Optimal Insurance Policies and Saving in a Temporal World

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Optimal Insurance Policies and Saving in a Temporal World.

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Abstract

We consider Pareto optimal risk sharing between a buyer and a seller of insurance contracts, as well as consumption substitution and saving in a two-period context. The separation of the time periods allows us to consider the substitution effect. We show that the classical result of Pareto optimal risk sharing between a customer and an insurer is robust, and remains so also with recursive utility. For both expected utility and recursive utility we obtain precautionary savings with prudence. With recursive utility we identify the connection between the coefficient of elasticity of substitution in consumption and optimal saving, both under certainty and uncertainty. The separation of consumption substitution from risk aversion is shown to be partial.

KEYWORDS: Pareto optimal risk sharing, two-period models, recursive utility, consumption substitution, separation, precautionary savings.
JEL-Code: G00, G22.

1 Introduction.

Much of the literature about Pareto optimal risk sharing between a customer and an insurer is set in a one period, time-less framework, where consumption takes place only at the end of the period.

In this article we allow for consumption in more than one period which means that we have a temporal setting. Here we can, for example, study consumption substitution between the present and the future, which we do and compare to optimal saving. We also consider Pareto optimal insurance contracts between the two parties in a two-period framework.

Our results show that the classical time-less results are robust regarding optimal risk sharing. For both expected additive and separable utility and recursive utility the Pareto optimal indemnity function satisfies an ordinary differential equation with no fundamental changes from the time-less world.

Optimal saving is different between expected utility and recursive utility. The result for the standard model is well-known and briefly reviewed. In this model the risk aversion and consumption substitution is governed by the same utility function, or perhaps by different utility functions which both measures risk aversion only. In either case this does lead to reasonable results as far as consumption substitution is concerned.

For recursive utility this is different. Consumption substitution is now governed by a parameter linked to the elasticity of intertemporal substitution in consumption (EIS). We show that when the agent's propensity to substitute consumption across time increases, as measured by the EIS-parameter, then the agent's saving increases, both under certainty and under uncertainty, and vice versa when EIS decreases.

We obtain precautionary saving for recursive when the function u, linked to risk aversion, has an associated absolute risk aversion function which is decreasing in wealth. This is an extension of the corresponding result for expected additive and separable utility to recursive utility.

The same assumption on u turns out to be sufficient for the substitution result. This shows why the separation between risk aversion and substitution substitution is only partial for recursive utility. We find the sharp connection between the impatience rate and the EIS for these results to be valid.

The premium of insurance contracts is calculated throughout using the principle of a market price by use of the state price deflator. In the insurance literature, both the actuarial and in insurance economics, this is rarely done.

The paper is organized as follows: In Section 2 we review the classical result of Pareto optimal risk sharing between an insurance customer and an insurer for the one-period (time-less) model. In Section 3 we treat two-period models, where we start with the standard one where the participants have additive and separable expected utility (EU) functions. In Section 4 we move to the situation where the participants have recursive utility (RU). In Section

5 we analyze optimal saving with recursive utility, and Section 6 concludes.

2 Pareto optimal risk sharing in the time-less world.

We consider the following situation. An individual with utility function $u(\cdot)$ and initial reserve w is faced with a loss $X \geq 0$, a random variable. The person can obtain insurance coverage $I(x)$ if the loss $X = x$ from an insurer, where the indemnity function I satisfies $0 \leq I(x) \leq x$ for any loss x. The insurer offers insurance coverage and charges a premium p of the indemnity I. She has utility function $u_F(\cdot)$ and risk-free reserves w_F, where $u_F' > 0$ and $u_F'' \leq 0$. In this situation we want to find the Pareto optimal risk exchange.

In doing so, the derivation is based upon the general risk sharing theory, where the premium is considered as an exogenous parameter p. The We denote the absolute risk aversion functions of the customer and the insurer by $A(w - x + I(x) - p)$ and $A_F(w_F - I(x) + p)$ respectively. The Pareto optimal indemnity function can be found as follows:

Proposition 1. *The Pareto optimal, real indemnity function I:* $\mathbb{R}_+ \to \mathbb{R}_+$, *satisfies the following ordinary differential equation*

$$\begin{cases} \frac{dI(x)}{dx} = \frac{A(w-p-x+I(x))}{A(w-p-x+I(x))+A_F(w_F+p-I(x))} \\ I(0) = 0, \end{cases} \tag{2.1}$$

where the functions $A = -\frac{u''}{u'}$, *and* $A_F = -\frac{u_F''}{u_F'}$ *are the absolute risk aversion functions of the insured and the insurer, respectively.*

Proof. The simplest way to demonstrate this result, is to start with Borch's Theorem: The first order conditions for Pareto optimum is given by

$$u'(w - x + I(x) - p) = \mu u_F'(w_F - I(x) + p), \tag{2.2}$$

where the positive constant $\mu = \lambda_2/\lambda_1$, the ratio of the agent weights. Notice that this equation does not depend on the probability distribution of the loss X. Differentiating this equation in x, we obtain

$$u''(w - x + I(x) - p)(-1 + I'(x)) = \mu\, u_F''(w_F - I(x) + p)(-I'(x)), \tag{2.3}$$

and dividing this equation by the equation (2.2), the result is

$$A(w - x + I(x) - p)(-1 + I'(x)) = A_F(w_F - I(x) + p)(-I'(x)).$$

Rearranging, we obtain the differential equation equation (2.1). $\square$

Moffet (1979) seems to have been the first to have used Borch's general theory of Pareto optimal risk exchange on the risk sharing problem between a customer and an insurer, see Borch (1960a-b, 1990), Arrow (1974), Aase (2002-04-08).

This differential equation is linear if and only if the two utility functions belong to the HARA-class with equal cautiousness parameter, in which case the continuity of the coefficients is sufficient for existence and uniqueness of the solution to the ODE (2.1). In this situation we know from the general theory (e.g., Wilson (1968)) that the optimal sharing rule is independent of the agent weights (here the positive constant μ).

In the general case the ODE is nonlinear, in which case existence and uniqueness must be investigated separately for each set of utility functions. In general Lipchitz conditions are sufficient. A particular Pareto optimal sharing rule may now depend on the agent weights.

The sharing rule is known to be independent of probability distributions as long as the agents have homogenous probability beliefs.

2.1 Some basic consequences of Proposition 1.

Some conclusions immediately follows from the general result in Proposition 1. If $u_F'' < 0$, we see that $0 < I'(x) < 1$ for all x, and together with the boundary condition $I(0) = 0$, by the mean value theorem we get that

$$0 < I(x) < x, \qquad \text{for all} \qquad x > 0,$$

stating that *full insurance is not Pareto optimal when both parties are strictly risk averse.* We notice that the natural restriction $0 \leq I(x) \leq x$ is not binding at the optimum for any $x > 0$, once the initial condition $I(0) = 0$ is employed.

We also notice that *contracts with a deductible D can not be Pareto optimal*, since such a contract means that $I_D(x) = x - D$ for $x \geq D$, and $I_D(x) = 0$ for $x \leq D$ for $D > 0$ a positive real number. Thus either $I_D' = 1$ or $I_D' = 0$, contradicting $0 < I_D'(x) < 1$ for all x.

Contracts with a deductible are known to be preferred by all risk averse insurance customers to all other contracts with same expected compensation

$E(I_p(X))$, in a framework of pure demand theory (there is no insurer in the model). Recall that Pareto optimal contracts are independent of probability distributions.

Furthermore, contracts with a cap M are not Pareto optimal either. These are defined by $I_M(x) = x$ for $x \in [0, M]$ and $I_M(x) = M$ for $x \geq M$, in which cases the derivatives are either 1 or 0.

Such contracts are known to be preferred to offer by all risk averse insurers to any other contract with the same expected compensation to the in a framework of pure supply of insurance (there is no customer in the model).

When $u''_F(y) = 0$ for all y so that the insurer is risk neutral, we notice that $I(x) = x$ for all $x \geq 0$. *Full insurance is optimal and the risk neutral part, the insurer, assumes all the risk.* Clearly, when A_F is uniformly much smaller than A, this will approximately be true even if $A_F > 0$.

This has no particular implications for the premium p. However, when the insurer is risk-neutral and there is *free competition* between many similar insurers, then the premium is known too be actuarially fair in standard economic theory.

Notice that it would be natural to solve the risk sharing problem under the constraint that $I(x) \in [0, x]$, but this is not what has been done in the above. The problem was solved without imposing this constraint, and it turned out that it was indeed satisfied by the unconstrained, optimal solution once we chose the integration constant of the differential equation (2.1) so that $I(0) = 0$. Thus we were justified in ignoring this constraint.

With insurance costs or other frictions like moral hazard, adverse selection, or non-verifiability, this constraint turns out to be binding at the optimum. With ex-ante costs, for example, this is shown to justify deductibles, but not policies with an upper limit (i.e., $I_B(x) = x$ if $x \leq B$; $I_B(x)$ is an increasing function for $x > B$), as shown by Raviv (1979), see also Blazenko (1985). Deductibles also follow with quasi-fixed costs, i.e., when a cost is incurred each time a claim is made, see Aase (2017). See also Spaeter and Roger (1997) for a general treatment of deductibles, as well as Gollier (1987). Holmstrőm (1979) adds moral hazard, caused by asymmetric information and Rothschild and Stiglitz (1976) adds adverse selection to the list.

When it comes to contracts with an upper cap M and a deductible D, this is not readily analyzed in the present framework, although such contracts emerge quite natural in a reinsurance setting, without really requiring any theory at all. An early reference to deductibles is Schlesinger (1981). With an exogenous upper policy cap M, Cummins and Mahul (2004) shows that a non-trivial deductible $D > 0$ appears. Aase (2017) generalizes this and shows that there is no upper limit B in this situation as well. With non-observability, meaning that a realized claim can be observed by both parties, but can not be proven in court, Doherty et.al. (2005) find that the optimal indemnity function contains an endogenous upper cap M as well as a strictly positive deductible D.

However, as we demonstrate below, nonlinear contracts, in the standard model presented above, can come close to explaining contracts with a cap.

Consider the following example of Proposition 1.

Example 1. Consider the Pareto optimal indemnity when both the insurer and the insured have the same preferences represented by CRRA utility with relative risk aversion $\gamma = 2$. The differential equation is then

$$\begin{cases} \frac{dI(x)}{dx} = \frac{w_F+p}{w_F+w-x} - \frac{I(x)}{w_F+w-x} \\ I(0) = 0. \end{cases} \tag{2.4}$$

In this case we can find the solution to the equation (2.4), and the Pareto optimal indemnity function is given by

$$I(x) = \frac{w_F + p}{w + w_F}\, x.$$

Here the optimal solution is linear in the loss x.

So far this does not depend on the probability distribution of the loss X. Now suppose X has distribution

x	0	1	10
prob.	0.5	0.3	0.2

Here $E(X) = 2.3$. For simplicity suppose the premium p is determined by $p = (1+\alpha)EI(X)$ where the loading $\alpha = 0.51$. In the insurance literature the loading α is usually interpreted as a risk premium (possibly containing costs as well). Suppose the relative risk aversion $\gamma = 2$ for both agents, the sure wealth of the customer is $w = 50$ and the insurer's reserves are $w_F = 65$. If there exists a risk-free asset in the model with simple return r, the premium p then solves the linear equation $p = (1 + \alpha)EI(X)$, which gives the value $p = 2.02$, and the indemnity function $I(x) = 0.57\, x$. □

Here the sharing rule is independent of the agent weights., which is the case when both parties in the exchange have HARA-utility functions with a common cautiousness parameter.

In the above example the price p is exogenous (in principle, but see below). This pricing principle is common in insurance, but lacks theoretical support. In economics prices (premiums) are determined endogenously. In this model with two agents, this can be done in this "mini-market" as follows: Assuming there exists a risk-free asset with simple return r. The premium p is in general calculated as

$$p = \frac{E(\pi_1(X_M)I(X))}{\pi_0}, \tag{2.5}$$

where $\pi_1(\cdot)$ is the state price deflator and X_M is the aggregate endowment of the two agents. The denominator $\pi_0 := (1+r)E(\pi_1(X_M))$, which follows from the market premium principle. If, for example, $I(X) = 1$ with probability 1, then $p = 1/(1+r)$, the price of a zero-cupon bond.[1] The state price deflator is here $\pi_1(X_M) = X_M^{-\gamma}$, the marginal utility of the representative agent at the aggregate endowment X_M, where $X_M = (w + w_F - X)$.[2]

Using this principle, assuming $r = 0$, we obtain $p = 2.02$. From this we have calculated the value of α such that the premium in the example corresponds to this market value.

The approach in Proposition 1 provides the basic principle of generating Pareto optimal sharing rules in the present situation, where the premium is computed after the optimal contract has been agreed upon, consistent with equilibrium, as in equation (2.5).

Let us end with an example where the indemnity function is not linear.

Example 2. We reconsider the situation where the customer has CRRA utility with relative risk aversion $\gamma = 2$, the insurer has logarithmic utility with relative risk aversion 1. The sure wealths $w_F = 65$ and $w = 50$ as before. The differential equation for the Pareto optimal indemnity function is given by

$$\begin{cases} \frac{dI(x)}{dx} = \frac{\gamma(w_F - I(x)+p)}{\gamma(w_F - I(x)+p)+1(w-p-x+I(x))} \\ I(0) = 0, \end{cases} \tag{2.6}$$

As can be seen, the optimal indemnity function depends on the premium. Let us set $p = 1.70$, reflecting a lower risk premium than in Example 1. In

[1] The market price price of a zero-cupon bond is $E(\pi_1)/\pi_0$.

[2] This follows from the sup-convolution theorem applied to risk sharing with expected utility.

this situation we can solve this non-linear differential equation, with these numerical values, and the solution is

$$I(x) = x + \frac{21\sqrt{-26680x + 3301489}}{580} - \frac{38157}{580}.$$

A graph of the solution is shown in Figure 1. The function $I(x)$ (blue) is concave, and approximately linear in the support of the random variable X. It is compared with full insurance (red line). The average slope in the support is about 0.73, to be compared to the 0.57 solution in Example 1. A less risk averse insurer has led to a higher insurance coverage.

We notice that with the values of w, w_F and p given, we obtain $I'(115) = 0$, and $I(x)$ is not valid beyond this point. This has the flavour of a cap at $x = 115$. $\square$

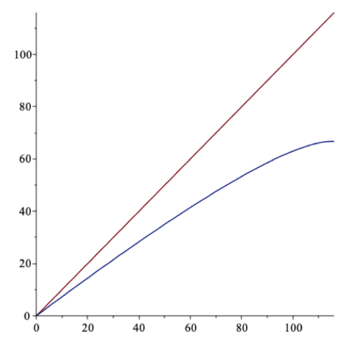

Fig. 1: The indemnity function $I(x)$ in Example 2.

Other examples of non-linear indemnity functions can, for example, be found in Aase (2025a).

2.2 Non-expected utility theory

Non-expected utility theory is a generalization of EU. One aim of this field is to examine important classical results from a more general point of view and determine which of these are robust to departures from the EU hypothesis and which are not.

To illustrate, consider a probability distribution $P = (x_1, p_1; x_2, p_2)$. Non-expected utility theory also follows the approach that individual preferences over such probability distributions can be represented by a preference function $V(P) = V(x_1, p_1; x_2, p_2)$. As with preferences over consumption bundles, the function V can be analyzed graphically by means of indifference curves, or analytically. With expected utility $V(P) = \sum_{i=1}^{2} u(x_i)p_i$.

In for example M. Machina (1995), it is shown that several central results in the economics of insurance are quite robust to dropping the EU hypothesis. In other words, the EU-hypothesis may not crucial for the insurance result under scrutiny, it would be true also with more general forms of preferences over probability distributions.

As an important example of this investigation, consider the results of Pareto optimal risk sharing in an insurance syndicate. These are some of the most powerful and interesting results in all of insurance economics. Not only does this theory allow us to study effectively markets for insurance and reinsurance, they can also be applied to the problem of optimal insurance purchasing in a model of two agents, an insurer and an insurance buyer.

It turns out that, for the one-period model, these results pass the robustness test. Karl Borch originally derived many of these results in a series of papers in the early 1960-ties using expected utility.

So far we have considered a time-less world. Below we move to two-period models. First we consider ordinary EU-theories, and next we move to preferences based on recursive utility. This is an example of a class of models that require at least two time points and is based on non-expected utility theory.

3 Two-period models.

In the time-less model considered above, there is no substitution effect of consumption between now and tomorrow. In this section we open up for this by assuming that the two agents consume at both time zero and time one. We start by using the additive and separable expected utility framework, and then move to recursive utility. The reason is that for the latter the substitution effect and risk aversion can be separated, unlike for expected utility, whee the same parameter explain both effects. These two properties of an individual's preferences are indeed different. We start with expected utility.

3.1 Additive and separable expected utility.

The problem of Pareto optimal risk sharing can be formulated as follows

$$(3.1)\quad \sup_{s,s_F,I}\Big\{E\Big[u(w-s-p)+\beta u((1+r)s-X+I(X)+Z)\Big] \\ +\mu E\Big[u_F(w_F-s_F+p)+\beta_F u_F((1+r)(s_F+p)-I(X)+Z_F)\Big]\Big\}.$$

Here $\mu = \lambda_1/\lambda_2$ is the ratio of the agent weights as before, s and s_F are the savings of the two agents, Z and Z_F represent uncertain incomes at date 1, and β and β_F are the utility discount factors of the two agents, respectively, and r is the interest rate. The uncertain incomes Z and Z_F are not subject to risk sharing in the following analysis, only the loss X is.

The premium paid up front in period 1 is subject to a return in the financial market, an important feature of real life insurance, which we can take into account in the present set up.

The first order conditions for Pareto optimal risk sharing and optimal saving are:

$$(3.2)\quad \beta u'(1+r)s-x+I(x)+z) = \mu\beta_F u_F'((1+r)(s_F+p)-I(x)+z_F), \quad (PO)$$

$$(3.3)\quad u'(w-s-p) = \beta(1+r)Eu'((1+r)s-X+I(X)+Z)$$

and

$$(3.4)\quad u_F'(w_F-s_F+p) = \beta_F(1+r)Eu_F'((1+r)(s_F+p)-I(X)+Z_F).$$

The insurance loss X and indemnity function $I(X)$ both appear only in the second period, which explains equation (3.2). As a consequence the optimal insurance contract does not depend on the agents consumption in period one.

As before, the lower case letters x, z and z_F in (3.2) represent the possible real values of the respective random variables at this stage.

First we focus on finding the Pareto optimal indemnity function I. We then have the following result:

Theorem 1. *The Pareto optimal, real indemnity function I: $\mathbb{R}_+ \to \mathbb{R}_+$, satisfies the following ordinary differential equation*

$$(3.5)\quad \begin{cases} \frac{dI(x)}{dx} = \frac{A((1+r)s-x+I(x)+z)}{A((1+r)s-x+I(x)+z)+A_F((1+r)(s_F+p)-I(x)+z_F)} \\ I(0)=0, \end{cases}$$

where the functions $A = -\frac{u''}{u'}$, *and* $A_F = -\frac{u''_F}{u'_F}$ *are the absolute risk aversion functions of the insured and the insurer, respectively.*

Proof. Differentiation in x the first order condition for Pareto optimal risk sharing in equation (3.2) gives

$$\beta u''((1+r)s - x + I(x) + z)(I'(x) - 1) =$$

$$\mu\beta_F u''_F((1+r)(s_F + p) - I(x) + z_F)(-I'(x)).$$

Dividing this equation by the equation (3.2) results in

$$A((1+r)s - x + I(x) + z)(I'(x) - 1) = A_F((1+r)(s_F + p) - I(x) + z_F)(-I'(x)).$$

Rearranging, this can be expressed as the differential equation for the indemnity function $I(x)$ given in the theorem. □

As noticed above, the optimal indemnity function depends on the agents consumption in the second period, but not in the first.

The basic structure of the former solution in equation (2.1) is maintained, except that only the last period parameters of the two parties now enter the differential equation for $I(x)$; the term $(1+r)s^*$ replaces the former time zero term $(w - p)$, and $(1+r)(s_F + p)$ replaces $(w_F + p)$ for the insurer.

Some simplifications can be considered: In the case where we only want to focus on the uncertainty represented by the loss X, we can disregard the two random variables Z and Z_F.

With this simplification, for CRRA utility the premium p can be written

$$p = \frac{E\{\big((1+r)(s + s_F + p) - X\big)^{-\gamma} I(X)\}}{(1+r)E\{\big((1+r)(s + s_F + p) - X\big)^{-\gamma}\}}, \tag{3.6}$$

where the optimal indemnity function is $I(X) = \frac{s_F + p}{s_F + s + p}X$, a "proportional" contract. When the customer's saving $s = 0$, full insurance is Pareto optimal.

Again the state price π_1 is the marginal utility of the representative agent at the aggregate endowment.

This is a consumption substitution effect, where $s = s^*$ and $s^F = s_F^*$ are the optimal savings, to which we not turn.

3.2 Optimal saving with additive and separable expected utility.

We focus attention on the customer's problem, where the optimal saving s^* satisfies the first order condition in (3.3):

$$u'(w - s^* - p) = \beta(1 + r)Eu'((1 + r)s - X + I(X) + Z).$$

The uncertainty affecting future consumption introduces a new motive for saving compared to a deterministic setting. The intuition is that it induces consumers to increase their wealth accumulation in order to prepare themselves to face future risk. This is the precautionary motive for saving, and it relies on the concept of *prudence*, to be defined shortly. The result can be derived by comparing s^* to optimal saving s^0 when uncertain future income $I(X) - X + Z$ is replaced by its expectation $E(I(X) - X + Z)$. The result is that $s^* \geq s^0$ whenever the function u' is convex, which is referred to as prudence. A sufficient condition is that the function u''' is positive. If this seems a lot to assume, it is equivalent that the absolute risk aversion function is decreasing in wealth. This is a common assumption, thought to reflect reasonably well how people behave faced with future uncertainty. This result is well-known, see e.g., Eeckhoudt et.al (2005), Ch 6, with references therein.

An analogous result is obtained for the insurer.

The above seems intuitive and reasonable, but since consumption substitution is involved, it is seems like a reasonable task to analyse these questions using recursive utility, which is what we do next.

4 Recursive utility.

The axiomatic basis of recursive utility (RU) dates back to Kreps and Porteus (1988). Consequences for consumption and asset returns is treated in Epstein and Zin (1989-91) as well as in Duffie and Epstein (19921a-b).

Recursive utility has several advantages in dynamic economics. It has the potential to explain known empirical and theoretical puzzles like the equity premium puzzle and the risk-free rate puzzle and other anomalies, see e.g., Aase (2016a-b), (2021). Its main advantage is the separation between the substitution effect and risk aversion, which opens up for the solution of many central problems in theoretical and empirical research.

This preference relation can at first sight appear cumbersome to work with, compared to the standard additive and separable expected utility (EU), but once the end results are derived, they provide interesting, additional insights and are often simple to interpret. RU seems to be the simplest non-trivial extension of EU. We start with a definition (see e.g., Skiadas (2009)):

Definition 1. *Recursive utility $U(c)$ is a dynamic utility for which there exists an aggregator f and a conditional certainty equivalent m such that for every consumption process C the utility process $U(c)$ is computed by backward recursion*

$$U_t(c) = f(c_t, m_t(U_{t+1}(c))), \quad t = 0, \ldots, T, \quad U_T(c_T) = c_T.$$

We consider Kreps-Porteus utility, where the conditional certainty equivalent $m_t = u^{-1}(E_t(u(\cdot)))$ for some increasing and continuous real function $u(\cdot)$.

We consider the following form of the aggregator f:

$$U_t(c) = v^{-1}(v(c_t) + \beta v(m_t)),$$

where the $v(\cdot)$ is a real, increasing and continuous function, and $\beta \in [0, 1]$ has the same interpretation as above. Notice that when $u = v$ we are back in the additive and separable expected utility framework (an ordinary equivalent).

This means that the insurance customer has utility function

$$U(s, I) = v^{-1}(v(w - s - p) + \beta v((1 + r)s + m))$$

where $(1 + r)s + m = u^{-1}(E(u((1 + r)s - X + I(X) + Z)))$, and the insurer has utility function

$$U_F(s_F, I) = v_F^{-1}(v_F(w_F - s_F + p) + \beta_F v_F((1 + r)(s_F + p) + m_F))$$

where $(1+r)(s_F+p)+m_F = u_F^{-1}(E(u((1+r)(s_F+p)-I(X)+Z_F)))$. Thus we have different recursive utility functions for the two agents.

4.1 Pareto optimal risk sharing with recursive utility.

In order to find the first order condition for Pareto optimal risk sharing, we first need to find the gradient of recursive utility. In Appendix 1 the general result is provided. For our two period problem, we have from Appendix 1 that the state price deflator for the insurance customer is

$$\pi_1 = f_c(c_0, m_1)\frac{f_m(c_0, m_1)}{u'(m_1)}u'(c_1),$$

with an analogue expression π_1^F for the insurer. Note that $U_1(c) = c_1$ for the agent at the end of the period for recursive utility and similarly for the insurer. The first order condition for Pareto optimal risk sharing with recursive utility is.

$$\pi_1 = \mu\pi_1^F, \ \ a.s. \text{ for } \mu > 0.$$

The aggregators of the present problem are as follows:

$$f(c_0, m_1) = v^{-1}(v(c_0) + \beta v(m_1))$$

where $m_1 = (1 + r)s + m$, and similarly for the insurer, $m_{F,1} = (1 + r)(s_F + p) + m_F$, the first order condition for Pareto optimal insurance contract I can be written

(4.1)

$$v_{c_0}^{-1}(v(w-s-p)+\beta v(m_1))v'(w-s-p)\frac{v_{m_1}^{-1}(v(w-s-p)+\beta v(m_1))\beta v'(m_1)}{u'(m_1)}\cdot$$
$$u'((1+r)s-x+I(x)+z) = \mu\, v_{F,c_0^F}^{-1}(v_F(w_F-s_F+p)+\beta_F v_F(m_{F,1}))v_F'(w_F-s_F+p)\cdot$$
$$\frac{v_{F,m_{F,1}}^{-1}(v_F(w_F-s_F+p)+\beta_F v_F(m_{F,1}))\beta_F v_F'(m_{F,1})}{u_F'(m_{F,1})}u_F'((1+r)(s_F+p)-I(x)+Z_F).$$

Here $c_0 = w - s - p$ and $c_0^F = w_F - s_F + p$ and subscripts on v^{-1} and v_F^{-1} mean partial derivatives.

From this we have the following result:

Theorem 2. *With recursive utility will the Pareto optimal, real indemnity function I:* $\mathbb{R}_+ \to \mathbb{R}_+$*, satisfiy the following ordinary differential equation*

(4.2)
$$\begin{cases} \frac{dI(x)}{dx} = \frac{A((1+r)s-x+I(x)+z)}{A((1+r)s-x+I(x)+z)+A_F((1+r)(s_F+p)-I(x)+z_F)} \\ I(0) = 0, \end{cases}$$

where the functions $A = -\frac{u''}{u'}$ *and* $A_F = -\frac{u_F''}{u_F'}$.

Proof. Differentiation in x of the first order condition for Pareto optimal risk sharing with recursive utility in equation (4.1) gives

(4.3)

$$v_{c_0}^{-1}(v(w-s-p)+\beta v(m_1))v'(w-s-p)\frac{v_{m_1}^{-1}(v(w-s-p)+\beta v(m_1))\beta v'(m_1)}{u'(m_1)}\cdot$$
$$u''((1+r)s-x+I(x)+z)(-1+I'(x))=$$
$$\mu\, v_{F,c_0^F}^{-1}(v_F(w_F-s_F+p)+\beta_F v_F(m_{F,1}))v_F'(w_F-s_F+p)\cdot$$
$$\frac{v_{F,m_{F,1}}^{-1}(v_F(w_F-s_F+p)+\beta_F v(m_{F,1}))\beta_F v_F'(m_{F,1})}{u_F'(m_{F,1})}\cdot$$
$$u_F''((1+r)(s_F+p)-I(x)+Z_F)(-I'(x)).$$

Dividing equation (4.3) by the equation (4.1) gives the following result

$$A((1+r)s-x+I(x)+z)(I'(x)-1)=A_F((1+r)(s_F+p)-I(x)+z_F)(-I'(x)).$$

Rearranging, this can be expressed as the differential equation for the indemnity function $I(x)$ given in equation (4.2) of the theorem. □

As we see, the end result is analogous (identical) to the one for two period model with additive and separable expected utility in Theorem 1, which in its turn has the same basic structure as the result of Proposition 1, the one-period model, except for the substitution effect.

In many ways this is a reassuring result regarding this basic problem of Pareto optimal risk exchange, and perhaps not so surprising once we recall that recursive utility is about the closest non-trivial extension of additive and separable expected utility (recall the $u=v$ criterion). In both representations the function u is related to risk aversion, while the function v is related to consumption substitution. The theorems state that the Pareto optimal risk sharing result of Proposition 1 is robust.

The premium p is as before given by

$$p=\frac{E(\pi_1 I(X))}{\pi_0}, \qquad (4.4)$$

where π_1 is the state price deflator. Here it is calculated as follows for recursive utility in the case when the function u is CRRA: The relevant part of the state price π_1 is the following: $\pi_1=\text{const}\cdot\big((1+r)s-X+I(X)\big)^{-\gamma}$. At a Pareto optimum this quantity equals $\mu\pi_1^F=\mu\cdot\text{const}\cdot\big((1+r)(s_F+p)-I(X)\big)^{-\gamma}$.

Also here $I(X) = \frac{s_F+p}{s_F+s+p}X$, the same expression as for the two-period EU-example above. Since $I(X) - X = -\frac{s}{s_F+s+p}X$, we notice that the market premium p can be written

$$(4.5)\quad p = \frac{E\{\big((1+r)s - \frac{s}{s_F+s+p}X\big)^{-\gamma} I(X)\}}{(1+r)E\{\big((1+r)s - \frac{s}{s_F+s+p}X\big)^{-\gamma}\}}$$
$$= \frac{E\{\big((1+r)(s+s_F+p) - X\big)^{-\gamma} I(X)\}}{(1+r)E\{\big((1+r)(s+s_F+p) - X\big)^{-\gamma}\}},$$

where the last expression is the same as with expected utility given in equation (3.6).

Alternatively, in (4.5) we could have started with $\mu\pi_1^F$ instead of π_1, and the result is easily shown to be the same as above.

For recursive utility we do not have the convenient result that the state price deflator is the marginal utility of the representative agent at the aggregate endowment, but despite of this we obtain the same market premium as for EU in this example, as we certainly should in this particular situation.

In the above theorem both saving parameters s and s_F are assumed to be optimal. What this implies we consider next. Here it turns out that the difference between these two types of dynamic utilities is more real, and to the advantage of recursive utility.

5 Optimal saving with recursive utility.

We now want to study optimal saving with recursive utility. We consider the customer only, since the analysis for the insurer is analogous. Starting with the utility function of the insurance customer

$$U(s, I) = v^{-1}(v(w - s - p) + \beta v(m_1))$$

we notice that this can be written

$$U(s, I) = v^{-1}(v(w - s - p) + \beta v(m_1)) =$$

$$v^{-1}\Big(v(w - s - p) + \beta v[u^{-1}\{E(u((1+r)s - X + I(X) + Z))\}]\Big) =$$

$$v^{-1}(v(w - s - p) + \beta v((1+r)s + m))$$

by the definition of the certainty equivalent of a constant plus a random variable. Here m depends on s. In order to simplify the notation, let us denote the sum of the random variables by $\tilde{z} = (-X + I(X) + Z)$.

The first order condition in the saving variable s is:

(5.1)
$$\frac{\partial U(s,I)}{\partial s} = v_{c_0}^{-1}(v(c_0) + \beta v(m_1))v'(m_1)(-1) + v_{m_1}^{-1}(v(c_0) + \beta v(m_1))\beta v'(m_1) \cdot \frac{(1+r)E(u'((1+r)s + \tilde{z}))}{u'(u^{-1}(E(u((1+r)s + \tilde{z}))))} = 0.$$

This can be written

$$\frac{\partial U(s,I)}{\partial s} = \frac{v'(m_1)(-1)}{v'(v^{-1}(v(c_0) + \beta v(m_1)))} + \frac{\beta v'(m_1)}{v'(v^{-1}(v(c_0) + \beta v(m_1)))} \cdot \frac{(1+r)E(u'((1+r)s + \tilde{z}))}{u'(u^{-1}(E(u((1+r)s + \tilde{z}))))} = 0. \quad (5.2)$$

Denoting the optimal value of s by s^*, we have the following relationship which s^* must satisfy:

$$v'(w - p - s^*) = \frac{\beta(1+r)v'(m_1)E(u'((1+r)s^* + \tilde{z}))}{u'(u^{-1}(E(u((1+r)s^* + \tilde{z}))))}. \quad (5.3)$$

So far we have that v and u are two real functions, both increasing and concave. In order to be able to discuss the substitution effect due to the function v in a simple way, we now introduce the elasticity of intertemporal substitution in consumption (EIS) via a constant parameter. This we can obtain by assuming that $v(c) = (c^{1-\rho} - 1)/(1-\rho)$ for $\rho \geq 0$ and $v(c) = \ln(c)$ when $\rho = 1$. Then the parameter ρ is the reciprocal of the EIS-parameter, i.e., $\rho = 1/EIS$. One interpretation is that when the parameter ρ increases, the individual becomes less willing to substitute consumption from one period to the next. This suggests that the individual may become less prone to save when ρ increases, a relationship we investigate below.

We also write the parameter $\beta = 1/(1+\delta)$ where the parameter δ is the impatience rate. Usually this parameter is set around 1% in applied work, and typically $\beta(1+r) = \frac{1+r}{1+\delta} > 1$.

Turning to the utility function u, we assume it has the property of decreasing absolute risk aversion. This is considered a reasonable property of an individual. Arrow (1970) for example, argued that wealthier people seem less willing to pay for the elimination of a fixed risk. Under this assumption the function $-u'$ is both increasing and concave, so it has the properties of a utility function. Moreover, an agent with this utility is more risk averse than an agent with utility function u. We then know that the certainty equivalent of the function $-u'$, let us call it m_2, is smaller than the certainty equivalent m_1 of u as defined above.

Using these assumptions, equation (5.3) can be written

$$\frac{1+\delta}{1+r}(w-p-s^*)^{-\rho} = \frac{m_1^{-\rho}}{u'(m_1)}E(u'((1+r)s^*+\tilde{z})) = \frac{m_1^{-\rho}}{u'(m_1)}u'(m_2) \geq m_1^{-\rho},$$

since $m_2 \leq m_1$ and $u'(\cdot)$ is a decreasing function. Accordingly

$$\frac{1+\delta}{1+r}\,\frac{1}{(w-p-s^*)^{\rho}} \geq \frac{1}{((1+r)s^*+m)^{\rho}},$$

which can be written

$$s^* + \frac{k(\rho)m(s^*)}{1+k(\rho)(1+r)} \geq \frac{w-p}{1+k(1+r)}, \tag{5.4}$$

where $k(\rho) = \left(\frac{1+\delta}{1+r}\right)^{\frac{1}{\rho}}$. Since $\frac{1+\delta}{1+r} \leq 1$, the function $k(\rho)$ is increasing in ρ.

On the other hand, if we replace m_1 in the function $m_1^{-\rho}$ by m_2, we obtain the inequality

$$k(\rho)(1+r)s^* + k(\rho)m_2(s^*) \leq l(\rho)((w-p)-s^*), \tag{5.5}$$

where $l(\rho) = \left(\frac{u'(m_2)}{u'(m_1)}\right)^{\frac{1}{\rho}}$. Since $u'(m_2) \geq u'(m_1)$, the function $l(\rho)$ is decreasing in ρ.

With these preparations we can show several results of interest. We start by demonstrating that we have precautionary savings for an agent under the above conditions.

5.1 Precautionary saving with recursive utility.

When we replace the random variable $\tilde{z}$ by its expectation $E(\tilde{z})$, the inequality becomes an equality and the optimal saving in absence of future risk, s^0, is given by the formula

$$s^0 = \frac{w-p-k(\rho)\,E(\tilde{z})}{1+k(\rho)(1+r)}. \tag{5.6}$$

Notice that the inequality (5.4) can be written

$$s^* \geq \frac{w - p - k(\rho)m(s^*)}{1 + k(\rho)(1 + r)}.$$

Since $m(s^*) \leq E(\tilde{z})$ (by Jensen's inequality), we notice that $s^* \geq s^0$, so we have in fact precautionary savings under the above conditions. We have proved the following result:

Theorem 3. *Under the above assumptions where the function u has a decreasing absolute risk aversion, we obtain precautionary savings for the recursive utility maximizer, that is, $s^* \geq s^0$.*

5.2 Consumption substitution: RU only.

We fiinally turn to consumption substitution. Recall that for the additive expected utility model, we had not much to say about this issue. If we make the additional assumption that the utility function u of final consumption is CRRA with relative risk aversion γ, then this single parameter has to play two roles, but it is not possible to reconcile these for one single parameter. It does not help much to use a different parameter for time zero utility, say $\rho < \gamma$. It does not lead to a reasonable result for consumption substitution, and moreover implies that the agent changes preferences with time, which is not a considered a reasonable assumption in economics.

With recursive utility this is different, since the function v is different from u and they have different interpretations. First notice that $k := k(\rho)$ is an increasing function of ρ. We then have the following:

Theorem 4. *The optimal saving under certainty, s^0, decreases as ρ increases, ceteris paribus. Similarly s^0 increases as ρ decreases.*

This supports the interpretation that ρ measures the agent's resistance to consumption substitution, in a world of certainty.

Next we turn to uncertainty. For this we need the following result:

Lemma 1. *Assume that the function u has a decreasing absolute risk aversion function in wealth. Then the certainty equivalent $m(s^*)$ is an increasing function of the optimal saving s^* under uncertainty.*

The prof is given in Appendix 1.

1) From the inequality (5.4), we notice the following. When ρ decreases, so does $k(\rho)$. Then the right-hand side of this inequality increases, and so does the left-hand side. This means that s^* or $m(s^*)$ or both must increase. Since $m(s^*)$ is increasing is s^*, it follows that s^* must increase.

2) From the inequality (5.5) we see the following: When ρ increases, so will $k(\rho)$. Then the left-hand side of the inequality also increases, and so will the right-hand side. Since $l(\rho)$ decreases, it follows that s^* must decrease.

In summary we obtain the substitution effect: as ρ increases, the agent's resistance to consumption substitution between now and the future increases, which materialises as the optimal amount of saving goes down. Similarly, when ρ decreases, the agent's resistance to consumption substitution goes down, which gives that the amount allocated to optimal saving increases.

We have shown the following result.

Theorem 5. *Assume that the function u's associated absolute risk aversion function decreases in wealth, and that the impatience rate $\delta \leq r$. Under these assumptions the substitution effect is measured by the parameter ρ, or equivalently, the reciprocal EIS-parameter. Referring to ρ, as this parameter increases, the optimal saving under uncertainty decreases. As the parameter ρ decreases, the optimal saving under uncertainty increases.*

We have noticed above that the property of substitution depends on the impatience rate. Recursive utility is known for the "separation" between risk aversion and consumption substitution, in that the functions u and v are different. From the above theorem we notice that this separation is only partial, since the substitution result depends on a property of the absolute risk aversion function $A(w) = -\frac{u''(w)}{u'(w)}$. For CRRA this property is satisfied, since here $A(w) = \frac{\gamma}{w}$ which is decreasing in w. Thus the Epstein-Zin parametrization (see Epstein and Zin (1989-91)) satisfies the conditions of Theorem 5 provided $\delta \leq r$.

In a multiperiod setting, however, basic risk sharing results from the one period model do no longer carry over with recursive utility in an exact manner, as in this article. In particular, the mutuality principle is not valid in its original form with recursive utility of the translation invariance type (see Aase (2025b)). However, the "spirit" of the basic original risk sharing results must, by and large, be said to remain.

6 Conclusions.

We have analyzed Pareto optimal risk sharing between a buyer and a seller of insurance contracts, as well as consumption substitution and saving in a two-period context, and compared to the time-less situation. The separation of the time periods allowed us to consider the substitution effect. We demonstrated that the classical result of Pareto optimal risk sharing between a customer and an insurer is robust, and, more importantly, remains so also with recursive utility. For both expected utility and recursive utility we obtained precautionary savings with prudence. With recursive utility we identified the connection between the coefficient of elasticity of substitution in consumption and optimal saving, both under certainty and uncertainty. The separation of consumption substitution from risk aversion was demonstrated to be partial. The basic risk-sharing property holds also in the dynamic setting of a two-period model.

Appendix 1.

The first order condition for Recursive Utility.

Consider the following version of recursive utility:

$$U_t(c) = f(c_t, m_t(U_{t+1}(c))), \quad t = 0, \dots, T, \quad U_T(c_T) = c_T.$$

Using directional derivatives and backward induction, it is shown in Aase (2021) that the utility gradient is given by the following expression

$$\text{(6.1)} \quad \bigtriangledown U(c;x) = \bigtriangledown U_0(c;x) = E\Big\{ \sum_{t=0}^{T} x_t \ f_c(c_t, m_{t+1}) \prod_{s=0}^{t-1} \frac{f_m(c_s, m_{s+1})}{h'(m_{s+1})} h'(U_{s+1}) \Big\}$$

from which it follows that the state price deflator is given as

$$\text{(6.2)} \qquad \pi_t = f_c(c_t, m_{t+1}) \prod_{s=0}^{t-1} \frac{f_m(c_s, m_{s+1})}{h'(m_{s+1})} h'(U_{s+1})$$

for $t = 0, 1, \cdots, T$. $\square$

Proof of Lemma 1 of Section 5.2.
Recall from equation 5.1 that $m_1 = (1+r)s + m$ and that

$$\frac{dm_1(s)}{ds} = \frac{(1+r)E(u'((1+r)s+\tilde{z}))}{u'(u^{-1}(E(u((1+r)s+\tilde{z}))))}.$$

As a consequence,

$$\frac{dm(s)}{ds} = \frac{(1+r)E(u'((1+r)s+\tilde{z}))}{u'(u^{-1}(E(u((1+r)s+\tilde{z}))))} - (1+r).$$

By the definition of m_1 and m_2 we have $E(u'((1+r)s+\tilde{z})) = u'((1+r)s+m_2)$ and $E(u((1+r)s+\tilde{z})) = u((1+r)s+m_1)$. Inserting these in the above equation gives

$$\frac{dm(s)}{ds} = \frac{(1+r)u'((1+r)s+m_2)}{u'((1+r)s+m_1)))} - (1+r).$$

Since the function $u'(\cdot)$ is decreasing and $m_2 \leq m_1$, we see that $\frac{dm(s)}{ds} \geq 0$.
□

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