exp\big(\lambda(T-t)\alpha_0\mu_1\big)$$

where we have again used the moment generating function of the Poisson random variable, it remains to calculate

$$E_t\Big\{\prod_{i=N_t+1}^{N_T}(1+\alpha_0 Z_i)\big(\sum_{i=N_t+1}^{N_T} Z_i\big)\Big\}.$$

By multiplying the product by the summands in the sum, term by term and using the independence structure as well as the fact that $E(Z_i) = \mu_1$,

$E(Z_i^2) = \mu_2$ so that $\text{var}(Z_i) = \mu_2 - \mu_1^2$, we obtain that the above equals:

$$E_t\big\{(N_T - N_t)\big(\mu_1 + \alpha_0\mu_2\big)\big[1 + \alpha_0\mu_1\big]^{N_T - N_t - 1}\big\} =$$

$$\big(\mu_1 + \alpha_0\mu_2)\big)\big(1 + \alpha_0\mu_1\big)^{-1}E_t\big\{N_{T-t}\big(1 + \alpha_0\mu_1\big)^{N_{T-t}}\big\}.$$

The latter conditional expectation can again best be handled using the moment generation function of the Poisson random variable. Consider the derivative of this function with respect to the parameter β: It is $\varphi'(\beta) = E(N_{T-t}e^{\beta N_{T-t}}) = e^{\lambda(T-t)(e^{\beta}-1)}(\lambda(T-t)e^{\beta})$. Using this and setting $e^{\beta} = (1 + \alpha_0\mu_1)$ we obtain

$$\big(\mu_1 + \alpha_0\mu_2\big)\big(1 + \alpha_0\mu_1\big)^{-1}e^{\lambda(T-t)(\alpha_0\mu_1)}(\lambda(T-t)(1 + \alpha_0\mu_1)) =$$

$$\big(\mu_1 + \alpha_0\mu_2)\big)e^{\lambda(T-t)(\alpha_0\mu_1)}\lambda(T-t).$$

Putting all this together, we finally obtain the forward formula, after division by Π:

$$F_t = \hat{V}_t + (\mu_1 + \alpha_0\mu_2)\frac{\lambda(T-t)}{\Pi}, \tag{80}$$

which is the forward price formula (67).

The positive jumps in the state price deflator drop out.

Now, suppose there are also positive jumps in the state price deflator, assumed to be a Compound Poisson process N' of frequency λ' with the two first central moments are μ_1' and μ_2', where the positive jumps are independent of the negative ones. The total number of jumps are $N'' = N + N'$.

The crucial new term in the numerator in the forward formula has accordingly changed to

$$\begin{aligned} E_t\Big\{\prod_{i=N_t''+1}^{N_T''}(1 + \alpha_0 Z_i)\Big(\sum_{i=N_t+1}^{N_T} Z_i\Big)\Big\} &= \\ E_t\big\{(N_T - N_t)\big(\mu_1 + \alpha_0\mu_2\big)\big[1 + \alpha_0\mu_1\big]^{N_T - N_t - 1}[1 - \alpha_0\mu_1']^{N_T' - N_t'}\big\} &= \\ \big(\mu_1 + \alpha_0\mu_2)\lambda(T-t)e^{\lambda(T-t)(\alpha_0\mu_1)} \cdot e^{-\lambda'(T-t)\alpha_0\mu_1'}&. \end{aligned}$$

Similarly, the crucial new term in the denominator has now changed to

$$E_t\Big\{\prod_{i=N_t+1}^{N_T}(1 + \alpha_0 Z_i)\prod_{i=N_t'+1}^{N_T'}(1 - \alpha_0 Z_i')\Big\} =$$

$$\exp\big(\lambda(T-t)\alpha_0\mu_1\big)\cdot e^{-\lambda'(T-t)\alpha_0\mu_1'}.$$

The last factor in each part of the ratio cancel, and the forward formula (80) again results, as claimed in the text.

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