Recursive utility and jump-diffusions

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DISCUSSION PAPER

NHH

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FOR 6/2025
ISSN: 2387-3000
February 2025

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January 29, 2025

Abstract

We consider agents in an exchange economy having preferences represented by scale invariant recursive utility, where the dynamics of both consumption and risky assets are given by jump-diffusions. In this setting we find state prices, where both diffusion and jump-size risk are priced. By including jumps, the theory has the potential to model insurance markets, as well as ordinary securities' markets. In the latter case, we derive the equilibrium, real interest rate and risk premiums. In the former case we consider catastrophe futures related to negative shocks in consumption. We use the stochastic maximum principle to analyze the model. This method uses forward/backward stochastic differential equations, and seems indispensable in this theory.

KEYWORDS: recursive utility, jump dynamics, the stochastic maximum principle, jump size risk, catastrophe futures.
JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction.

We consider a market model in financial economics in the setting of continuous time, where the dynamics of economic variables are modelled by jump-diffusions.

From a theoretical point of view a most important step in our analysis is the internalization of the probability distributions, or more precisely, stochastic processes for the market and the wealth portfolios. They are determined in equilibrium from the primitives of the underlying economic model. These

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are the stochastic process for future utility (preferences) and the process determining the dynamics of the growth rate of aggregate consumption (the given endowment process). We use a general method of optimization, the stochastic maximum principle, together with the theory of forward/backward stochastic differential equations, which allows for an extension to jump dynamics.

The recursive model has several interesting features when jumps are allowed in the dynamics of the aggregate consumption process as well as in the recursive utility process. In addition to the nonlinear terms that are introduced, it also gives a new parameter for the risk aversion related to jump size risk. Both together, and in isolation, these features may yield added insights in explaining real data.

In addition to ordinary securities markets, where the jump structure can be approximated to have the same basic structure as the continuous components, in other markets the jump components can be more innovative. In particular this type of model can be used in the securitisation of catastrophe insurance futures. We present an application where insurance risk is priced, and the model represents an adequate economic framework for many problems in insurance. This application is related to Weitzman's "dismal theorem" (Weitzman (2009)) in connection with risks associated with climate change. In contrast to the background for this theorem, we offer a solution: Risk sharing via the world's financial markets.

In an appendix (Appendix 2) we have addressed the well-known empirical regularities of the conventional asset pricing model in financial- and macro economics, where our extended model may calibrate, with a few simplifications, to reasonable values of the preference parameters. In doing so, we consider the situation where the market portfolio is not a proxy for the wealth portfolio.

This latter part has a long history, which we do not repeat here. It started with the well-known papers by Mehra and Prescott (1985) and Hansen and Singleton (1983). The former paper, for example, chose the parameters of the endowment process to match the sample mean, variance and the annual growth rate of per capita consumption in the years 1889-1978. The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

In the present paper we consider recursive utility in a continuous-time model including jump dynamics along the lines of Øksendal and Sulem (2014). This is an extension of the model developed by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994) which elaborates the foundational work by Kreps and Porteus (1978) and Epstein and Zin (1989) of recursive utility in dynamic models.

In Aase (2016) it was shown that the recursive model in continuous time with diffusion driven uncertainty may have better explanatory properties than many of the alternative theories. While jump dynamics has been introduced in the conventional model in order to, among other things, throw some light on the puzzles (see Aase and Lillestøl (2015)), in the present recursive model jump dynamics may play additional roles. It can, for example, illustrate market behaviour after crashes, or could be used as a fundament of models aimed at analyzing catastrophe insurance contracts used in securitizations of insurance markets, see for example Aase (1999).

It has been a goal in the modern theory of asset pricing to internalize probability distributions of financial assets. To a large extent this has been achieved in our approach. As with the one-agent Lucas-style model, aggregate consumption is in equilibrium equal to the agent's endowment process, given by jump/diffusion process in our approach. The solution of a backward stochastic differential equations (BSDE) provides the main characteristics in the probability distributions of future utility. With existence of a solution to the BSDE, market clearing finally determines the characteristics in the wealth portfolio from the corresponding characteristics of the utility and aggregate consumption processes. However, the primitives of the model include probability distributions of jump sizes. In the model these are exogenous, and must be estimated in applications. One strength of the present model is that these are priced in equilibrium.

The paper is organized as follows: In Section 2 we introduce jump-diffusions. In Section 3 we explain certain problems with the conventional, expected utility model, including jump dynamics. Here we illustrate some effects of deviation from the standard mean/variance analysis in financial economics. Section 4 contains a preview of these results. Section 5 contains basics of stochastic differentiable utility. Section 6 derives the first order conditions, Section 7 details the financial market, and Section 8 presents the dynamics of the state price deflator, with risk premiums and the real interest rate. Section 9 summarizes the above results for the risk premiums and the short-term interest rate. In Section 10 a catastrophe futures contract is considered on aggregate consumption growth rates, and Section 11 concludes. The paper has two appendices, where Appendix 1 contains a US-data set which also has wealth data, while Appendix 2 has some calibrations.

2 Jump-diffusions.

The stochastic processes in this paper are jump-diffusions. Such a process (sometimes called an Itô-Lévy process) is of the form

$$dX(t) = \mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}^l} \gamma(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T, \tag{1}$$

where $\mu : [0,T] \to \mathbb{R}^N$, $\sigma : [0,T] \to \mathbb{R}^{N\times d}$ amd $\gamma : [0,T] \times \mathbb{R}^l \to \mathbb{R}^{N\times l}$ are predictable processes such that the integrals exist. Here $B(t)$ is a d-dimensional Brownian motion and

$$\begin{aligned}\tilde{N}(dt,dz)' &= (\tilde{N}_1(dt,dz_1),\cdots,\tilde{N}_l(dt,dz_l)) \\ &= (N_1(dt,dz_1) - \nu_1(dz_1)dt,\cdots,N_l(dt,dz_l) - \nu_l(dz_l)dt)\end{aligned} \tag{2}$$

where prime means transpose, $z = (z_1, z_2, \cdots, z_l)$ and B_t and $\tilde{N}(dt,dz)$ are independent martingales. Here $\{N_j\}$ are independent Poisson random measures with Lévy measures ν_j coming from l independent (one-dimensional) Lévy prodesses $\eta_1, \cdots \eta_l$ satisfying $E[\eta_i^2(t)] < \infty$ for all $t \in [0,T]$, $i = 1, 2, \cdots, l$.

We let $\mathcal{F}_t^B$ be the σ-algebra generated by $B(s); s \le t$ and we let $\mathcal{F}_t^N$ be the σ-algebra generated by $N(ds,dz); s \le t$. Finally we set $\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{F}_t^N$, the σ-algebra generated by $\mathcal{F}_t^B$ and $\mathcal{F}_t^N$.

Both risky assets, consumption dynamics and indexes will be of this form in what follows. For example could (1) be the model of a stock market consisting of N risky assets. If there exists a risk-less asset, its dynamics is given by

$$dX_0(t) = r_t X_0(t)dt, \quad X_0 = x_0 > 0$$

where r_t is the short rate and x_0 is a constant.

Consider an arbitrary security with strictly positive price process

$$dX_i(t) = \mu_i(t)dt + \sigma_i(t)dB(t) + \int_{\mathbb{R}^l} \gamma_i(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T.$$

Here $\sigma_i(t)$ and $B(t)$ are d-dimensional vectors, and $\gamma_i(t,z)$ and $\tilde{N}(dt,dz)$ are both l-dimensional vectors. The cumulative-return process of this security is an Itô-Lévy process R_i defined by $R_i(0) = 0$ and

$$dR_i(t) = \mu_R(t)dt + \sigma_R(t)dB(t) + \int_{\mathbb{R}^l} \gamma_R(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T,$$

where $\mu_R(t) = \mu_i(t)/X_i(t)$ is the rate of return on the risky asset, $\sigma_R(t) = \sigma_i(t)/X_i(t)$ is the return volatility of the continuous part, and $\gamma_R(t,z) = \gamma_i(t,z)/X_i(t)$ is the corresponding quantity associated to the jumps.

Consumption processes $c(t)$ are here one-dimensional and would typically have dynamics

$$\frac{dc(t)}{c(t^-)} = \mu_c(t)dt + \sigma_c(t)dB(t) + \int_{\mathbb{R}^l} \gamma_c(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T, \tag{3}$$

where $\sigma_c(t)$ and $B(t)$ are d-dimensional vectors, and $\gamma_c(t,z)$ and $\tilde{N}(dt,dz)$ are both l-dimensional vectors. With this interpretation the term $\mu_c(t)$ would be the growth rate of consumption at time t.

Smooth functions of jump-diffusions are again of this type, a result of Itô's lemma for such processes. We have also a version of Girsanov's theorm for jump-diffusions, important for the pricing of derivatives in such markets.

3 The problems with the conventional model.

The conventional asset pricing model in financial economics, the consumption-based capital asset pricing model (CCAPM) of Lucas (1978) and Breeden (1979), assumes a representative agent with a utility function of consumption that is the expectation of a sum, or a time integral, of future discounted utility functions. The model has been criticized for several reasons. First, it does not perform well empirically. Second, the usual specification of utility can not separate the risk aversion from the elasticity of intertemporal substitution, while it would clearly be advantageous to disentangle these two conceptually different aspects of preference.

In the conventional model the utility $U(c)$ of a consumption stream c_t is given by $U(c) = E\{\int_0^T u(c_t,t)\,dt\}$, where the felicity index u has the separable form $u(c,t) = \frac{1}{1-\gamma}c^{1-\gamma}\,e^{-\delta\,t}$. The parameter γ is the representative agent's relative risk aversion and δ is the utility discount rate, or the impatience rate, and T is the time horizon. These parameters are assumed to satisfy $\gamma > 0$, $\delta \ge 0$, and $T < \infty$.

When jumps are included the risk premium $(\mu_R - r)$ of any risky security labeled R (for "risky") is given by

$$\mu_R(t) - r_t = \gamma\,\sigma_{Rc}(t) - \int_{\mathcal{Z}} \big((1+\gamma_c(t,\zeta))^{-\gamma} - 1\big)\gamma_R(t,\zeta)\nu(d\zeta). \tag{4}$$

Here r_t is the equilibrium real interest rate at time t, and the term $\sigma_{Rc}(t) = \sum_{i=1}^d \sigma_{R,i}(t)\sigma_{c,i}(t)$ is the covariance rate between returns of the risky asset and the growth rate of aggregate consumption at time t, a measurable and adaptive process satisfying standard conditions. The dimension of the Brownian motion is $d > 1$. Underlying the jump dynamics we have $\{N_j\}$, $j = 1, 2, \cdots, l$ independent Poisson random measures with Lévy measures ν_j coming from l independent (1-dimensional) Lévy processes. The possible time inhomogeneity in the jump processes is expressed through the terms denoted $\gamma_{R,j}(t, \zeta_j)$ for the risky asset under consideration, and $\gamma_{c,j}(t, \zeta_j)$ for the aggregate consumption process, both measuring the jump sizes. The jump frequencies at time t are embedded in the Lévy measures. The "mark space" $\mathcal{Z} = \mathbb{R}^l$ in this paper, where $\mathbb{R} = (-\infty, \infty)$. Thus the above term in (4) is short-hand notation for the following

$$\int_{\mathcal{Z}} \big((1 + \gamma_c(t,\zeta))^{-\gamma} - 1\big)\gamma_R(t,\zeta)\nu(d\zeta)$$
$$= \sum_{j=1}^{l} \int_{\mathbb{R}} \big((1 + \gamma_{c,j}(t,\zeta_j))^{-\gamma} - 1\big)\gamma_{R,j}(t,\zeta_j)\nu_j(d\zeta_j).$$

This is a continuous-time version of the consumption-based CAPM, allowing for jumps at random time points. Similarly the expression for the risk-free, real interest rate is

$$r_t = \delta + \gamma\,\mu_c(t) - \frac{1}{2}\gamma\,(\gamma + 1)\,\sigma_c'(t)\sigma_c(t)$$
$$- \Big(\gamma \int_{\mathcal{Z}} \gamma_c(t,\zeta)\,\nu(d\zeta) + \int_{\mathcal{Z}} \big((1 + \gamma_c(t,\zeta))^{-\gamma} - 1\big)\nu(d\zeta)\Big). \quad (5)$$

In the risk premium (4) the last term stems from the jump dynamics of the risky asset and aggregate consumption, while in (5) the last two terms have this origin. These results follow from Aase (1999a,b).

One quick way to derive these results, is to realize that the product of the state price deflator π_t and any risky asset's price process S_t, $S_t\pi_t$, is a martingale and hence has drift term equal to zero. Also, the state price deflator is equal to $\pi_t = u'(c_t, t)$ in equilibrium, where u' is the marginal felicity index in the expected utility representation. First, using Ito's lemma for jump diffusions, the multidimensional version (see Øksendal and Sulem (2019), Theorem 1.16, or Gihman and Skorohod (1983)), we obtain the following expression for the risk premium

$$\mu_R(t) - r_t = -\frac{1}{\pi_t}\sigma_R(t)\sigma_\pi(t) - \frac{1}{\pi_t}\int_R \gamma_\pi(t,z)\gamma_R(t,z)\nu(dz),$$

where $r_t = -\mu_\pi(t)/\pi_t$. If $\sigma_R(t) = \gamma_\pi(t, z) = 0$ for all t and z, then $\mu_R(t) = r_t$, and the risk premium of such an asset is 0. So the short-term riskless rate process, if there is one, must be r_t. The representations in (4) and (5) are obtained from this expression by use of Ito's lemma for jump diffusions, using (3).

If the consumption process were as volatile as the stock market index, the jump dynamics could potentially contribute to giving a better explanation of empirical regularities than the continuous model can alone. However, because of the relatively small sizes of the potential jumps in the consumption process, it is unlikely that the last terms in these two relationships move these quantities enough in the right direction. As with the continuous model, the problem stems from the low covariance rate between consumption and the market index.

The process $\mu_c(t)$ is the annual growth rate of aggregate consumption and $(\sigma_c'(t)\sigma_c(t))$ is the annual variance rate of the consumption growth rate, both at time t, again dictated by the Ito-isometry. Both these quantities are measurable and adaptive stochastic processes, satisfying usual conditions. The return processes as well as the consumption growth rate process in this paper are also assumed to be such that statistical estimation makes sense (ergodic processes or constant coefficients).

Notice that in the model is the instantaneous correlation coefficient between returns and the consumption growth rate given by

$$\kappa_{Rc}(t) = \frac{\sigma_{Rc}(t)}{||\sigma_R(t)|| \cdot ||\sigma_c(t)||} = \frac{\sum_{i=1}^{d} \sigma_{R,i}(t)\sigma_{c,i}(t)}{\sqrt{\sum_{i=1}^{d} \sigma_{R,i}(t)^2}\sqrt{\sum_{i=1}^{d} \sigma_{c,i}(t)^2}},$$

and similarly for other correlations given in this model. Here $-1 \leq \kappa_{Rc}(t) \leq 1$ for all t. With this convention we can equally well write $\sigma_R'(t)\sigma_c(t)$ for $\sigma_{Rc}(t)$, and the former does *not* imply that the instantaneous correlation coefficient between returns and the consumption growth rate is equal to one. Prime means transpose.

Similarly the term $\sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{R,j}(t, \zeta_j)\gamma_{c,j}(t, \zeta_j)\nu_j(d\zeta_j)$ is the covariance rate at time t between returns of the risky asset and the growth rate of aggregate consumption stemming from the discontinuous dynamics. We use the short-hand notation $\int_{\mathcal{Z}} \gamma_R(t, \zeta)\gamma_c(t, \zeta)\nu(d\zeta)$ for this term as well.

Using a Taylor series expansion, the risk premium is approximately

$$\mu_R(t) - r_t = \gamma \Big(\sigma_{Rc}(t) + \int_{\mathcal{Z}} \gamma_R(t,\zeta)\gamma_c(t,\zeta)\,\nu(d\zeta)\Big) - \frac{1}{2}\gamma(\gamma+1)\int_{\mathcal{Z}} \gamma_R(t,\zeta)\gamma_c^2(t,\zeta)\,\nu(d\zeta) + \cdots \quad (6)$$

and an approximation for the interest rate is

$$r_t = \delta + \gamma\mu_c(t) - \frac{1}{2}\gamma(1+\gamma)\Big(\sigma_c'(t)\sigma_c(t) + \int_{\mathcal{Z}} \gamma_c^2(t,\zeta)\nu(d\zeta)\Big) + \frac{1}{6}\gamma(\gamma+1)(\gamma+2)\int_{\mathcal{Z}} \gamma_c^3(t,\zeta)\nu(d\zeta) - \cdots \quad (7)$$

Here the term $\int_{\mathcal{Z}} \gamma_c^2(t,\zeta)\nu(d\zeta)$ is the variance rate of the consumption growth rate at time t, stemming from the discontinuous dynamics, so that the total consumption variance rate is $(\sigma_c'(t)\sigma_c(t) + \int_{\mathcal{Z}} \gamma_c^2(t,\zeta)\nu(d\zeta))$ at time t. Similarly the total covariance rate between returns of the risky asset and the consumption growth rate is $(\sigma_{Rc}(t) + \int_{\mathcal{Z}} \gamma_R(t,\zeta)\gamma_c(t,\zeta)\nu(d\zeta))$.

In Table 1 we present the key summary statistics of the data in Mehra and Prescott (1985), of the real annual return data related to the S&P-500, denoted by M, as well as for the annualized consumption data, denoted c, and the government bills, denoted b [1].

	Expectat.	Standard dev.	Covariances
Consumption growth	1.83%	3.57%	$\text{cov}(M,c) = .002226$
Return S&P-500	6.98%	16.54%	$\text{cov}(M,b) = .001401$
Government bills	0.80%	5.67%	$\text{cov}(c,b) = -.000158$
Equity premium	6.18%	16.67%	

Table 1: Key US-data for the time period 1889-1978. Discrete-time compounding.

Here we have, for example, estimated the covariance between aggregate consumption and the stock index directly from the data set to be .00223. This gives the estimate .3770 for the correlation coefficient [2].

Since our development is in continuous time, we have carried out standard adjustments for continuous-time compounding, from discrete-time compounding. The results of these operations are presented in Table 2. This

[1] There are of course newer data by now, but these retain the same basic features. If our model can explain the data in Table 1, it can explain any of the newer sets as well.

[2] The full data set was provided by Professor Rajnish Mehra.

gives, e.g., the estimate $\hat{\kappa}_{Mc} = .4033$ for the instantaneous correlation coefficient $\kappa(t)$. The overall changes are in principle small, and do not influence our comparisons to any significant degree, but are still important.

	Expectation	Standard dev.	Covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 2: Key US-data for the time period 1889-1978. Continuous-time compounding.

Interpreting the risky asset R as the value weighted market portfolio M corresponding to the S&P-500 index, equations (6) and (7) are two equations in two unknowns that can provide estimates of the two preference parameters by the "method of moments". Ignoring the higher order terms in each of these equations, the result is $\gamma = 26.3$ and $\delta = -.015$, i.e., a relative risk aversion of about 26 and an impatience rate of minus 1.5%.

The jump terms might mitigate these numbers somewhat, since the jump model can, under certain assumptions, produce a larger equity premium than the continuous model can alone. It is an empirical question to estimate these quantities (e.g., Ait Sahalia and Jacod (2009-11)), but see below.

3.1 Deviations from normality in the standard model.

In the conventional model we may use jump dynamics to study the effects of deviations from normality. This we have done by using the pure jump model alone to fit the data summarized in Table 1, and its logarithmic version (Table 4). In doing so we have fixed the frequency of "jumps" to one per year on the average. The advantage with this approach is that we do not have to separate the jump dynamics from the continuous part in the data. We have modeled the simultaneous jumps in the Lévy-measure $\nu(d\zeta_1, d\zeta_2)$ by a joint Normal Inverse Gaussian (NIG)-distribution. This distribution measures heavy tails, kurtosis, skewness, etc, often found in financial stock market data. It fits fat-tailed and skewed data very well and is analytically tractable. This distribution was brought to the attention of workers in empirical finance by Barndorff-Nielsen (1997).

The result of this analysis weakened the puzzle somewhat, using the above model when calibrated to the data (for details, see Aase and Lillestøl (2015)).

By maximum likelihood estimators for the NIG-parameters, we obtain the same estimates of the moments as given in Table 1 (and Table 4), from which we obtain the following calibrated values: $(\gamma, \delta) = (22.2, 0.0083)$. Moreover, by varying the NIG-estimates, one by one, within the bounds given by sampling errors, and using resampling techniques, the puzzle was further weakened to $(17.7, 0.058)$.

As a comparison, under joint normality we get $(\gamma, \delta) = (24.3, \text{-.044})$. Jumps alone move the risk premium down somewhat relative to the diffusion model, deviations from normality accounts for the rest.

The result of this is encouraging for the task we now set out to do, namely to include jumps in the recursive model.

4 Preview of results.

Turning to recursive utility, one more parameter occurs in its most basic form. It is the reciprocal of the EIS denoted by ρ. It measures the degree of resistance to intertemporal substitution in consumption. In the form we consider, the parameter $\psi = 1/\rho$ is the elasticity of intertemporal substitution in consumption (EIS), which we simply refer to as the EIS-parameter. In the conventional Eu-model $\gamma = \rho$, but relative risk tolerance $(1/\gamma)$ is something quite different from EIS.

We show that the standard recursive model extended to include jump dynamics takes the following form: For $\rho \neq 1$ and with the same notation as above

$$\mu_R(t) - r_t = \rho\, \sigma_c(t)'\sigma_R(t) + (\gamma - \rho)\, \sigma_V(t)'\sigma_R(t) + \\ \int_{\mathcal{Z}} \Big\{ \gamma_0 K_V(t,\zeta) - \Big(\big(\frac{1 + K_V(t,\zeta)}{1 + \gamma_c(t,\zeta)} \big)^{\rho} - 1 \Big) \Big(1 - \gamma_0 K_V(t,\zeta) \Big) \Big\} \gamma_R(t,\zeta)\nu(d\zeta). \tag{8}$$

Here the term $K_V(t,\cdot)$ signify the jump sizes in the future utility process V, and $\gamma_c(t,\zeta)$ is the corresponding quantity for the growth rate of aggregate consumption, both parts of the primitives of the economic model.

The jump size function of the wealth portfolio is then determined in equilibrium as

$$1 + \gamma_W(t,\zeta) = \frac{1 + K_V(t,\zeta)(1 - \gamma_0 - \gamma_0 K_V(t,\zeta))}{1 - \gamma_0 K_V(t,\zeta) + \big(\big(\frac{1 + K_V(t,\zeta)}{1 + \gamma_c(t,\zeta)} \big)^{\rho} - 1 \big) \big(1 - \gamma_0 K_V(t,\zeta) \big)} \tag{9}$$

where the equality holds $\nu(\cdot)$ a.e.

Also the volatility of the wealth portfolio, $\sigma_W(t)$, is determined as a linear combination of corresponding utilities of future utility and the growth rate of aggregate consumption, $\sigma_V(t)$ and $\sigma_c(t)$ respectively, as follows

$$\sigma_W(t) = (1-\rho)\sigma_V(t) + \rho\sigma_c(t), \tag{10}$$

i.e., also from primitives of the model. The jump term in (8) reduces to the jump term in (4) when $K_V(t,\cdot) = 0$ a.e., so K_V has strictly to do with recursive utility. Similarly if $\sigma_V(t) = 0$ a.e., we obtain the risk premium of the conventional model for the continuous part, so this term has also to do with recursive utility. The equation (9) is seen to be linear in $\gamma_W(t,\cdot)$, and can be seen to reduce to simpler forms in special cases.

The short term real interest rate is given by

$$\begin{aligned} r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\rho(\rho+1)\,\sigma_c'(t)\sigma_c(t) \\ - \rho(\gamma-\rho)\sigma_c(t)'\sigma_V(t) - \frac{1}{2}(\gamma-\rho)(1-\rho)\,\sigma_V'(t)\sigma_V(t) \\ - \int_{\mathcal{Z}} \Big\{ \frac{1}{2}(1+\rho)\gamma_0 K_V'(t,\zeta)K_V(t,\zeta) + \Big(\big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\big)^{\rho} - 1 \Big) \Big(1 - \gamma_0 K(t,\zeta) \Big) \\ + \rho\gamma_c(t,\zeta) - \rho\, K_V(t,\zeta) \Big\}\, \nu(d\zeta). \end{aligned} \tag{11}$$

In the model the covariances are assumed to be measurable, adaptive, stochastic processes satisfying standard conditions. The parameter γ_0 we interpret as the agent's relative risk aversion related to jump size risk. When there are no jumps, we obtain the standard recursive model. When $\rho = \gamma$ the latter model reduces to the conventional, additive EU-model. When $K_V = 0$ and $\sigma_V = 0$ the standard model with jumps, presented in the previous section, results.

4.1 The pure jump part.

In order to study the effects from the nonlinearities caused by the jump dynamics, we may remodel the jump part slightly by letting $y := \gamma_W(t,\zeta)$, $c = \gamma_c(t,\zeta)$, $v = K_V(t,\zeta)$ and $x = \gamma_R(t,\zeta)$. Assuming a stationary distribution for the jumps, for $l = 1$ the equation (9) can be written

$$1 + y = \frac{1 + v(1-\gamma_0-\gamma_0 v)}{1-\gamma_0 v + \big(\big(\frac{1+v}{1+c}\big)^{\rho} - 1\big)(1-\gamma_0 v)}. \tag{12}$$

Similarly $\nu(d\zeta) = \nu(d\zeta_1, d\zeta_2, d\zeta_3))$ can be remodeled as $\lambda_t dH_t(c,v,x)$ for a jump frequency λ_t and a cumulative distribution function $H_t(c,v,x)$ for the jump parts of the consumption growth rate, utility growth rate, and the return rate on the risky security R. The transformation (12) gives the following connection in terms of the variables y, c and x. The jump contribution to the risk premium can be written

$$\int_{-1}^{\infty}\int_{-1}^{\infty}\int_{-1}^{\infty}\{\gamma_0 v - ((\frac{1+v}{1+c})^{\rho}-1)(1-\gamma_0 v)\}x\lambda_t dF_t(c,y,x) = \tag{13}$$

$$\iiint_{\mathcal{Z}'}\{\gamma_0 v - ((\frac{1+v}{1+c})^{\rho}-1)(1-\gamma_0 v)\}x\lambda_t dH_t(c,v,x)$$

where $F(c,y,x)$ is the joint probability distribution of the jump parts of the consumption growth rate, the return rate on the market (wealth) portfolio, and the return rate on the risky security R. Assuming F has a probability density function $f(c,y,x)$, the connection to the given H, with density h, is that $h(c,v,x) = J(c,v,x)f(c,y(c,v),x)$. Here $y = y(c,v)$ is given in (12), and $J(c,v,x)$ is the Jacobian associated with the change of variables from (c,v,x) to (c,y,x), given by

$$J(c,v,x) = \text{mod}\begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial y}{\partial c} & \frac{\partial y}{\partial v} & 0 \\ 0 & 0 & 1 \end{vmatrix} = |\frac{\partial y}{\partial v}|$$

where "mod"means the absolute value of the expression following it. Here $\frac{\partial y}{\partial v}$ can be written

$$\frac{\partial y}{\partial v} = \Bigg(\big(1-\gamma_0-2\gamma_0 v\big)\big(1-\gamma_0 v + ((\frac{1+v}{1+c})^{\rho}-1)(1-\gamma_0 v)\big) - \big(1+v(1-\gamma_0-\gamma_0 v)\big)\Big(\Big(\frac{1+v}{1+c}\Big)^{\rho}\Big(\frac{(\rho-\gamma_0)-\gamma_0(1+\rho)v}{1+v}\Big)\Bigg)\Bigg/ \Bigg(1-\gamma_0 v + ((\frac{1+v}{1+c})^{\rho}-1)(1-\gamma_0 v)\Bigg)^2 \tag{14}$$

For this to be well defined, the Jacobian must be different from zero in the relevant domain, where the set $\mathcal{Z}'$ is the image of $(-1,+\infty)\times(-1,+\infty)\times(-1,+\infty)$ under the change of variables.

This version contains the higher order terms in addition to the extra parameter γ_0 for the jump size risk. As for the conventional model, one can also consider deviations from normality in this framework. A similar rewriting can be formulated for the jump part of the interest rate r_t.

Notice the logic of the the equation (13): From the probability distribution H governing the 'primitives' of the model, which include (the jump parts of) consumption and utility, the probability distribution F is determined in equilibrium by the transformation (12). Turning this around, by the same relationship we also connect the mostly 'unobservable' H to the partly 'observable' F.

4.2 The CAPM++: $\rho = 0$.

When $\rho = 0$ the equation (9) takes on the simple form

$$\gamma_W(t,\zeta) = K(t,\zeta) \quad \text{for all } t \text{ and } \zeta \in \mathcal{Z},$$

and the relationship (10) reduces to $\sigma_W(t) = \sigma_V(t)$, in which case we have perfect substitutability of consumption across time. This corresponds to a dynamic version of the classical one-period CAPM:

$$\mu_R(t) - r_t = \gamma \sigma_W'(t)\sigma_R(t) + \gamma_0 \int_{\mathcal{Z}} \gamma_W'(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) \tag{15}$$

and

$$r_t = \delta - \frac{1}{2}\gamma\, \sigma_W'(t)\sigma_W(t) - \frac{1}{2}\gamma_0 \int_{\mathcal{Z}} \gamma_W'(t,\zeta)\gamma_W(t,\zeta)\,\nu(d\zeta). \tag{16}$$

Notice that these results are exact. We denote the dynamic version of the CAPM model based on recursive utility by CAPM++.

4.3 The second order approximation.

If we disregard moments of order three and higher, the expressions for the risk premiums and the real rate can be simplified for any non-negative value of $\rho \neq 1$ to the following:

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\,\sigma_c(t)'\sigma_R(t) + \frac{\gamma-\rho}{1-\rho}\,\sigma_W(t)'\sigma_R(t) +$$
$$\frac{\rho(1-\gamma_0)}{1-\rho}\int_{\mathcal{Z}} \gamma_c(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) + \frac{\gamma_0-\rho}{1-\rho}\int_{\mathcal{Z}} \gamma_W(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) + \cdots \tag{17}$$

and

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho(1-\rho\gamma)}{1-\rho}\,\sigma_c'(t)\sigma_c(t) + \frac{1}{2}\,\frac{\rho-\gamma}{1-\rho}\sigma_W(t)'\sigma_W(t)$$
$$-\frac{1}{2}\,\frac{\rho(1-\rho\gamma_0)}{1-\rho}\int_{\mathcal{Z}}\gamma_c(t,\zeta)\gamma_c(t,\zeta)\,\nu(d\zeta) + \frac{1}{2}\,\frac{\rho-\gamma_0}{1-\rho}\int_{\mathcal{Z}}\gamma_W(t,\zeta)\gamma_W(t,\zeta)\,\nu(d\zeta)$$
$$+\cdots \quad (18)$$

The possibility that γ_0 is different from γ gives the recursive model an extra degree of freedom in these relationships.

In the above the jump sizes in the wealth portfolio is approximately internalized as follows

$$\gamma_W(t,\zeta) = (1-\rho)K_V(t,\zeta) + \rho\gamma_c(t,\zeta) + \cdots .$$

This is an approximation derived from (9) disregarding higher order terms. In the above we have used (10) as it stands.

These results show that our jump/diffusion version (8)-(11) is a natural extension of the continuous recursive model, just as (6) and (7) show that (4) and (5) is a natural extension to jump/diffusions of the conventional Eu-model with continuous dynamics only.

The covariance rates between consumption and the market index (interpreting R as M) on the right-hand side of (17) are rather small, as in the conventional model, and let us just ignore them for now. The variance rates of M are more significant. In order for this model to explain a large risk premium, consider, for example $\gamma > \rho$, $\rho < 1$ and $\gamma_0 > \rho$. Then the risk premium can be as large as one pleases by letting ρ be close enough to 1, for otherwise reasonable values of γ and γ_0. Or, consider instead $\gamma < \rho$,$\rho > 1$ and $\gamma_0 < \rho$. Again the corresponding terms are positive, and can be as large as we please by letting ρ be close enough to 1, but notice that his combination may lead to values of γ and γ_0 that are too low to be plausible.

Turning to the interest rate (18), in order for the model to explain a small short rate, consider the terms associated with the variance rates of the wealth portfolio. When $\gamma > \rho$, $\rho < 1$ and $\gamma_0 > \rho$ these two terms are negative, and the variance of the wealth portfolio is not quite as small as the corresponding variance rate of the consumption growth rate. The same argument again shows that the interest rate rate may be made as small as we please by letting ρ be close enough to 1. Not surprisingly, this gives us a reasonable fit for the parameters γ, γ_0 and ρ, where also the parameter $\delta \geq 0$, and of reasonable size.

In the standard model the third term in (18) does not adjust enough for the relatively large second term $\rho\mu_c(t)$ (when $\rho = \gamma$ and γ is large). Here the reciprocal of the EIS parameter ρ takes on a reasonable value. Thus these terms account for the adjustment necessary for δ to be non-negative.

With recursive utility, the variance rate of wealth terms are reasonably interpreted to yield *precautionary savings*: When $\gamma > \rho$, $\rho < 1$ and $\gamma_0 > \rho$, these terms are seen to be negative, so that in response to an increase of the volatility of the wealth portfolio, the agent 'saves' and the interest rate must fall. In the conventional model this concept is linked to the variance rate of consumption, but savings really has more to do with wealth.

5 Recursive Stochastic Differentiable Utility.

In this section we give a brief introduction to recursive, stochastic differential utility in the continuous-time model including jumps, along the lines of Øksendal and Sulem (2014). The starting point for this theory for the continuous model is Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994). Our approach based on Øksendal and Sulem (2014) includes jump dynamics.

We are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0,T], P)$ satisfying the 'usual' conditions, and a standard model for the stock market with Lévy-process driven uncertainty, N risky securities and one riskless asset (Section 6 provides more details). Consumption processes are chosen from the space L of square integrable progressively measurable processes with values in R_+. The agent has utility function U, to be specified below, and an endowment process $e \in L$.

The stochastic differential utility $V : L \to \mathbb{R}$ is defined as follows by three primitive functions: $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $A : \mathbb{R} \to \mathbb{R}$ and $A_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The function $f(t, c_t, V_t, \omega)$ corresponds to a felicity index at time t, A is associated with a measure of absolute risk aversion related to the continuous dynamics, while A_0 is connected to a similar measure related to jump size risk. Both the latter two terms may also depend on t. In addition to current consumption c_t, the function f also depends on utility V_t.

The utility process V for a given consumption process c that we consider, satisfying $V_T = 0$, is given by the representation

$$V_t = E_t\Big\{ \int_t^T \big(f(s, c_s, V_s) - \frac{1}{2} A(V_s)\, Z(s)' Z(s) \\ - \frac{1}{2} \int_{\mathcal{Z}} A_0(V_s, \zeta) K'(s,\zeta) K(s,\zeta) \nu(d\zeta)\big) ds \Big\}, \quad t \in [0,T] \quad (19)$$

where $E_t(\cdot)$ denotes conditional expectation given $\mathcal{F}_t$, and $Z(t)$ as well as $K(t,\cdot)$ are square-integrable progressively measurable processes, to be determined in our analysis. The term $Z(t)'Z(t)dt = d[V^c, V^c]_t$ where $[V^c, V^c]_t$ is the quadratic variation of the continuous part of V, and

$$\int_{\mathcal{Z}} K'(t,\zeta)K(t,\zeta)N(d\zeta, dt) = d[V^{di}, V^{di}]_t$$

is the quadratic variation of the discontinuous part of V. The Brownian motion B_t has dimension d, and $K(t,\cdot)$ is an l dimensional vector. We think of V_t as the utility for c at time t, conditional on current information $\mathcal{F}_t$. The term $A(V_t)$ is penalizing for risk in the continuous model, while the term $A_0(V_t,\cdot)$ penalizes for jump size risk.

Recall the *timeless* situation with a mean zero risk X having variance σ^2, where the certainty equivalent m is defined by $Eu(w+X) := u(w-m)$ for a constant wealth w. Then the Arrow-Pratt approximation to m, valid for "small" risks, is given by $m \approx \frac{1}{2}A(w)\sigma^2$, where $A(\cdot)$ is the absolute risk aversion associated with u. This approximation is often good also when risks are not necessarily "small". The financial risks in this paper we consider small enough.

If, for each consumption process c_t, there is a well-defined utility process V, the stochastic differential utility V is defined by $V(c) = V_0$, the initial utility. The triplet (f, A, A_0) generating V is called an aggregator.

Since $V_T = 0$ and $\int Z(t)dB_t$ and $\int\int_{\mathcal{Z}} K(t,\zeta)\tilde{N}(dt, d\zeta)$ are assumed to be martingales, (19) has the stochastic differential equation representation

$$dV_t = \Big(- f(t, c_t, V_t) + \frac{1}{2}A(V_t)\, Z(t)'Z(t) + \\ \frac{1}{2}\int_{\mathcal{Z}} A_0(V_t,\zeta)K'(t,\zeta)K(s,\zeta)\nu(d\zeta)\Big)\, dt + Z(t)\, dB_t + \int_{\mathcal{Z}} K(t,\zeta)\tilde{N}(dt, d\zeta). \tag{20}$$

Here $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$ is an l-dimensional compensated Poisson random measure of the underlying l-dimensional Lévy process, and $B(t)$ is an independent d dimensional, standard Brownian motion.

If terminal utility different from zero is of interest, like for applications to life insurance, then V_T may be different from zero. We may think of A and A_0 as associated with functions $h, h_0 : \mathbb{R} \to \mathbb{R}$ such that $A(v) = -\frac{h''(v)}{h'(v)}$, where h is two times continuously differentiable, and similarly for h_0. U is monotonic and risk averse if $A(\cdot) \geq 0$, $A_0(\cdot,\cdot) \geq 0$ and f is jointly concave and increasing in consumption.

The preference ordering represented by recursive utility is usually assumed to satisfy A1: Dynamic consistency (in the sense of Johnsen and Donaldson (1985)), A2: Independence of past consumption, and A3: State independence of time preference (see Skiadas (2009a)).

The version we consider has the Kreps-Porteus CES utility representation in discrete time, which here corresponds to the aggregator with the specification

$$f(c,v) = \frac{\delta}{1-\rho}\frac{c^{1-\rho} - v^{1-\rho}}{v^{-\rho}}, \quad A(v) = \frac{\gamma}{v} \text{ and } A_0(v,\zeta) = \frac{\gamma_0}{v}, \ \forall \zeta \in R \tag{21}$$

If, for example, $A(v) = A_0(v) = 0$ for all v, this means that the recursive utility agent is risk neutral.

Here $\rho \geq 0, \rho \neq 1, \delta \geq 0, \gamma \geq 0, \gamma_0 \geq 0$. The elasticity of intertemporal substitution in consumption is $\psi := 1/\rho$. The parameter ρ we call the reciprocal of the EIS parameter. When $\rho \neq \gamma$ or $\rho \neq \gamma_0$ the desired disentangling of risk aversion from consumption substitution results.

General existence results for our application do not yet exist, although several partial results are available in the BSDE literature.[3]

It is instructive to recall that the conventional additive and separable utility has aggregator

$$f(c,v) = u(c) - \delta v, \quad A = 0, \quad A_0 = 0, \tag{22}$$

in the present framework (an ordinally equivalent one). As can be seen, even if $A = A_0 = 0$, the agent of the conventional model is not risk neutral.

[3]For example, Duffie and Epstein (1992b) applies a general filtration but requires that there is an ordinally equivalent version in which the aggregator is not a function of the volatility terms. Also required is a Lipschitz restriction on the aggregator. Pardoux (1997) showed an existence result for the case of Poisson random measures with a Lipschitz restriction. For the aggregator of the Kreps and Porteus type, a Lipschitz condition in the above reference related to the drift term of the BSDE is not satisfied, however, existence and uniqueness has then been proven in Duffie and Lions (1992) for diffusion processes. One may conjecture that this can be extended to jump-diffusions, since it is the drift term that poses the problem.

5.1 Homogeniety.

The following result will be made use of in sections 7.3-4. For a given consumption process c_t we let $(V_t^{(c)}, Z_t^{(c)}, K_t(\zeta)^{(c)})$ be the solution of the BSDE

$$\begin{cases} dV_t^{(c)} = \Big(- f(t, c_t, V_t^{(c)}) + \frac{1}{2} A(V_t^{(c)})\, Z(t)'^{(c)} Z(t)^{(c)} + \\ \frac{1}{2}\int_{\mathcal{Z}} A_0(V_t^{(c)}, \zeta) K'(t,\zeta)^{(c)} K(s,\zeta)^{(c)} \nu(d\zeta)\Big)\, dt + Z(t)^{(c)}\, dB_t \\ + \int_{\mathcal{Z}} K(t,\zeta)^{(c)} \tilde{N}(dt, d\zeta); \qquad 0 \le t \le T \\ V_T^{(c)} = 0 \end{cases} \tag{23}$$

Theorem 1 *Assume that, for all* $\lambda > 0$,
(i) $\lambda f(t, c, v) = f(t, \lambda c, \lambda v)$; $\forall$ t, c, v, ω
(ii) $A(\lambda v) = \frac{1}{\lambda} A(v)$; $\forall$ v
(iii) $A_0(\lambda v) = \frac{1}{\lambda} A_0(v)$; $\forall$ v
Then

$$V_t^{(\lambda c)} = \lambda V_t^{(c)},\ Z_t^{(\lambda c)} = \lambda Z_t^{(c)} \text{ and } K_t^{(\lambda c)}(\zeta) = \lambda K_t^{(c)}(\zeta);\ \forall \zeta,\ t \in [0, T]. \tag{24}$$

Proof: Assuming uniqueness of the solution of the BSDEs of the type (23), all we need to do is to verify that the triple $(\lambda V_t^{(c)}, \lambda Z_t^{(c)}, \lambda K_t(\cdot)^{(c)})$ is a solution of the BSDE (23) with c_t replaced by λc_t, i.e. that

$$\begin{cases} d(\lambda V_t^{(c)}) = \Big(- f(t, \lambda c_t, \lambda V_t^{(c)}) + \frac{1}{2} A(\lambda V_t^{(c)})\, \lambda Z(t)'^{(c)} \lambda Z(t)^{(c)} + \\ \frac{1}{2}\int_{\mathcal{Z}} A_0(\lambda V_t^{(c)}, \zeta) \lambda K'(t,\zeta)^{(c)} \lambda K(s,\zeta)^{(c)} \nu(d\zeta)\Big)\, dt + \lambda Z(t)^{(c)}\, dB_t \\ + \int_{\mathcal{Z}} \lambda K(t,\zeta)^{(c)} \tilde{N}(dt, d\zeta); \qquad 0 \le t \le T \\ \lambda V_T^{(c)} = 0 \end{cases} \tag{25}$$

By (i), (ii) and (iii) and the quadratic variation interpretations of $Z'Z$ and $K'Kd\nu$, the BSDE (25) can be written

$$\begin{cases} \lambda dV_t^{(c)} = \Big(- \lambda f(t, c_t, V_t^{(c)}) + \frac{1}{2}\frac{1}{\lambda} A(V_t^{(c)})\, \lambda^2 Z(t)'^{(c)} Z(t)^{(c)} + \\ \frac{1}{2}\int_{\mathcal{Z}} \frac{1}{\lambda} A_0(V_t^{(c)}, \zeta) \lambda^2 K'(t,\zeta)^{(c)} K(s,\zeta)^{(c)} \nu(d\zeta)\Big)\, dt + \lambda Z(t)^{(c)}\, dB_t \\ + \lambda \int_{\mathcal{Z}} K(t,\zeta)^{(c)} \tilde{N}(dt, d\zeta); \qquad 0 \le t \le T \\ \lambda V_T^{(c)} = 0 \end{cases} \tag{26}$$

But this is exactly the equation (23) multiplied by the constant λ. Hence (26) holds and the proof is complete. □

Remarks 1) Note that the system need not be Markovian in general, since we allow

$$f(t,c,v,\omega);\ \ (t,\omega)\in[0,T]\times\Omega$$

to be an adapted process, for each fixed c, v.
2) Similarly, we can allow A_0 and A to depend on t as well[4].

Corollary 1 *Define* $U(c) = V_0^{(c)}$*. Then* $U(\lambda c) = \lambda U(c)$ *for all* $\lambda > 0$*.*

Notice that the aggregator in (21) satisfies the assumptions of the theorem.

5.2 Itô's Lemma for Jump-Diffusions.

Also jump-diffusion processes have the pleasant property that smooth functions of such processes are again jump-diffusions, which is of great importance in applications. We will use the following one-dimensional extension of Itô's formula to jump-diffusions:. Consider a jump-diffusion model of the following form:

$$dX(t) = \mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}^l} \gamma(t,z)\tilde{N}(dt,dz), \quad 0 \le t \le T. \tag{27}$$

Let $f \in C^{1,2}([0,T]\times\mathbb{R}^2;\mathbb{R})$ and define $Y(t) = f(t, X(t))$. Then $Y(t)$ is again a jump-diffusion with the representation

$$\begin{aligned} dY(t) = \frac{\partial f}{\partial t}(t,X(t))dt &+ \frac{\partial f}{\partial x}\big(t,X(t)\big)\big(\mu(t)dt + \sigma(t)dB(t)\big) \\ &+ \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial x^2}\big(t,X(t)\big)dt \\ + \int_{\mathbb{R}} \Big\{ f\big(t,X(t^-)+\gamma(t,z)\big) &- f(t,X(t^-)) - \frac{\partial f}{\partial x}\big(t,X(t^-)\big)\gamma(t,z)\Big\}\nu(dz)dt \\ &+ \int_{\mathbb{R}} \Big\{ f\big(t,X(t^-)+\gamma(t,z)\big) - f\big(t,X(t^-)\big)\Big\}\tilde{N}(dt,dz). \end{aligned} \tag{28}$$

As with ordinary diffusion processes, there is also a convenient extension of this result to the multidimensional case, see for example Gihman and Skorohod (1979) or Øksendal and Sulem (2019).

6 The First Order Conditions.

In the following we solve the consumer's optimization problem using the stochastic maximum principle and forward/backward stochastic differential

[4]not common in economics

equations. We have the specification in (20) and (21) in mind, formulated in the previous section, where the $\tilde{f}$ to appear below is the drift term in (20). However, in principle the analysis is valid for any f, A and A_0 satisfying the stated conditions. The representative agent's problem is to solve

$$\sup_{c\in L} U(c)$$

subject to

$$E\Big\{\int_0^T c_t\pi_t dt\Big\} \leq E\Big\{\int_0^T e_t\pi_t dt\Big\},$$

where e is the endowment process of the agent. Here $V_t = V_t^c$, and $(V_t, Z(t), K(t,\cdot))$ is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dV_t = -\tilde{f}(t, c_t, V_t, Z(t), K(t,\zeta))\, dt + Z(t)\, dB_t + \int_{\mathcal{Z}} K(t,\zeta)\tilde{N}(dt, d\zeta) \\ V_T = 0 \end{cases} \tag{29}$$

where

$$\tilde{f}(t, c, v, z, k) = f(c, v) - \frac{1}{2}A(v)z'z - \frac{1}{2}\int_{\mathcal{Z}} A_0(v,\zeta)k'(t,\zeta)k(t,\zeta)\nu(d\zeta) \tag{30}$$

with f, A and A_0 given in (21).

For $\alpha > 0$ we define the Lagrangian

$$\mathcal{L}(c; \lambda) = U(c) - \alpha\, E\Big(\int_0^T \pi_t(c_t - e_t)dt\Big).$$

The volatility $Z(t)$ as well as the jump size quantity $K(t,\zeta)$ are both part of the solution, together with the dynamics of utility V. Market clearing combined with properties of recursive utility in Theorem 1 will be used to internalize the corresponding quantities for "prices", by connecting these to Z and K.

In order to set down the first order condition for the representative consumer's problem, we use Kuhn-Tucker and either directional derivatives in function space, or the stochastic maximum principle. Both these methods are rather robust. The problem is well posed since U is increasing and concave and the constraint is convex.

Below we utilize the stochastic maximum principle (see Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Øksendal and Sulem (2014), Hu and Peng (1995), or Peng (1990)): We then have a forward backward stochastic differential equation (FBSDE) system consisting of the simple FSDE $dX(t) = 0; X(0) = 0$ and the BSDE (29). The Hamiltonian for this problem is

$$H(t,c,v,z,k,y) = y_t\, \tilde{f}(t,c_t,v_t,z_t,k_t) - \alpha\, \pi_t(c_t - e_t), \tag{31}$$

Here y_t refers to the adjoint variable to be defined shortly. Let $\bigtriangledown_k \tilde{f}$ denote the Frechet derivative of $\tilde{f}$ with respect to k, and $\frac{d\bigtriangledown_k \tilde{f}}{d\nu}(\zeta)$ denote its Radon-Nikodym derivative with respect to ν. The adjoint equation is then

$$\begin{cases} dY_t = Y(t\text{-})\big\{\big(\frac{\partial f}{\partial v}(t,c_t) - \frac{1}{2}(\frac{\partial}{\partial v}A(V_t))\, Z'(t)Z(t) \\ -\frac{1}{2}\int_{\mathcal{Z}}(\frac{\partial}{\partial v}A_0(V_t,\zeta))\, K'(t,\zeta)K(t,\zeta)\nu(d\zeta)\big)dt \\ -\frac{1}{2}\frac{\partial}{\partial z}\big(A(V_t)Z_t'Z_t\big)dB_t + \int_{\mathcal{Z}} \frac{d\bigtriangledown_k \tilde{f}}{d\nu}(t,c_t,V_t,Z_t,K(t,\cdot))(\zeta)\tilde{N}(dt,d\zeta)\big\}, \\ Y_0 = 1. \end{cases}$$

With a general form of $A_0(v,\zeta)$ as in (19), we see that the Frechet derivative, $\bigtriangledown_k \tilde{f}$, is the linear operator

$$h \to (\bigtriangledown_k \tilde{f})(h) = -\int_{\mathcal{Z}} A_0(v,\zeta)k'(\zeta)h(\zeta)\nu(d\zeta); \;\; h \in L^2(\nu).$$

Therefore, as a random measure we have that $\bigtriangledown_k \tilde{f} << \nu$, with Radon-Nikodym derivative

$$\frac{d \bigtriangledown_k \tilde{f}}{d\nu}(\zeta) = -A_0(v,\zeta)k(\zeta).$$

Based on this, the adjoint equation can be written

$$dY_t = Y(t\text{-})\big\{\big(\frac{\partial f}{\partial v}(t,c_t) + \frac{1}{2}\frac{\gamma}{V_t^2}Z'(t)Z(t) + \frac{1}{2}\int_{\mathcal{Z}}\frac{\gamma_0}{V_t^2}K'(t,\zeta)K(t,\zeta)\nu(d\zeta)\big)dt \\ - \frac{\gamma}{V_t}Z(t)dB_t - \int_{\mathcal{Z}}\frac{\gamma_0}{V_t}K(t,\zeta)\tilde{N}(dt,d\zeta)\big\}, \tag{32}$$

where $Y(0) = 1$. The adjoint equation is seen to depend on primitives of the economy only. We now use Itô's lemma to "solve" this stochastic differential equation. Letting $M(t) = \ln(Y(t))$ we first find a stochastic differential equation for M and then invert to find $Y(t) = Y(0)\exp(M(t))$. The result is

$$
\begin{aligned}
Y_t = \exp\Big(\int_0^t \big(& \frac{\partial f}{\partial v}(s, c_s) + \frac{1}{2}\frac{\gamma(1-\gamma)}{V_s^2} Z'(s)Z(s) \\
& + \frac{1}{2}\int_{\mathcal{Z}} \frac{\gamma_0}{V_{s-}^2} K'(s,\zeta)K(s,\zeta)\nu(d\zeta)\big)ds - \int_0^t \frac{\gamma}{V_s} Z(s)dB_s \\
& + \int_0^t \int_{\mathcal{Z}} \{\ln(1 - \frac{\gamma_0}{V_s}K(s,\zeta)) + \frac{\gamma_0}{V_s}K(s,\zeta)\}\nu(d\zeta)ds \\
& + \int_0^t \int_{\mathcal{Z}} \ln(1 - \frac{\gamma_0}{V_s}K(s,\zeta))\tilde{N}(ds, d\zeta)\Big).
\end{aligned}
\tag{33}
$$

The interpretation of Y_t is a shadow price; the marginal value as of time zero of an additional "unit of utility" at time t.

Sufficient conditions for a unique, optimal solution using the stochastic maximum principle are the same as the corresponding conditions for the existence and uniqueness of a solution to the BSDE (29).

Maximizing the Hamiltonian with respect to c gives the first order equation

$$y \frac{\partial \tilde{f}}{\partial c}(t, c^*, v, z, k) - \alpha\, \pi = 0$$

or

$$\alpha\, \pi_t = Y(t)\frac{\partial \tilde{f}}{\partial c}(t, c_t^*, V(t), Z(t), K(t,\cdot)) \quad \text{a.s. for all } t \in [0,T]. \tag{34}$$

where c^* is optimal. The state price deflator π_t at time t formally depends, through the adjoint variable Y_t, on the entire optimal paths $(c_s^*, V_s, Z(s), K(s,\cdot))$ for $0 \leq s \leq t$, which means that marginal value at time t depends in theory on the consumption history.

For the representative agent equilibrium the optimal consumption process is the given aggregate consumption c in society, and for this consumption process the utility V_t is optimal at each time t.

We now have the first order conditions for the scale invariant recursive utility in the jump/diffusion world. Before we proceed to a solution of the problem, we return to the financial market model.

7 The financial market.

Having established the general recursive utility form of interest, in his section we specify our model for the financial market. The model is much like the one used by Duffie and Epstein (1992a), except that we do not assume any unspecified factors in our model. In addition we include general jump-diffusion processes.

Let $\lambda_R(t) \in R^N$ denote the vector of expected rates of return of the N given risky securities in excess of the riskless instantaneous return r_t, and let $\sigma(t)$ denote the $N \times d$-matrix of diffusion coefficients of the risky asset prices, normalized by the asset prices, so that $\sigma(t)\sigma(t)'$ is the instantaneous covariance matrix for the continuous part of asset returns. The jumps in the various assets are captured by the $N \times l$-matrix $\gamma(t, \zeta)$ and a vector valued, compensated random measure

$$\tilde{N}(dt, d\zeta)' = (\tilde{N}_1(dt, d\zeta_1), \cdots, \tilde{N}_l(dt, d\zeta_l)) = \\ (N_1(dt, d\zeta_1) - \nu_1(d\zeta_1)dt, \cdots, N_l(dt, d\zeta_l) - \nu_l(d\zeta_l)dt),$$

where $\{N_j\}$ are independent Poisson random measures with Lévy measures ν_j coming from l independent (1-dimensional) Lévy processes.

The representative consumer's problem is, for each initial level w of wealth to solve

$$\sup_{(c,\varphi)} U(c) \tag{35}$$

subject to the intertemporal budget constraint

$$dW_t = \big(W_t(\varphi_t' \cdot \lambda_R(t) + r_t) - c_t\big)dt + W_t\varphi_t' \cdot \sigma(t)dB_t \\ + W_t\varphi_t' \cdot \int_{R^l} \gamma(t, \zeta)\tilde{N}(dt, d\zeta). \tag{36}$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ are the fractions of total wealth W_t held in the risky securities. The processes $\nu_R(t)$, $\sigma(t)$ and $\gamma(t)$ are progressively measurable processes.

Market clearing requires that $\varphi_t'\sigma(t) = (\delta_t^M)'\sigma(t) = \sigma_M(t)$ and $\varphi_t'\gamma(t, \cdot) = (\delta_t^M)'\gamma(t, \cdot) = \gamma_M(t, \cdot)$ in equilibrium, where $\sigma_M(t)$ is the volatility of the return on the market (wealth) portfolio, $\gamma_M(t, \cdot)$ is the corresponding jump size function, and δ_t^M are the fractions of the different securities, $j = 1, \cdots, N$ held in the value-weighted market portfolio. That is, the representative agent must hold the market portfolio in equilibrium, by construction.

The model is a pure exchange economy where the aggregate endowment process e_t in society is exogenously given, and prices are determined such that in equilibrium the single agent optimally consumes $c_t = e_t$ in every period. The main issue is then the determination of prices, including risk premiums and the interest rate, consistent with this behavior.

8 The dynamics of the state price deflator $\pi(t)$.

We now turn our attention to pricing restrictions relative to the given optimal consumption plan. Recall the first order conditions are given in (34).

It is convenient to use the notation $Z(t)/V_t := \sigma_V(t)$ and $K(t,\cdot)/V(t-) := K_V(t,\cdot)$, where V_{t-} means the value of V just before a possible jump at time t, assuming $V \neq 0$. By Theorem 1, $\sigma_V(t)$ and $K_V(t,\cdot)$ are both homogeneous of degree zero in c. With this convention the utility process V_t satisfies the following backward equation

$$\frac{dV_t}{V_{t-}} = \Big(- \frac{\delta}{1-\rho}\frac{c_t^{1-\rho} - V_t^{1-\rho}}{V_t^{-\rho+1}} + \frac{1}{2}\gamma \sigma_V'(t)\sigma_V(t) \\ + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0\, K_V'(t,\zeta) K_V(t,\zeta)\nu(d\zeta)\Big) dt \\ + \sigma_V(t) dB_t + \int_{\mathcal{Z}} K_V(t,\zeta)\tilde{N}(dt,d\zeta), \quad (37)$$

where $V(T) = 0$. The short-hand notation for the integrals with jump dynamics is as explained in Section 2. Since the jump times have Lebesgue measure zero, $V_t = V_{t-}$ a.e. on $[0,T]$.

Aggregate consumption is exogenous, with dynamics on of the form

$$\frac{dc_t}{c_{t-}} = \mu_c(t)\, dt + \sigma_c(t)\, dB_t + \int_{\mathcal{Z}} \gamma_c(t,\zeta)\tilde{N}(dt,d\zeta), \quad (38)$$

where $\mu_c(t)$, $\sigma_c(t)$ and $\gamma_c(t,\cdot)$ are measurable, $\mathcal{F}_t$ adapted stochastic processes, satisfying appropriate integrability conditions. We assume these processes to be estimable.

Under these conditions the adjoint variable Y has dynamics given in (32). From the FOC in equation (34) we derive the dynamics of the state price deflator. We then seek the connection between V_t, $\sigma_V(t)$ and $K_V(t,\cdot)$ and the rest of the economy. Towards this end, we first normalize to $\alpha = 1$ without loss of generality, and then use the multivariate version of Ito's lemma (see

Gihman and Skorohod (1979)). This gives

$$d\pi_t = f_c(c_t, V_t)\, dY_t + Y_t\, df_c(c_t, V_t) + d[Y, f_c(c, V)](t), \tag{39}$$

since $\tilde{f}_c = f_c$, where $[X, Y](t)$ is the quadratic covariation of the processes X and Y given by

$$[X, Y](t) = \int_0^t \big(\sigma_X(s)\sigma_Y(s) + \int_{\mathcal{Z}} \gamma_X(s, \zeta)\gamma_Y(s, \zeta)\nu(d\zeta)\big)ds \\ + \int_0^t \int_{\mathcal{Z}} \gamma_X(s, \zeta)\gamma_Y(s, \zeta)\tilde{N}(ds, d\zeta).$$

By the dynamics of the adjoint and the backward equations, this can be written, again using using Ito's formula

$$\begin{aligned}
d\pi_t = {} & Y_t\, f_c(c_t, V_t) \Big(\{f_v(c_t, V_t) + \frac{1}{2}\gamma\sigma'_V(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 K'_V K_V \nu(d\zeta)\}dt \\
& -\gamma\sigma_V(t)dB_t - \int_{\mathcal{Z}} \gamma_0 K_V(t, \zeta)\tilde{N}(dt, d\zeta)\Big) + Y_t\, \frac{\partial f_c}{\partial c}(c_t, V_t)(c_t\mu_c(t)dt + c_t\sigma_c(t)dB_t) \\
& + Y_t\, \frac{\partial f_c}{\partial v}(c_t, V_t)\Big(\{-f(c_t, V_t) + \frac{1}{2}\gamma V_t\sigma'_V(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} V_{t-}\gamma_0 K'_V K_V \nu(d\zeta)\}dt \\
& V_t\sigma_V(t)dB_t\Big) + Y_t\Big(\frac{1}{2}\frac{\partial^2 f_c}{\partial c^2}(c_t, V_t)\, c_t^2\sigma'_c(t)\sigma_c(t) + \frac{\partial^2 f_c}{\partial c\partial v}(c_t, V_t)\sigma'_c(t)\sigma_V(t) \\
& + \frac{1}{2}\frac{\partial^2 f_c}{\partial v^2}(c_t, V_t)\, V_t^2\sigma'_V(t)\sigma_V(t)\Big)dt + Y_t\Big(\int_{\mathcal{Z}} \{f_c(c_{t-}(1+\gamma_c(t, \zeta)), V_{t-}(1+K_V(t, \zeta)) \\
& - f_c(c_{t-}, V_{t-}) - \gamma_c(t, \zeta)c_{t-}\frac{\partial f_c}{\partial c}(c_t, V_t) - K_V(t, \zeta)V_{t-}\frac{\partial f_c}{\partial v}(c_t, V_t)\}\nu(d\zeta)dt \\
& + \int_{\mathcal{Z}} \{f_c(c_{t-}(1 + \gamma_c(t, \zeta)), V_{t-}(1 + K_V(t, \zeta)) - f_c(c_{t-}, V_{t-})\}\tilde{N}(dt, d\zeta)\Big) \\
& - \gamma\sigma_V(t)Y_t\{c_t\sigma_c(t)\frac{\partial f_c}{\partial c}(c_t, V_t) + V_t\sigma_V(t)\frac{\partial f_c}{\partial v}(c_t, V_t)\}dt \\
& - Y_t \int_{\mathcal{Z}} \gamma_0 K_V(t, \zeta)\{f_c(c_{t-}(1+\gamma_c(t, \zeta)), V_{t-}(1+K_V(t, \zeta)) - f_c(c_{t-}, V_{t-})\}\nu(d\zeta)dt \\
& - Y_t \int_{\mathcal{Z}} \gamma_0 K_V(t, \zeta)\{f_c(c_{t-}(1+\gamma_c(t, \zeta)), V_{t-}(1+K_V(t, \zeta)) - f_c(c_{t-}, V_{t-})\}\tilde{N}(dt, d\zeta).
\end{aligned} \tag{40}$$

Here

$$f_c(c, v) := \frac{\partial f(c, v)}{\partial c} = \delta c^{-\rho}v^{\rho},\ f_v(c, v) := \frac{\partial f(c, v)}{\partial v} = -\frac{\delta}{1-\rho}(1 - \rho c^{1-\rho}v^{\rho-1}),$$

$$\frac{\partial f_c(c,v)}{\partial c} = -\delta\rho c^{-(1+\rho)} v^{\rho}, \quad \frac{\partial f_c(c,v)}{\partial v} = \delta\rho v^{\rho-1} c^{-\rho},$$

$$\frac{\partial^2 f_c}{\partial c^2}(c,v) = \delta\rho(\rho+1) v^{\rho} c^{-(\rho+2)}, \quad \frac{\partial^2 f_c}{\partial c\,\partial v}(c,v) = -\delta\rho^2 v^{\rho-1} c^{-(\rho+1)},$$

and

$$\frac{\partial^2 f_c}{\partial v^2}(c,v) = \delta\rho(\rho-1) v^{\rho-2} c^{-\rho}.$$

From the canonical representation of the state price deflator

$$d\pi_t = \mu_\pi(t)dt + \sigma_\pi(t)dB_t + \int_{\mathcal{Z}} \gamma_\pi(t,\zeta)\tilde{N}(dt,d\zeta),$$

from (40) we find the key characteristics of π. They are

$$\begin{aligned}
\mu_\pi(t) = Y_t\Big(& f_c(c_t,V_t)\left(f_v(c_t,V_t) + \frac{1}{2}\gamma\sigma_V'(t)\sigma_V(t)\right.\\
&\left. + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 K_V' K_V \nu(d\zeta)\right) + \frac{\partial f_c}{\partial c}(c_t,V_t)c_t\mu_c(t)\\
& + \frac{\partial f_c}{\partial v}(c_t,V_t)\Big\{-f(c_t,V_t) + \frac{1}{2}\gamma V_t\sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} V_{t-}\gamma_0 K_V' K_V \nu(d\zeta)\Big\}\\
& + \frac{1}{2}\frac{\partial^2 f_c}{\partial c^2}(c_t,V_t)\, c_t^2\sigma_c'(t)\sigma_c(t) + \frac{\partial^2 f_c}{\partial c \partial v}(c_t,V_t)\sigma_c'(t)\sigma_V(t)\\
& + \frac{1}{2}\frac{\partial^2 f_c}{\partial v^2}(c_t,V_t)\, V_t^2\sigma_V'(t)\sigma_V(t) + \int_{\mathcal{Z}}\{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta))\\
& - f_c(c_{t-},V_{t-}) - \gamma_c(t,\zeta)c_{t-}\frac{\partial f_c}{\partial c}(c_t,V_t) - K_V(t,\zeta)V_{t-}\frac{\partial f_c}{\partial v}(c_t,V_t)\}\nu(d\zeta)\\
& - \gamma\sigma_V(t)\{c_t\sigma_c(t)\frac{\partial f_c}{\partial c}(c_t,V_t) + V_t\sigma_V(t)\frac{\partial f_c}{\partial v}(c_t,V_t)\}\\
& - \int_{\mathcal{Z}}\gamma_0 K_V(t,\zeta)\{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-},V_{t-})\}\nu(d\zeta)\Big),
\end{aligned} \tag{41}$$

$$\sigma_\pi(t) = Y_t\Big(-f_c(c_t,V_t)\gamma\sigma_V(t) + \frac{\partial f_c}{\partial c}(c_t,V_t)c_t\sigma_c(t) + V_t\sigma_V(t)\frac{\partial f_c}{\partial v}(c_t,V_t)\Big) \tag{42}$$

and

$$\begin{aligned}
\gamma_\pi(t,\zeta) = Y_t\Big(& f_c(c_t,V_t)(-\gamma_0 K_V(t,\zeta))\\
& + \{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-},V_{t-})\}\\
& - \gamma_0 K_V(t,\zeta)\{f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-},V_{t-})\}\Big).
\end{aligned} \tag{43}$$

8.1 Some basic features of the economy.

With recursive utility the conditional distribution of future consumption may depend on history in complicated ways. The stochastic maximum principle allows us to derive some optimality conditions without explicitly specifying the dependence.

With the above analysis in place, we can demonstrate this by exploring the nature of the state price deflator $\{\pi_t, 0 \leq t \leq T\}$. Its dynamics is given by

$$
\begin{aligned}
d\pi_t = \mu_\pi(t, c, V, \sigma_V, K_V)dt + \sigma_\pi(t, c, V, \sigma_V, K_V))dB_t \\
+ \int_{\mathcal{Z}} \gamma_\pi(t, c, V, \sigma_V, K_V(t,\zeta); \zeta)\tilde{N}(dt, d\zeta), \quad (44)
\end{aligned}
$$

where the basic variables are consumption c and the quantities characterizing utility (V, σ_V, K_V). From the expressions (41), (42) and (43) we notice that the triplet characterizing our jump/diffusion process $(\mu_\pi(t), \sigma_\pi(t), \gamma_\pi(t, \cdot))$ depends not only on the present values of these basic variables, but also on their entire past from time 0 to time t. This is seen when we notice that they all contain the adjoint variable Y_t as a factor, which is given in (33) and repeated here:

$$
\begin{aligned}
Y_t = \exp\Big(\int_0^t \big(\frac{\partial f}{\partial v}(s, c_s) + \frac{1}{2}\frac{\gamma(1-\gamma)}{V_s^2} Z'(s)Z(s) \\
+ \frac{1}{2}\int_{\mathcal{Z}} \frac{\gamma_0}{V_{s-}^2} K'(s,\zeta)K(s,\zeta)\nu(d\zeta)\big)ds - \int_0^t \frac{\gamma}{V_s} Z(s)dB_s \\
+ \int_0^t \int_{\mathcal{Z}} \{\ln(1 - \frac{\gamma_0}{V_s}K(s,\zeta)) + \frac{\gamma_0}{V_s}K(s,\zeta)\}\nu(d\zeta)ds \\
+ \int_0^t \int_{\mathcal{Z}} \ln(1 - \frac{\gamma_0}{V_s}K(s,\zeta))\tilde{N}(ds, d\zeta)\Big).
\end{aligned}
\quad (45)
$$

Accordingly, the process π_t may be a Markov process.

In the continuous-time model with no jumps, Duffie and Epstein (1992a,b) restricted attention to an ordinally equivalent version of recursive utility with the aggregator

$$
f_2(c, v) = \frac{\delta}{1-\rho} \frac{c^{1-\rho} - ((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \qquad A_2(v) = 0. \quad (46)
$$

Compared to the aggregator (21) that we use, the one in (46) can not readily be associated with the required separation between risk aversion and consumption substitution; by changing A or A_0 leaving f fixed. (With no jumps Duffie and Epstein (1992a,b) show that the individual with the largest A is the most risk averse, keeping f fixed.)

Nevertheless, with no jump dynamics the version (46) is an ordinally equivalent representation, which leads to the same economic consequences as (21) with $A_0 = 0$. Note that even if $A_2(v) = 0$ for all v, this agent is not risk-neutral for the aggregator in (46). The reasons this version was used seems two-fold: (i) It was most convenient to show existence of the solution to the associated BSDE; (ii) dynamic programming was convenient when analyzing this version.

When the process π_t is considered in isolation, detached from the other variables, it is a Markov process when $\mu_c(t)$, $\sigma_c(t)$ and $\sigma_V(t)$ are all deterministic. If this assumption is used, dynamic programming is appropriate.

It is difficult to see how the representation (46) has any counterpart in the present model with jumps. Hence our method of using the stochastic maximum principle seems indispensable in the present setting.

8.2 The risk premiums.

The risk premium of any risky security with return process R is given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t}\,\sigma'_\pi(t)\,\sigma_R(t) - \frac{1}{\pi_t}\int_{\mathcal{Z}} \gamma_\pi(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta) \qquad (47)$$

where the last term follows from Aase (1993a,b). Since $\pi_t = Y_t f_c(c_t, V_t)$, it is a consequence of the expressions in (42) and (47) that the risk premium of any risky security is given by

$$\mu_R(t) - r_t = \Big(- \frac{\frac{\partial f_c}{\partial c}(c_t, V_t)}{f_c(c_t, V_t)} c_t \sigma'_c(t)\,\sigma_R(t) + \Big(\gamma - \frac{\frac{\partial f_c}{\partial v}(c_t, V_t)}{f_c(c_t, V_t)} V_t\Big)\sigma'_V(t)\,\sigma_R(t)\Big)$$
$$+ \int_{\mathcal{Z}} \Big(\gamma_0 K_V(t,\zeta) - \frac{1}{f_c(c_t,V_t)}\big(f_c(c_{t-}(1+\gamma_c(t,\zeta)), V_{t-}(1+K_V(t,\zeta)) - f_c(c_{t-}, V_{t-})\big)\cdot$$
$$\Big(1 - \gamma_0 K_V(t,\zeta)\Big)\Big)\gamma_R(t,\zeta)\nu(d\zeta). \qquad (48)$$

This is our basic result for risk premiums. We now substitute in for f given in (21) and the various partial derivatives derived above. This gives

$$\mu_R(t) - r_t = \rho\,\sigma_c(t)'\sigma_R(t) + (\gamma-\rho)\,\sigma_V(t)'\sigma_R(t)$$
$$+\int_{\mathcal{Z}}\Big(\gamma_0\,K_V(t,\zeta) - \Big(\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^{\rho} - 1\Big)\Big(1-\gamma_0\,K_V(t,\zeta)\Big)\Big)\gamma_R(t,\zeta)\nu(d\zeta). \tag{49}$$

It remains to connect the characteristics of consumption and the wealth portfolio to the fundamentals σ_V and K_V, which we do below. Before that we turn to the interest rate.

8.3 The equilibrium interest rate.

The equilibrium short-term, real interest rate r_t is given by the formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \tag{50}$$

The real interest rate at time t can be thought of as the expected exponential rate of decline of the representative agent's marginal utility, which is π_t in equilibrium.

In order to find an expression for r_t in terms of the primitives of the model, we use (41). Using the expression for f and its various partial derivatives, we obtain the expression

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\rho(\rho+1)\,\sigma_c'(t)\sigma_c(t)$$
$$-\,\rho(\gamma-\rho)\sigma_c(t)'\sigma_V(t) - \frac{1}{2}\,(\gamma-\rho)(1-\rho)\,\sigma_V'(t)\sigma_V(t)$$
$$-\int_{\mathcal{Z}}\Big\{\frac{1}{2}(1+\rho)\gamma_0 K_V'(t,\zeta)K_V(t,\zeta) + \Big(\big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\big)^{\rho} - 1\Big)\Big(1-\gamma_0 K(t,\zeta)\Big)$$
$$+\,\rho\gamma_c(t,\zeta) - \rho\,K_V(t,\zeta)\Big\}\,\nu(d\zeta), \tag{51}$$

first presented in (11) of Section 2, which is our basic result for the equilibrium short rate.

8.4 The determination of the volatility and jump characteristics of the market portfolio.

In order to determine $\sigma_W(t)$, and $\gamma_W(t,\cdot)$ from the primitives of the model, which in this case involve $\sigma_V(t)$, $K_V(t,\cdot)$, $\sigma_c(t)$ and $\gamma_c(t,\cdot)$, first notice that the wealth at any time t is given by

$$W_t = \frac{1}{\pi_t} E_t\Big(\int_t^T \pi_s c_s \, ds \Big), \tag{52}$$

where c is optimal. By the definition of directional derivatives (the Frechet derivative) we have that

$$\begin{aligned}\bigtriangledown U(c;c) &= \lim_{\alpha \downarrow 0} \frac{U(c+\alpha c) - U(c)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{U(c(1+\alpha)) - U(c)}{\alpha} \\ &= \lim_{\alpha \downarrow 0} \frac{(1+\alpha)U(c) - U(c)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{\alpha U(c)}{\alpha} = U(c),\end{aligned}$$

where the third equality uses that U is homogeneous of degree one as shown in Theorem 1. By the Riesz representation theorem and dominated convergence theorem it follows from the linearity and continuity of the directional derivative that

$$\bigtriangledown U(c;c) = E\Big(\int_0^T \pi_t c_t \, dt \Big) = W_0 \pi_0 \tag{53}$$

where W_0 is the wealth of the representative agent at time zero, and the last equality follows from (52) for $t = 0$. Thus $U(c) = \pi_0 W_0$.

Let $V_t(c)$ denote future utility at the optimal consumption for our representation. Since this function is also homogeneous of degree one and is continuously differentiable, by Riesz' representation theorem and the dominated convergence theorem, the same type of basic linear relationship holds here for the associated directional derivatives at any time t, i.e.,

$$\bigtriangledown V_t(c;c) = E_t\Big(\int_t^T \pi_s^{(t)} \, c_s \, ds \Big) = V_t(c)$$

where the Riesz representation $\pi_s^{(t)}$ for $s \geq t$ is the state price deflator at time $s \geq t$, as of time t. As for the discrete time model, it follows by results in Skiadas (2009a) that with assumption A2, implying that this quantity is independent of past consumption, the consumption history in the adjoint variable Y_t is 'removed' from the state price deflator π_t, so that $\pi_s^{(t)} = \pi_s / Y_t$

for all $t \leq s \leq T$. By this it follows that

$$V_t = \frac{1}{Y_t}\pi_t W_t. \tag{54}$$

This gives us the dynamics of V in terms of the other variables, which determines the relationship between primitives and the endogenous variables. By the product rule,

$$dV_t = d\big(Y_t^{-1}\big)(\pi_t W_t) + Y_t^{-1} d(\pi_t W_t) + d[Y_t^{-1}, (\pi_t W_t)](t) \tag{55}$$

where

$$d(\pi_t W_t) = W_t d\pi_t + \pi_t dW_t + d[\pi_t, W_t](t). \tag{56}$$

Ito's lemma gives

$$\begin{aligned} d\Big(\frac{1}{Y_t}\Big) = -\frac{1}{Y_{t-}}\Big(f_v(c_t, V_t) + \frac{1}{2}\gamma\sigma_V'(t)\sigma_V(t) + \frac{1}{2}\int_{\mathcal{Z}} \gamma_0 K_V'(t,\zeta) K_V(t,\zeta)\nu(d\zeta)\Big)\, dt \\ + \frac{\gamma^2}{Y_{t-}}\sigma_V'(t)\sigma_V(t)\, dt + \frac{1}{Y_{t-}}\gamma\sigma_V(t)\, dB_t \\ + \int_{\mathcal{Z}} \Big\{\frac{1}{Y_{t-}(1 - A_0(t,\zeta)K(t,\zeta))} - \frac{1}{Y(t-)} - \frac{1}{Y(t-)}A_0(t,\zeta)K(t,\zeta)\Big\}\nu(d\zeta)dt \\ + \int_{\mathcal{Z}} \Big\{\frac{1}{Y_{t-}(1 - A_0(t,\zeta)K(t,\zeta))} - \frac{1}{Y(t-)}\Big\}\tilde{N}(dt, d\zeta). \end{aligned} \tag{57}$$

From the equations (55)-(57) it follows by the market clearing condition $\varphi_t' \cdot \sigma(t) = \sigma_W(t)$ that

$$V_t\sigma_V(t) = \frac{1}{Y_t}\Big(\pi_t W_t \gamma\sigma_V + \pi_t W_t \sigma_M(t) - \pi_t W_t\big(\rho\sigma_c(t) + (\gamma - \rho)\sigma_V(t)\big)\Big) \tag{58}$$

From the expression (54) for V_t we obtain the following equation for σ_V

$$\sigma_V(t) = \gamma\sigma_V(t) + \sigma_W(t) - (\rho\sigma_c(t) + (\gamma - \rho)\sigma_V(t))$$

from which it follows that

$$\sigma_W(t) = (1 - \rho)\sigma_V(t) + \rho\sigma_c(t)). \tag{59}$$

This relationship internalizes this important quantity in equilibrium. The relationship can also be written

$$\sigma_V(t) = \frac{1}{1 - \rho}(\sigma_W(t) - \rho\sigma_c(t)), \tag{60}$$

which, when inserted into (49) and (51) gives the model of Section 3 for the continuous part of the dynamics.

The version treated by Duffie and Epstein (1992a) is an ordinally equivalent one, which was claimed to be better suited for dynamic programming, the solution method used by them. One assumption which is difficult to avoid in order to solve the associated Bellman equation, is that the volatilities involved are constants. In the conventional model the result of this is that the volatility of the consumption growth rate must equal the volatility of the wealth portfolio. In the recursive model in continuous time, this happens only if $\rho = 1$. In any case, this assumption does not seem well supported by data.

Under the assumptions of this paper and without any jump dynamics, the two ordinally equivalent versions give the same expressions for the risk premiums and the real interest rate (see Aase (2016)).

We turn to the equilibrium determination of $\gamma_W(t,\cdot)$. In relation to equation (57) we use the notation

$$\gamma_{Y^{-1}}(t,\zeta) := \frac{\gamma_0 K_V(t,\zeta)}{1-\gamma_0 K_V(t,\zeta)}.$$

From the equations (54)-(57), using the market clearing condition $\varphi_t'\gamma(t,\cdot) = \gamma_M(t,\cdot)$, it follows that

$$\begin{aligned} K_V(t,\zeta) = \gamma_{Y^{-1}}(t,\zeta) &+ \big(\gamma_\pi(t,\zeta)/\pi_t + \gamma_W(t,\zeta) + (\gamma_\pi(t,\zeta)/\pi_t)\gamma_W(t,\zeta)\big) \\ &+ \gamma_{Y^{-1}}(t,\zeta)\big(\gamma_\pi(t,\zeta)/\pi_t + \gamma_W(t,\zeta) + (\gamma_\pi(t,\zeta)/\pi_t)\gamma_W(t,\zeta)\big), \end{aligned}$$

or

$$\begin{aligned} K_V(t,\zeta) = \gamma_{Y^{-1}}(t,\zeta) &+ \\ &\big(\frac{\gamma_\pi(t,\zeta)}{\pi_t} + \gamma_W(t,\zeta) + \frac{\gamma_\pi(t,\zeta)}{\pi_t}\gamma_W(t,\zeta)\big)\big(1+\gamma_{Y^{-1}}(t,\zeta)\big) \end{aligned}$$

We now use the expression for $\gamma_\pi(t,\cdot)/\pi_t$ found in (49):

$$\gamma_\pi(t,\zeta) = \pi_t\Big(-\gamma_0 K_V(t,\zeta) + \big(\big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\big)^\rho - 1\big)\big(1-\gamma_0 K_V(t,\zeta)\big)\Big).$$

This gives the following relationship between $\gamma_W(t,\cdot)$, $K_V(t,\cdot)$ and $\gamma_c(t,\cdot)$:

$$K_V(t,\zeta)\Big(1-\gamma_0-\gamma_0 K_V(t,\zeta)\Big)=\gamma_W(t,\zeta)+$$
$$\Big(1+\gamma_W(t,\zeta)\Big)\Big(-\gamma_0 K_V(t,\zeta)+\big((\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)})^{\rho}-1\big)\big(1-\gamma_0 K_V(t,\zeta)\big)\Big) \tag{61}$$

ν a.e[5]. This relationship is seen to be "linear" in $\gamma_W(t,\zeta)$, and reduces to

$$1+\gamma_W(t,\zeta)=\frac{1+K_V(t,\zeta)(1-\gamma_0-\gamma_0 K_V(t,\zeta))}{\big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\big)^{\rho}\big(1-\gamma_0 K_V(t,\zeta)\big)}. \tag{62}$$

where the equality holds $\nu(\cdot)$ a.e. The equation is seen to be non-linear in K_V, which is to be determined in terms of γ_c, γ_W and the parameters ρ and γ_0.

This proves the results of Section 3.1, which we formulate in the next section.

9 The results.

We can now summarize our results so far in the following

Theorem 2 *For the standard recursive model where $\rho \neq 1$ and with jump dynamics included, in equilibrium the risk premium of any risky asset R is given by*

$$\mu_R(t)-r_t=\frac{\rho(1-\gamma)}{1-\rho}\,\sigma_c(t)'\sigma_R(t)+\frac{\gamma-\rho}{1-\rho}\,\sigma_W(t)'\sigma_R(t)+$$
$$\int_{\mathcal{Z}}\Big(\gamma_0\,K_V(t,\zeta)-\Big(\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big)^{\rho}-1\Big)\Big(1-\gamma_0\,K_V(t,\zeta)\Big)\Big)\gamma_R(t,\zeta)\nu(d\zeta),$$

[5] By starting with the identity $Y_t V_t=\pi_t W_t$ instead of using (54), these computations can be made somewhat easier.

and the real interest rate by

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho(1-\rho\gamma)}{1-\rho}\,\sigma_c'(t)\sigma_c(t) + \frac{1}{2}\,\frac{\rho-\gamma}{1-\rho}\sigma_W(t)'\sigma_W(t)$$
$$-\int_{\mathcal{Z}}\Big\{\frac{1}{2}(1+\rho)\gamma_0 K_V'(t,\zeta)K_V(t,\zeta) + \Big((\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)})^{\rho} - 1\Big)\Big(1-\gamma_0 K(t,\zeta)\Big)$$
$$+\,\rho\gamma_c(t,\zeta) - \rho\,K_V(t,\zeta)\Big\}\,\nu(d\zeta).$$

The volatility of the wealth portfolio, $\sigma_W(t)$, is connected to the volatilities $\sigma_V(t)$ and $\sigma_c(t)$) via equation (60), and $K_V(t,\zeta)$ is given in terms of $\gamma_W(t,\zeta)$ and $\gamma_c(t,\zeta)$ by the equation (62).

Under the reformulation in Section 3 regarding the jump part, the relationship (62) can be integrated in the formula for the risk premium and the real rate, by a change of variables. Let $y := \gamma_M(t,\zeta)$, $c = \gamma_c(t,\zeta)$, $v = K_V(t,\zeta)$ and $x = \gamma_R(t,\zeta)$. As explained in Section 3, for $l = 1$ the relationship (62) with this notation is

$$1 + y = \frac{1 + v(1-\gamma_0-\gamma_0 v)}{1-\gamma_0 v + \big((\frac{1+v}{1+c})^{\rho} - 1\big)(1-\gamma_0 v)}. \tag{63}$$

For the Lévy-measure $\nu(d\zeta, d\zeta_2, d\zeta_3))$ on the form $\lambda_t dH_t(y,c,x)$ the risk premium and the real rate can be written respectively

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\,\sigma_c(t)'\sigma_R(t) + \frac{\gamma-\rho}{1-\rho}\,\sigma_W(t)'\sigma_R(t) +$$
$$\iiint_{\mathcal{Z}'}\big\{\gamma_0 v - ((\frac{1+v}{1+c})^{\rho} - 1)(1-\gamma_0 v)\big\}x\,\lambda_t dH_t(c,v,x) \tag{64}$$

and

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho(1-\rho\gamma)}{1-\rho}\,\sigma_c'(t)\sigma_c(t) + \frac{1}{2}\,\frac{\rho-\gamma}{1-\rho}\sigma_W(t)'\sigma_W(t)$$
$$-\iiint_{\mathcal{Z}'}\Big(\frac{1}{2}\gamma_0(1+\rho)v^2 + ((\frac{1+v}{1+c})^{\rho} - 1)(1-\gamma_0 v) + \rho c - \rho v\Big)\lambda_t dH_t(c,v,x). \tag{65}$$

where $h_t(x,v,x) = J(c,v,x)f_t(c,y(c,v),x)$ is the pdf of H_t. As pointed out in Section 3, $F_t(c,y,x)$, the joint probability distribution function of the jump sizes in consumption, the wealth portfolio and the risky asset under consideration, is internalised this way, assuming F_t has a probability density function $f_t(c, y, x)$.

Further transformations of variables suitable for computations are sometimes needed, as we show below.

9.1 The case $\rho = 0$.

This corresponds to high substitutability of consumption across time, which with dynamics yields the CAPM. We first consider the diffusion part. The result follows directly from the general equations above: The diffusion part of the risk premium for any risky asset is

$$\mu_R(t) - r(t) = \gamma \sigma_{W,R}(t), \qquad (\rho = 0),$$

and the corresponding diffusion risk-free interest rate is

$$r(t) = \delta - \frac{1}{2}\gamma \sigma'_W(t)\sigma_W(t), \qquad (\rho = 0).$$

Turning to the pure jump part of the risk premium and the interest rate in Theorem 2, the expression (62) becomes particularly simple when $\rho = 0$. In this situation the agent is neutral to consumption substitution across time, where

$$-\frac{\gamma_\pi(t,\zeta)}{\pi_t} = \gamma_0 K_V(t,\zeta) \quad \text{for all } t \text{ and for all } \zeta \in \mathcal{Z}, \tag{66}$$

which means that the pure jump expressions for the risk premium and the real interest rate are

$$\mu_R(t) - r_t = \gamma_0 \int_{\mathcal{Z}} K'_V(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta),$$

and

$$r_t = \delta - \frac{1}{2}\gamma_0 \int_{\mathcal{Z}} K'_V(t,\zeta) K_V(t,\zeta)\, \nu(d\zeta)$$

respectively, and where K_V satisfies the quadratic equation

$$K_V^2(t,\zeta) - \big(\frac{1}{\gamma_0} + \gamma_W(t,\zeta)\big)K_V(t,\zeta) + \frac{\gamma_W(t,\zeta)}{\gamma_0} = 0$$

for all t and ζ. This gives the two solutions $K_V(t,\zeta)_1 = \frac{1}{\gamma_0}$ or $K_V(t,\zeta)_2 = \gamma_W(t,\zeta)$. With risk aversion, only the latter solution makes sense, and we have for the jump part

$$\mu_R(t) - r_t = \gamma_0 \int_{\mathcal{Z}} \gamma'_W(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta),$$

and

$$r_t = \delta - \frac{1}{2}\gamma_0 \int_{\mathcal{Z}} \gamma'_W(t,\zeta)\gamma_W(t,\zeta)\,\nu(d\zeta).$$

There is full agreement between the continuous and the jump parts as far as these results are concerned.

This model can be considered as a dynamic version of the classical CAPM of Mossin (1966). The CAPM is derived in a time-less setting, where there is consumption only on the last time point, so that the risk-free interest rate has no real meaning. In contrast, our model is valid in a dynamic setting with recursive utility, and has an associated real, equilibrium risk-free rate of interest as given above.

As an illustration of the pure jump model, it can be seen to fit the data summarized in Table 1, see Appendix 2 below, by modelling the discrete data by a marked point process of frequency one per year, on the average. The result of this calibration is: $\gamma = 2.38$, and $\delta = .038$. As we have seen, this is the same result as obtained using the continuous model with no jumps, and also using the combined model when $\gamma_0 = \gamma$.

9.2 The case of $\rho = 1$.

Starting with the continuous part, we have the relationship $\sigma_W(t) = (1-\rho)\sigma_V(t) + \rho\sigma_c(t)$, where $\sigma_V(t)$ is the volatility of future utility at the optimal consumption path. When $\rho = 1$ we obtain that $\sigma_W(t) = \sigma_c(t) = \sigma_V(t)$ a.s. for all t. From this we obtain

$$\mu_R(t) - r(t) = \gamma\sigma_{W,R}(t), \qquad (\rho = 1),$$

for the risk premium of any risky asset, and

$$r(t) = \delta + \mu_c(t) - \gamma\sigma'_c(t)\sigma_c(t), \qquad (\rho = 1),$$

for the risk-free interest rate of the diffusion part.

Turning to the pure jump part for $\rho = 1$, in this situation risk premiums are given by

$$\mu_R(t) - r_t =$$
$$\int_{\mathcal{Z}} \Big(\gamma_0\, K_V(t,\zeta) - \Big(\Big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\Big) - 1\Big)\Big(1-\gamma_0\, K_V(t,\zeta)\Big)\Big)\gamma_R(t,\zeta)\nu(d\zeta)$$

and the short rate is

$$r_t = \delta + \mu_c(t) - \gamma_0 \int_{\mathcal{Z}} \Big\{ K_V'(t,\zeta)K_V(t,\zeta) + \Big(\big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\big) - 1\Big)\Big(1-\gamma_0 K(t,\zeta)\Big)$$
$$+ \gamma_c(t,\zeta) - K_V(t,\zeta)\Big\}\, \nu(d\zeta).$$

Now equation (62) is satisfied when $K_V(t,\zeta) = \gamma_c(t,\zeta)$, for all t and ζ in their respective domains, in which case it also follows that $\gamma_W(t,\zeta) = \gamma_c(t,\zeta)$, for all t and ζ. As a consequence of this, we obtain the following results:

$$\mu_R(t) - r_t = \gamma_0 \int_{\mathcal{Z}} \gamma_W(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta)$$

and the short rate is

$$r_t = \delta + \mu_c(t) - \gamma_0 \int_{\mathcal{Z}} \gamma_c'(t,\zeta)\gamma_c(t,\zeta)\nu(d\zeta).$$

As in the above situation where $\rho = 0$, also here there is full harmony between the continuous and jump parts, and no approximations have been used in these two situations.

In the literature there seems to be disagreement about the most plausible value of the EIS-parameter $1/\rho$; some claim larger than 1, others smaller than 1. However, a happy compromise value may be 1, although this value may not be supported by the data we have.

In the next section we demonstrate that the exact results of this section are approximately true also in the general model under certain conditions.

9.3 An approximation for the pure jump part for general $\rho \neq 1$.

Let us consider the relationship (62) and expand the power function in the denominator in a Taylor series, retaining up to second order terms. This gives

$$1+\gamma_W(t,\zeta)\approx$$
$$\frac{1+K_V(t,\zeta)(1-\gamma_0-\gamma_0K_V(t,\zeta))}{1-\gamma_0K_V(t,\zeta)+\rho(K_V(t,\zeta)-\gamma_c(t,\zeta)-\rho\gamma_c(t,\zeta)K_V(t,\zeta))(1-\gamma_0K_V(t,\zeta))}.$$

Disregarding terms of higher order in the denominator, this can be written

$$1+\gamma_W(t,\zeta)\approx\big(1+K_V(t,\zeta)(1-\gamma_0-\gamma_0K_V(t,\zeta))\big)\big(1+\gamma_0K_V(t,\zeta)+ \rho\gamma_c(t,\zeta)-\rho K_V(t,\zeta)\big)+\cdots$$

or

$$\gamma_W(t,\zeta)\approx K_V(t,\zeta)(1-\rho)+\rho\gamma_c(t,\zeta)+\cdots\quad \text{for all } t \text{ and for all } \zeta\in\mathcal{Z}, \tag{67}$$

an approximate internalization of $\gamma_M(t,\cdot)$. This reduces to (66) when $\rho=0$. Inverting this we also have

$$K_V(t,\zeta)\approx\frac{1}{1-\rho}\big(\gamma_W(t,\zeta)-\rho\gamma_c(t,\zeta)\big)+\cdots. \tag{68}$$

These relationship can be compared to the corresponding between the volatilities $\sigma_M(t)$, $\sigma_V(t)$ and $\sigma_c(t)$ given in Theorem 2.

The jump part of the expression for the risk premium we approximate as follows

$$\int_{\mathcal{Z}}\Big(\gamma_0K_V(t,\zeta)-\big((1+\rho K_V(t,\zeta))(1-\rho\gamma_c(t,\zeta))-1\big)(1-\gamma_0K_V(t,\zeta)\Big)\gamma_R(t,\zeta)\nu(d\zeta).$$

Retaining second order moments only, this expression can be written

$$\int_{\mathcal{Z}}\Big((\gamma_0-\rho)K_V(t,\zeta)+\rho\gamma_c(t,\zeta)\Big)\gamma_R(t,\zeta)\nu(d\zeta).$$

Inserting the expression for $K_V(t,\zeta)$ from (68), we obtain an approximation to the risk premium for the pure jump model as follows

$$\mu_R(t)-r_t=\frac{\rho(1-\gamma_0)}{1-\rho}\int_{\mathcal{Z}}\gamma_c(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta)+ \frac{\gamma_0-\rho}{1-\rho}\int_{\mathcal{Z}}\gamma_W(t,\zeta)\gamma_R(t,\zeta)\nu(d\zeta)+\cdots \tag{69}$$

We notice that up to the first two moments this expression has the same structure as the corresponding formula for the continuous part, except that γ_0 now replaces γ.

Turning to the interest rate, we proceed as follows. The jump part of the interest rate can be written

$$-\int_{\mathcal{Z}}\Big\{\frac{1}{2}(1+\rho)\gamma_0 K_V'(t,\zeta)K_V(t,\zeta)+\Big(\big(\frac{1+K_V(t,\zeta)}{1+\gamma_c(t,\zeta)}\big)^{\rho}-1\Big)\Big(1-\gamma_0 K_V(t,\zeta)\Big)$$
$$+\rho\gamma_c(t,\zeta)-\rho\, K_V(t,\zeta)\Big\}\,\nu(d\zeta)$$

To obtain the same order of approximation as above, we must include one more term in the Taylor series expansion for the power term. This gives

$$-\int_{\mathcal{Z}}\Big\{\frac{1}{2}(1+\rho)\gamma_0 K_V'(t,\zeta)K_V(t,\zeta)+\Big(1-\gamma_0 K_V(t,\zeta)\Big)\cdot$$
$$\Big(\big(1+\rho K_V(t,\zeta)+\frac{1}{2}\rho(\rho-1)K_V^2(t,\zeta)\big)\big(1-\rho\gamma_c(t,\zeta)+$$
$$\frac{1}{2}\rho(\rho+1)\gamma_c^2(t,\zeta)\big)-1\Big)+\rho\gamma_c(t,\zeta)-\rho\, K_V(t,\zeta)\Big\}\,\nu(d\zeta). \quad (70)$$

First we focus on the term $K_V'K_V$ which leads to $\gamma_W'\gamma_W/(1-\rho)^2$ using (68). Examining (70), we see that three terms contribute to to the coefficient of $\gamma_W'(t,\zeta)\gamma_W(t,\zeta)$: They are

$$-\Big(\frac{1}{2}\frac{\gamma_0(1+\rho)}{(1-\rho)^2}-\frac{\gamma_0\rho}{(1-\rho)^2}+\frac{1}{2}\frac{\rho(\rho-1)}{(1-\rho)^2}\Big)=\frac{1}{2}\frac{\rho-\gamma_0}{1-\rho}. \quad (71)$$

From Theorem 2, or (65), we see that this is the coefficient multiplying the corresponding $\sigma_W'(t)\sigma_W(t)$-term in the continuous part of r_t.

Next we focus on the term that gives $\gamma_w'(t,\zeta)\gamma_c(t,\zeta)$, which we obtain from $K_V'(t,\zeta)\gamma_c(t,\zeta)$. In addition to the previous component, the following contributes directly from (70) to this product:

$$-\big(\gamma_0\rho-\rho^2\big)\frac{1}{1-\rho}\big(\gamma_W'(t,\zeta)\gamma_c(t,\zeta)-\rho\gamma_c'(t,\zeta)\gamma_c(t,\zeta)\big) \quad (72)$$

where we have used (68). Recall that (71) is obtained as the first part of

$$K_V'(t,\zeta)K_V(t,\zeta)=\frac{1}{(1-\rho)^2}\big(\gamma_W'(t,\zeta)\gamma_W(t,\zeta)-2\rho\gamma_W'(t,\zeta)\gamma_c(t,\zeta)+$$
$$\rho^2\gamma_c'(t,\zeta)\gamma_c(t,\zeta)\big) \quad (73)$$

Taking these two terms into account, the coefficient in question is is:

$$\Big(\frac{1}{2}\frac{\rho-\gamma_0}{1-\rho}(-2\rho)-\big(\gamma_0\rho-\rho^2\big)\frac{1}{1-\rho}\Big)\gamma_W'(t,\zeta)\gamma_c(t,\zeta)=0,$$

in agreement with Theorem 2, or (65): No such term appears in the interest rate. Finally we turn to the term $\gamma_c'(t,\zeta)\gamma_c(t,\zeta)$. We obtain from (72) the contribution

$$-\frac{\gamma_0\rho-\rho^2}{1-\rho}(-\rho)\gamma_c'(t,\zeta)\gamma_c(t,\zeta).$$

Directly from (70) we have

$$-\frac{1}{2}\rho(\rho+1)\gamma_c'(t,\zeta)\gamma_c(t,\zeta).$$

From (71) and (73) we get

$$\frac{1}{2}\frac{\rho-\gamma_0}{1-\rho}\rho^2\gamma_c'(t,\zeta)\gamma_c(t,\zeta).$$

Adding these three terms gives the following result

$$-\frac{1}{2}\frac{\rho(1-\rho\gamma_0)}{1-\rho}\gamma_c'(t,\zeta)\gamma_c(t,\zeta).$$

This proves that the pure jump part of the real interest rate is given by

$$r_t=\delta+\rho\mu_c(t)-\frac{1}{2}\,\frac{\rho(1-\rho\gamma_0)}{1-\rho}\int_{\mathcal{Z}}\gamma_c'(t,\zeta)\gamma_c(t,\zeta)\,\nu(d\zeta)\\
+\frac{1}{2}\,\frac{\rho-\gamma_0}{1-\rho}\int_{\mathcal{Z}}\gamma_W'(t,\zeta)\gamma_W(t,\zeta)\,\nu(d\zeta)+\cdots$$

to the required order of approximation. Thus we have shown (17) and (18) in Section 3.2.

9.4 Summary of the approximative model.

Combining the results in Section 3.1 and 8.2 with the formulation of the jump model Section 8, the risk premiums and the real interest rate can be written

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\,\sigma_c(t)'\sigma_R(t) + \frac{\gamma-\rho}{1-\rho}\,\sigma_W(t)'\sigma_R(t) + \\ \frac{\rho(1-\gamma_0)}{1-\rho}\int_{-1}^{\infty}\int_{-1}^{\infty} c\,x\,\lambda_t\,dF_t^{(1)}(c,x) + \frac{\gamma_0-\rho}{1-\rho}\int_{-1}^{\infty}\int_{-1}^{\infty} y\,x\,\lambda_t\,dF_t^{(2)}(y,x) \tag{74}$$

and

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho(1-\rho\gamma)}{1-\rho}\,\sigma_c'(t)\sigma_c(t) + \frac{1}{2}\,\frac{\rho-\gamma}{1-\rho}\sigma_W(t)'\sigma_W(t) \\ -\frac{1}{2}\,\frac{\rho(1-\rho\gamma_0)}{1-\rho}\int_{-1}^{\infty}\int_{-1}^{\infty} c^2\,\lambda_t\,dF_t^{(1)}(c,x) + \frac{1}{2}\,\frac{\rho-\gamma_0}{1-\rho}\int_{-1}^{\infty}\int_{-1}^{\infty} y^2\,\lambda_t\,dF_t^{(2)}(y,x) \tag{75}$$

respectively. Here $F_t^{(1)}(c,x)$ and $F_t^{(2)}(y,x)$ are the marginal distributions of $F_t(c,y,x)$. We assume $F_t^{(1)}(c,x)$ has density $f_t^{(1)}(c,x)$ and $F_t^{(2)}(y,x)$ has density $f_t^{(2)}(c,x)$.

This model may be used for several purposes. The continuous parts can be interpreted as the standard risk premiums and short rate, while the terms stemming from the jumps can be interpreted as the economic effects of major events, such as catastrophes, market crashes, pandemics etc.

These major events, as parts of the economic life, will be reflected in the market quantities and be part of the market data. To separate these effects in the data can be involved, and we do not attempt such an investigation here. Some market data has been calibrated using the continuous model in Aase (2016), and the results were promising. The model could be consistent with reasonable values of the preference parameters δ, ρ and γ.

In the following it will be an advantage to consider the model in exponential, rather than in the stochastic exponential form. We therefore make the substitution $1+y=e^{z_W}$, $1+c=e^{z_c}$ and $1+x=e^{z_R}$ which leads to the following expressions

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\,\sigma_c(t)'\sigma_R(t) + \frac{\gamma-\rho}{1-\rho}\,\sigma_W(t)'\sigma_R(t) + \\ \frac{\rho(1-\gamma_0)}{1-\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{z_c}-1)(e^{z_R}-1)\,\lambda_t\,dG_t^{(1)}(z_c,z_R) + \\ \frac{\gamma_0-\rho}{1-\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{z_W}-1)(e^{z_R}-1)\,\lambda_t\,dG_t^{(2)}(z_W,z_R) \tag{76}$$

and

$$
\begin{aligned}
r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho(1-\rho\gamma)}{1-\rho}\,\sigma_c'(t)\sigma_c(t) + \frac{1}{2}\,\frac{\rho-\gamma}{1-\rho}\sigma_W(t)'\sigma_W(t) \\
- \frac{1}{2}\,\frac{\rho(1-\rho\gamma_0)}{1-\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{z_c}-1)^2\,\lambda_t\,dG_t^{(1)}(z_c,z_W) \\
+ \frac{1}{2}\,\frac{\rho-\gamma_0}{1-\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{z_W}-1)^2\,\lambda_t\,dG_t^{(2)}(z_W,z_W) \quad (77)
\end{aligned}
$$

where $G^{(1)}$ has density function $g^{(1)}$ given by

$$
g^{(1)}(z_c,z_R) = f^{(1)}(c(z_c),x(z_R))J^{(1)}(z_c,z_R)
$$

and where the $J^{(1)}$ is the Jacobian

$$
J^{(1)}(z_c,z_R) = \operatorname{mod}\begin{vmatrix} e^{z_c} & 0 \\ 0 & e^{z_R} \end{vmatrix} = e^{z_c+z_R}
$$

and similarly for $G^{(2)}(z_W,z_R)$. This form we will make use of in the calibrations of Appendix 2.

9.5 Some related issues.

From theories that come out from the classical CAPM, much of financial analysis has traditionally been based on various forms of mean square analysis. Including jumps in the analysis represents a break with this tradition. The general results presented in this paper for scale invariant (SI) recursive utility, shows that focus on only the two first moments in financial data can be insufficient. Closed form solutions are helpful for intuition, and for that reason we have pointed out simplifications in the general theory sufficient to obtain such solutions.

With translation invariant (TI) recursive utility, which is based on negative exponential utility instead of power utility, we obtain exact results for the jump parts in the CCAPM (see, for example, Aase (2024)). This model allows us to study the deviations from the local mean square analysis in detail. Another advantage with the TI-model is that it aggregates.

A two agent model with the SI-version and diffusion dynamics only has been derived (see, for example, Aase (2019)). It was used to study the CCAPM in the situation when it was known that a certain fraction of the population did not participate in the stock market. In this situation the SI-model worked well, and exact formulas were derived.

The topic of separating the jump part from the continuous part of a data set is dealt with by Ait Sahalia and Jacod (2009-11), and is far from trivial.

In the construction a *security-spot market equilibrium* (for a definition, see, for example, Duffie (2001)), we can fix an Arrow-Debreu equilibrium of the type that we have considered in this paper, and examine a spanning condition on the gains processes, or the price processes adjusted for dividends, of the risky securities under which there exists a security-spot market equilibrium with the same consumption allocation. The result is that the dynamic spanning condition requires at least as many securities as there are independent sources of risk. In the pure diffusion case with a risk-less asset and d independent Brownian motions, precisely $d + 1$ securities turns out to be both necessary and sufficient. With a jump component added the conventional wisdom seems to be that the model is generally incomplete. In the recent paper (Aase (2021)) it is demonstrated why this is an hastened conclusion. The clue is to consider a full market model with as many assets as there are sources of risk. This means that when the jump components are market point processes with a finite support, we need a finite number of risky asset in order to make the resulting market model complete.

On the other hand, results like the consumption based capital asset pricing theory and extensions, as in the present article, can be derived without the spanning condition. In order to study financial derivatives, it is usually required that the model is complete. The irony is that when this is the case, such contracts are also redundant.

In the present model we study a catastrophe forward (or futures) contract based on the jump part of the model. As mentioned, this part can be made to be complete provided the jump size distributions are discrete and finite. This latter part we simply ignore by assuming that these distributions are continuous, which results in closed form solutions. We think that these are of interest in their own right, where this is the topic of the last section of this paper.

10 Catastrophe futures and forward contracts.

The model we have developed above can be utilized for other purposes than the ones we have so far described. In particular this is the case for the jump part of the model.

In an article based EU using jump-dynamics, futures and forward contracts on catastrophe indices were treated in Aase (1999). For events triggered by climate change, one may think of basing such contracts on negative shocks to consumption, or wealth. This we claim can be considered as a constructive attempt to deal with such events, described in the literature below.

10.1 Relation to Weitzman's "dismal theorem".

Weitzman (2009) expressed, with climate change as a prime example, worries about the implications of structural uncertainty for economics of low probability, high-impact catastrophes. He focused on the conditional expected value of the marginal rate of substitution, which could become large, even infinite, in such cases. As an illustration, he considered a perturbation of the volatility of consumption. However, such a perturbation has no "direction", since it may lead to positive shocks as well as negative ones. A more concise model is one containing unpredictable jumps, positive as well as negative, at random time points, as in the present paper. With a catastrophe futures index related to, for example, consumption, one would be able to transfer some of the consumption risks to the world's financial markets. This could be a step in the right direction, since it might trigger incentives to do something about the underlying problems leading to such negative events.

Weitzman's analyses have been discussed by several authors. One way to react to a scenario as he describes, it has been suggested that society should be willing to exchange today's consumption to the future at a large rate (even an infinite rate), a form of ex post insurance. But, for many reasons, this does not seem very likely to happen in such a scenario. One is the lack of institutions do deal with this kind of risk. To use the capacity of the world's financial markets seems appropriate in situations of catastrophes, since it is well known how such markets can be established. The Chicago Board of Trade is one important example.

Below we briefly describe a model for pricing of forward and futures contracts on negative consumption shocks, based on the jump part of the present paper. Other means must be made to reduce the likelihood of such negative events caused by climate change. In insurance terminology this is related to risk management.

10.2 A catastrophe insurance forward contract.

In this section we consider a catastrophe insurance futures/forward contract, where jump processes play a natural and central role.

In Aase (1999) an equilibrium model for a catastrophe insurance futures contract was analyzed using expected utility. The model for the aggregate claims is $V(t) = \int_0^t \int_{\mathbb{R}_+} \zeta N(ds, d\zeta)$ by time t, reported to, say, a statistical agent associated to a catastrophe insurance exchange (like the Chicago Board of Trade). The idea with these derivatives on various loss indexes is to transfer some of the catastrophe risk from the reinsurance markets to the financial markets, where the capacity is presumably much larger.

This quantity can alternatively be written $V(t) = \sum_{k=1}^{N(t)} Y_k$, where $N(t)$ is a counting process recording the number of claims by time t and $Y_1, T_2, \ldots$ are the consecutive claim sizes. The loss-ratio index is assumed to be of the form $\hat{V}(t) = V(t)/\Pi$ where Π is a normalizing constant, like the total premiums for the next quarter.

We give a demonstration illustrating how the theory in the present paper can be used in this regard. In particular a closed form forward formula is derived.

In the present model the claims are related to negative jumps in the consumption growth rates, which are in the domain $(-1, 0)$, and therefore the range of the Z_i-values are in the subset $(0, 1)$ of the real numbers, with 1 the most serious value, which would mean that aggregate consumption in the next period is 0.

Assuming we have a loss ratio index of this type, the forward price at time t with expiration time $T > t$ in the future is of the following form

$$F(t) = F(t) = \frac{E_t(\pi_T \hat{V}(T))}{E_t(\pi_T)}, \tag{78}$$

where π is the state price deflator.

Under the assumption that $V(t)$ is a compound Poisson Process, we find a closed form of the forward price $F(t)$.

Towards this end, we start with the state price deflator in this paper and focus on the jump part. We limit the investigation to the situation where the parameter $\rho = 1$. From the results in Section 8.4 we have that

$$\gamma_\pi(t, z) = -\pi_t \gamma_0 \gamma_c(t, z). \tag{79}$$

Recalling that $d\pi_t = \int_{\mathbb{R}} \gamma_\pi(t, z) \tilde{N}(dt, dz)$, let us consider an equation of the form

$$d\pi_t = \pi_{t-} \int_{\mathcal{Z}} c(t, z) \tilde{N}(dt, dz)$$

for an appropriate function $c(t, \zeta)$. By Itô's generalized lemma we then have

$$\pi_t = \exp\Big\{ \int_0^t \int_{\mathcal{Z}} \big(\ln(1 + c(s, z)) - c(s, \zeta)\big) \nu(dz) ds + \int_0^t \int_{\mathcal{Z}} \ln\big(1 + c(s, z)\big) \tilde{N}(ds, dz) \Big\}. \tag{80}$$

When $\rho = 1$ we can consider consumption directly, so let $\gamma_c(t,z) = z$, where $z \in \mathbb{R}$. When the consumption z approaches $-\infty$, then the state price becomes very large. The relevant part of the state price deflator in the cat insurance context is given by

$$\pi_t = \exp\Big\{ \int_0^t \int_{-\infty}^0 \gamma_0 z \nu(dz) ds + \int_0^t \int_{-\infty}^0 \ln(1 - \gamma_0 z) N(ds, dz) \Big\}$$

for negative consumption jumps in $(-\infty, 0)$. The positive jumps are of no concern in this setting. Thus the relevant part of the state price deflator can be expressed as

$$\pi_t = \exp\Big\{ \int_0^t \int_0^\infty \ln(1 + \gamma_0 z) N(ds, dz) \Big\}$$

If z in this formula is large, then the state price becomes large as well, in fact it approaches $+\infty$ as $z \to \infty$, since this corresponds to no consumption after such a sequence of negative jumps.

This may be related to Weitzman's dismal theorem, but in this situation the present theory attempt to do something about the problem, by distributing the risk across the World's financial markets.

From this the forward price at any time $t < T$, for recursive utility of the scale invariant type and $\rho = 1$, equals

$$F(t) = \frac{E_t\Big\{\exp\Big(\int_0^T \int_0^\infty \ln(1 + \gamma_0 z) N(dt, dz) \Big) \hat{V}(T) \Big\}}{E_t\Big\{\exp\Big(\int_0^T \int_0^\infty \ln(1 + \gamma_0 z) N(dt, dz) \Big) \Big\}}. \tag{81}$$

Suppose that for the jump-size distribution on $(0, \infty)$, the two moments $\mu_1 = E\big(Z_1\big)$ and $\mu_2 = E\big(Z_1^2\big)$ both exist. We then have the following theorem:

Theorem 3 *Under the above assumptions the forward price of the catastrophe index regarding consumption growth rates is given by the following formula*

$$F(t) = \hat{V}_t + (\mu_1 + \gamma_0 \mu_2) \frac{\lambda(T-t)}{\Pi}, \qquad \rho = 1. \tag{82}$$

Proof: Assuming that $\sum_{i=1}^{N_T} Z_i$ is a compound Poisson process, where the frequency of the Poisson process N_T is λ and the claim sizes Z_i are i.i.d. with values in $(0, \infty)$, all independent of the Poisson process $N_t, t \geq 0$, let us start with the denominator. Recalling that the positive jumps in the state-price deflator cancel, by independence, in the forward formula, we have to compute the following:

$$E_t\{\exp\big(\int_0^T\int_0^1 \ln\big[1+\gamma_0\zeta\big]\big)N(dt,d\zeta)\big)\} =$$

$$E_t\{\exp\big(\sum_{i=1}^{N_T}\ln(1+\gamma_0 Z_i)\big)\} = E_t\{\prod_{i=1}^{N_T}(1+\alpha_0 Z_i)\} =$$

$$\prod_{i=1}^{N_t}(1+\gamma_0 Z_i^{(obs)})E_t\{\prod_{i=N_t+1}^{N_T}(1+\gamma_0 Z_i)\} =$$

$$\prod_{i=1}^{N_t}(1+\gamma_0 Z_i^{(obs)})E_t\big(1+\gamma_0\mu_1\big)^{N_{T-t}} =$$

$$\prod_{i=1}^{N_t}(1+\gamma_0 Z_i^{(obs)})\exp\big(\lambda(T-t)\gamma_0\mu_1\big).$$

Here $Z_i^{(obs)}$ are the observed values of Z_i by time t. We have used iterated expectation, the independence structure and the time homogeneity of the Poisson process to replace $(N_T - N_t)$ by N_{T-t}. The last expression follows from the moment generating function of the Poisson random variable given by $\varphi(\beta) = E(e^{\beta N_{T-t}}) = e^{\lambda(T-t)(e^{\beta}-1)}$.

Moving to the numerator, we need to compute for $V(T) = \sum_{i=1}^{N_T} Z_i$ the following

$$E_t\{\exp\big(\int_0^T\int_0^1 \ln\big[1+\gamma_0 Z_i)\big]\big)N(dt,d\zeta)\big)V(T)\} =$$

$$\prod_{i=1}^{N_t}(1+\gamma_0 Z_i^{(obs)})E_t\Big\{\prod_{i=N_t+1}^{N_T}(1+\gamma_0 Z_i\big(V_t+\sum_{i=N_t+1}^{N_T} Z_i\Big\}.$$

Since

$$E_t\{\prod_{i=N_t+1}^{N_T}(1+\gamma_0 Z_i)V_t\} = V_t E_t\big(1+\gamma_0\mu_1\big)^{N_{T-t}} = V_t\exp\big(\lambda(T-t)\gamma_0\mu_1\big),$$

where we also have used the moment generating function of the Poisson random variable, it remains to calculate

$$E_t\Big\{\prod_{i=N_t+1}^{N_T}(1+\gamma_0 Z_i)\big(\sum_{i=N_t+1}^{N_T} Z_i\big)\Big\}.$$

By multiplying the product by the summands in the sum, term by term and using the independence structure as well as the fact that $E(Z_i)=\mu_1$, $E(Z_i^2)=\mu_2$, we obtain the following:

$$E_t\big\{(N_T-N_t)\big(\mu_1+\gamma_0\mu_2\big)\big[1+\gamma_0\mu_1\big]^{N_T-N_t-1}\big\}=$$

$$\big(\mu_1+\alpha_0\mu_2)\big)\big(1+\gamma_0\mu_1\big)^{-1}E_t\big\{N_{T-t}\big(1+\gamma_0\mu_1\big)^{N_{T-t}}\big\}.$$

The latter conditional expectation can again best be handled using the moment generation function of the Poisson random variable. Consider the derivative of this function with respect to the parameter β: It is $\varphi'(\beta)=E(N_{T-t}e^{\beta N_{T-t}})=e^{\lambda(T-t)(e^{\beta}-1)}(\lambda(T-t)e^{\beta})$. Using this and setting $e^{\beta}=(1+\alpha_0\mu_1)$ we obtain

$$\big(\mu_1+\gamma_0\mu_2\big)\big(1+\alpha_0\mu_1\big)^{-1}e^{\lambda(T-t)(\gamma_0\mu_1)}(\lambda(T-t)(1+\gamma_0\mu_1))=$$

$$\big(\mu_1+\gamma_0\mu_2)\big)e^{\lambda(T-t)(\gamma_0\mu_1)}\lambda(T-t).$$

Putting all this together, we finally obtain the forward formula, after division by Π:

$$F_t=\hat{V}_t+(\mu_1+\gamma_0\mu_2)\frac{\lambda(T-t)}{\Pi}, \tag{83}$$

which is the forward price formula (82).

The positive jumps in the state price deflator drop out:

Now, suppose there are also positive jumps in the state price deflator, assumed to be a Compound Poisson process N' of frequency λ' with the two first central moments are μ_1' and μ_2', where the positive jumps are independent of the negative ones. The total number of jumps are $N''=N+N'$.

The crucial new term in the numerator in the forward formula has accordingly changed to

$$E_t\Big\{\prod_{i=N''_t+1}^{N''_T}(1+\gamma_0 Z_i)\big(\sum_{i=N_t+1}^{N_T} Z_i\big)\Big\} =$$

$$E_t\big\{(N_T-N_t)\big(\mu_1+\gamma_0\mu_2\big)\big[1+\gamma_0\mu_1\big]^{N_T-N_t-1}[1-\gamma_0\mu'_1]^{N'_T-N'_t}\big\} =$$

$$\big(\mu_1+\gamma_0\mu_2\big)\lambda(T-t)e^{\lambda(T-t)(\gamma_0\mu_1)}\cdot e^{-\lambda'(T-t)\gamma_0\mu'_1}.$$

Similarly, the crucial new term in the denominator has now changed to

$$E_t\big\{\prod_{i=N_t+1}^{N_T}(1+\gamma_0 Z_i)\prod_{i=N'_t+1}^{N'_T}(1-\gamma_0 Z'_i)\big\} =$$

$$e^{\lambda(T-t)\gamma_0\mu_1}\cdot e^{-\lambda'(T-t)\gamma_0\mu'_1}.$$

The last factor in each part of the ratio cancel, and the forward formula (82) again results, as claimed in the text. □

Under risk neutrality, when $\gamma_0 = 0$, $F_t = \hat{V}_t + E(Z_1)\frac{\lambda(T-t)}{\Pi}$, as matters should be.

As a numerical illustration of this theorem, let us consider the parameters in Example 1 of Aase (1999) assume that the claim sizes are Gamma-distributed with parameters n and μ. This means that $E(Z) = \frac{n}{\mu}$ and $\text{var}(Z) = \frac{n)n+1)}{\mu^2}$. Suppose $n = 10, \mu = 10^{-6}, \lambda = 10, (T-t) = 0.25$, and $\Pi = \$26417200$. Then the latter term in the above formula (82) (the loss ratio) is 0.99840.

10.3 An interpretation of the market price of insurance risk.

Some readers may be used to the so-called equivalent martingale measure Q in pricing contexts. This also has a natural analogue here, which we now demonstrate.

Starting with the stochastic model we have chosen, the Radon-Nikodym derivative $\xi(T) = \frac{dQ}{dP}$ and its associated density process $\xi(t) = E(\xi(T) \mid \mathcal{F}_t)$ can alternatively be represented as (see, for example, Bremaud (1981), T10

Theorem, p241)[6]

$$\xi(t) = \Big(\prod_{n\geq 1} \kappa v(Z_n) 1(\tau_n \leq t)\Big) \exp\Big\{\int_0^t \int_{\mathcal{Z}} (1-\kappa v(y))\lambda F_Z(dz)ds + \int_0^t r(s)ds\Big\}, \tag{84}$$

holding for any $t \in [0, T]$, where τ_n represent the time points of jumps. Here $F_Z(dz)$ is the probability distribution of the claim sizes Z_i, κ is a positive constant and $v(z)$ is a function satisfying

$$\int_0^\infty v(z) F_Z(dz) = 1\,, \qquad P\text{- a.s.} \tag{85}$$

The term $v(z)F_Z(dz)$ is accordingly the probability distribution of the claim sizes Z_i under the probability measure Q, whereas $\kappa\lambda$ is the corresponding frequency of the Poisson process under Q. We interpret κ as the market price of frequency risk, and $v(z)$ as the market price of claim size risk. The function v is non-negative and strictly positive on the support of F_Z.

The state price deflator for positive claim sizes, corresponding to negative jumps in the growth rates of consumption, when $\rho = 1$, is, in terms of the variable Z,

$$\pi_t = \exp\Big(\int_0^t \int_0^1 \ln(1+\gamma_0 z) N(dt, dz)\Big)$$

and using the connection between this quantity and the density process ξ_t, which is

$$\pi_t = \xi_t \exp\Big(-\int_0^t r(t)dt\Big)\pi_0,$$

we obtain the following ($\pi_0 = 1$):

$$\xi_t = \exp\Big(\int_0^t \int_0^1 \ln(1+\gamma_0 z) N(ds, dz) + \int_0^t r(s)ds\Big) = \prod_{i=1}^{N_t} (1+\gamma_0 Z_i) e^{\int_0^t r(s)ds}. \tag{86}$$

We now assume, for simplicity, that the interest rate process $r(s)$ is independent of the claim sizes, for example, r has no jump component, in which case the forward price equals the futures price. By this assumption we can ignore r in what follows. The other term in the exponent of (84) can likewise be

[6] In the literature this sometimes goes under the heading of likelihood ratios, or Radon-Nikodym derivatives

ignored since it is a non-random constant.

Comparing the above expression with (84), we obtain

$$\kappa v(z) = 1 + \gamma_0 z.$$

Let the probability density function of Z be $f_Z(z)$. Recall that the insurance claims by time t are $V(t) = \sum_{i=1}^{N_t} Z_i$ The jump size distribution of Z under Q, $f^Q(z)dz$, is given by

$$f_Z^Q(z)dz = \frac{1}{\kappa}(1 + \gamma_0 z) f_Z(z)dz.$$

Since both f_Z^Q and f_Z are bona fide probability distributions, this implies that

$$1 = \int_0^\infty f_Z^Q(z)dz = \int_0^\infty \frac{1}{\kappa}(1 + \gamma_0 z) f_Z(dz) = \frac{1}{\kappa}(1 + \gamma_0 \mu_1),$$

where $\int_0^\infty z f_Z(z)dz = \mu_1$. As a consequence $\kappa = 1 + \gamma_0 \mu_1$, and $v(z) = \frac{1+\gamma_0 z}{1+\gamma_0 \mu_1}$. Both the adjustment to frequency risk and jump size risk are determined.

Note that under Q the frequency of the claims is larger than under P since $\kappa > 1$, and the claim size distribution has a larger expected value under Q than under P. The latter follows, since $E^Q(Z) = \int_0^\infty z f_Z^Q(z)dz) = \frac{1}{\kappa}\int_0^\infty z(1 + \gamma_0 z) f_Z(z)dz = (\mu_1 + \gamma_0 \mu_2)/(1 + \gamma_0 \mu_1)$, where $\mu_2 = E(Z^2)$. By polynomial division this can be written

$$E^Q(Z) = \mu_1 + \frac{\gamma_0(\mu_2 - \mu_1^2)}{1 + \alpha_0 \mu_1} > \mu_1 = E(Z),$$

since the variance of Z is strictly positive.

A comparison between the two claim size distributions Q and P can perhaps best be made by the Monotone Likelihood Ratio (MLR) order.[7] Here Z^Q (Z under Q) is larger than Z^P according to monotone likelihood ratio ordering if $\frac{f_Z^Q(z)}{f_Z(z)}$ is nondecreasing is z, see Lehmann (1955). This condition holds since $\frac{f_Z^Q(z)}{f_Z(z)} = \frac{1}{\kappa}(1 + \gamma_0 z)$ is increasing in z. In the present context this may be interpreted as "claim sizes are more risky" under Q than under P.

Under the probability measure Q we have the following pricing rule:[8]

$$F(t) = E^Q(\hat{V}_T|\mathcal{F}_t) = \hat{V}_t + E^Q(\hat{V}_T - \hat{V}_t|\mathcal{F}_t), \qquad (87)$$

[7] sometimes used in the principal agent literature.

[8] Q is formally (mathematically) a probability measure, but the expectation under Q in reality a linear pricing functional.

which can readily be found from the above.[9] Using that $E^Q(Z)) = \frac{1}{\kappa}\big(\mu_1 + \gamma_0\mu_2)\big)$, we obtain that the forward price is

$$F(t) = \hat{V}_t + \lambda\kappa(T-t)\frac{1}{\kappa}\big(\mu_1 + \alpha_0\mu_2\big)\frac{1}{\Pi}, \qquad \rho = 1,$$

which is our previous result, since κ cancels. That is, formula (82) is demonstrated by the method of an equivalent martingale measure.

11 Conclusions.

We have considered a market model in financial economics in the setting of continuous time, where the dynamics of economic variables are modelled by jump-diffusions. From a theoretical point of view a most important step in our analysis is the internalization of the probability distributions, or more precisely, stochastic processes for the market and the wealth portfolios. They are determined in equilibrium from the primitives of the underlying economic model. These are the stochastic process for future utility (preferences) and the process determining the dynamics of the growth rate of aggregate consumption (the given endowment process). We use a general method of optimization, the stochastic maximum principle, together with the theory of forward/backward stochastic differential equations, which allows for an extension to jump dynamics.

The recursive model has several interesting features when jumps are allowed in the dynamics of the aggregate consumption process as well as in the recursive utility process. In addition to the nonlinear terms that are introduced, it also gives a new parameter for the risk aversion related to jump size risk. Both together, and in isolation, these features may yield added insights in explaining real data.

In addition to ordinary securities markets, where the jump structure can be approximated to have the same basic structure as the continuous components, in other markets the jump components can be more innovative. In particular this type of model can be used in the securitisation of catastrophe insurance futures. In this application insurance risk is priced, and the model presents an adequate economic framework for insurance. Our application is related to Weitzman's "dismal theorem" in connection with risks associated with climate change. In contrast to the background for this theorem, we offer a solution: Risk sharing via financial markets.

In an appendix (Appendix 2) we have addressed the well-known empirical

[9] This relationship explains why Q is called the "risk neutral" probability measure.

regularities of the conventional asset pricing model in financial- and macro economics, where our extended model may calibrate, with a few simplifications, to reasonable values of the preference parameters. In doing so, we consider the situation where the market portfolio is not a proxy for the wealth portfolio.

Appendix 1

The US-data set of the period 1960-2015.

S&P 500	Growth	Std error	Ln growth	Std error
Mean	7.07%	2.21%	5.56%	2.24%
Std dev	16.36%		16.62%	
Variance	2.68%	0.51%	2.76%	0.53%
$\sigma_{(W,M)}$	0.09%		0.10%	
$\kappa_{(W,M)}$	18.18%		21.45%	

Table 3: Key US-data for the time period 1960-2015: Real S&P 500 data.

Consumption	Growth	Std error	Ln growth	Std error
Mean	2.90%	2.21%	2.85%	0.21%
Std dev	1.59%		1.55%	
Variance	0.03%	0.00%	0.02%	0.00%
$\sigma_{(c,M)}$	0.09%		0.10%	
$\kappa_{(c,M)}$	9.75%		13.41%	

Table 4: Key US-data for the time period 1960-2015: Real consumption non-durables and services.

Real r_f	Growth	Std error	Ln growth	Std error
Mean	.91%	0.28%	0.89%	0.28%
Std dev	2.09%		2.07%	
Variance	0.044%	0.01%	0.043%	0.01%
$\sigma_{(c,b)}$	0.02%		0.02%	
$\kappa_{(c,b)}$	46.92%		47.08%	

Table 5: Key US-data for the time period 1960-2015: Real riskfree interest rate (government bills).

Real national wealth	Growth	Std error	Ln growth	Std error
Mean	1.11%	0.40%	1.06%	0.40%
Std dev	2.96%		2.94%	
Variance	0.09%	0.02%	0.09%	0.02%
$\sigma_{(c,W)}$	0.03%		0.03%	0.008%
$\kappa_{(c,W)}$	70.84%		71.24%	

Table 6: Key US-data for the time period 1960-2015: Real national wealth.

Appendix 2.

Some calibrations.

In this section we calibrate the recursive model to the data summarized in Table 1. Some calibrations of the recursive model with only continuous diffusion dynamics are shown in Table 2. The risk premium was first derived by Duffie and Epstein (1992a), while the interest rate was first derived in Aase (2016), and also follows from our approach in the present paper. In the calibration we have fixed the time impatience rate δ and solved the two equations (8) and (11) in the two remaining unknowns γ and ρ, for values of δ between 1.5 and 3.8 per cent[10].

The values obtained for γ and ρ seem reasonable, in particular the ones corresponding to $\delta \geq 0.022$. For comparisons, the results for the conventional additive model is given in the first line of Table 3, and for the pure jump model with joint NIG-distributed jump sizes in the second line. In applied economics values of the impatience rate between 1 and 4 per cent seem common. One reason for this is of course that the conventional, additive Eu-model is often taken for granted, and from the expression for the interest rate in (5) (disregarding the jump terms) one simply does not obtain reasonable values for the short rate unless δ is in this range, or smaller.

Weil (1989) did not have expressions like (17) and (18) to his disposal, but used numerical analysis on the same two-state Markov chain employed by Mehra and Prescott (1985). By varying the preference parameters, he tried to reproduce the moments estimated by the latter. As it turns out, he missed the most interesting solution in his calibrations. For details, see Aase (2016)).

With the jump terms included, we may expect some changes. The above continuous model gives interesting results in itself. One might conjecture

[10]The system of equations sometimes has another solution, which can be close to the one given by the conventional model.

	γ	ρ	EIS
Conventional Eu-Model			
$\delta = -.015$	26.37	26.37	.037
Conventional Eu-model with jumps only (NIG)			
$\delta = .0083$	22.2	22.2	.045
Continuous recursive model			
$\delta = .015$	.46	1.34	.74
$\delta = .020$	.90	1.06	.94
$\delta = .025$	1.33	.78	1.28
$\delta = .030$	1.74	.48	2.08
$\delta = .035$	2.14	.18	5.56
$\delta = .038$ CAPM+	2.38	.00	$+\infty$

Table 7: Calibrations of the continuous recursive model

that only minor adjustments may be required, which the discontinuous part could provide.

To investigate this, we employ the model on the form summarized in Section 8.3. Since this requires a transformation to log returns, the relevant statistics is summarized in Table 4. Notice that this table is not a mere transformation of Table 1, but developed from the the original data set used in the Mehra and Prescott (1985)-study, by taking logarithms of the relevant yearly quantities, and basing the statistical analysis on these transformed data points. As an illustration, of the total annual variation of .02509 in

	Expectat.	Standard dev.	Covariances
Consumption growth	1.75%	3.55%	$\text{cov}(M, c) = .002268$
Return S&P-500	5.53%	15.84%	$\text{cov}(M, b) = .001477$
Government bills	0.64%	5.74%	$\text{cov}(c, b) = -.000149$
Equity premium	4.89%	15.95%	

Table 8: Key US-data for the time period 1889-1978 in terms of log returns of discrete-time compounding.

the stock market, measured as variance, suppose we allocate .0126 to jumps. Similarly, the total annual variance of the consumption growth rate of .00126 is divided in two so that the jump part retains .00063. The expected growth rate of the jump part of the consumption variable is set to .01 and the jump parts contribution to the annual return on the market portfolio is set to .028. Higher order terms are ignored, so it is the new aspect of $\gamma_0 \neq \gamma$, as well as the existence of a full, joint probability distribution for the jump sizes that is investigated here. The latter distribution is assumed to be joint lognormal, and as a consequence both $G^{(1)}$ and $G^{(2)}$ are both joint normal probability distribution functions. Some tentative calculations are presented in Table 5. As can be seen from this table, with the new elements added, the model

	γ	ρ	γ_0	EIS
Recursive model including jump dynamics				
$\delta = .005$	1.06	1.02	.95	.98
$\delta = .010$	1.04	1.02	.95	.98
$\delta = .015$	1.04	1.03	.90	.97
$\delta = .020$	.98	.96	1.10	1.04
$\delta = .025$	1.04	.87	1.30	1.15
$\delta = .030$	1.14	.55	2.00	1.82
$\delta = .035$	1.71	.69	1.50	1.45
$\delta = .040$	2.60	.57	1.30	1.75
$\delta = .045$	2.97	.62	1.10	1.61

Table 9: Calibrations of the model including jump dynamics

explains the data also for small values of the impatience rate δ, (as well as for large ones). When the impatience rate becomes small enough, the continuous model does not fit the data well in the present version. With jumps included, and the risk aversion on jump size risk γ_0 added as a new and independent parameter which is allowed to differ from γ, we obtain more plausible results, although values of the risk aversion around one may be considered a bit low. It can be noticed that log utility in the conventional model, the so-called Kelly-criterion, is known to have certain advantages as an objective in the long run (see e.g., Breiman (1960)).

CAPM++: $\rho = 0$.

When the reciprocal of the EIS parameter $\rho = 0$ no approximations are involved. This model corresponds to the dynamic version of the classical one-period CAPM, which we denote by CAPM++. Some results are presented in Table 10 below. For the upper six rows in the table the assumptions are as in the last section. For the last three rows we have set the return rate to minus one percent annually for the jump part. This is to check the casual observation that jumps in the stock market often seem associated with slumps, or even market crashes.

CAPM++	γ	ρ	γ_0
Recursive model with $\rho = 0$ including jump dynamics			
$\delta = .038$	2.38	.00	2.38
$\delta = .038$	1.25	.00	3.00
$\delta = .038$	2.42	.00	2.00
$\delta = .038$	4.18	.00	.50
$\delta = .038$	3.59	.00	1.00
$\delta = .038$	3.01	.00	1.50
$\delta = .038$	4.77	.00	.00
$\delta = .038$	5.27	.00	-.50
$\delta = .038$	5.76	.00	-1.00

Table 10: Calibrations of the model including jump dynamics

The parameters γ and γ_0 are seen to supplement each other; when one is large the other is small, and vice versa, and both the impatience rate and the risk aversions calibrate to plausible values.

For the last three rows of the table jumps are associated with negative shocks in the stock market, so we may check if *loss aversion* is supported by the model by choosing a low, or even a negative value of γ_0 (since loss aversion is associated with risk proclivity for losses). The ordinary risk aversion γ is then a between four and six, and the impatience rate is still reasonable at 3.8 per cent. This indicates a utility based connection to loss aversion (see e.g., Kahneman and Tversky (1979)). However, since this brings us outside the the range where the first order conditions yields a maximum, this connection is not well-founded. We do not advocate models based on non-rational choice.

An empirical example: US-data.

In this section we test the empirical relevance of the above theory when the market portfolio is not assumed to be a proxy for the wealth portfolio. This we do by calibrating our resulting model to US-market data that can be found in Appendix 3. There we present the key summary statistics of the real annual return data related to the S&P-500, denoted by M, as well as for the annualized consumption data, denoted c, the government bills, denoted b and wealth denoted W.

We have data related to the wealth portfolio from 1960 to 2015 (see Asghar and Mortensen (2017)). The wealth portfolio include capital that is measurable in units of account: (i) human capital; (ii) real capital; (iii) financial capital ; (iv) natural resources. For the whole period about 80 per cent of the national wealth can be attributed to human capital.

Below we take the pure jump model to represent the data, with frequency one per year on the average, as explained in Section 2.1. The advantage of this approach is that we do not need to separate jumps from the continuous part of the data paths. This way we may study the deviations from the local mean square analysis in isolation, and take advantage of the joint probability distribution of jump sizes. The model of Section 8.3 then takes the form

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma_0)}{1-\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{z_c}-1)(e^{z_R}-1)\,\lambda\,dG^{(1)}(z_c, z_R) + \frac{\gamma_0-\rho}{1-\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{z_W}-1)(e^{z_R}-1)\,\lambda\,dG^{(2)}(z_W, z_R) \quad (88)$$

and

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\frac{\rho(1-\rho\gamma_0)}{1-\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{z_c}-1)^2\,\lambda\,dG^{(1)}(z_c, z_W) + \frac{1}{2}\frac{\rho-\gamma_0}{1-\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{z_W}-1)^2\,\lambda\,dG^{(2)}(z_W, z_W) \quad (89)$$

where W signify the wealth portfolio, and R will be used for the market portfolio M. Below we set $\mu_W(t) = .01$, $\sigma_W(t) = .03$, and $\kappa_{W,M} = .80$. We assume a joint lognormal distribution F for the the various variables c, W and M, in which case the two marginal distributions $G^{(1)}$ and $G^{(2)}$ are both joint normal distribution functions. The rest of the data is provided in Appendix 1.

The results of the calibrations are given in Table 7. As can be seen, they correspond to plausible values of the various parameters.

γ_0	ρ	EIS	δ	β
1.50	.994	1.01	0.01	.994
2.00	.988	1.01	0.01	.994
2.50	.982	1.02	0.01	.994
3.00	.976	1.02	0.01	.994
3.50	.970	1.03	0.01	.995
4.00	.964	1.04	0.01	.995
4.50	.958	1.04	0.01	.955
5.00	.952	1.05	0.01	.995

Table 11: Calibrations of the pure jump, recursive model to the US-economy.

The result is a fairly patient agent, and plausible values for the other parameters.

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