Pareto Optimal Insurance Policies: Kinks with or without frictions

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Abstract

We analyze optimal risk sharing between a customer and an insurer, and present alternative explanations for the prevalence of kinks in Pareto optimal contracts, like deductibles and upper bounds as in XL-contracts. Linear indemnity functions have primarily been considered in the literature. We focus on nonlinear contracts, which can be explained on the basic of different preferences held by the parties involved. In this setting we derive Pareto optimal contracts with "near" deductibles and "near' caps, which we illustrate by examples. Lastly we consider a model based on non-verifiability where the insurer is risk-neutral. We change to a setting where both the cedent and the reinsurer are strictly risk averse. This rationalizes both an endogenous upper cap and a deductible, retaining compensations for risk bearing.

KEYWORDS: Pareto optimal risk sharing; nonlinear contracts; XL-contracts, non-verifiability.

JEL-Code: G00, G22.

1 Introduction.

It seems broadly accepted that deductible policies give the best tradeoff between risk sharing and economizing on costly claim settlements. The presence of insurance costs are often considered as the "best" and most straightforward explanation of deductibles occurring in insurance contracts. There are other explanations, usually involving models of asymmetric information, like moral hazard (for example Holmstrøm (1979)) or adverse selection (for example Rothschild and Stiglitz 1976). These models are much more complex than simply introducing ex-post costs in the classical model of risk sharing. In these models deductibles appear more or less as a by-product of the analysis. When, for example, moral hazard is present, it is socially optimal that the insured keeps more of the risk than when moral hazard is absent in order to get the incentives right. When the insurer is risk neutral and the classical recipe is that full insurance is Pareto optimal, with moral hazard genuine risk sharing is optimal (second best). When there is adverse selection, the good risks can not be offered full coverage because of the presence of the bad risks. The latter, on the other hand, obtain full insurance (when this is optimal). In both cases the insurance customers will end up taking more risk than in the classical case. Whether this risk-sharing takes the form of a deductible, or some other forms of coinsurance is not a central point.

In this paper we add to the list of situations where "kinks" can occur, without introducing costs or other frictions. These kinks will not be of the types where the indemnity function is not differentiable. We know from the general theory that such points simply does not exist under the standard smoothness assumptions of preferences. By looking at a nonlinear indemnity function, we can identify situations where an insurance customer is willing to take the risk of smaller losses against the possibility of having more major losses covered by the insurer. Although this does not give a "straight" deductible with a flat part followed by a kink, it describes a contract with the same economic features, which we refer to as an "almost", or "near" deductible. Similarly with caps, which occur in the other end of the loss distribution. When the insurer's risk tolerance decreases because of an increasing indemnity function and limited reserves, this indicates an area of losses where no further insurance will be offered, and the economic effect will be similar to a cap. Although this is not a "straight" cap with a kink followed by a flat part, it describes an insurance indemnity function with similar features; if the customer seeks further insurance coverage, this has to be bought from other insurers, or the insurer passes the liability over to a reinsurer.

The framework of Pareto optimal risk sharing between an insurer and an insurance buyer is built on Borch's pioneering theory (Borch (1960a-b, 1990), where Moffet (1979) was the first to treat this problem in its present form. Aase (2002-04-08-17-21-22) developed this theory further along different lines. Deductibles have also been analyzed in the framework of pure demand theory, such as in Arrow (1970-74), Schlesinger (1981), Karni (1983) and Pashigian et.al (1966). Raviv's analysis of Pareto optimal deductibles in the presence of ex-post insurance costs is the classical one (Raviv (1979)), and is the first analysis connecting deductibles directly to these costs. See also Blazenko (1985). For example are some of the results of Arrow clarified through the analysis of Raviv. Pareto optimal contracts contain deductibles when ex-post costs are continuous and variable, however it induces no upper limit contracts or contracts with a cap.

Spaeter and Roger (1997) give a unifying treatment by moving the problems to a more general setting, using Gateau derivatives and topology. Gollier (1978) treat various aspects of cost accounting, where he obtains what is termed partially disappearing deductible. His analysis also agrees with one of Raviv's conclusions: There is no deductible when costs are constant.

In Aase (2017) it is shown that when there are fixed costs triggered whenever a claim is made, deductibles appear in the Pareto optimal policies even if there are no variable costs. With this paper the case of quasi-fixed costs can be added to the list of hypotheses that conduct to deductibles.

Excess of loss contracts (XL-contracts) are common in insurance and reinsurance practice. It is not hard to explain why: Imagine a reinsurance syndicate, where a ceding company typically takes part of a risk (the "slip") on own account up to a certain level, then cedes the remaining risk to a reinsurer, except that there is some upper bound, F, beyond which this reinsurer is not responsible. Seen from the reinsurer's point of view, this is an XL-contract.

By this arrangement the reinsurer only keeps a certain 'layer" of the risk. This process can be extended to several reinsurers, each responsible for a certain layer, until the slip is fully signed. A 100 million dollar risk could, for example, be divided in 10 equal parts among 10 reinsurers including the cedent, responsible for 10 millions each. The premiums will then reflect the probabilities of losses falling in the various layers, normally decreasing towards the tail of the loss distribution. That schemes like this make common sense, goes without saying. Our task is to identify conditions under which this is indeed optimal in a theory involving preferences of the parties in the exchange, within the economics framework of Pareto optimal risk sharing.

Perhaps surprisingly, this simple arrangement turns out to be hard to explain within the standard framework of contract theory. In this framework preferences of the participants play an essential role, where the standard assumption imply that these are smooth. As a result, any "kink" in the contract is difficult to explain without further restrictions.

One approach is to consider risk sharing between the insurer and the insurance buyer, where the optimal contracts are nonlinear. This happens in the standard model when the utility functions are outside the HARA-class, or when the two utility functions are of HARA-class but with unequal cautiousness (slope).[1] In Section 5 of the paper we present examples of this kind, where the Pareto optimal contracts come close to contracts with deductibles, or even to XL-contracts.

Another approach to explain XL-contracts is to introduce imperfect or asymmetric information. This is what Doeherty et.al. (2005) do. They assume that the damage can be observed by both parties, but not be proven in court. This situation calls for the use of game theory.

In the last section of the paper we consider an extension of this model, where we deviate in that we consider the situation with risk averse insurers instead of risk neutral ones. Doing this we can justify a strictly positive expected profit in each period, a premium larger than the actuarially fair one, which is instrumental in this approach. This modification implies that the Pareto optimal indemnity function between the kinks satisfies a differential equation, and the resulting contract deviates from full insurance.

In order to illustrate various uses of optimal risk sharing, some of which have reached textbooks by now, in Section 2 we present some examples, where we point out which contract designs are consistent with Pareto optimal risk sharing, and which does not satisfy these requirements. Here we present examples of "pure demand" theory, where we show that optimal contracts in such a setting are not Pareto optimal in the standard interpretation of this concept, although they have been presented as such in parts of the literature and in textbooks.

The paper is organized as follows. In Section 2 we discuss some aspects of pure demand theory. Section 3 treats various constraints that could (and should) have been used in the previous section. Section 4 looks at Pareto optimal risk sharing in the time-less model. Section 5 treats non-linear con-

[1] In the context of a mutual insurance company, concave utility functions are non-problematic. For corporations, this may be used to model the rationale for corporate risk management, see for example Froot et.al. (1993).

tracts by several examples, aiming to explain contracts of the excess of loss type by looking at different preferences by the parties involved. Section 6 explains XL-reinsurance contracts by asymmetric information. Section 7 is a brief discussion of the robustness of the EU-assumption in setting of the paper, while Section 8 concludes.

2 The theory of "pure demand".

By pure demand theory we mean a situation where only the insurance customer is actually in the model, whereas the supply side is not explicitly modelled. A situation like this is treated in Arrow (1974), and in this section we present another example discussed by Mossin (1968), which is well-known text-book material in insurance economics. We briefly discuss various aspects of this model, and demonstrate that the contract is not Pareto optimal according to the standard definition.

The situation is as follows: An individual with utility function $u(\cdot)$ and initial reserve is faced with a loss $X \geq 0$, a random variable. The person can obtain insurance coverage $I(x)$ if the loss $X = x$, where the indemnity function I satisfies $0 \leq I(x) \leq x$ for any loss x. The premium p is assumed to be on the form $p = (1 + \alpha)E(I(X)$ where $\alpha \geq 0$ is a loading parameter. It can measure deadweight transaction costs, but a risk premium can also in principal be contained in this parameter. Mossin assumed that the indemnity function $I(\cdot)$ is linear, i.e., $I(x) = \beta x$ for all x, for some parameter $\beta \in [0, 1]$. This restriction is of course a limitation, but is not our only concern.[2] Assuming preferences are represented by expected utility, the problem in this example is to solve

$$\max_{\beta \in [0,1]} Eu(W_\beta),$$

where $u(W_\beta) = u(w - X + \beta X - p_\beta)$ and $p_\beta = \beta p_0$ for $p_0 = (1 + \alpha)E(X)$. Here w is a positive constant signifying the customers risk-free wealth.

Let us define the real function $f(\beta) := Eu(w - X + \beta X - p_\beta)$, where it is assumed that $u' > 0$ and $u'' < 0$. Because of strict monotonicity and strict risk aversion the first order condition is both necessary and sufficient for optimality. The derivative for f with respect to β is

$$f'(\beta) = E\big(u'(W_\beta)(X - p_0)\big) = 0.$$

[2] If both the customer and the insurer has utility functions of the HARA class with equal cautiousness parameter, then Pareto optimal sharing rules are known to be affine.

We notice that $f'(1) = u'(w - p_0)(-\alpha E(X)) < 0$, while the sign of $f'(0) = E\big(u'(w-X)(X-p_0)\big)$ is not directly determined. This gives us two cases to consider:

(1) Suppose $f'(0) > 0$. Because the function f is smooth, it will then have an interior maximum at say β^*, and it is easy to see that this is a unique optimum.

If the loading $\alpha = 0$, the optimal $\beta^* = 1$, which is full insurance (the second order condition holds since $u'' < 0$). With zero loading, the premium is actuarially fair. This can be interpreted as a situation with a a risk neutral insurer offering insurance at "fair" terms, and consequently making no profit in the direct underwriting business.

(2) Suppose $f'(0) \leq 0$. This means that no insurance is optimal, $\beta^* = 0$.

For a slight change of scenery, suppose we change to the premium functional to $p = E\big(I(X)\big) + c$ where $c \geq 0$ is a constant (an administrative cost or a risk premium or both). In order to find the optimal insurance contract in this situation, one has to solve the problem

$$\max_{0 \leq I(x) \leq x} Eu(w - X + I(X) - p),$$

where we do not constrain I to be linear.

By use of Jensen's inequality we have the following:

$$\begin{aligned} Eu(w - X + I(X) - E(I(x) - c) &\leq u(w - EX + EI(X) - EI(X) - c) \\ &= u(w - EX - c). \end{aligned}$$

Here the right-hand side does not depend upon the indemnity function I and is thus an upper bound for all I. It is easy to see that the contract $I(x) = x$ for all $x \geq 0$ attains this upper bound, and is thus optimal.

We can conclude that unless the constant c is so large that the customer does not want any insurance at all, full insurance is optimal with this type of premium. Despite some shortcomings for this premium principle (the premium functional is not linear), with a positive value of c it is certainly more realistic than the loading principle with $\alpha = 0$. One would think that an insurer can not maintain his business for very long with the latter premium principle, unless it happens to be the return on the premium reserve, or another source, that gives the company its main income.

This example illustrates that "optimal demand theory" in insurance is not necessarily all that clear cut, it clearly depends on the form of the premium functional as well as other features. But more importantly it is not rigged for the important principle of Pareto optimality.

3 Constrained optimal risk sharing.

Let us return to Mossin's problem and introduce an insurer in this model with utility function $u_F(\cdot)$ and risk-free reserves w_F, where $u_F' > 0$ and $u_F'' \leq 0$. In order to discuss Pareto optimality, all the agents must be explicitly present.

We first let the insurer be strictly risk averse, and consider the determination of a Pareto optimal contract, constrained to be linear. Such contracts are solutions to the following problem

$$\max_I Eu(w - X + I(X) - p) \quad \text{subject to} \quad E(u_F(w_F - I(X) + p) \geq k$$

for some real constant k, where it is assumed that $I(x) = \beta x$ for some constant $\beta \in (0,1)$. When the constant k varies in a certain range, this spans out a constrained Pareto optimal frontier.

If we retain the form of the premium functional $p = (1+\alpha)E\big(I(X)\big)$ for some loading $\alpha > 0$, the Lagrangian of the problem can be written

$$\mathcal{L}(\beta) = E\big(u(w - X + \beta X - \beta p_0)\big) - \mu\Big(k - E\big(u_F(w_F - \beta X + \beta p_0)\big)\Big),$$

where the constant $\mu > 0$ is the Lagrangian multiplier. The first order condition is given by the equation

$$(3.1)\quad E\big(u'(w - X + \beta X - \beta p_0)(X - p_0)\big) = \mu E\big(u_F'(w_F - \beta X + \beta p_0)(X - p_0)\big).$$

For this solution to be the same as for the pure demand problem of the previous section, the right-hand-side must be zero for $\beta = \beta^*$, where β^* is the same as found earlier. But this does not happen.

Consider for example the case when the insurer is risk neutral. In this situation $u_F(x) = x$ for all $x \geq 0$, and the first order condition reduces to

$$(3.2)\quad E\big(u'(w - X + \beta X - \beta p_0)(X - p_0)\big) = \mu E(X - p_0) = -\mu\alpha EX < 0,$$

provided $\alpha > 0$. Under the above assumptions this implies that the constrained Pareto optimal value $\beta > \beta^*$, which shows that our previous optimum is not Pareto optimal in this meaning, even with a risk-neutral insurer.

This result is a consequence of two issues: (i) the restriction to linear contracts, and (ii) the form of the premium functional.

When the loading $\alpha = 0$, we notice that Mossin's contract is Pareto optimal in the above definition, and only then.

The model is perhaps best illustrated by an example.

Example 1. Consider a loss X with probability distribution

x	0	1	10
prob.	0.5	0.3	0.2

Here $E(X) = 2.3$, and let the loading $\alpha = 0.1$, which means that $p_0 = 2.53$. Suppose the utility function of the insurance customer is CRRA with relative risk aversion $\gamma = 2$ and sure wealth $w = 50$. The insurer is risk neutral with reserves $w_F = 65$, and let us first consider the sharing rule corresponding to $\mu = 1$.

In the pure demand theory the solution for the customer is $\beta^* = 0.66$. Including the insurer in the model, the Pareto optimal solution is full insurance, i.e., $\beta = 1$. (If the solution for β is larger than one, this means full insurance is optimal. If $\mu = 0.00045$ we obtain $\beta = 1$.)

Suppose that the insurer is risk averse, but with a lower relative risk aversion than the insurer. Here let $u_F(x) = \ln(x)$, so the insurer's relative risk aversion is 1. The constrained Pareto optimal solution is then $\beta = 0.90 > \beta^*$ when $\mu = 1$.

If the insurer has CRRA utility with the same relative risk aversion as the customer, the constrained Pareto optimal solution is on the other hand $\beta = 0.60 < \beta^*$ when $\mu = 1$. □

In the general case where the insurer is strictly risk averse, the optimal contract will depend on the the parameters of the problem, in particular the relationship between the two risk aversions as well as the reserves w_F and the insurance customer's expected wealth $w - EX$. If the risk aversion of the insurer is lower than the one of the customer, the (constrained) Pareto optimal value of β is larger than β^*. When the risk aversions and the wealths are comparable, we have seen that β can be strictly smaller than the pure demand solution β^*. In general is the constrained PO solution different from β^*, and will only agree for rather special choices of values of the parameters.

Finally notice (i); the optimal value of β depends on the probability distribution of the loss X, and (ii); even when the two agents have the same utility functions, the optimal solution depends on the Lagrange multiplier μ.

As it turns out, neither (i), nor (ii) will hold for a general unconstrained Pareto optimal insurance contract, as we demonstrate next.

Also, premiums should in general be calculated according to the economic market value principle, which is not the case for the above examples.

4 Pareto optimal risk sharing.

In this section we turn to Pareto optimal risk sharing. The derivation is based upon the general economic risk sharing theory, where the premium p must be determined endogenously. This is different from the above approach, where the premium is an exogenous parameter.

In general is the premium, determined in the market, is a linear functional of the following form

$$p = \frac{1}{\pi_0} E(\pi_1 I(X)) = \frac{1}{\pi_0} E(I(X))E(\pi_1) + \frac{1}{\pi_0} \text{cov}(I(X), \pi_1). \tag{4.1}$$

where π_1 is the state price deflator, or the Arrow-Debreu state price in units of probability. In an equilibrium setting it can be expressed as $\pi_1 = u'_\lambda(X_M)$, where u'_λ is the marginal utility of the representative agent (if this were to exist) and X_M is the aggregate endowment in the market. Supposing there exists a risk-free asset in the economy with simple return r, then $\pi_0 = (1+r)E(\pi_1)$, in which case the first term on the right-hand side in the above equation can be written $E(I(X))/(1-r)$, the expected present value of the insurance indemnity, while the last term can be considered as a risk-premium, a compensation for risk-bearing.

The output of a linear functional is a real number, so $p \in \mathbb{R}_+$. Since we do not want to specify a complete market model at this stage, a natural assumption is to let the premium be represented by the positive real number p. Once a Pareto optimal risk exchange has been established, the premium must then be determined in the market, and based upon this, the parties will decide wether or not to enter the contract.

Denoting the absolute risk aversion functions of the customer and the insurer by $A(w - x + I(x) - p)$ and $A_F(w_F - I(x) + p)$ respectively, we have the following result:

Theorem 1. *The Pareto optimal, real indemnity function* I*:* $\mathbb{R}_+ \to \mathbb{R}_+$*,* *satisfies the ordinary differential equation (ODE)*

$$\begin{cases} \frac{dI(x)}{dx} = \frac{A(w-p-x+I(x))}{A(w-p-x+I(x))+A_F(w_F+p-I(x))} \\ I(0) = 0, \end{cases} \tag{4.2}$$

where the functions $A = -\frac{u''}{u'}$, *and* $A_F = -\frac{u''_F}{u'_F}$ *are the absolute risk aversion functions of the insured and the insurer, respectively.*

Proof. The simplest way to demonstrate this result, is to start with Borch's Theorem: The first order conditions for Pareto optimum is given by

$$u'(w - x + I(x) - p) = \mu u'_F(w_F - I(x) + p), \tag{4.3}$$

where the positive constant $\mu = \lambda_2/\lambda_1$, the ratio of the agent weights. Notice that this equation does not depend on the probability distribution of the loss X. Differentiating this equation in x, we obtain

$$u''(w - x + I(x) - p)(-1 + I'(x)) = \mu\, u''_F(w_F - I(x) + p)(-I'(x)), \tag{4.4}$$

and dividing this equation by the equation (4.3), the result is

$$A(w - x + I(x) - p)(-1 + I'(x)) = A_F(w_F - I(x) + p)(-I'(x)).$$

Rearranging, we obtain equation (4.2) $\square$

Moffet (1979) seems to have been the first to have used Borch's general theory of Pareto optimal risk sharing in a syndicate of N members on the risk sharing problem between a customer and an insurer ($N = 2$).

This differential equation is linear if and only if the two utility functions belong to the HARA-class with equal cautiousness parameter, in which case the continuity of the coefficients is sufficient for existence and uniqueness of the solution to the ODE (4.2). In this situation we know from the general theory (e.g., Wilson (1968)) that the evaluation measure of the insurance market is independent of the agent weights (here the positive constant μ).

In the general case the ODE is nonlinear, in which case existence and uniqueness must be investigated separately for each set of utility functions. In general Lipchitz conditions are sufficient. In this case the evaluation measure will in general depend on the on the agent weights. However, here we have the special situation the Pareto optimal sharing rule does not depend on the agent weights, since the constant μ disappeared form the above differential equation. Also, the sharing rule is seen to be independent of probability distributions as long as the agents have homogenous probability beliefs.

4.1 Some basic consequences of the theorem

Some conclusions immediately follows from this theorem: If $u_F'' < 0$, we see that $0 < I'(x) < 1$ for all x, and together with the boundary condition $I(0) = 0$, by the mean value theorem we obtain that

$$0 < I(x) < x, \qquad \text{for all} \qquad x > 0,$$

stating that *full insurance is not Pareto optimal when both parties are strictly risk averse*. We notice that the natural restriction $0 \leq I(x) \leq x$ is not binding at the optimum for any $x > 0$, once the initial condition $I(0) = 0$ is employed.

We also notice that *contracts with a deductible D can not be Pareto optimal*, since such a contract means that $I_D(x) = x - D$ for $x \geq D$, and $I_D(x) = 0$ for $x \leq D$ for $D > 0$ a positive real number. This contradicts $0 < I_D'(x)$ for all x.

Contracts with a deductible are known to be preferred by all risk averse insurance customers to all other contracts with same expected compensation $E(I_p(X))$, in a framework of pure demand theory (there is no insurer in the model). This methodology requires the probability distribution to be given.

Below we indicate that with nonlinear contracts, we can come close to Pareto contracts with deductibles in the standard model.

Furthermore, contracts with an upper cap M are not Pareto optimal either. These are defined by two constants K and M such that $I_M(x) = K$ for $x \geq M > 0$ where the constants $K \leq M$. In this case $I_M'(x) = 0$ for $x \geq M$, which violates $0 < I'(x)$ for all x.

Such contracts are known to be preferred to offer by all risk averse insurers to any other contract with the same expected compensation to the insured, in a framework of pure supply theory (there is no customer in the model).

With nonlinear contracts allowed, below we indicate how the standard model can come close to Pareto optimal contracts with a cap.

When $u_F''(y) = 0$ for all y so that the insurer is risk neutral, we notice that $I(x) = x$ for all $x \geq 0$. *Full insurance is optimal and the risk neutral part, the insurer, assumes all the risk*. Clearly, when A_F is uniformly much smaller than A, this will approximately be true even if $A_F > 0$.

Our approach does not put any constraints on the premium p. When the insurer is risk neutral, and there is *competition* among many similar insurers, then the premium is known to be actuarially fair in an idealized world.

It would have been natural to solve the risk sharing problem under the constraint that $I(x) \in [0, x]$, but this is not what has been done in the above. The problem was solved without imposing this constraint, and it turned out that it was indeed satisfied by the unconstrained, optimal solution once we chose the integration constant of the differential equation (4.2) so that $I(0) = 0$. Thus we were justified in ignoring this constraint. With insurance costs or other frictions like moral hazard, adverse selection, or non-verifiability, this constraint turns out to be binding at the optimum.

Since the approach illustrated in Example 1 and the above method are rather different, the results can not be directly compared. Let us try to clarify this by an example.

Example 2. Consider the Pareto optimal indemnity from Theorem 1 when both the insurer and the insured have the same preferences represented by CRRA utility with relative risk aversion $\gamma = 2$. The differential equation for the indemnity function I is then

$$\begin{cases} \frac{dI(x)}{dx} = \frac{w_F+p}{w_F+w-x} + \frac{I(x)}{w_F+w-x} \\ I(0) = 0. \end{cases} \tag{4.5}$$

In this case we can find the solution to the equation (4.5), and the Pareto optimal indemnity function is given by

$$I(x) = \frac{w_F + p}{w + w_F}\, x.$$

Here the optimal solution is linear, so let us compare it to the solution in Example 1 where $\mu = 1$, and $\beta = 0.60$ when $p = 1.52$. In the present case this gives the contract $I(x) = 0.58\, x$ with the same value for the premium p. However, if we set $0.6 = (w_F + p)/(w + w_F)$, this corresponds to $p = 4$. □

Quite generally, Pareto optimal sharing rules do not depend on probability distributions. The theory leading to Example 1 violates this general principle.

In the approach behind Example 1 the constrained Pareto optimal indemnity function depends on the Lagrange multiplier μ, and as this constant varies, the corresponding optimal contract vary. This is not true for the contract of Theorem 1. More importantly, the sharing rule depended on the probability distribution of the loss X, which is not the case in the present theory.

The premium in Example 1 is of a particular form depending on an actuarial premium principle, and is thus exogenous. However, prices in economic analyses ought to be determined endogenously. Let us briefly see how this can be done in the above "mini-market".

Let us instead use the economic premium principle in equation (4.1) introduced at the beginning of this section. The denominator in the premium formula, $\pi_0 := (1+r)E(\pi_1(X_M))$, which follows from the market premium principle. If, for example, $I(X) = 1$ with probability 1, then $p = 1/(1+r)$, the price of a zero-cupon bond.[3] The state price deflator $\pi_1(X_M) = X_M^{-\gamma}$, the marginal utility of the representative agent at the aggregate endowment X_M, where $X_M = (w + w_F - X)$.[4]

Using this principle, assuming for simplicity that $r = 0$, we obtain $p = 2.02$. From this we can, for example, calculate the value of α such that the premium in Example 1 corresponds to this market value, which gives that $\alpha = 0.5$. This is different from the premium of Example 1, which is 2.53 in the constrained Pareto optimal situation, and 1.67 in the pure demand theory, in both cases depending on the ad hoc choice of $\alpha = 0.1$.

The approach given in Theorem 1 is the general principle of generating Pareto optimal sharing rules. Together with the premium principle (4.1) which is in principle based on equilibrium, this constitutes a fairly complete theory of optimal risk sharing.

4.2 HARA utilities with equal slopes.

When both the insurance customer and the insurer have HARA-utility with identical cautiousness parameters, the equation (4.2) becomes an ordinary, linear, inhomogeneous differential equation of first order that can be solved by quadrature.

In this case it is convenient to express the preferences of the two parties by their risk tolerance functions instead of the absolute risk aversion functions. Since these are just reciprocals of each other, this change is straight-forward. Consider the risk tolerance $\rho(x) = \alpha + \beta x$ for the insurance customer, and $\rho_F(x) = \alpha_F + \beta_F x$ for the insurer, where α, α_F are two positive constants, and $\beta = \beta_F$ are the same constant (β can be negative, zero or positive, and should not be confused with the β in Example 1).

[3]The market price price of a zero-cupon bond is $E(\pi_1)/\pi_0$.

[4]This follows from the sup-convolution theorem applied to risk sharing with expected utility.

Expressed with risk tolerances instead of absolute risk aversion functions, the differential equation (4.2) takes the general form

$$(4.6)\qquad \begin{cases} \frac{dI(x)}{dx} = \frac{\rho_F(w_F - I(x)+p)}{\rho(w-x+I(x)-p)+\rho_F(w_F-I(x)+p)} \\ I(0) = 0, \end{cases}$$

which is a first order *nonlinear* differential equation. By using the HARA-specification with $\beta_F = \beta$, we obtain

$$(4.7)\qquad \begin{cases} \frac{dI(x)}{dx} = \frac{\alpha_F + \beta(w_F - I(x)+p)}{\alpha+\alpha_F+\beta(w+w_F)-\beta x} \\ I(0) = 0. \end{cases}$$

We then have the following solution:

Theorem 2. *The Pareto optimal, real indemnity function $I := I_p$: $R_+ \rightarrow R_+$, for HARA-utility functions with equal cautiousness parameters is linear and given by*

$$(4.8)\qquad I_p(x) = \frac{\alpha_F + \beta(w_F + p)}{\alpha + \alpha_F + \beta(w + w_F)}\, x.$$

Proof: First observe that the differential equation can be written in the standard form

$$I_p'(x) + \frac{\beta I_p(x)}{k - \beta x} = \frac{k_F}{k - \beta x}$$

where prime means derivative with respect to x, and where $k_F = \alpha_F + \beta(w_F + p)$, and $k = \alpha + \alpha_F + \beta(w + w_F)$. The general solution of this equation is given by

$$I_p(x) = e^{-\int_0^x \frac{\beta}{k-\beta y}dy}\Big(c + \int_0^x \frac{k_F}{k-\beta y} e^{\int_0^y \frac{\beta}{k-\beta z}dz} dy\Big),$$

where c is a constant. The initial condition requires $c = 0$. We obtain

$$I_p(x) = (1 - \frac{\beta}{k}x) \int_0^x \frac{k k_F}{(k-\beta y)^2} dy = \frac{k_F}{k} x$$

by elementary calculus. □

A simple, but important example of Theorem 2 was given by Example 2 above.

This is an example where there is unanimity within a syndicate about the syndicate's attitude towards risk. To explain this, consider the attitude of agent j, $j = 1, 2, \dots n$ to the syndicate's aggregate risk Z. Let us call the generic state by z. We consider n agents facing risks Z_j, where $Z = \sum_{j=1}^{n} Z_j$. Denote the optimal allocations by the function $c_j(z_1, z_2, \dots, z_n)$ for member j, where $z = \sum_{j=1}^{n} z_j$. It follows from Borch's theorem that the functions $c_j(\cdot, \cdot, \dots, \cdot)$ depend on one variable only, namely z, the aggregate risk in the syndicate; that is, $c_j(\cdot) = c_j(z)$ for all $j = 1, 2, \dots, n$. Also $\sum_{i=1}^{n} c_i(z) = z$.

Consider individual j's *implicit utility* function defined by $v_j(\cdot)$, where $v_j(z) := u_j(c_j(z))$ and $u_j(\cdot)$ is agent j's utility function.

With HARA-utility the risk tolerance function of each member is linear in risk (consumption) and given by $\rho_i(z_j) = \alpha_i + \beta_j z_j$, $j = 1, 2, \dots n$. Then we know from this theory that in a Pareto optimum the risk tolerance $\rho(z)$ of the representative agent, also referred to as the central planner or just the syndicate, is given as the sum of the individual risk tolerances after a Pareto optimal exchange has taken place: $\rho(z) = \sum_{i=1}^{n} \rho_i(c_i(z))$. This result does not depend on equal cautiousness (slope) parameters β_j.

It follows that with equal slopes, the risk tolerances of the individual members based on the implicit utility functions $v_j(z)$ are given as follows:

$$-\frac{v_j'(z)}{v_j''(z)} = \rho(z) = \alpha + \beta z, \text{ for all } j = 1, 2, \cdots, n.$$

Thus, after risk sharing each member's attitude towards their own risk is the same for every member, and equal to the pool's attitude. The task of managing the risk facing the pool can, in principal, be allocated to any one of the pool's members.

These properties hold only in the special case of linear risk tolerances with the same slope β. For groups where preferences do not satisfy this condition, it is always possible to find a risky choice problem for which some members of the pool disagree on the risk policy followed by the planner.

This is demonstrated this as follows. With equal slopes, the optimal allocations of risk are given as solutions of the following system of differential equations

$$\begin{cases} \frac{d}{dz} c_i(z) = \frac{\alpha_i + \beta c_i(z)}{\sum_{j=1}^{n} (\alpha_j + \beta c_j(z))}, \\ c_{i,0} = k_i \end{cases} \tag{4.9}$$

for $i = 1, 2, \dots, n$. This is the n agent analogue to equation (4.6). The k_i's are constants, with interpretations depending upon the problem at hand.

With these preparations, let us compute the risk tolerance of agent i's implicit utility function $v_i(z) = u_i(c_i(z))$. First we need to find

$$v_i'(z) = u_i'(c_i(z))c_i'(z) = u_i'\big(c_i(z)\big)\frac{\alpha_i + \beta c_i(z)}{\alpha + \beta z},$$

where $\alpha = \sum_{j=1}^n \alpha_j$. Next we must also calculate

$$v_i''(z) = u_i''(c_i(z))(c_i'(z))^2 + u_i'(c_i(z))\frac{d}{dz}\big(\frac{\alpha_i + \beta c_i(z)}{\alpha + \beta z}\big).$$

Here

$$\frac{d}{dz}\big(\frac{\alpha_i + \beta c_i(z)}{\alpha + \beta z}\big) = \frac{\beta c_i'(z)(\alpha + \beta z) - \beta(\alpha_i + \beta c_i(z))}{(\alpha + \beta z)^2} = 0.$$

Consequently, the risk tolerance of agent i, with reference to his implicit utility function, is given by

$$-\frac{v_i'(z)}{v_i''(z)} = -\frac{u_i'(c_i(z))c_i'(z)}{u_i''(c_i(z))(c_i'(z))^2} = \frac{\alpha_i + \beta c_i(z)}{c_i'(z)} = \frac{\alpha_i + \beta c_i(z)}{\alpha_i + \beta c_i(z)}(\alpha + \beta z) = \alpha + \beta z,$$

for $i = 1, 2, \ldots, n$, where we have used equation (4.9); regardless of the initial wealths they are all equal to the risk tolerance of the central planner.

An example of this is called for.

Example 3. Consider an insurance customer with negative exponential utility function $u(x) = -e^{-\frac{x}{\alpha}}$, and an insurer with utility function $u_F(x) = -e^{-\frac{x}{\alpha_F}}$. Here α and α_F are the respective risk tolerances, where $\beta = 0$ for both parties. So this is an example of agents with HARA-utilities with equal slope.

The Pareto optimal indemnity function $I(x) = \frac{\alpha_F}{\alpha + \alpha_F}x$ when the loss $X = x$, which follows from Theorem 2. The aggregate risk is $Z = w + w_F - X$ so that $x = w_F + w - z$ for the state variable z. Here $c(z) = w - x + I(x) - p =$ and $c_F(z) = w - I(x) + p$ as before. These must explicitly be expressed as functions of the state z. It follows that $c(z) = w - \frac{\alpha}{\alpha + \alpha_F}(w_F + w - z) - p$ and $c_F(z) = w_F - \frac{\alpha_F}{\alpha + \alpha_F}(w_F + w - z) + p$.

From this the implicit utility functions are given by

$$v(z) = u\big(c(z)\big) = -e^{-\frac{1}{\alpha}(w - \frac{\alpha}{\alpha + \alpha_F}(w_F + w - z) - p)}$$

and

$$v_F(z) = u_F\big(c_F(z)\big) = -e^{-\frac{1}{\alpha_F}(w_F - \frac{\alpha_F}{\alpha + \alpha_F}(w_F + w - z) + p)}$$

respectively. Based on these preparations it is an easy exercise to finally show that $-\frac{v'(z)}{v''(z)} = \alpha + \alpha_F$ and $-\frac{v'_F(z)}{v''_F(z)} = \alpha + \alpha_F$, in agreement with the above result. $\square$

One may, perhaps, be led to think that optimal insurance exchange does not depend on the individual initial wealths, only on the aggregate wealth, since $z = w + w_F - x$. However, we can not reexamine the risk sharing problem between the two parties based on the implicit utilities, as the solution to the original risk sharing problem has already been utilized in deriving these utilities.

The "wealth effect" is a special topic in insurance, which will not be discussed here. It suffices to mention that in the pure demand theory of insurance, a result of Mossin (1968) says that under decreasing absolute risk aversion (DARA), provided the probability distribution of the loss X does not depend on wealth w, the reservation premium of the insurance customer is a decreasing function of wealth. The realism of this result has, however, been discussed in the insurance economics literature.

5 Nonlinear indemnity functions : Kinks without frictions.

We now leave the subject of the HARA utility class with equal cautiousness parameter, and consider utility functions that lead to non-linear indemnity functions.

Despite the clear consequences of Theorem 1, that deductibles and caps are ruled out in the pure theory of risk sharing, we demonstrate that with nonlinear contracts we can come fairly close. We label such situations by "near", or "almost" kinks in the indemnity function $I(\cdot)$.

It is easy to find examples where the Pareto optimal indemnity function is close to linear for reasonable choices of the preferences of the two parties.

Here we are interested in situations where the indemnity function departs from linearity. This is, perhaps, best illustrated by some examples, and we start with a situation that results in a near cap. This example can be directly compared to the linearized version discussed in Example 1.

Example 3. We consider a situation where the customer has CRRA utility with relative risk aversion $\gamma = 2$, the insurer has logarithmic utility with relative risk aversion 1. The sure wealth $w = 50$ and the insurer's reserves are $w_F = 65$, while we here simply set the premium $p = 2.28$. The differential equation for the Pareto optimal indemnity function is given by

$$
\begin{cases} \frac{dI(x)}{dx} = \frac{\gamma(w_F - I(x)+p)}{\gamma(w_F - I(x)+p)+1(w-p-x+I(x))} \\ I(0) = 0, \end{cases} \tag{5.1}
$$

For these values of the parameters we have solved this nonlinear differential equation, and the solution is given by the following expression

$$
I(x) = x - \frac{5436501}{84100} + \frac{1193\sqrt{-168200x + 20766249}}{84100}.
$$

It's graph shown in Figure 1.

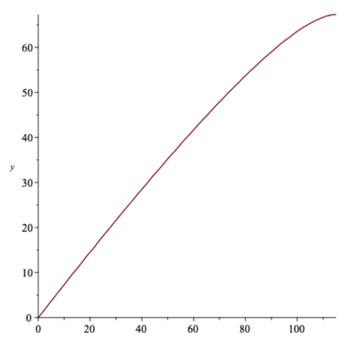

Fig. 1: The indemnity function $I(x)$ in Example 3.

The function $I(x)$ is seen to be concave in the range given, where $I''(x) < 0$ in this region.

Assuming the random loss X has the same distribution as in Example 1, the average slope is about 0.61 in its support, to be compared to the slope 0.90 in Example 1, where the preferences, endowments and the premium are the same as here. In addition, the Pareto optimal contract is non-linear.

The indemnity function has derivative at $I'(115) = 0$ for the above parameter values, which we interpret to mean that coverage beyond $I(115)$ will not be offered. The contract can accordingly be said to have a "near cap" at $x = 115$.

Here the preferences of the two parties are fairly reasonable for a risk sharing situation, where the insurer has the largest reserves and highest risk tolerance. $\square$

We can conclude that the linear approximation in Example 1 to the Pareto optimal indemnity function is quite different from the exact solution of the problem. One important difference is that the contracts illustrated in Figure 1 does not depend on probability distribution, while the linearized version in Example 1 does. We can explore the solution in any range of interest, quite independent of the particular probability distribution function of the loss X.

We know that an exogenous cap requires a non-zero deductible in Pareto optimal risk exchange (see for example Aase (2017)). The "near" cap we have here is endogenous, and not a cap in its traditional definition, so we do not have a counterexample to established theory; however, still an interesting border line situation.

If the customer is a ceding insurer, it seems natural to seek out other reinsurers for risks above the loss point of 115, with supports risk coverage also beyond this point.

Let us illustrate by another example.

Example 4. Consider a situation where the customer has relative risk aversion $\gamma = 0.01$, risk-free wealth $w = 60$ and where the premium $p = 10$. The insurer has relative risk aversion $\gamma_F = 1$ and reserves $w_F = 100$. The differential equation for the indemnity function is again given by equation (5.1), and a graph for these parameter values is illustrated if Figure 2.

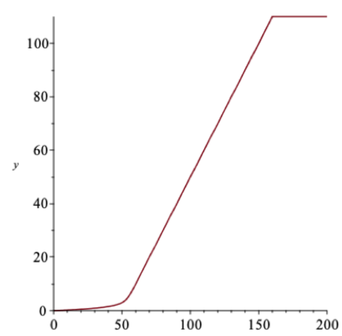

Fig. 2: The indemnity function $I(x)$ in Example 4.

It has the characteristics of a near XL-contract. Here the customer's sure wealth, the insurer's reserves as well as the premium are all reasonable, while the odd feature is that the customer is more risk tolerant than the insurer. □

We illustrate below by three examples how nonlinear contracts with a near deductible can come about, and where the parameters of the risk sharing problem is more reasonable than in the last example.

When the buyer has a low risk aversion, we see from Figure 2 that this may give a near deductible, but this is not quite satisfactory. Next a different specification is given, where preferences better correspond to the risk exchange situation at hand.

We look at a situation where the insurer is more risk tolerant in the major part of the range of the loss, but where the insurance customer is the most risk tolerant for small values of the loss.

In this situation it is convenient to specify the preferences by risk tolerance functions instead of absolute risk aversion functions. As in Theorem 2, the differential equation for $I(x)$ in Theorem 1 can alternatively be written

$$\begin{cases} \frac{dI(x)}{dx} = \frac{\alpha_F + \beta_F(w_F - I(x) + p)}{(\alpha_F + \beta_F(w_F - I(x) + p)) + (\alpha + \beta(w - p - x + I(x)))} \\ I(0) = 0. \end{cases} \tag{5.2}$$

Example 5. In this example it is again convenient to describe the preferences of the two parties by risk tolerance functions. The insurer has HARA-class utility with risk tolerance function $\rho_F(x) = \alpha_F + \beta_F x$, the customer has HARA-class utility with risk tolerance function $\rho(x) = \alpha + \beta x$. When $\beta_F \neq \beta$ we have HARA-class utilities with unequal slopes, which gives rise to a nonlinear indemnity function, here given by equation (5.2).

For the following parameters we obtain the Pareto optimal indemnity function illustrated in Figure 4. The insurer has $\beta_F = 0$, $\alpha_F = 1$. This means that the insurer has exponential utility function with risk tolerance 1. The insurance customer has sure wealth $w = 11$, and HARA utility with parameters $\alpha = 0$ and $\beta = 1$. The premium $p = 2$. This means that the customer has CRRA utility with relative risk aversion $\gamma = 1$, i.e., the customer has logarithmic utility, with risk tolerance $\rho(w - x + I(x) - p) = 9 + I(x) - x$ to be compared to the insurer's risk tolerance $\rho_F(w_F - I(x) + p) = 1$ for all x.

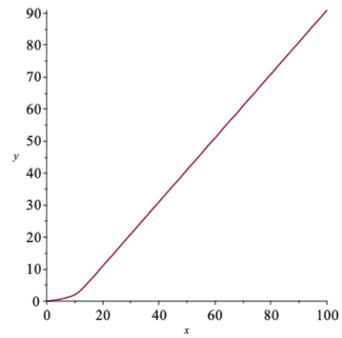

Fig. 3: The indemnity function $I(x)$ in Example 5.

Now recall that $I(x) \leq x$ for all x. When the loss $x > I(x) + 8$, then $\rho_F > \rho(w - x + I(x) - p)$ in which case the insurer is the most risk tolerant, which is the normal situation, and is satisfied in this example for larger values of the losses. This is where the indemnity function $I(x)$ in Figure 4 has the typical regular, here almost linear, form to the right in the figure. This requires $x - I(x) > 8$, which here means that $x > 12$. However, when $x \leq 12$ then $\rho_F \leq \rho(w - x + I(x) - p)$ and the customer is the most risk tolerant. This is where the near deductible occurs. Here the customer feels confident and takes most of the risk compared to the insurer, and consequently does not require much coverage, for values of losses comparable to, and smaller than, the customer's sure wealth. □

Consider next an example where the customer has exponential utility and the insurer has quadratic utility. This means that the two risk tolerances have the following forms: $\rho(x) = \alpha$ for all x, and $\rho_F(x) = \alpha_F - x$, since here $\beta_F = -1$. Since risk tolerances must both be positive, this means that $\alpha > 0$ and $x \leq \alpha_F$. The nonlinear differential equation (5.2) still applies.

Example 6. With the above two preferences, exponential utility for the customer and quadratic utility for the insurer, let $\alpha = .9$, $\beta = 0$, $\alpha_F = 1100.1$, $\beta_F = -1$, $w_F = 1000$ and $p = 100$. Here the risk tolerance of the insurer is larger than the one of the customer whenever $\rho_F(w_F - I(x) + p) = \alpha_F - (w_F + p - I(x)) \geq \alpha$. This opens up for an interesting situation also here, where the insurer may be more risk averse than the insurance customer in a limited range of x-values just above 0. This is again reasonable as long as the insurer is more risk tolerant than the customer for larger values of the loss. For these parameter values this happens when $I(x) \geq 0.8$. Using the functional form of the indemnity function, [5] this is equivalent to $x \geq 2.78$.

The nonlinear differential equation is also here given by equation (5.2). In this situation the differential equation takes the following form

$$\frac{dI(x)}{dx} = \frac{a_2 + I(x)}{a_1 + I(x)} \tag{5.3}$$

where $a_1 = -\frac{\alpha+\alpha_F}{\beta_F} - (w_F + p)$ and $a_2 = -\frac{\alpha_F}{\beta_F} - (w_F + p)$. This equation is separable, meaning that we can separate the variables as follows

$$\int_0^{I(x)} \frac{du}{\frac{a_2+u}{a_1+u}} = \int_0^x dz = x$$

and carry out the integration to provide an implicit equation for $I(x)$ as follows

$$I(x) + (a_1 - a_2)\ln(I(x) + a_2) = x + (a_2 - a_1)\ln(a_2) \tag{5.4}$$

First, notice that since $I(0) = 0$, the constant a_2 must be strictly positive for the solution I to be real-valued. As the wealth can, in principal, come close to 0, in order for the risk tolerance of the insurer to be positive, it must be the case that $\alpha_F > 0$. It follows that $\beta_F < 0$. We have chosen $\beta_F = -1$ consistent with this, which means that that the insurer has quadratic utility.

The equation (5.4) can be illustrated by implicit plotting, and the indemnity function is illustrated for the above parameter values in Figure 4.

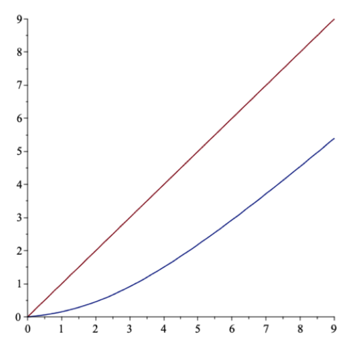

Fig. 4: The second indemnity function $I(x)$ in Example 6.

It is clearly nonlinear, and has the characteristics of a contract with a deductible around $x = 2$. The reason for this is also here that in the low

[5]Here the solution can be represented as $I(x) = \frac{9}{10}\ \mathrm{LambertW}\left(\frac{1}{9}e^{\frac{10x}{9}+\frac{1}{9}}\right) - \frac{1}{10}$.

range of losses, the customer is more risk tolerant than the insurer, while at larger values of the loss, here $x \geq 2.78$, the insurer is the most risk tolerant. Thus the customer can better carry smaller losses, and require real insurance cover only when losses are larger, as indicated. This is the same basic explanation for deductibles as in the previous example, but with different types of preferences held by the two parties. $\square$.

Example 7. Consider the following example where preferences are of the same type as in the last example above, but where $\alpha = 19.2$, $\alpha_F = 12$, $w_F = 10$ and $p = 1.8$. The risk tolerance of the insurer is $\rho_F(w_F - I(x) + p) = \alpha_F - (w_F + p - I(x)) \geq \alpha$ if and only if $x \geq x_0$. The insurer is more risk averse than the insurance customer in a limited range of x-values around $[0, 100]$. As in Example 6, this can again be reasonable as long as the insurer is more risk tolerant than the customer for larger values of the loss, which is the case in this example. For these parameter values this happens when $I(x) \geq 19$, which means that the loss $x \geq 110$. The following figure illustrates:

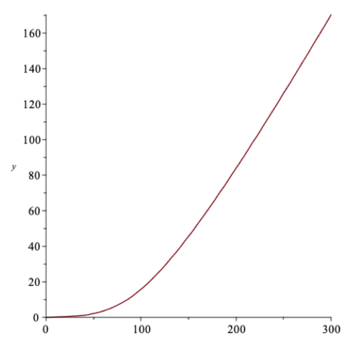

Fig. 5: The indemnity function $I(x)$ in Example 7.

Here we have a situation with a near deductible for lower values of the loss x for reasonable values of the parameters. $\square$

These examples indicate that, by moving to non-liner contracts, we do not have to go further than HARA-class utilities with unequal cautiousness parameters, to obtain some interesting features reflecting contracts occurring in real life (re)-insurance. We have demonstrated that contracts with a near deductible, meaning low coverage at low level of losses, can arise with reasonable assumptions regarding the preferences.

Contracts with near caps can also arise from mere assumptions about the preferences of the two parties. These may be a bit harder to defend, but can arise, perhaps surprisingly, without an associated near deductible, under fairly reasonable assumptions about the preferences of the two parties, and the rest of the parameters of the problem,

We also have an example of a near XL-contract, a contract with both a near deductible and a near cap. This example, giving rise to Figure 2, is fairly interesting and reasonable in terms of the sure wealth w and the reserves w_F, despite of its shortcomings when it comes to the two risk aversions. It can arise under various constellations between the insurer's and the customer's risk tolerances in the right-end of the loss distribution.

Last but not least, Example 3 clarifies some of the problems with the insurance contract described in Example 1. With the parameters of this example, the exact Pareto optimal contract is demonstrated to be nonlinear, and does not depend on the probability distribution of the loss. The premium p, a constant, must be determined in the market, after the basic risk-sharing arrangement has been established and the loss characteristics are taken into account, as we have demonstrated earlier. Only after the numerical value of p has been established, the parties decide whether the contract is to be underwritten or not.

When the utility functions of the two parties have unequal slopes, we do not have the property that there is unanimity on the management of risk followed by the representative agent. With linear risk tolerances of the same slope, it will always be possible to agree on the optimal risk sharing. In practice this means that it may be more difficult to find an insurance contract that both parties can agree upon, for the examples considered in this section.

An advantage with the above approach is that only basic assumption about the preferences of the participants are used, where no external costs or other frictions occur. The contracts are accordingly determined by the primitives of the model only.

6 Pareto optimal contracts with frictions.

Deductibles have been internalized by introducing costs, as mentioned. In the previous section we have indicated that mere risk-sharing, without frictions, can lead to contracts of this basic type. Indemnity functions capped after a certain level of the loss have also been studied by introducing frictions as well, but as noticed above, the pure theory of risk sharing can also lead to such contracts, by a variety of different specifications of the preferences and the other parameters of the model, i.e., by looking at the primitives of the model only.

When we solved the Pareto optimal risk sharing problem in Theorem 1, it turned out to be no need to impose the constraint $I(x) \in [0, x]$, since the optimal contract turned out to satisfy this natural constraint once the initial condition $I(0) = 0$ was imposed.

With insurance costs or other frictions like moral hazard, adverse selection, etc. this constraint turns out to be binding at the optimum. With ex-ante costs, for example, this is shown to justify deductibles, but not policies with an upper limit (i.e., $I_B(x) = x$ if $x \leq B$; $I_B(x)$ is an increasing function for $x > B$), as shown by Raviv (1979). Deductibles also follow with quasi-fixed costs, i.e., when a cost is incurred each time a claim is made, see Aase (2017). Also Spaeter and Roger (1997) can be consulted for a general treatment of deductibles, as well as Gollier (1978). Holmstrøm (1979) adds moral hazard, caused by asymmetric information, to the list.

When it comes to contracts with an upper cap M and a deductible D, this has not been readily analysed in the present framework, although such contracts emerge quite natural in a reinsurance setting, without really requiring any theory at all. In the above, however, we have made some progress in this direction by introducing nonlinear contracting. With an exogenous upper policy cap M, Cummins and Mahul (2004) shows that a non-trivial deductible $D > 0$ appears, and Aase (2017) generalizes this and shows that there is no upper limit B (in the above definition of upper limit) in this situation as well. With non-observability, meaning that a realized claim can be observed by both parties, but can not be proven in court, Doherty et.al. (2005) find that the optimal indemnity function contains an endogenous upper cap M as well as a strictly positive deductible D.

Huberman, Mayers and Smith (1983) show that an upper limit (here meaning a cap) can be optimal when policyholders are protected by limited liability, for example by third party insurance, provided the indemnity function is nondecreasing. Moral hazard can also generate a cap if the option to rebuild has a positive value (Garratt and Marshall (1996)). This may also occur if audit costs can be manipulated and auditors are infinitely risk averse, see Picard (2000).

In this last section we take a look at the non-verifiability condition of Doherty et.al. (2005), where we introduce a few, but we think important modifications to their approach. One is that the insurers are risk averse, which imply that there will be an equilibrium risk premium to compensate for risk bearing. As with Doherty et. al., the analysis requires the insurers to be competitive in order to obtain equality in the constraint generating the cap. In the conventional model, when insurers are risk neutral, they compete away all rents and offer insurance at an actuarially fair premium, i.e, p is given by the expected indemnity, $p = E(I(X))$. This is not the case with risk averse insurers, competition will still leave a strictly positive risk premium, which turns out to be instrumental in obtaining an incentive compatible coverage schedule, which eventually leads to the XL-insurance indemnity function.

Their paper contains many extensions and interesting theory, and should be consulted by the interested reader. Here we will just give a brief reconstruction of the part that induces an XL-contract also under our assumptions.

6.1 Non-verifiability and XL-contracts.

When losses are not verifiable, competitive insurers can deviate form the promised claims payment. It must be in the insurers self-interest to make the promised payment, which means that the coverage schedule has to be incentive compatible. An insurer's optimal strategy in a one-period game is to not pay any compensations to its customers, and the latter, realizing this, will not purchase any insurance. The only Nash equilibrium of the one-period stage game is accordingly no insurance.

Therefore the need to consider a multi-period model. In such a setting the insurer would loose future business if he shirks, under the premise that customers can switch to other insurers without extra costs. Given this strategy, it is optimal for the insurer to pay the indemnity provided it does not exceed the present value of the insurer's profit from future business. Suppose the claims $x \in [0, B]$ where B is a finite positive constant. In a dynamic, equilibrium setting the market premium of any risk $I_{t+1}(X_{t+1})$ at time t is given by the linear functional of the following form

$$(6.1) \quad p_t = \frac{1}{\pi_t} E_t(I_{t+1}(X_{t+1})\pi_{t+1}) = \frac{1}{1+r_t} E_t(I_{t+1}(X_{t+1})) + \frac{1}{\pi_t} \mathrm{cov}_t(I_{t+1}(X_{t+1}), \pi_{t+1})$$

where π_t is a state price deflator (Arrow-Debreu price in units of probability) satisfying $(1+r_t)E_t(\pi_{t+1}) = \pi_t$, and r_t is the simple return at time t on a risk-free asset. In equilibrium, $\pi_t = u'_\lambda(e_t, t)$, the marginal utility of a representative agent at the aggregate endowment $e_t = w_t - Y_t$, where w_t are aggregate reserves in the market and Y_t represent aggregate claims. Notice that if a risk X_t is positively correlated with the aggregate risk Y_t, the last term in the formula for the premium p_t in equation (6.1), the risk premium will typically be non-negative, since the marginal utility function is decreasing and the indemnity function $I_t(\cdot)$ is non-decreasing. We assume there is a strictly positive risk premium in this market, compensating insurers for risk bearing.

With this in place, given that the insurance customer terminates the business relation if the insurer shrinks, a contract is incentive compatible if, for all $x \in [0, B]$, the present value of future profits from continued business is at least as large as the required claims payment:

$$I_t(x) \leq PV_t(FRP), \tag{6.2}$$

where $PV_t(FRP)$ is the present value at time t of future risk premiums. Condition (6.2) is satisfied for all indemnities if it is satisfied for the maximum indemnity $I^M = \max_{x \in [0,B]} I(x)$.

The dynamic problem generating a Pareto optimal insurance contract is then the following:

$$\sup_{(I,p)\in C} E \sum_{t=0}^{\infty} \big(u(c_u(t), t) + \lambda v(c_v(t), t)\big). \tag{6.3}$$

Here the set C summarizes the restrictions described above, the function u is the utility index of the insurance customer, v of the insurer.

Because of the additive nature of expected utility, we can solve this problem separately for each t and state $x \in \mathbb{R}$, assuming the random variable X_t has a probability density $f_t(x)$.

Assuming stationarity the problem can be formulated as follows:

$$\sup_{(I,p)\in C} E\big(u(w_u - X + I(X) - p) + \lambda v(w_v - I(X) + p)\big). \tag{6.4}$$

The utility discount factors have been included in the agent weights λ_1 and λ_2, and we continue to denote $\lambda_2/\lambda_1 = \lambda$ without loss of generality. The constraints contained in the set C are the following two:

(i) $I^M \leq PV_t(FRP)$,

(ii) $0 \leq I(x) \leq I^M, \ \forall x \in [0, B]$.

By our stationarity assumption, the constraint (i) can be rewritten. The present value of all future profits look the same from any time point t since we have an infinite horizon, and is given by $PV_t(FRP) = \big(p - E(I(X))\big)/R$ where R is a risk adjusted discount rate. With this observation Condition (i) takes the form

(i') $p \geq E(I(X)) + RI^M$.

We now have a one-period problem with two constraints, (i') and (ii). This problem is most conveniently solved by use of the maximum principle. The Hamiltonian of the problem is (see, for example, Seierstad and Sydsæter (1987))

$$\mathcal{H}(I, p; \lambda) = \big(u(w_u - X + I(X) - p) + \lambda v(w_v - I(X) + p)\big) f(x), \tag{6.5}$$

and the Lagrangian is

$$\mathcal{L}(I, p, I^M; \lambda, \mu, \xi(x), \zeta(x)) = \mathcal{H}(I, p; \lambda) + \mu(p - I(x) - RI^M) f(x) + \xi(x) I(x) + \zeta(x)(I^M - I(x)), \tag{6.6}$$

where μ, $\xi(x)$ and $\zeta(x)$ are Lagrange multipliers associated with the constraints (i') and (ii), and λ is the agent weight of Pareto optimal risk sharing.

Suppose the constraints (i') and (ii) are not binding. In this case μ, $\xi(x)$ and $\zeta(x)$ are all zero for all x, the Lagrangian is equal to the Hamiltoian and $\frac{\partial \mathcal{H}(I,\lambda)}{\partial I} = 0$ implies that

$$\frac{dI(x)}{dx} = \frac{A(w - p - x + I(x))}{A(w - p - x + I(x)) + A_F(w_F + p - I(x))}. \tag{6.7}$$

This equation for the indemnity function has the same form as the standard one, except that the premium p can not be competitive in (6.7). In principle we can not add the initial condition $I(0) = 0$ either, since then the left part of (ii) would be binding.

With competition between the insurers, but where (ii) is not binding, $\xi(x) = \zeta(x) = 0$ for all x and the first order condition $\frac{\partial \mathcal{L}(I;\lambda)}{\partial I} = 0$ is equivalent to

$$u'(w_u - x + I(x) - p) - \lambda v'(w_v - I(x) + p) = \mu. \tag{6.8}$$

This implies that $I(x)$ satisfies the following differential equation

$$\frac{dI(x)}{dx} = \frac{A(w-p-x+I(x))}{A(w-p-x+I(x)) + A_F(w_F+p-I(x))/(1+\frac{\mu}{\lambda v'})}, \tag{6.9}$$

where $v' = v'(w_F + p - I(x))$. Since $v' > 0$ and both the constants λ and μ are positive, the Pareto optimal compensation I will depend on the agent weight λ, unlike in the standard situation. Moreover, the compensation to the insured may now be larger for larger losses compared to the situation described by the equation (6.7), depending on deductibles, since I' is now larger.

Equation (6.9) holds for all x such that $0 < I(x) < I^M$, and we refer to this as the interior solution with non-verifiability. A contract with a deductible, a cap and following this differential equation in between, we refer to as an XL-contract. It follows from the differential equation (6.9) that the interior solution is strictly increasing. The effect of competition lowers the premium and may also increase the compensation to the insured. As is normally the case, competition between insurers tends to benefit the customers.

In the general case the first order conditions for Pareto optimality are given by

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial I}(I, p, I^M; \lambda, \mu, \xi(x), \zeta(x))) = (u'(w_u - x + I(x) - p) \\ - \lambda v'(w_v - I(x) + p) - \mu) f(x) + \xi(x) - \zeta(x) = 0,\end{aligned} \tag{6.10}$$

$$\frac{\partial \mathcal{L}}{\partial p}(I, p, I^M; \lambda, \mu, \xi(x), \zeta(x)) = -u'(w_u - x + I(x) - p) + \lambda v'(w_v - I(x) + p) + \mu = 0, \tag{6.11}$$

and

$$\frac{\partial \mathcal{L}}{\partial I^M}(I, p, I^M; \lambda, \mu, \xi(x), \zeta(x)) = -\mu R f(x) + \zeta(x) = 0. \tag{6.12}$$

Here I^M is endogenous as well, and the last equation is the associated necessary first order condition.

Integrating this latter relationship, we obtain

$$\int_0^B \zeta(x) dx = \mu R > 0. \tag{6.13}$$

This shows that the function $\zeta(x)$ is not identically zero in $[0, B]$.

Combining equation (6.10) with (6.11) and (6.13), we obtain

$$\int_0^B \xi(x)dx = \mu R > 0, \tag{6.14}$$

which shows that the function $\xi(x)$ is not identically zero in $[0, B]$ either.

Putting this together, since $R > 0$, condition (6.14) implies that there exists a subset of $[0, B]$ for which $\xi(x) > 0$, that is for which $I(x) = 0$. Since the interior solution $I(x)$ is strictly increasing by (6.9), this implies that this subset is of the form $[0, D]$ for some strictly positive constant D. Thus we have a contract with a deductible $D > 0$.

In the same vein, condition (6.13) implies that there exists a subset of $[0, B]$ for which $\zeta(x) > 0$, i.e., for which $I(x) = I^M$. Again, since the interior solution $I(x)$ is strictly increasing, this subset is of the form $[M, B]$ for some $M < B$. Thus the Pareto optimal contract has a cap M.

Summarizing, the final solution $I(x)$ with non-verifiability has a strictly positive deductible $D > 0$, then follows the differential equation (6.9) and thus increases until $x = M$, from which point I is constant for $x > M$, that is the Pareto optimal contract is an XL-contract.

7 Non-expected utility theory.

Non-expected utility theory is a generalization of EU. One aim of this field is to examine important classical results from a more general point of view and determine which of these are robust to departures from the EU hypothesis and which are not.

To illustrate, consider a probability distribution

$$P = (x_1, p_1; x_2, p_2; \cdots ; x_n, p_n).$$

Non-expected utility theory also follows the approach that individual preferences over such probability distributions can be represented by a preference function $V(P) = V(x_1, p_1; x_2, p_2; \cdots ; x_n, p_n)$. As with preferences over consumption bundles, the function V can be analyzed graphically by means of indifference curves, or analytically. With expected utility $V(P) = \sum_{i=1}^n u(x_i)p_i$.

In for example M. Machina (1995), it is shown that several central results in the economics of insurance are quite robust to dropping the EU hypothesis.

In other words, the EU-hypothesis may not crucial for the insurance result under scrutiny, it would be true also with more general forms of preferences over probability distributions.

As an important example of this investigation, consider the results of Pareto optimal risk sharing in an insurance syndicate. These are some of the most powerful and interesting results in all of insurance economics. Not only does this theory allow us to study effectively markets for insurance and reinsurance, they can also be applied to the problem of optimal insurance purchasing in a model of two agents, an insurer and an insurance buyer.

It turns out that these results pass the robustness test. Karl Borch originally derived the fundaments behind such results in a series of papers in the early 1960-ties using expected utility. In particular, Aase (2025b) used a two-period model with recursive utility of the scale invariant type, and demonstrated that the basic risk sharing results still remain. However, this framework opens up for other issues, like consumption substitution and optimal saving, in which case new and more interesting results can be derived than the EU-theory can provide. In a multiperiod setting, however, basic risk sharing results from the one period model do no longer hold with recursive utility. For example, the *mutuality principle* is not valid with recursive utility of the translation invariance type (see, for example, Aase (2025a)).

The theory based on expected utility theory gives a standard, against which alternative theories can be compared. Some results in this theory may be robust compared to results based on alternative preferences, but certainly not all. For example are empirical puzzles based on micro-based macro theory well known, where alternative theory provide clear improvements.

8 Conclusions.

We have analyzed optimal risk sharing between a customer and an insurer, and presented alternative explanations for the prevalence of kinks in Pareto optimal contracts, like deductibles and upper bounds as in XL-contracts. Linear indemnity functions have primarily been considered in the literature. We focused on nonlinear contracts, which can be explained on the basic of different preferences held by the parties involved. In this setting we derive Pareto optimal contracts with "near" deductibles and "near' caps, which we illustrate by examples. Lastly we considered a model based on non-verifiability where the insurer is risk-neutral. We changed to a setting where both the cedent and the reinsurer are strictly risk averse. This rationalizes both an endogenous upper cap and a deductible, retaining compensations for risk bearing.

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